DISCRETIZED ROTATION HAS INFINITELY MANY PERIODIC ORBITS

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ABSTRACT. For a fixed $\lambda \in (-2, 2)$, we study a family of discretized rotation on $\mathbb{Z}^2$ defined by

$$(x, y) \mapsto (y, -\lfloor x + \lambda y \rfloor).$$

We prove that this reversible dynamics has infinitely many periodic orbits.

1. INTRODUCTION

Space discretization of dynamical systems attracted considerable interests among researchers [9, 27, 24, 18, 6]. One motivation is to understand the distance between original dynamics and its computer simulation through discretized model. In this paper, we are interested in a discretized planar rotation. It is very simple but we know surprisingly little on this discretized system. We start with a conjecture studied by many authors, for e.g., in [20, 25, 11] and from a point of view of shift radix system in [1].

Conjecture . *For all fixed $-2 < \lambda < 2$, all integer sequences (a_n) defined by*

$$(1) \qquad\qquad 0 \le a_{n+2} + \lambda a_{n+1} + a_n < 1$$

with initial value $(a_0, a_1) \in \mathbb{Z}^2$ are periodic.

For $(x, y) = (a_n, a_{n+1})$, we have

$$(a_{n+1}, a_{n+2}) = (y, -\lfloor x + \lambda y \rfloor),$$

and it defines a map $F : (x, y) \mapsto (y, -\lfloor x + \lambda y \rfloor)$ on $\mathbb{Z}^2$. In other words, we are interested in the dynamics F on $\mathbb{Z}^2$:

$$(2) \qquad\qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \langle \lambda y \rangle \end{pmatrix},$$

where $\langle x \rangle = x - \lfloor x \rfloor$. Since the eigenvalues of the matrix are two conjugate complex numbers of modulus one, this dynamics can be regarded as a rotation having invariant confocal ellipses, acting on the lattice $\mathbb{Z}^2$. After the rotation of angle θ with $\lambda = -2 \cos \theta$, we translate by a small vector to make the image lie in $\mathbb{Z}^2$. An affine equivalent formulation using Euclidean

rotation is found in the next section. From the shape of the inequality in (1), the dynamics (2) is reversible, i.e., $F(x,y) = (y,z)$ implies $F(z,y) = (y,x)$. Thus we have $\phi F^{-1} = F\phi$ with $\phi(x,y) = (y,x)$ and F is a bijection on $\mathbb{Z}^2$. In other words, F is a composition of the two involutions $F\phi$ and ϕ.

The conjecture is supported by numerical experiments [24, 1]. It is also expected from a heuristic ground: cumulation of errors of F^n from the exact $n\theta$ rotation is expected to be small and seemingly impossible to avoid hitting the same lattice points. The cumulative error bound is discussed in [18, 26]. However this problem is notorious, and our knowledge is limited. We only know that Conjecture 1 holds for 11 values $\lambda = 0, \pm 1, (\pm 1 \pm \sqrt{5})/2, \pm\sqrt{2}, \pm\sqrt{3}$, see [2, 20, 1]. Apart from three trivial cases $0, \pm 1$, the proof is highly non trivial and uses the self-inducing structure found in the associated planar piecewise isometry when θ/π is rational and λ is quadratic. If θ/π is rational, then we can embed the problem into piecewise isometry acting on a certain higher dimensional torus (see [20, 17], and also [4, 5] for connection to digital filters). Piecewise isometries have zero entropy [12], but we know little on their periodic orbits [14]. It is noteworthy that a certain piecewise isometry generated by 7-fold rotation in the plane is governed by several self-inducing structures [19, 15, 3], but it is irrelevant to the map F. If λ is a rational number whose denominator is the power of a prime, then the dynamics is understood as the composition of p-adic rotation and symbolic shift in [10], but it seems difficult to extract information on periodic orbits through this embedding. At this stage, we are interested in giving a non trivial general statement for this dynamics. In this note, we will show

Theorem 1. *For all fixed $\lambda \in (-2, 2)$ there are infinitely many periodic orbits of the dynamics (2) on $\mathbb{Z}^2$.*

More precisely, we prove that there are infinitely many *symmetric* periodic orbits (see §3 for the definition). See Corollary 3 in §6, for a qualitative statement. Theorem 1 is new for all λ except the above 11 values, and gives another support of the conjecture. Note that the idea of observing symmetric periodic orbits dates back to G. Birkhoff, who showed the existence of infinitely many symmetric periodic orbits for the restricted three body problem [7, 8], whose dynamics is composed of two involutions.

