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by

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AVERAGE TREE SOLUTION AND CORE FOR COOPERATIVE GAMES WITH GRAPH STRUCTURE

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Abstract. This paper considers cooperative transferable utility games with graph structure, called graph games. A graph structure restricts the set of possible coalitions of players, so that players are able to cooperate only if they are connected in the graph. Recently the average tree solution has been proposed for arbitrary graph games by Herings et al. The average tree solution is the average of some specific marginal contribution vectors, and was shown to belong to the core if the game exhibits link-convexity. In this paper the main focus is placed on the relationship between the core and the average tree solution, and the following results were obtained. Firstly, it was shown that some marginal contribution vectors do not belong to the core even though the game is link-convex. Secondly, an alternative condition to link-convexity was given. Thirdly, it was proven that for cycle-complete graph games the average tree solution is an element of the core if the game is link-convex.

1. Introduction

In many settings of cooperative games players gain more benefits by cooperating rather than by acting on their own. A subgroup of players is called a *coalition* and the total profit they can obtain from cooperation is called its *worth*. If the players are able to divide the worth of a coalition(transferable utility), there arises a question of how to allocate the worth to each player. A classical set-valued solution is the core, the set of payoff vectors which satisfy the following conditions. First, the worth of the whole set of players (the grand coalition) is distributed among the players (efficiency),

$$\sum_{i \in N} x_i = v(N).$$

Next, no coalition receives less than its worth (non-domination),

$$\sum_{i \in S} x_i \geq v(S), \quad \text{for all} \quad S \subseteq N \text{ with } S \neq \emptyset.$$

If a payoff vector is an element of the core, no coalitions can do better than that by their own. Thus, the payoff vector prevents the collapse of the grand coalition. The core, however, has two possible problems: it might be empty, and it might contain many elements. To overcome this problem, several single-valued solutions have been introduced.

The Shapley value is the most well-known single-valued solution, see Shapley [10]. At the Shapley value each player is promised the average of all his marginal contributions to any coalition

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that he joins. The Shapley value is an element of the core if the game exhibits convexity. However, it is not always true that any coalition S can form and achieve worth $v(S)$. In many cases cooperation among players rely on their communication structure.

We study cooperative games with limited communication structure represented by an undirected graph. These so-called graph games were introduced by Myerson [8]. A group of players is only able to cooperate if they are connected in the graph. The best known single-valued solution for such games is the Myerson value, which is characterized by efficiency and fairness. The Myerson value coincides with the Shapley value when the underlying graph is complete. Van den Nouweland and Borm [7] showed that the Myerson value lies in the core if the game exhibits convexity and the underlying graph is cycle-complete.

Herings et al. [5] proposed the average tree solution on the class of cycle-free graph games. The average tree solution is the average of marginal contribution vectors over a set of rooted spanning trees. Herings et al. proved that the corresponding solution is in the core if the game exhibits superadditivity, while the Myerson value or the position value may not. The condition of superadditivity was relaxed to a weaker one by Talman and Yamamoto [12].

In Herings et al. [6] the average tree solution was generalized for the class of arbitrary graph games. They constructed a specific set of rooted spanning trees, called admissible spanning trees. The generalized average tree solution coincides with the Shapley value when the underlying graph is complete and with the average tree solution as defined by Herings et al. [5] when the underlying graph is cycle-free. They also introduced the notion of link-convexity for graph games. For games with complete graph, link-convexity coincides with convexity, but in general the condition is weaker than convexity. For games with a cycle-free graph, link-convexity is even weaker than superadditivity. Herings et al. also claimed that the average tree solution is in the core if the game is link-convex. Baron et al. [1] defined the average tree solution with respect to trees constructed by Depth First Search (DFS) and Breadth First Search (BFS). When the underlying graph is complete, the average tree solution with respect to DFS trees coincides with the Shapley value and the solution with respect to BFS trees yields the equal surplus division.

In this paper we discuss the relationship between the core and the average tree solution. We first show that some link-convex graph games have some marginal vectors not in the core. Secondly, we refine link-convexity to the condition that ensures the average tree solution belongs to the core in arbitrary graph games. Thirdly, we prove that for the class of cycle-complete graph games satisfying link-convexity the average tree solution is an element of the core.

This paper is organized as follows. Section 2 is a preliminary section on games with graph structure. Section 3 introduces the average tree solution for arbitrary games with graph structure. Section 4 relates the average tree solutions to the core. Section 5 gives some conclusions.

