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THE STRUCTURE OF THE ASSUMED MODEL THROUGH THE DISCRETIZED LIKELIHOOD ESTIMATOR

Dedicated to Professor Tokitake Kusama on his 60th birthday

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In the presence of a nuisance parameter the asymptotic deficiency of the discretized likelihood estimator (DLE) relative to the bias-adjusted maximum likelihood estimator is obtained under the assumed model. It consists of two parts. One is the loss of information associated with the DLE of the parameter to be estimated. Another, is that due to the "incorrectness" of the assumed model. Some examples on the normal and Weibull type distributions are given.

Key words: Nuisance parameter, Asymptotic deficiency, Discretized likelihood estimator, Maximum likelihood estimator, Assumed model, Loss of information.

1. Introduction

A previous paper by Akahira (1989) showed that in the presence of a nuisance parameter, the jackknife estimator has an asymptotic deficiency of zero relative to the bias-adjusted maximum likelihood estimator (MLE) under true and assumed models. This means that the estimators are asymptotically equivalent up to the third order in the sense that their asymptotic distributions are equal up to the order n^{-1} under the models. The asymptotic deficiency of the MLE or the jackknife estimator under the assumed model relative to that under the true model is also given.

It is shown that there does not exist a uniformly third order asymptotically efficient estimator in some class C. That is, the bound for third order asymptotic distributions of the all estimators in the class C is not uniformly attained (e.g. see Akahira (1986)). In the one parameter case Akahira and Takeuchi (1979, 1981) showed that, for any fixed point, the discretized likelihood estimator (DLE) is third order asymptotically efficient at the point in the class C. The second order asymptotic comparison of the DLE with asymptotically efficient estimators was also made by Akahira (1990) in the double exponential case. In the presence of a nuisance parameter it is useful to consider the DLE under the assumed model. It is interesting to clarify a structure of the assumed model by comparison of the DLE and the MLE through the concept of asymptotic deficiency.

The useful results on the asymptotic deficiency are summarized as follows (see Akahira (1986) for details). Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed random variables with a density $f(x, \theta)$, where θ is a real valued parameter. Under suitable regularity conditions, it can be proved that the maximum likelihood estimator (MLE) $\hat{\theta}_n^*$ of θ is asymptotically expanded into the form

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$$\sqrt{n}\,(\hat{\theta}_n^*-\theta)=\frac{Z_1}{I}+\frac{1}{\sqrt{n}}Q_0+o_p\left(\frac{1}{\sqrt{n}}\right),$$

and that for any asymptotically efficient estimator $\hat{\theta}_n$ which admits the same type of expansion

$$\sqrt{n}\,(\hat{\theta}_n-\theta)=\frac{Z_1}{I}+\frac{1}{\sqrt{n}}Q+o_p\left(\frac{1}{\sqrt{n}}\right)$$

we have $V_\theta(Q)\geq V_\theta(Q_0)$, where $Q_0=O_p(1)$, $Q=O_p(1)$,

$$Z_1=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f(X_i,\theta),$$

and

$$I=E_\theta\left[\left\{\frac{\partial}{\partial\theta}\log f(X,\theta)\right\}^2\right].$$

This implies that one can construct a "bias-adjusted" estimator (relative to $\hat{\theta}_n$) of the form

$$\hat{\theta}_n^{**}=\hat{\theta}_n^*+\frac{1}{n}h(\hat{\theta}_n^*)$$

so that $\hat{\theta}_n^{**}$ has the same asymptotic bias as $\hat{\theta}_n$ up to the order n^{-1} and

$$P_{\theta,n}\{\sqrt{n}\,|\hat{\theta}_n^{**}-\theta|\leq t\}\geq P_{\theta,n}\{\sqrt{n}\,|\hat{\theta}_n-\theta|\leq t\}+o\left(\frac{1}{n}\right)$$

for all $t>0$ and all θ. Moreover, if m_n is defined so that the bias-adjusted MLE $\hat{\theta}_{m_n}^{**}$ (with Q_0^*) based on the sample of size m_n has the same asymptotic distribution as that of the estimator $\hat{\theta}_n$ (with Q) based on the sample size n up to the order n^{-1}, we have

$$\lim_{n\to\infty}(n-m_n)=I\{V_\theta(Q)-V_\theta(Q_0^*)\}. \tag{1.1}$$

The left-hand side of (1.1) is called the asymptotic deficiency of $\hat{\theta}_n$ relative to $\hat{\theta}_n^{**}$ (see Hodges and Lehmann, 1970, and Akahira, 1986). It is noted that the right-hand side of (1.1) does not necessarily mean the difference of the variances of the estimators, but of the asymptotic variances. Hence we do not need to bother about the remaining terms. The above results can be extended to the presence of nuisance parameters.

In this paper, in the presence of a nuisance parameter, the asymptotic deficiency of the DLE relative to the bias-adjusted MLE is obtained under the assumed model. It consists of the losses of information on the parameter to be estimated and due to the "incorrectness" of the assumed model. Some examples on the normal and Weibull type distributions are given.

2. Notations and assumptions

Suppose that $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) real random variables with a density function $f(x, \theta, \xi)$ with respect to a σ-finite measure μ, where θ is a real-valued parameter to be estimated and ξ is a real-valued nuisance parameter. We assume the following conditions (A.1) to (A.5).

(A.1) The set $\{x: f(x, \theta, \xi)>0\}$ does not depend on θ and ξ.

(A.2) For almost all $x[\mu]$, $f(x, \theta, \xi)$ is three times continuously differentiable in θ and ξ.

(A.3) For each θ and each ξ

$$0<I_{00}(\theta,\xi)=E[\{l_0(\theta,\xi,X)\}^2]=-E[l_{00}(\theta,\xi,X)]<\infty\ ,$$
$$0<I_{11}(\theta,\xi)=E[\{l_1(\theta,\xi,X)\}^2]=-E[l_{11}(\theta,\xi,X)]<\infty\ ,$$

where $l_0(\theta,\xi,x)=(\partial/\partial\theta)l(\theta,\xi,x)$, $l_{00}(\theta,\xi,x)=(\partial^2/\partial\theta^2)l(\theta,\xi,x)$, $l_1(\theta,\xi,x)=(\partial/\partial\xi)l(\theta,\xi,x)$ and $l_{11}(\theta,\xi,x)=(\partial^2/\partial\xi^2)l(\theta,\xi,x)$ with $l(\theta,\xi,x)=\log f(x,\theta,\xi)$.