Adding a counting technique of lattice points in number theory, we can generalize Theorem 1 to the sequences generated by:

$$-\eta \leq a_{n+2} + \lambda a_{n+1} + a_n < 1 - \eta$$

for a fixed $\eta \in \mathbb{R}$, as in §6, Theorem 2. It covers a significant class of discretized rotations, but we do not know how large is this two parameter family within the set of all invertible discretized rotations in $\mathbb{Z}^2$, up to conjugacy.

We say $p = p(x,y) > 0$ is the *period* of $(F^n(x,y))_{n\in\mathbb{Z}}$, if it is the smallest positive integer p with $F^p(x,y) = (x,y)$. If there is no such p, then $p(x,y)$

is not defined. It is remarkable that the distribution of periodic orbits drastically changes by whether θ/π is irrational or rational. We have

Lemma 1. *Let θ/π be irrational and p is a positive integer. Then there are only finitely many periodic orbits of period p.*

This fact follows from Theorem 2.1 of [24] but we give a quick proof in §5. On the other hand, if θ/π is rational, in view of the above torus embedding, it is natural to obtain infinitely many periodic orbits of period p, which fall into the same period cell. Theorem 4.3 in [1] gives a concrete example of infinite periodic orbits of period p for $\theta = (1 - 1/p)\pi$ with an odd prime p.

By Theorem 1 and Lemma 1, we know that there exist arbitrary large periods, if θ/π is irrational. We expect the same to hold for all $\lambda \neq 0, \pm 1$, but there are proofs only for the above eight quadratic cases.

2. Setting and strategy

Let $\lambda = -2\cos\theta$ where θ is a real number in $(0, \pi)$ and let $Q = \begin{pmatrix} -\sin\theta & \cos\theta \\ 0 & 1 \end{pmatrix}$.
Our transformation on $\mathbb{Z}^2 : (x, y) \mapsto (X, Y)$ is written as

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \mu \end{pmatrix}$$

with $\mu \in [0, 1)$. Since

$$Q \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} Q^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & -\lambda \end{pmatrix},$$

we view this algorithm as

$$(3) \qquad Q^{-1}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} Q^{-1} \begin{pmatrix} x \\ y \end{pmatrix} + Q^{-1}\begin{pmatrix} 0 \\ \mu \end{pmatrix}.$$

Thus it is the dynamics acting on the lattice $\mathcal{L} = \begin{pmatrix} -\csc\theta \\ 0 \end{pmatrix}\mathbb{Z} + \begin{pmatrix} \cot\theta \\ 1 \end{pmatrix}\mathbb{Z}$
written as the composition of the Euclidean rotation of angle θ followed by a small translation

$$\mathbf{v} \mapsto \mathbf{v} + \mu \begin{pmatrix} \cot\theta \\ 1 \end{pmatrix}$$

with $\mu \in [0, 1)$. Let R be a positive real number and $B(R)$ be a ball of radius R centered at the origin. Define a *trap region* $T(R)$ by

$$T(R) = \left\{ x + y\begin{pmatrix} \cot\theta \\ 1 \end{pmatrix} \;\middle|\; x \in B(R), y \in [0, 1) \right\} \setminus B(R).$$

The situation is demonstrated in Figure 1.