2. TU-games with communication structure

We consider cooperative transferable utility games with graph structure, called *graph games* introduced by Myerosn [8]. A graph game is represented by a triple (N, v, L) where N is a set of n players, $v : 2^N \to \mathbb{R}$ a characteristic function that assigns the worth to coalitions, and $L \subseteq \{ \{i, j\} \mid i \neq j, i, j \in N \}$ is a collection of communication links between players. The pair (N, L) is called an undirected *graph* with N the set of *nodes*, being the players of the game, and L the collection of *edges* (links) between the nodes. In case $L = \{ \{i, j\} \mid i \neq j, i, j \in N \}$ the game $\{N, v, L\}$ is said to have full communication structure and is simply denoted by (N, v). A *payoff vector* $\boldsymbol{x} \in \mathbb{R}^n$ is an n-dimensional vector giving payoff x_i to player $i \in N$. For simplicity we denote $x(S) = \sum_{i \in S} x_i$ for $S \in 2^N$.

Next we give several notations for an undirected graph. For a graph (N, L) and a subset $K \subseteq N$, the set $L(K)$ is given by

$$L(K) = \{ \ \{i, j\} \in L \mid i, j \in K \ \}.$$

A sequence of different nodes $P = \{i_1, i_2, \ldots, i_m\}$ is a *path* from i_1 to i_m in the graph $(K, L(K))$ if $\{i_k, i_{k+1}\} \in L(K)$ for all $k \in \{1, 2, \ldots, m-1\}$. The path P is denoted by $i_1 \sim_K i_m$, where $i_1 \sim i_m$ stands for a path in (N, L). Two nodes $i, j \in N$ are *connected* in $(K, L(K))$ if either $i = j$ or there exists a path $i \sim_K j$. For a path $P = \{i_1, i_2, \ldots, i_m\}$ the number, $m - 1$, is said to be the *length* of P. A path between i_1 and i_m is *shortest* if the length is minimal among all paths connecting i_1 to i_m. A graph (N, L) is *connected* if any two nodes $i, j \in N$ are connected in (N, L). A subset of nodes $K \subseteq N$ is said to be a *connected subset* of N when the subgraph $(K, L(K))$ is connected. The collection of all connected subsets of K in $(K, L(K))$ is denoted by $C^L(K)$, i.e., $C^L(K) := \{ S \mid S \subseteq K \text{ is a connected subset of } K \}$. A subset K' of K is called a *connected component* of $(K, L(K))$ if K' is maximally connected, that is, K' is connected but the set $K' \cup \{j\}$ is not connected for any $j \in K \setminus K'$. The collection of all connected components of $(K, L(K))$ is denoted by $\hat{C}^L(K)$, i.e., $\hat{C}^L(K) := \{ S \mid S \subseteq K \text{ is a connected component of } K \}$. A sequence of nodes $\{i_1, i_2, \ldots, i_m\}$ is called a *cycle* in a graph (N, L) if

(i) $m \geq 3$,
(ii) all nodes $i_1, i_2, \ldots, i_m$ are different,
(iii) $i_{m+1} = i_1$,
(iv) $\{i_k, i_{k+1}\} \in L$ for $k = 1, 2, \ldots, m$.

A graph is said to be a *tree* if it is connected and does not contain any cycle. A *spanning tree* of (N, L) is a tree containing all the nodes N. A graph is said to be *cycle-free* when it does not contain any cycle. A graph is said to be *complete* when any two of its nodes are connected by an edge. A graph is said to be *cycle-complete* if the following holds: if $\{i_1, i_2, \ldots, i_m, i_1\}$ is a cycle in the graph (N, L) then $\{i_k, i_h\} \in L$ for all distinct $k, h \in \{1, 2, \ldots, m\}$. Since a cycle-free graph does not contain any cycle, they trivially satisfy the requirement of cycle-completeness. The class of complete graphs is another class of graphs that are cycle-complete.

In this paper it is assumed without loss of generality that in a graph game (N, v, L), (N, L) is always connected in the graph (N, L), i.e., $N \in C^L(N)$. We also assume that players of a coalition $S \in 2^N$ are able to cooperate only if all players of S can communicate directly or indirectly with each other, i.e., $S \in C^L(N)$. For $S \in C^L(N)$, the worth $v(S)$ is the maximum amount of payoff a coalition S can obtain for its players.

Superadditivity and convexity are defined below, referring the definitions in Talman and Yamamoto [12]. Convexity of (N, v, L), however, is originally defined in this paper.