(A.4) The parameters are defined to be "orthogonal" in the sense that

$$E[l_{01}(\theta,\xi,X)]=0$$

where $l_{01}(\theta,\xi,x)=(\partial^2/\partial\theta\partial\xi)l(\theta,\xi,x)$.

Note that the condition (A.4) is not necessarily restricted, because otherwise we can redefine the parameter $\eta=g(\theta,\xi)$ so that we have the above orthogonality.

(A.5) There exist

$$J_{000}=E[l_{00}(\theta,\xi,X)l_0(\theta,\xi,X)],\qquad J_{001}=E[l_{00}(\theta,\xi,X)l_1(\theta,\xi,X)],$$
$$J_{010}=E[l_{01}(\theta,\xi,X)l_0(\theta,\xi,X)],\qquad J_{011}=E[l_{01}(\theta,\xi,X)l_1(\theta,\xi,X)],$$
$$J_{110}=E[l_{11}(\theta,\xi,X)l_0(\theta,\xi,X)],\qquad K_{000}=E[\{l_0(\theta,\xi,X)\}^3],$$
$$K_{001}=E[\{l_0(\theta,\xi,X)\}^2l_1(\theta,\xi,X)],$$
$$M_{0000}=E[\{l_{00}(\theta,\xi,X)\}^2]-I_{00}^2,$$
$$M_{0001}=E[l_{00}(\theta,\xi,X)l_{01}(\theta,\xi,X)],$$
$$M_{0101}=E[\{l_{01}(\theta,\xi,X)\}^2],$$

and the following holds.

$$E[l_{000}(\theta,\xi,X)]=-3J_{000}-K_{000},\qquad E[l_{001}(\theta,\xi,X)]=-J_{010},$$
$$E[l_{011}(\theta,\xi,X)]=-J_{011},$$

where $l_{000}(\theta,\xi,x)=(\partial^3/\partial\theta^3)l(\theta,\xi,x)$, $l_{001}(\theta,\xi,x)=(\partial^3/\partial\theta^2\partial\xi)l(\theta,\xi,x)$ and $l_{011}(\theta,\xi,x)=(\partial^3/\partial\theta\partial\xi^2)l(\theta,\xi,x)$.

From the condition (A.5) it is noted that $K_{001}=-J_{010}-J_{001}$. We put

$$Z_0=(1/\sqrt{n})\sum_{i=1}^{n}l_0(\theta,\xi,X_i)\ ,\qquad Z_1=(1/\sqrt{n})\sum_{i=1}^{n}l_1(\theta,\xi,X_i)\ ,$$
$$Z_{00}=(1/\sqrt{n})\sum_{i=1}^{n}\{l_{00}(\theta,\xi,X_i)+I_{00}\}\ ,\qquad Z_{01}=(1/\sqrt{n})\sum_{i=1}^{n}l_{01}(\theta,\xi,X_i)\ .$$

3. The discretized likelihood estimator under the assumed model

In Akahira (1989) the asymptotic deficiency of the jackknife estimator relative to the bias-adjusted maximum likelihood estimator under the true model where $\theta=\theta_0$ and $\xi=\xi_0$ and the assumed model where $\theta=\theta_0$ and $\xi=0$ was obtained. In this section we calculate the asymptotic deficiency of the discretized likelihood estimator relative to the maximum likelihood estimator. Henceforth, for simplicity we denote by (θ,ξ) and $(\theta,0)$ the true model (θ_0,ξ_0) and the assumed model $(\theta_0,0)$ omitting subscript 0, respectively. We assume $\xi=t/\sqrt{n}$ under the true model (θ,ξ).

Let $\hat{\theta}_{ML}$ be the maximum likelihood estimator (MLE) of θ based on a sample $X_1, \ldots, X_n$ of size n under the assumed model $(\theta, 0)$. Then we have the following.

THEOREM 3.1. *Assume that the conditions* (A.1) *to* (A.5) *hold. Then, under the assumed model* $(\theta, 0)$, *the MLE* $\hat{\theta}_{ML}$ *of* θ *has the following stochastic expansion.*

$$\sqrt{n}(\hat{\theta}_{ML}-\theta)=\frac{Z_0}{I_{00}}+\frac{1}{\sqrt{n}}Q_0+\frac{1}{\sqrt{n}}(L-c)+\frac{1}{n}R_0+o_p\left(\frac{1}{n}\right),$$

where

$$Q_0=\frac{1}{I_{00}^2}\left(Z_{00}Z_0-\frac{3J_{000}+K_{000}}{2I_{00}}Z_0^2\right),$$

$$L=\frac{t}{I_{00}}\left(\frac{J_{010}}{I_{00}}Z_0-Z_{01}\right), \qquad c=\frac{J_{011}}{2I_{00}}t^2,$$

and $O_p(R_0)=1$, *and* $o_p(\cdot)$ *is taken under the distribution* $P_{\theta,\xi}$ *with the density* $f(x,\theta,\xi)$.

The proof is omitted since the theorem and its proof are given as Theorem 4.1 in Akahira (1989).

In Akahira and Takeuchi (1979, 1981) and Akahira (1986), the discretized likelihood estimator was defined. In a similar way to those we define it as follows: A $\sqrt{n}$-consistent estimator $\hat{\theta}_n=\hat{\theta}_n(X_1, \ldots, X_n)$ based on a sample $(X_1, \ldots, X_n)$ of size n is called a discretized likelihood estimator (DLE) of θ if, for each real number r, $\hat{\theta}_n^*$ satisfies the discretized likelihood equation

$$\sum_{i=1}^{n} l(\hat{\theta}_n^*+rn^{-1/2}, 0, X_i)-\sum_{i=1}^{n} l(\hat{\theta}_n^*, 0, X_i)=a_n(\hat{\theta}_n^*, 0, r),$$

where $a_n(\theta,\xi,r)$ is a function of θ and ξ, and also dependent on n and r. The function $a_n(\theta,\xi,r)$ is not now defined but will be determined in the sequel so that the solution obtained in the above equation will be k-th order asymptotically unbiased, i.e., $E_{\theta,\xi}(\hat{\theta}_n^*)=\theta+o(n^{-k/2})$. A concrete construction of such a function $a_n(\theta,\xi,r)$ will be given in the proof of THEOREM 3.2. Note that the DLE depends on r. In the one parameter case it is shown in Akahira (1986) that the DLE has an asymptotically best property at the point depending on r in the sense that its asymptotic distribution attains the bound for asymptotic distributions of estimators in some class at the point up to the order $o(1/n)$. A related result will be given from the viewpoint of the concept of asymptotic deficiency in Remark 3.2. We further assume the following condition.