Now we explain the strategy of the proof. It is clear from the description of the dynamics, that every unbounded orbit starting from a point in $\mathcal{L} \cap B(R)$ must visit at least once the trap region $T(R)$. Assume that there are only finitely many periodic orbits of (3). Since we are dealing with dynamics on the lattice $\mathcal{L}$, periodicity of an orbit is equivalent to its boundedness. We
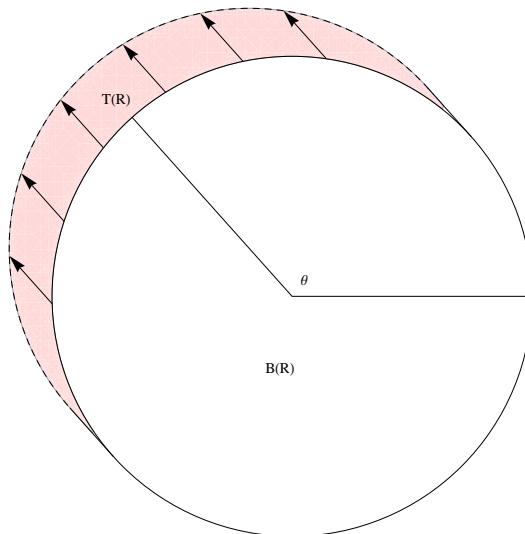
FIGURE 1. Trap Region

argue by contradiction: the assumption that there are just a finite number of periodic orbits would imply that, for $R \gg 1$, the number of lattice points in $T(R)$ are strictly less than the number of symmetric unbounded orbits starting from $B(R)$, which is impossible.

3. LOWER BOUND OF UNBOUNDED ORBITS

Symmetric periodic orbits of time-reversal dynamics had been studied for a long time. We shall make use of a well-known property which may belong to a folklore. Let (a_n) be a bi-infinite integer sequence and b be an integer. We say that (a_n) is periodic, if $a_{n+b} = a_n$ hold for all n. The sequence (a_n) is *symmetric* at $b/2$, if $a_{b-n} = a_n$ holds for all n, and it is *doubly symmetric* if it is symmetric at $b_1/2$ and $b_2/2$ with $b_1 \neq b_2$.

Lemma 2. *If a sequence is doubly symmetric then it is periodic. Moreover symmetric and periodic sequences are doubly symmetric.*

Proof. Assume that (a_n) is symmetric at b_1 and b_2 with $b_1 \neq b_2$. Then $a_{n+b_2-b_1} = a_{b_1-n} = a_n$. Let (a_n) be periodic of period b and symmetric at c. Then $a_{c-b-n} = a_{n+b} = a_n$. □

In fact, this lemma is a restatement of Theorems 1 and 2 in [13], where symmetric periodic orbits of time reversal dynamics composed of two involutions are studied. The reader finds precise description on such periodic orbits there.

Let $(x, y) \in \mathbb{Z}^2$. To the bi-infinite orbit $(F^n(x, y))_{n \in \mathbb{Z}}$ we can associate uniquely the bi-infinite sequence (a_n) consisting of the 1-st coordinates of the elements of the orbit. It is clear that $(F^n(x, y))$ is periodic if and only if

(a_n) is periodic. Hereafter we identify the orbit $(F^n(x,y))$ and the bi-infinite sequence (a_n) and say that an orbit $(F^n(x,y))$ is symmetric if (a_n) is so. Assume that (a_n) is symmetric at $b/2$. If b is odd, then $a_{(b-1)/2} = a_{(b+1)/2}$ and the orbit is of the form:

$$\ldots, c_3, c_2, c_1, X, X, c_1, c_2, c_3, \ldots$$

with some $X \in \mathbb{Z}$ and a sequence $(c_n) \subset \mathbb{Z}$. Clearly (c_n) is determined by X. Such orbits are in $\mathrm{Fix}(\phi)$, the set of orbits fixed by the involution ϕ. If b is even, then $a_{b/2-1} = a_{b/2+1}$ and the orbit is of the form

$$\ldots, c_3, c_2, c_1, X, Y, X, c_1, c_2, c_3, \ldots$$

for some $X, Y \in \mathbb{Z}$ and $(c_n) \subset \mathbb{Z}$. Of course (c_n) is determined by X and Y. These orbits belong to $\mathrm{Fix}(F\phi)$, the set fixed by the other involution $F\phi$. By Lemma 2, each periodic orbit of $(F^n(x,y))$ is doubly symmetric. It belongs to the same fix point set if the period is even, and to the different fix point sets if the period is odd.

Remark 1. Not all orbits are symmetric. For e.g., if $\lambda = (1 + \sqrt{5})/2$ then we have

$$(-1, 4) \to (4, -6) \to (-6, 5) \to (5, -3) \to (-3, -1) \to (-1, 4).$$

We do not know a way to estimate from below the number of asymmetric orbits.

By our assumption of reductio ad absurdum, there exist only a finite number, say C_1, of periodic orbits. For any R, the number of points in $\mathcal{L} \cap B(R)$ whose orbits are periodic is less than C_1.

3.1. Unbounded orbits in $\mathrm{Fix}(\phi)$.
Let (a_n) be an unbounded orbit in $\mathrm{Fix}(\phi)$. By Lemma 2, (a_n) can not be doubly symmetric, which implies that two orbits starting from different fixed points in $\mathcal{L}$ never intersect. Apart from a finite number of exceptions, points of the form

$$X \begin{pmatrix} -\csc\theta \\ 0 \end{pmatrix} + X \begin{pmatrix} \cot\theta \\ 1 \end{pmatrix} \in \mathcal{L} \cap B(R)$$

generate distinct unbounded orbits. From

$$X^2 \left(-\csc\theta + \cot\theta\right)^2 + X^2 \leq R^2,$$

we conclude that there are at least $2R\cos(\theta/2) - C_1$ unbounded orbits in $\mathrm{Fix}(\phi)$ starting from $\mathcal{L} \cap B(R)$.

3.2. Unbounded orbits in $\mathrm{Fix}(F\phi)$.
Similarly, by Lemma 2, (a_n) can not be doubly symmetric. So, by (1), our task is to count the number of the pairs (X, Y) which satisfy

$$(4) \qquad\qquad 0 \leq X + \lambda Y + X < 1$$

and

$$(5) \qquad\qquad X \begin{pmatrix} -\csc\theta \\ 0 \end{pmatrix} + Y \begin{pmatrix} \cot\theta \\ 1 \end{pmatrix} \in B(R).$$

For this computation, we substitute the inequality (4) by

$$(6) \qquad\qquad -1 \leq X + \lambda Y + X < 1$$

and count the number of pairs (X, Y) satisfying (6) and (5). It is clear that for a fixed Y, there is a unique X which satisfies (6). Since $X = Y \cos \theta + \varepsilon$ with $|\varepsilon| \leq 1/2$, we have

$$(7) \qquad\qquad (Y \cot \theta - X \csc \theta)^2 + Y^2 = Y^2 + \frac{\varepsilon^2}{\sin^2 \theta} \leq R^2$$

from (5), and we have at least $2R - C_2$ such points with a non negative constant C_2.

If λ is irrational, then there is no $(X, Y) \neq (0, 0)$ which satisfies either

$$-1 = 2X + \lambda Y \quad \text{or} \quad 0 = 2X + \lambda Y.$$

Using the symmetry $(X, Y) \leftrightarrow (-X, -Y)$, we see that the number of (X, Y) with (5) and

$$-1 \leq X + \lambda Y + X < 0$$

is exactly one less than the number of (X, Y) with (4) and (5), which counts the origin. Thus the number of (X, Y) having (4) and (5) is at least $R - C_2/2$.

If λ is rational, then we additionally have to take care of the points (X, Y) on the line $-1 = 2X + \lambda Y$ and $0 = 2X + \lambda Y$. However we can easily show that the number of (X, Y) with (5) on the line $-1 = 2X + \lambda Y$ and the one on the line $0 = 2X + \lambda Y$ differ only by some constant. Thus in any case, there are at least $R - C_3$ unbounded orbits in $\mathrm{Fix}(F\phi)$ starting from $\mathcal{L} \cap B(R)$ with a non negative constant C_3.