(A.6) For given function $a_n(\theta,\xi,r)$

$$\sum_{i=1}^{n} l(\theta+rn^{-1/2}, 0, X_i)-\sum_{i=1}^{n} l(\theta, 0, X_i)-a(\theta, 0, r)$$

is locally monotone in θ with probability larger than $1-o(n^{-1})$.

The motivation for the definition of the DLE is the following. When we test the hypothesis $\theta=\theta_0+rn^{-1/2}$, $\xi=0$ against $\theta=\theta_0$, $\xi=0$, the most powerful test is given by rejecting the hypothesis if

$$\sum_{i=1}^{n} l(\theta_0+rn^{-1/2}, 0, X_i)-\sum_{i=1}^{n} l(\theta_0, 0, X_i)<k_n,$$

for some constant k_n. Hence if an estimator $\hat{\theta}^r$ is defined so that the event $\hat{\theta}^r>\theta_0$ is equivalent to the above inequality (at least asymptotically up to some order), then $\hat{\theta}^r$ is effici-

ent (asymptotically up to some order) for a specified choice of r.

Let C' be the class of the all bias-adjusted best asymptotically normal estimators $\hat{\theta}_n$ which are second order asymptotically unbiased and asymptotically expanded as

$$\sqrt{n}(\hat{\theta}_n-\theta)=\frac{Z_0}{I_{00}}+\frac{1}{\sqrt{n}}Q+\frac{1}{n}R+o_p\left(\frac{1}{n}\right),$$

where $Q=O_p(1)$, $R=O_p(1)$ and the distribution of $\sqrt{n}(\hat{\theta}_n-\theta)$ admits the Edgeworth expansion up to the order n^{-1}. If an estimator $\hat{\theta}_n$ belongs to the class C', then we call it C'-estimator.

Now we obtain the stochastic expansion of the DLE under the assumed model $(\theta, 0)$. The following additional assumption is made.

(A.7) $$J_{011}=0 \quad \text{and} \quad J_{001}=0 .$$

The condition (A.7) is necessary to adjust the bias due to the "incorrectness" of the assumed model up to the second order through the function $a_n(\theta, \xi, r)$. The condition $J_{011}=0$ in (A.7) holds true if, for example, θ is a location parameter, i.e., $f(x, \theta, \xi)=f_0(x-\theta, \xi)$ $a.a.x[\mu]$ and $f_0(x, \xi)$ has the symmetric property, i.e., $f_0(x, \xi)=f_0(-x, \xi)$ $a.a.x[\mu]$. Here, in the case when f_0 with respect to the Lebesgue measure μ has the above property, a sufficient condition for $J_{001}=0$ is the following:

(3.1) $$\int_{-\infty}^{\infty}\{(\partial/\partial\theta)\log f_0(x-\theta, \xi)\}\{(\partial/\partial\xi)f_0'(x-\theta, \xi)\}d\mu=0 ,$$

and

(3.2) $$\lim_{|x|\to\infty} f_0'(x-\theta, \xi)\frac{(\partial/\partial\xi)f_0(x-\theta, \xi)}{f_0(x-\theta, \xi)}=0 ,$$

where $f_0'(x, \xi)=(\partial/\partial x)f_0(x, \xi)$. Indeed, since

$$l_{00}=(\partial^2/\partial\theta^2)\log f_0(x-\theta, \xi)=\frac{f_0''(x-\theta, \xi)}{f_0(x-\theta, \xi)}-\left\{\frac{f_0'(x-\theta, \xi)}{f_0(x-\theta, \xi)}\right\}^2$$

$$l_1=(\partial/\partial\xi)\log f_0(x-\theta, \xi)=\frac{(\partial/\partial\xi)f_0(x-\theta, \xi)}{f_0(x-\theta, \xi)} ,$$

it follows that

(3.3) $$J_{001}=E(l_{00}l_1)=\int\frac{f_0''(x-\theta, \xi)}{f_0(x-\theta, \xi)}(\partial/\partial\xi)f_0(x-\theta, \xi)d\mu$$
$$-\int\left\{\frac{f_0'(x-\theta, \xi)}{f_0(x-\theta, \xi)}\right\}^2(\partial/\partial\xi)f_0(x-\theta, \xi)d\mu .$$

From (3.1) to (3.3) we have

$$J_{001}=E(l_{00}l_1)=\int\{(\partial/\partial\theta)\log f_0(x-\theta, \xi)\}\{(\partial/\partial\xi)f_0'(x-\theta, \xi)\}d\mu=0 .$$

Then we have the following.