4. LATTICE POINTS IN THE TRAP REGION

By construction, if the trap region $T(R)$ and the line $x = y \cot \theta + c$ has non empty intersection, then it is a half-open interval of length $\csc \theta$. Thus if the line $x = y \cot \theta + c$ intersects $\mathcal{L} \cap T(R)$ then it is a single point. The lattice $\mathcal{L}$ is covered by a family of parallel lines:

$$\Xi = \{ x = y \cot \theta - k \csc \theta \mid k \in \mathbb{Z} \}.$$

We easily see that the distance between adjacent lines of Ξ is 1. Thus there are exactly $2\lfloor R \rfloor + 1$ points in $\mathcal{L} \cap T(R)$.

Remark 2. If $\theta < 2\pi/3$, then $2R \cos(\theta/2) - C_1 + R - C_3 > 2R + 1$ holds for sufficiently large R and we immediately obtain the desired contradiction.

Let us take into account the reversibility of F. Since we are dealing with unbounded symmetric orbits starting from $\mathcal{L} \cap B(R)$, if $(a_n, a_{n+1}) = (C, D)$ then there is an index m such that $(a_m, a_{m+1}) = (D, C)$. Let $\Phi : \mathcal{L} \mapsto \mathcal{L}$ be defined as follows

$$\Phi : x \begin{pmatrix} -\csc \theta \\ 0 \end{pmatrix} + y \begin{pmatrix} \cot \theta \\ 1 \end{pmatrix} \mapsto y \begin{pmatrix} -\csc \theta \\ 0 \end{pmatrix} + x \begin{pmatrix} \cot \theta \\ 1 \end{pmatrix}.$$

If an orbit visits $\Phi(T(R)) \cap T(R)$ then the number of visits is at least two. In other words, we only have to count the number of lattice points up to this symmetry by Φ in $T(R)$.

The mapping Φ is the reflection with respect to the vector $\begin{pmatrix} -\csc\theta + \cot\theta \\ 1 \end{pmatrix}$, because the two vectors $\begin{pmatrix} -\csc\theta \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \cot\theta \\ 1 \end{pmatrix}$ have the same length. Thus the reflection Φ leaves the vector $\begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} = \begin{pmatrix} -\csc\theta + \cot\theta \\ 1 \end{pmatrix} \cos(\theta/2)$ invariant.

To make computation easy, we rotate $T(R)$ and $\mathcal{L}$ by $-\theta$ and present the situation in Figure 2. $T(R)'$, $\mathcal{L}'$ are the images by this rotation and Ψ is
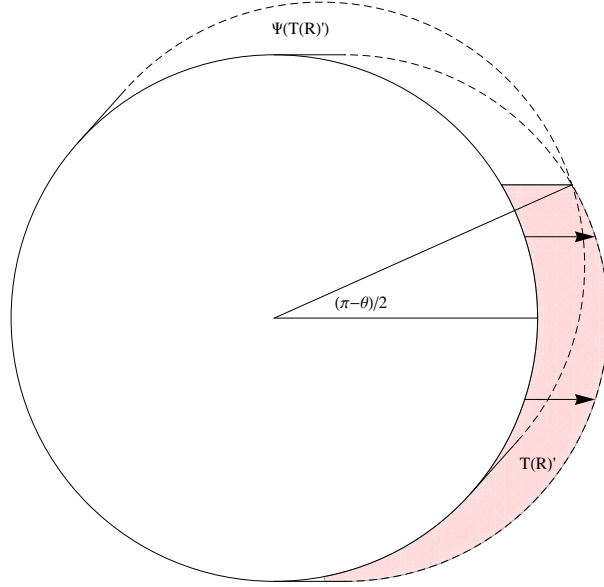

FIGURE 2. Symmetry of the trap region

the corresponding reflection. Then every line $y = k$ with $k \in \mathbb{Z} \cap [-R, R]$ contains a single point in $\mathcal{L}' \cap T(R)'$ and the reflection Ψ leaves the vector

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} = \begin{pmatrix} \sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$$

invariant.