THEOREM 3.2. *Assume that the conditions* (A.1) *to* (A.7) *hold. Then, under the assumed model* $(\theta, 0)$, *the DLE* $\hat{\theta}^r_{DL}$ *of* θ *has the following stochastic expansion.*

$$\sqrt{n}(\hat{\theta}^r_{DL}-\theta)=\frac{Z_0}{I_{00}}+\frac{r}{2I_{00}\sqrt{n}}\left(Z_{00}-\frac{J_{000}}{I_{00}}Z_0\right)+\frac{1}{\sqrt{n}}L+\frac{1}{\sqrt{n}}(Q_0-a)+\frac{1}{n}R_0^*+o_p\left(\frac{1}{n}\right),$$

where L *and* Q_0 *are given in* THEOREM 3.1, $R_0^*=O_p(1)$ *and* $a=-(J_{000}+K_{000})/(2I_{00}^2)$. *Further the asymptotic deficiency* $D_t(u, r)$ *of the DLE* $\hat{\theta}^r_{DL}$ *relative to the bias-adjusted MLE* $\hat{\theta}^0_{ML}$ *in* C' *under the assumed model* $(\theta, 0)$ *is given by*

$$D_t(u, r)=\frac{r}{4I_{00}^2}\left(r+\frac{2u}{\sqrt{I_{00}}}\right)(I_{00}M_{0000}-J_{000}^2)-\frac{rt}{I_{00}^2}(I_{00}M_{0001}-J_{000}J_{010}) \tag{3.4}$$

so that the DLE $\hat{\theta}^r_{DL}$ *has the same asymptotic distribution as the bias-adjusted MLE* $\hat{\theta}^0_{ML}$ *at a point* u *up to the order* n^{-1}, *i.e.*,

$$P_{\theta,\xi}\{\sqrt{nI_{00}}\,(\hat{\theta}^r_{DL}-\theta)\le u+(c_0/n)\}=P_{\theta,\xi}\{\sqrt{nI_{00}}(\hat{\theta}^0_{ML}-\theta)\le u\}+o(1/n),$$

where

$$c_0=\sqrt{I_{00}}\,(\mu_2^*-\mu_2)-\frac{r}{4I_{00}^{5/2}}(I_{00}M_{0000}-J_{000}^2) \quad \textit{with} \quad \mu_2=E(R_0) \quad \textit{and} \quad \mu_2^*=E(R_0^*).$$

The proof is given in section 4.

REMARK 3.1. In THEOREM 3.2, the asymptotic deficiency (3.4) can be interpreted as follows. The asymptotic deficiency consists of two parts. The first term of the RHS of (3.4) is the loss of information associated with the DLE of the parameter to be estimated. The second term of the RHS of (3.4) is that due to the "incorrectness" of the assumed model. Indeed, the loss of information associated with any statistic $T=T(X_1, \ldots, X_n)$ is given by

$$E\left[V\left(\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f(X_i, \theta, \xi)\,\middle|\,T\right)\right], \tag{3.5}$$

where $V(\cdot|T)$ denotes the conditional variance given T (see Fisher (1925) and Rao (1961)). A straightforward calculation of (3.5) with a stochastic expansion of the DLE yields the first term of the RHS of (3.4). In a similar way, it can be also shown that the second one of (3.4) is derived from (3.5) under the assumed model.

COROLLARY 3.1. *Assume that the conditions* (A.1) *to* (A.6) *hold. Suppose that* $J_{001}=0$ *and* θ *is a location parameter, i.e.,* $f(x, \theta, \xi)=f_0(x-\theta, \xi)$ *a.a.x* $[\mu]$ *and* $f_0(x, \xi)$ *has the symmetric property, i.e.,* $f_0(x, \xi)=f_0(-x, \xi)$ *a.a.x* $[\mu]$. *If* $\xi=t/\sqrt{n}$, *then, under the assumed model* $(\theta, 0)$, *the stochastic expansions of the DLE* $\hat{\theta}^r_{DL}$ *and the MLE* $\hat{\theta}_{ML}$ *are given by*

$$\sqrt{n}(\hat{\theta}^r_{DL}-\theta)=\frac{Z_0}{I_{00}}+\frac{1}{\sqrt{n}}L+\frac{1}{\sqrt{n}}Q_0^*+\frac{r}{2I_{00}\sqrt{n}}Z_{00}+\frac{1}{n}R_0^*+o_p\left(\frac{1}{n}\right),$$

$$\sqrt{n}(\hat{\theta}_{ML}-\theta)=\frac{Z_0}{I_{00}}+\frac{1}{\sqrt{n}}L+\frac{1}{\sqrt{n}}Q_0^*+\frac{1}{n}R_0+o_p\left(\frac{1}{n}\right),$$

where L, R_0 *and* R_0^* *are given in* THEOREMS 3.1, 3.2 *and* $Q_0^*=Z_0Z_{00}/I_{00}^2$. *Further, the asymptotic deficiency* $D_t(u, r)$ *of the DLE relative to the MLE under the assumed model* $\theta, 0)$ *is given by*

$$D_t(u, r)=\frac{r}{4I_{00}}\left(r+\frac{2u}{\sqrt{I_{00}}}\right)M_{0000}$$

so that the DLE $\hat{\theta}^r_{DL}$ *has the same asymptotic distribution as the MLE* $\hat{\theta}_{ML}$ *at a point* u *up to the order* n^{-1}, *i.e.*

$$P_{\theta,\xi}\{\sqrt{nI_{00}}\,(\hat{\theta}^r_{DL}-\theta)\leq u+(c_0^*/n)\}=P_{\theta,\xi}\{\sqrt{nI_{00}}\,(\hat{\theta}_{ML}-\theta)\leq u\}+o(1/n)\ ,$$

where

$$c_0^*=\sqrt{I_{00}}\,(\mu_2^*-\mu_2)-\frac{r}{4I_{00}^{3/2}}M_{0000}$$

with $\mu_2=E(R_0)$ *and* $\mu_2^*=E(R_0^*)$.

The proof is given in section 4. It is noted that the symmetric property of f_0 implies unnecessity of a bias-correction of the MLE.

REMARK 3.2. When $u=-r\sqrt{I_{00}}$, under the same conditions as in COROLLARY 3.1, the asymptotic deficiency of the DLE $\hat{\theta}^r_{DL}$ relative to the MLE $\hat{\theta}_{ML}$ is given by

$$D_t(-r\sqrt{I_{00}}\,,\ r)=-\frac{r^2}{4I_{00}}M_{0000}\leq 0\ .$$

Since the value is non-positive, it is seen that the DLE $\hat{\theta}^r_{DL}$ is asymptotically better than $\hat{\theta}_{ML}$ at $u=-r\sqrt{I_{00}}$ up to the order $o(1/n)$.

4. Examples

Under the previous framework, we now give examples on the normal and Weibull type distributions.

EXAMPLE 4.1. (Normal case). Let $X_1, \ldots, X_n$ be i.i.d. random variables with a normal density function

$$f_0(x-\theta,\ \xi)=\frac{1}{\sqrt{2\pi(\xi+1)}}\exp\left\{-\frac{(x-\theta)^2}{2(\xi+1)}\right\}$$

for $-\infty<x<\infty$; $-\infty<\theta<\infty$, $-1<\xi$. Since $\xi=t/\sqrt{n}$, it is noted that

$$I_{00}=I_{00}(\theta,\ \xi)=I_{00}(\theta,\ 0)+o(1)=1/(\xi+1)+o(1)=1+o(1)\ ,$$
$$E_{\theta,\xi}[l_{00}^2(\theta,\ \xi,\ X)]=E_{\theta,0}[l_{00}^2(\theta,\ 0,\ X)]+o(1)=1/(\xi+1)^2+o(1)=1+o(1)\ ,$$

hence

$$M_{0000}=M_{0000}(\theta,\ \xi)=M_{0000}(\theta,\ 0)+o(1)=1+o(1)\ .$$

Then it follows from COROLLARY 3.1 that the asymptotic deficiency of the DLE relative to the MLE under the assumed model $(\theta,\ 0)$ is equal to $r(r+2u)/4+o(1)$.

EXAMPLE 4.2. (Weibull type case). Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables with a Weibull type density function

$$f_0(x-\theta,\ \xi)=\frac{\beta}{2(\xi+1)\Gamma((\alpha+1)/\beta)}\left|\frac{x-\theta}{\xi+1}\right|^{\alpha}\exp\left\{-\left|\frac{x-\theta}{\xi+1}\right|^{\beta}\right\}\ ,$$

for $-\infty<x<\infty$; $-\infty<\theta<\infty$, $-1<\xi$, $0<\alpha$, $0<\beta$. Since $\xi=t/\sqrt{n}$, it is noted that

$$(4.1)\qquad I_{00}=I_{00}(\theta,\xi)=I_{00}(\theta,0)+o(1)=\frac{\Gamma((\alpha-1)/\beta)}{\Gamma((\alpha+1)/\beta)}\{1+\beta(\alpha-1)\}+o(1)\ ,$$

and for $\alpha>3$

$$E_{\theta,\xi}[\{l_{00}(\theta,\xi,X)\}^2]=E_{\theta,\xi}[\{l_{00}(\theta,0,X)\}^2]+o(1)$$
$$=\frac{\Gamma((\alpha-3)/\beta)}{\Gamma((\alpha+1)/\beta)}\{\alpha^2+\alpha(\alpha-3)(\beta^2-1)+(\alpha-3)(\beta-1)^2(\beta-3)\}+o(1)\ ,$$

hence

$$(4.2)\qquad M_{0000}(\theta,\xi)=M_{0000}(\theta,0)+o(1)$$
$$=\frac{\Gamma((\alpha-3)/\beta)}{\Gamma((\alpha+1)/\beta)}\{\alpha^2+\alpha(\alpha-3)(\beta^2-1)+(\alpha-3)(\beta-1)^2(\beta-3)\}$$
$$-\left\{\frac{\Gamma((\alpha-1)/\beta)}{\Gamma((\alpha+1)/\beta)}\right\}^2\{1+\beta(\alpha-1)\}^2+o(1)$$

for $\alpha>3$. A restriction on α and β yields from the condition $J_{001}=0$ in (A.7) as follows. Since

$$l_{00}(\theta,\xi)=-\frac{\alpha}{(x-\theta)^2}-\frac{\beta(\beta-1)}{(\xi+1)^\beta}|x-\theta|^{\beta-2}\ ,$$

$$l_1(\theta,\xi)=\frac{(\partial/\partial\xi)C_\xi}{C_\xi}-\frac{\alpha}{\xi+1}+\frac{\beta}{(\xi+1)^{\beta+1}}|x-\theta|^\beta\ ,$$

it follows from the assumption $\xi=t/\sqrt{n}$ that

$$(4.3)\qquad J_{001}(\theta,\xi)=J_{001}(\theta,0)+o(1)$$
$$=E_{0,0}\left[\left\{-\frac{\alpha}{X^2}-\beta(\beta-1)|X|^{\beta-2}\right\}(-1-\alpha+\beta|X|^\beta)\right]+o(1)$$
$$=\alpha(\alpha+1)E_{0,0}\left(\frac{1}{X^2}\right)-\alpha\beta E_{0,0}(|X|^{\beta-2})+(\alpha+1)\beta(\beta-1)E_{0,0}(|X|^{\beta-2})$$
$$-\beta^2(\beta-1)E(|X|^{2\beta-2})+o(1)\ ,$$

where $C_\xi=\beta/\{2(\xi+1)\Gamma((\alpha+1)/\beta)\}$.
Since

$$E_{0,0}\left(\frac{1}{X^2}\right)=\frac{\Gamma((\alpha-1)/\beta)}{\Gamma((\alpha+1)/\beta)}\ ,\qquad E_{0,0}(|X|^{\beta-2})=\frac{(\alpha-1)\Gamma((\alpha-1)/\beta)}{\beta\Gamma((\alpha+1)/\beta)}$$

$$E_{0,0}(|X|^{2\beta-2})=\frac{(\alpha-1)(\alpha+\beta-1)\Gamma((\alpha-1)/\beta)}{\beta^2\Gamma((\alpha+1)/\beta)}\ ,$$

it follows from (4.3) that

$$J_{001}=\{\Gamma((\alpha-1)/\beta)/\Gamma((\alpha+1)/\beta)\}\{2\alpha-(\alpha-1)(\beta-1)(\beta-2)\}+o(1)\ .$$

Hence $J_{001}=0$ if and only if $2\alpha-(\alpha-1)(\beta-1)(\beta-2)=0$, i.e.,

$$\alpha=\frac{(\beta-1)(\beta-2)}{(\beta-1)(\beta-2)-2}\ .$$

Further, $\alpha>3$ if and only if

$$\alpha=\frac{(\beta-1)(\beta-2)}{(\beta-1)(\beta-2)-2} \quad \text{and} \quad 3<\beta<\frac{3+\sqrt{13}}{2}. \tag{4.4}$$

Therefore it follows from COROLLARY 3.1 that the asymptotic deficiency of the DLE $\hat{\theta}_{DL}^{r}$ relative the MLE $\hat{\theta}_{ML}$ under the assumed model $(\theta, 0)$ is equal to the value

$$\frac{r}{4I_{00}}\left(r+\frac{2u}{\sqrt{I_{00}}}\right)M_{0000},$$

where I_{00} and M_{0000} are given by (4.1) and (4.2) with parameters α and β satisfying (4.4).