So our task is to estimate from above the number of lattice points in $\mathcal{L}' \cap T(R)'$ which are below the line $y = x \cot(\theta/2)$. The intersection of the line and the boundary of $T(R)'$ with the largest y-coordinate is

$$\left(\sqrt{R^2 \sin^2\left(\frac{\theta}{2}\right) - \frac{1}{4}} + \frac{1}{2}\tan\frac{\theta}{2}, \cot\frac{\theta}{2}\sqrt{R^2\sin^2\left(\frac{\theta}{2}\right) - \frac{1}{4}} + \frac{1}{2} \right)$$

and we have

$$\cot \frac{\theta}{2} \sqrt{R^2 \sin^2 \left(\frac{\theta}{2} \right) - \frac{1}{4}} + \frac{1}{2} = R \cos \left(\frac{\theta}{2} \right) + \frac{1}{2} + O \left(\frac{1}{R} \right).$$

We count the number of points whose y-coordinate does not exceed this value, i.e., the points in the shaded part in Figure 2. Thus the number of lattice points up to symmetry in $\mathcal{L} \cap T(R)$ is bounded from above by $R + R \cos(\theta/2) + C_4$ with a non negative constant C_4.

5. Proof of Theorem 1 and Lemma 1

From the assumption that there are only finitely many periodic orbits, we derived several estimates in the previous sections. By Lemma 2, unbounded orbits in $\mathrm{Fix}(\phi)$ and those in $\mathrm{Fix}(F\phi)$ have no intersection. Thus

$$2R \cos(\theta/2) - C_1 + R - C_3$$

distinct unbounded orbits must visit $T(R)$ and there are only

$$R + R \cos(\theta/2) + C_4$$

lattice points in $\mathcal{L} \cap T(R)$ up to symmetry. However

$$2R \cos(\theta/2) - C_1 + R - C_3 \leq R + R \cos(\theta/2) + C_4$$

does not hold for sufficiently large R. The proof of Theorem 1 is finished.

Let us show Lemma 1. Assume that there are infinitely many (x, y) that $F^p(x, y) = (x, y)$ with $p > 2$. By induction using (3), we have

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos p\theta & -\sin p\theta \\ \sin p\theta & \cos p\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \sum_{i=1}^{p} \mathbf{v}_i$$

where $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{L}$ and $\|\mathbf{v}_i\| \leq \csc \theta$. Here $\|\cdot\|$ is the Euclidean norm. However if $\begin{pmatrix} u \\ v \end{pmatrix}$ is sufficiently large, then

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \cos p\theta & -\sin p\theta \\ \sin p\theta & \cos p\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\| \geq c \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| > p \csc \theta.$$

Here c is computed as the operator norm:

$$c = \frac{1}{\left\| \begin{pmatrix} 1 - \cos p\theta & \sin p\theta \\ -\sin p\theta & 1 - \cos p\theta \end{pmatrix}^{-1} \right\|} = 2 \left| \sin \left(\frac{p\theta}{2} \right) \right|$$

which is positive since θ/π is irrational. This gives a contradiction.

6. Generalization

One can generalize the result to the sequences defined by:

$$-\eta \le a_{n+2} + \lambda a_{n+1} + a_n < 1 - \eta,$$

with $\eta \in \mathbb{R}$. This kind of interval shifts are studied, for e.g., in [17, 21]. In complete analogy to our main result, we have

Theorem 2. *For a fixed $\lambda \in (-2, 2)$ and $\eta \in \mathbb{R}$, there are infinitely many periodic orbits of the dynamics*

$$(8) \qquad (x, y) \to (y, -\lfloor \lambda y + x + \eta \rfloor)$$

on $\mathbb{Z}^2$.