If, in particular $\alpha=0$ and $\beta=2$, then the density $f_0(x-\theta, \xi)$ is a normal one with mean θ and variance $(\xi+1)^2/2$, which is reduced to Example 4.1. If $\alpha=0$ and $\beta>1$, then

$$J_{001}=\beta(\beta-1)E[|X|^{\beta-2}]-\beta^2(\beta-1)E[|X|^{2\beta-2}]+o(1)=-\beta(\beta-1)(\beta-2)\Gamma((\beta-1)/\beta)\Gamma(1/\beta),$$

hence $J_{001}=0$ if and only if $\beta=2$, i.e., the normal case.

5. Proofs

Here, the proofs of the theorem and the corollary in section 3 are given.

PROOF OF THEOREM 3.2. Let $\hat{\theta}_n$ be the DLE of θ under the assumed model. By the Taylor expansion around the true value (θ, ξ), we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n} l_0(\hat{\theta}_n, 0, X_i) \\
&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n} l_0(\theta, \xi, X_i)+\frac{1}{n}\sum_{i=1}^{n} l_{00}(\theta, \xi, X_i)\sqrt{n}\,(\hat{\theta}_n-\theta) \\
&\quad-\frac{1}{n}\sum_{i=1}^{n} l_{01}(\theta, \xi, X_i)\sqrt{n}\,\xi+\frac{1}{2n\sqrt{n}}\sum_{i=1}^{n} l_{000}(\theta, \xi, X_i)\{\sqrt{n}\,(\hat{\theta}_n-\theta)\}^2 \\
&\quad+\frac{1}{2n\sqrt{n}}\sum_{i=1}^{n} l_{011}(\theta, \xi, X_i)n\xi^2-\frac{1}{n\sqrt{n}}\sum_{i=1}^{n} l_{001}(\theta, \xi, X_i)n(\hat{\theta}_n-\theta)\xi+o_p\left(\frac{1}{\sqrt{n}}\right) \\
&=Z_0+\frac{1}{\sqrt{n}}(Z_{00}-\sqrt{n}\,I_{00})\sqrt{n}\,(\hat{\theta}_n-\theta)-\frac{1}{\sqrt{n}}Z_{01}\sqrt{n}\,\xi \\
&\quad-\frac{1}{2\sqrt{n}}(3J_{000}+K_{000})\{\sqrt{n}\,(\hat{\theta}_n-\theta)\}^2-\frac{J_{011}}{2\sqrt{n}}n\xi^2 \\
&\quad+\frac{J_{010}}{\sqrt{n}}n(\hat{\theta}_n-\theta)\xi+o_p\left(\frac{1}{\sqrt{n}}\right).
\end{aligned} \tag{5.1}$$

In a similar way to the above we obtain

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} l_{00}(\hat{\theta}_n, 0, X_i)&=\frac{1}{n}\sum_{i=1}^{n} l_{00}(\theta, \xi, X_i)+\frac{1}{n\sqrt{n}}\sum_{i=1}^{n} l_{000}(\theta, \xi, X_i)\sqrt{n}\,(\hat{\theta}_n-\theta) \\
&\quad-\frac{1}{n\sqrt{n}}\sum_{i=1}^{n} l_{001}(\theta, \xi, X_i)\sqrt{n}\,\xi+o_p\left(\frac{1}{\sqrt{n}}\right) \\
&=\frac{1}{\sqrt{n}}Z_{00}-I_{00}-\frac{1}{\sqrt{n}}(3J_{000}+K_{000})\sqrt{n}\,(\hat{\theta}_n-\theta) \\
&\quad+\frac{1}{\sqrt{n}}J_{010}\sqrt{n}\,\xi+o_p\left(\frac{1}{\sqrt{n}}\right);
\end{aligned} \tag{5.2}$$

$$\frac{1}{n}\sum_{i=1}^{n} l_{000}(\hat{\theta}_n, \xi, X_i)=\frac{1}{n}\sum_{i=1}^{n} l_{000}(\theta, \xi, X_i)+o_p(1)=-(3J_{000}+K_{000})+o_p(1). \tag{5.3}$$

Since the DLE $\hat{\theta}_n$ satisfies the equation

$$\sum_{i=1}^{n} l(\hat{\theta}_n + rn^{-1/2}, 0, X_i) - \sum_{i=1}^{n} l(\hat{\theta}_n, 0, X_i) = a_n(\hat{\theta}_n, 0, r)$$

and $\xi = t/\sqrt{n}$, it follows from (5.1) to (5.3) that

(5.4)
$$\begin{aligned} a_n(\hat{\theta}_n, 0, r) &= rZ_0 + \frac{r}{\sqrt{n}}(Z_{00} - \sqrt{n}\, I_{00})\sqrt{n}\,(\hat{\theta}_n - \theta) - \frac{rt}{\sqrt{n}} Z_{01} \\ &\quad - \frac{r}{2\sqrt{n}}(3J_{000} + K_{000})\{\sqrt{n}\,(\hat{\theta}_n - \theta)\}^2 - \frac{rt^2}{2\sqrt{n}} J_{011} + \frac{rt}{\sqrt{n}} J_{010}\sqrt{n}\,(\hat{\theta}_n - \theta) \\ &\quad + \frac{r^2}{2\sqrt{n}} Z_{00} - \frac{r^2}{2} I_{00} - \frac{r^2}{2\sqrt{n}}(3J_{000} + K_{000})\sqrt{n}\,(\hat{\theta}_n - \theta) \\ &\quad + \frac{r^2 t}{2\sqrt{n}} J_{010} - \frac{r^3}{6\sqrt{n}}(3J_{000} + K_{000}) + o_p\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

If $a_n(\theta, \xi, r) - (I_{00}r^2/2) = o(1)$, then, putting $T_n = \sqrt{n}\,(\hat{\theta}_n - \theta)$, we have from (5.4)

(5.5)
$$\begin{aligned} T_n &= -\frac{1}{I_{00}r} a_n(\hat{\theta}_n, 0, r) + \frac{Z_0}{I_{00}} + \frac{1}{I_{00}\sqrt{n}} Z_{00} T_n - \frac{t}{I_{00}\sqrt{n}} Z_{01} \\ &\quad - \frac{1}{2I_{00}\sqrt{n}}(3J_{000} + K_{000}) T_n^2 - \frac{t^2}{2I_{00}\sqrt{n}} J_{011} + \frac{t}{I_{00}\sqrt{n}} J_{010} T_n \\ &\quad + \frac{r}{2I_{00}\sqrt{n}} Z_{00} - \frac{r}{2} - \frac{r}{2I_{00}\sqrt{n}}(3J_{000} + K_{000}) T_n + \frac{rt}{2I_{00}\sqrt{n}} J_{010} \\ &\quad - \frac{r^2}{6I_{00}\sqrt{n}}(3J_{000} + K_{000}) + o_p\left(\frac{1}{\sqrt{n}}\right) \\ &= -\frac{1}{I_{00}r} a_n(\theta, \xi, r) - \frac{1}{I_{00}r\sqrt{n}} \left\{ \left(\frac{\partial}{\partial \theta} a_n(\theta, \xi, r) \right) T_n - \left(\frac{\partial}{\partial \xi} a_n(\theta, \xi, r) \right) t \right\} \\ &\quad + \frac{Z_0}{I_{00}} - \frac{r}{2} + \frac{1}{I_{00}^2\sqrt{n}} Z_0 Z_{00} - \frac{t}{I_{00}\sqrt{n}} Z_{01} - \frac{1}{2I_{00}^2\sqrt{n}}(3J_{000} + K_{000}) Z_0 \\ &\quad - \frac{t^2}{2I_{00}\sqrt{n}} J_{011} + \frac{tJ_{010}}{I_{00}^2\sqrt{n}} Z_0 + \frac{r}{2I_{00}\sqrt{n}} Z_{00} - \frac{r}{2I_{00}^2\sqrt{n}}(3J_{000} + K_{000}) Z_0 \\ &\quad + \frac{rt}{2I_{00}\sqrt{n}} J_{010} - \frac{r^2}{6I_{00}\sqrt{n}}(3J_{000} + K_{000}) + o_p\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial}{\partial \theta} a_n(\theta, \xi, r) &= -\frac{r^2}{2}\left\{ \frac{\partial}{\partial \theta} I_{00}(\theta, \xi) \right\} + o(1) = -\frac{r^2}{2}\{2J_{000}(\theta, \xi) + K_{000}(\theta, \xi)\} + o(1) \\ &= -\frac{r^2}{2}(2J_{000} + K_{000}) + o(1) \quad \text{(say)}; \\ \frac{\partial}{\partial \xi} a_n(\theta, \xi, r) &= -\frac{r^2}{2}\left\{ \frac{\partial}{\partial \xi} I_{00}(\theta, \xi) \right\} + o(1) = -\frac{r^2}{2}\{2J_{010}(\theta, \xi) + K_{001}(\theta, \xi)\} + o(1) \\ &= -\frac{r^2}{2}(2J_{010} + K_{001}) + o(1) \quad \text{(say)} \\ &= -\frac{r^2}{2}(J_{010} - J_{001}) + o(1), \end{aligned}$$

putting $a_n(\theta, \xi, r) = -(I_{00}r^2/2) + (b/\sqrt{n}) + o(1/\sqrt{n})$, we have

$$(5.6)\qquad E(T_n)=-\frac{b}{I_{00}r\sqrt{n}}+\frac{J_{000}}{I_{00}^2\sqrt{n}}-\frac{3J_{000}+K_{000}}{2I_{00}^2\sqrt{n}}-\frac{t^2J_{011}}{2I_{00}\sqrt{n}}-\frac{r^2}{6I_{00}\sqrt{n}}(3J_{000}+K_{000})+\frac{rtJ_{001}}{2I_{00}\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right).$$

In order that $E(T_n)=o(1/\sqrt{n})$, from (5.6) we take as b the following:

$$b=\frac{J_{000}}{I_{00}}r-\frac{J_{000}+K_{000}}{2I_{00}}r-\frac{J_{011}}{2}rt^2-\frac{1}{6}(3J_{000}+K_{000})r^3+\frac{J_{001}}{2}r^2t\,.$$

By the condition (A.7) we have

$$(5.7)\qquad a_n(\theta,\xi,r)=-\frac{I_{00}}{2}r^2-\frac{r}{\sqrt{n}}\left\{\frac{J_{000}+K_{000}}{2I_{00}}-\frac{1}{6}(3J_{000}+K_{000})r^2\right\}+o\left(\frac{1}{\sqrt{n}}\right).$$

From (5.5) and (5.7) we obtain

$$\begin{aligned}T_n&=\frac{J_{000}+K_{000}}{2I_{00}^2\sqrt{n}}-\frac{t}{I_{00}\sqrt{n}}\left(Z_{01}-\frac{J_{010}}{I_{00}}Z_0\right)+\frac{r}{2I_{00}\sqrt{n}}\left(Z_{00}-\frac{J_{000}}{I_{00}}Z_0\right)\\&\quad+\frac{Z_0}{I_{00}}+\frac{Z_0}{I_{00}^2\sqrt{n}}\left(Z_{00}-\frac{3J_{000}+K_{000}}{2I_{00}}Z_0\right)+\frac{1}{n}R_0^*+o_p\left(\frac{1}{n}\right)\\&=\frac{Z_0}{I_{00}}+\frac{r}{2I_{00}\sqrt{n}}\left(Z_{00}-\frac{J_{000}}{I_{00}}Z_0\right)+\frac{1}{\sqrt{n}}L+\frac{1}{n}R_0^*+o_p\left(\frac{1}{n}\right),\end{aligned}$$

where $a=-(J_{000}+K_{000})/(2I_{00}^2)$ and $R_0^*=O_p(1)$. From THEOREM 3.1 we see that the stochastic expansion of the bias-adjusted MLE $\hat{\theta}_{ML}^0$ is given by

$$\sqrt{n}\,(\hat{\theta}_{ML}^0-\theta)=\frac{Z_0}{I_{00}}+\frac{1}{\sqrt{n}}L+\frac{1}{\sqrt{n}}(Q_0-a)+\frac{1}{n}R_0+o_p\left(\frac{1}{n}\right)\quad\text{with}\quad R_0=O_p(1)\,.$$