We can deduce a qualitative statement:

Corollary 3. *There is a positive constant C depending on λ and η such that within $B(R)$, there are at least CR periodic orbits of (8).*

Hereafter we sketch the proof of Theorem 2 and Corollary 3. Putting $\kappa = \eta/(2 + \lambda)$, the inequality becomes

$$0 \le (a_{n+2} + \kappa) + \lambda(a_{n+1} + \kappa) + (a_n + \kappa) < 1.$$

Therefore by substituting $\mathcal{L}$ with $\mathcal{L}' = \mathcal{L} + Q^{-1} \begin{pmatrix} \kappa \\ \kappa \end{pmatrix}$, our algorithm has exactly the same shape as (3). Though the error term becomes worse than the one in §4, we can show that the number of lattice points of $\mathcal{L}'$ up to symmetry within the trap region is

$$R + R\cos(\theta/2) + O(R^{2/3+\epsilon})$$

for any positive constant ϵ. Here we used the method of Vinogradov to count the number of lattice points in the cylindrical region bounded by curves of positive curvature, for e.g., see p.8-22 of [23] or [22, 16].

Similarly to §3, there are $2R\cos(\theta/2) - C_1$ unbounded orbits in $\text{Fix}(\phi)$. We count unbounded orbits in $\text{Fix}(F\phi)$, i.e., the number of (X, Y), which satisfy:

$$(9) \qquad \lambda Y/2 \bmod 1 \cap [-\eta/2, (1-\eta)/2) \ne \emptyset$$

and (7).

If λ is irrational, then $(\lambda/2)Y \bmod 1$ is uniformly distributed and the number of such Y's is $R + o(R)$. When λ is rational, put $\lambda/2 = p/q$ with $(p, q) = 1$. Then $(\lambda/2)Y \equiv i/q \bmod 1$ for $i \in \{0, 1, \ldots, q-1\}$ with the same frequency $1/q$.

Let us study the case that q is even. Since

$$\{i/q \bmod 1\} \cap [-\eta/2, (1-\eta)/2)$$

has cardinality $q/2$, the number of points with (9) and (7) is again $R + o(R)$. Once we have this estimate $R + o(R)$ then

$$2R\cos(\theta/2) - C_1 + R + o(R) \le R + R\cos(\theta/2) + o(R^{2/3+\epsilon})$$

does not hold for sufficiently large R. Here C_1 appears only once in the left side, because it is the number of periodic orbits of the system. We obtain the contradiction for $R \gg 1$.

It remains to show the case when q is odd. Then

$$\{i/q \bmod 1\} \cap [-\eta/2, (1-\eta)/2)$$

has cardinality either $(q-1)/2$ or $(q+1)/2$ depending on η. Thus the number of unbounded orbits in $\mathrm{Fix}(F\phi)$ is bounded from below by $R - R/q + o(R)$. Thus we have to show that

$$2R\cos(\theta/2) - C_1 + R - R/q + o(R) > R + R\cos(\theta/2) + o(R^{2/3+\epsilon})$$

for large R. This is valid because

$$\cos(\theta/2) > 1/q$$

holds for $q > 2$, since $\cos(\theta/2) = \sqrt{(1+\cos(\theta))/2} = \sqrt{(1-p/q)/2}$.

The validity of statement of Corollary 3 is invariant under affine transformations fixing the origin, up to appropriate changes the constant C. To prove Corollary 3, note that we may take

$$C_1 = (1-\varepsilon)R\left(\cos(\theta/2) - 1/q\right)$$

with a small $\varepsilon > 0$ in the above proof to get the same contradiction. Here the term $-1/q$ is necessary only in the last case of the proof.

7. Acknowledgments

References

[1] S. Akiyama, H. Brunotte, A. Pethő, and W. Steiner, *Remarks on a conjecture on certain integer sequences*, Periodica Math. Hungarica **52** (2006), 1–17.

[2] ———, *Periodicity of certain piecewise affine planar maps*, Tsukuba J. Math. **32** (2008), no. 1, 1–55.

[3] S. Akiyama and E. Harriss, *Pentagonal domain exchange*, submitted.

[4] P. Ashwin, *Elliptic behaviour in the sawtooth standard map*, Phys. Lett. **A 232** (1997), 409–416.

[5] P. Ashwin, W. Chambers, and G. Petrov, *Lossless digital filters overflow oscillations: approximations of invariant fractals*, Inter. J. Bifur. Chaos **7** (1997), 2603–2610.