It is seen in a similar way to THEOREM 2.3.1 in Akahira (1986) that for $-\infty<u<\infty$

$$P_{\theta,\xi}\{\sqrt{nI_{00}}(\hat{\theta}_{DL}^r-\theta)\le u+(c_0/n)\}=P_{\theta,\xi}\{\sqrt{nI_{00}}(\hat{\theta}_{ML}^0-\theta)\le u\}+o(1/n)$$

if and only if the asymptotic deficiency $D_t(u,r)$ of $\hat{\theta}_{DL}^r$ relative to $\hat{\theta}_{ML}^0$ is given by

$$\begin{aligned}(5.8)\qquad D_t(u,r)&=I_{00}\left\{V\left(\frac{r}{2I_{00}}\left(Z_{00}-\frac{J_{000}}{I_{00}}Z_0\right)+L+Q_0-a\right)-V(L+Q_0-a)\right\}\\&\quad+\frac{u\sqrt{I_{00}}}{2}\left\{E\left[Z_0\left(\frac{r}{2I_{00}}\left(Z_{00}-\frac{J_{000}}{I_{00}}Z_0\right)+L+Q_0-a\right)^2\right]\right.\\&\quad\left.-E[Z_0(L+Q_0-a)^2]\right\},\end{aligned}$$

where $c_0=\sqrt{I_{00}}\,(\mu_2^*-\mu_2)-\frac{r}{4I_{00}^{5/2}}(I_{00}M_{0000}-J_{000}^2)$ with $\mu_2=E(R_0)$ and $\mu_2^*=E(R_0^*)$, and $V(\cdot)$ denotes the asymptotic variance (see also Akahira, 1991). Since

$$\begin{aligned}&E[Z_{00}(L+Q_0)]=-\frac{t}{I_{00}^2}(I_{00}M_{0001}-J_{000}J_{010})+o(1);\\&E[Z_0(L+Q_0)]=o(1)\,,\end{aligned}$$

it follows that

$$
\begin{aligned}
(5.9)\qquad & V\left(\frac{r}{2I_{00}}\left(Z_{00}-\frac{J_{000}}{I_{00}}Z_0\right)+L+Q_0-a\right)-V(L+Q_0-a) \\
&=E\left[\frac{r^2}{4I_{00}^2}\left(Z_{00}-\frac{J_{000}}{I_{00}}Z_0\right)^2\right]+\frac{r}{I_{00}}E\left[\left(Z_{00}-\frac{J_{000}}{I_{00}}Z_0\right)(L+Q_0-a)\right] \\
&=\frac{r^2}{4I_{00}^2}\left(M_{0000}-\frac{J_{000}^2}{I_{00}}\right)+\frac{r}{I_{00}}E[Z_{00}(L+Q_0)]-\frac{rJ_{000}}{I_{00}^2}E[Z_0(L+Q_0)] \\
&=\frac{r^2}{4I_{00}^3}(I_{00}M_{0000}-J_{000}^2)-\frac{rt}{I_{00}^3}(I_{00}M_{0001}-J_{000}J_{010})+o(1)\ .
\end{aligned}
$$

Since

$$
\begin{aligned}
&E\left[Z_0\left(Z_{00}-\frac{J_{000}}{I_{00}}Z_0\right)^2\right]=o(1); \\
&E\left[Z_0\left(Z_{00}-\frac{J_{000}}{I_{00}}Z_0\right)(L+Q_0-a)\right]=\frac{t}{I_{00}}E\left[\left(Z_0Z_{00}-\frac{J_{000}}{I_{00}}Z_0^2\right)\left(\frac{J_{010}}{I_{00}}Z_0-Z_{01}\right)\right] \\
&\qquad\qquad+\frac{1}{I_{00}^2}E\left[\left(Z_0Z_{00}-\frac{J_{000}}{I_{00}}Z_0^2\right)\left(Z_0Z_{00}-\frac{3J_{000}+K_{000}}{2I_{00}}Z_0^2\right)\right] \\
&\qquad\qquad=\frac{1}{I_{00}}\left\{E(Z_0^2Z_{00}^2)-\frac{3J_{000}+K_{000}}{2I_{00}}E(Z_0^3Z_{00})\right. \\
&\qquad\qquad\quad\left.-\frac{J_{000}}{I_{00}}E(Z_0^3Z_{00})+\frac{J_{000}(3J_{000}+K_{000})}{2I_{00}^2}E(Z_0^4)\right\}+o(1) \\
&\qquad\qquad=\frac{1}{I_{00}^2}(I_{00}M_{0000}-J_{000}^2)+o(1)\ ,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
(5.10)\qquad & E\left[Z_0\left\{\frac{r}{2I_{00}}\left(Z_{00}-\frac{J_{000}}{I_{00}}Z_0\right)+L+Q_0-a)\right\}^2\right]-E[Z_0(L+Q_0-a)^2] \\
&=\frac{r}{I_{00}^3}(I_{00}M_{0000}-J_{000}^2)+o(1)\ .
\end{aligned}
$$

From (5.8) to (5.10) we see that

$$
D_t(u,r)=\frac{r}{4I_{00}^2}\left(r+\frac{2u}{\sqrt{I_{00}}}\right)(I_{00}M_{0000}-J_{000}^2)-\frac{rt}{I_{00}^2}(I_{00}M_{0001}-J_{000}J_{010})+o(1)\ ,
$$

which completes the proof.

PROOF of COROLLARY 3.1. Since the density function has the symmetric property, it follows that $J_{000}=K_{000}=M_{0001}=0$. Hence the conclusion easily follows from THEOREMS 3.1 and 3.2.

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