[6] V. Berthé and B. Nouvel, *Discrete rotations and symbolic dynamics*, Theoret. Comput. Sci. **380** (2007), no. 3, 276–285.

[7] G. D. Birkhoff, *The restricted problem of three bodies*, Rendiconti del Circolo Matematico di Palermo, **39** (1915) no. 1, 265–334.

[8] G. D. Birkhoff, *Dynamical systems*, With an addendum by Jurgen Moser. American Mathematical Society Colloquium Publications, Vol. IX, American Mathematical Society, Providence, R.I., 1966.

[9] M. Blank, *Pathologies generated by round-off in dynamical systems*, Phys. D **78** (1994), no. 1-2, 93–114.

[10] D. Bosio and F. Vivaldi, *Round-off errors and p−adic numbers*, Nonlinearity **13** (2000), 309–322.

[11] H. Bruin, A. Lambert, G. Poggiaspalla, and S. Vaienti, *Numerical analysis for a discontinuous rotation of the torus*, Chaos **13** (2003), no. 2, 558–571.

[12] J. Buzzi, *Piecewise isometries have zero topological entropy*, Ergodic Theory Dynam. Systems **21** (2001), no. 5, 1371–1377.

[13] R. DeVogelaere, *On the structure of symmetric periodic solutions of conservative systems, with applications*, Contributions to the theory of nonlinear oscillations, Vol. IV, Annals of Mathematics Studies, no 41, Princeton University Press, Princeton, N.J., 1958, pp. 53–84.

[14] A. Goetz, *Piecewise isometries - An emerging area of dynamical systems*, Fractals in Graz 2001 (P. Grabner and W. Woess, eds.), Birkhäuser, 2002, pp. 135–144.

[15] A. Goetz and G. Poggiaspalla, *Rotations by $\pi/7$*, Nonlinearity **17** (2004), no. 5, 1787–1802.

[16] M. N. Huxley, *The area within a curve*, Proc. Indian Acad. Sci. Math. Sci. **97** (1987), no. 1-3, 111–116 (1988).

[17] K. Koupstov, J. H. Lowenstein, and F. Vivaldi, *Quadratic rational rotations of the torus and dual lattice maps*, Nonlinearity **15** (2002), 1795–1842.

[18] V. Kozyakin, N. Kuznetsov, A. Pokrovskii, and I. Vladimirov, *Some problems in analysis of discretizations of continuous dynamical systems*, Proceedings of the Second World Congress of Nonlinear Analysts, Part 2 (Athens, 1996), vol. 30, 1997, pp. 767–778.

[19] J. H. Lowenstein, K. L. Koupstov, and F. Vivaldi, *Recursive tiling and geometry of piecewise rotations by $\pi/7$*, Nonlinearity **17** (2004), 371–395.

[20] J.H. Lowenstein, S. Hatjispyros, and F. Vivaldi, *Quasi-periodicity, global stability and scaling in a model of hamiltonian round-off*, Chaos **7** (1997), 49–56.

[21] P. Surer, *ϵ-shift radix systems and radix representations with shifted digit sets*, Publ. Math. Debrecen **74** (2009), no. 1-2, 19–43.

[22] I. M. Vinogradov, *Elements of number theory*, Dover Publications Inc., New York, 1954, Translated by S. Kravetz.

[23] _____ , *Selected works*, Springer-Verlag, Berlin, 1985.

[24] F. Vivaldi, *Periodicity and transport from round-off errors*, Experiment. Math. **3** (1994), no. 4, 303–315.

[25] _____ , *The arithmetic of discretized rotations*, p-adic mathematical physics, AIP Conf. Proc., vol. 826, Amer. Inst. Phys., Melville, NY, 2006, pp. 162–173.

[26] F. Vivaldi and I. Vladimirov, *Pseudo-randomness of round-off errors in discretized linear maps on the plane*, Int. J. of Bifurcations and Chaos **13** (2003), 3373–3393.

[27] I. Vladimirov, *Quantized linear systems on integer lattice: frequency-based approach*, (1996), 1–37, Centre for Applied Dynamical Systems and Environmental Modeling.

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