- ▷ (N, v) is superadditive if

$$v(S) + v(T) \leq v(S \cup T)$$
$$\text{for all } S, T \in 2^N \text{ satisfying } S \cap T = \emptyset.$$

- ▷ (N, v, L) is superadditive if

$$v(S) + v(T) \leq v(S \cup T)$$
$$\text{for all } S, T \in C^L(N) \text{ satyisfying } S \cup T \in C^L(N) \text{ and } S \cap T = \emptyset.$$

- ▷ (N, v) is convex if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$
$$\text{for all } S, T \in 2^N.$$

  It is equivalent to

$$v(S) - v(S \cup \{i\}) \leq v(T) - v(T \cup \{i\})$$
$$\text{for all } S, T \subseteq N \setminus \{i\} \text{ satyisfying } S \subseteq T.$$

- ▷ (N, v, L) is convex if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$
$$\text{for all } S, T \in C^L(N) \text{ satyisfying } S \cup T \in C^L(N) \text{ and } S \cap T \in C^L(N) \cup \{\emptyset\}.$$

For a graph game (N, v, L) a payoff vector $\boldsymbol{x}$ is said to be *efficient* if $x(N) = v(N)$. The core, denoted by $\mathrm{Core}(N, v, L)$, of a graph game (N, v, L) is the set of efficient payoff vectors that are not dominated through any connected coalition,

$$\mathrm{Core}(N, v, L) := \{\, \boldsymbol{x} \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \quad \text{for all } S \in C^L(N)\,\}.$$

The core of a game (N, v) with full communication is denoted by $\mathrm{Core}(N, v)$, i.e.,

$$\mathrm{Core}(N, v) := \{\, \boldsymbol{x} \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \quad \text{for all } S \in 2^N \,\}.$$

Given a graph game (N, v, L), Myerson [8] defined the restricted game (N, v^L) as

$$v^L(S) = \sum_{T \in \hat{C}^L(S)} v(T), \quad \text{for } S \in 2^N.$$

Notice that the core of a graph game $\mathrm{Core}(N, v, L)$ equals the core $\mathrm{Core}(N, v^L)$ of the restricted game (N, v^L) with full communication.

3. The average tree solution

In this section we provide two definitions of the average tree solution, the solution given by Herings et al. [6] and the solution constructed by Depth First Search algorithm. To describe the average tree solution we first give some definitions of directed graph.

3.1. **Definition of directed graph.** A graph (N, A) is *directed* if $A \subseteq N \times N$, i.e., A is a set of ordered pairs of nodes. An ordered pair of nodes is called an *arc*. For a graph (N, A) a sequence $\{i_1, i_2, \dots, i_m\}$ is a *directed path* if $(i_k, i_{k+1}) \in A$ for all $k \in \{1, 2, \dots, m-1\}$. For $A \subseteq N \times N$ let $L(A) = \{\, \{i, j\} \mid (i, j) \in A \,\}$, i.e., undirected version of A. A directed graph (K, T) is said to be a *rooted tree* if the undirected graph $(K, L(T))$ induced by T is a tree and each node has at most one arc entering the node. Clearly, a rooted tree has exactly one node that no arc enters, which is called the *root*, and there is a unique directed path from the root to every node. A *rooted spanning tree* (N, T) is a rooted tree containing all the nodes N. For a given rooted spanning tree (N, T) and a subset $K \subseteq N$, the set $T(K)$ is given by

$$T(K) := \{\, (i, j) \in T \mid i, j \in K \,\}.$$

For a given rooted spanning tree (N, T), for each node $i \in N$ we define its sets of successors and descendants as

$$suc^T(i) = \{\, j \in N \mid (i, j) \in T \,\},$$
$$des^T(i) = \{\, j \in N \mid j = i \text{ or there is a directed path from } i \text{ to } j \text{ in } (N, T) \,\},$$

respectively. A node $j \in N$ is said to be an ancestor of $i \in N$ if $j \neq i$ and there is a directed path from j to i.

3.2. **Admissible coalitions.** To generalize the average tree solution to the class of arbitrary graph games, Herings et al. [6] consider a collection of admissible coalitions constructed as follows.

Definition 3.1 (Admissible Coalitions)**.** For a graph (N, L), $\mathcal{B} = \{B_1, B_2, \dots, B_n\}$ of n subsets of N is a collection of admissible coalitions if it satisfies the following conditions:

(i) For all $i \in N$, $i \in B_i$, and for some $j \in N$, $B_j = N$;

(ii) For all $i \in N$ and $K \in \hat{C}^L(B_i \setminus \{i\})$, $K = B_j$ and $\{i, j\} \in L$ for some $j \in N$.

Definition 3.2. For a graph (N, L), let $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be a collection of admissible coalitions. Define the directed graph $(N, T^{\mathcal{B}})$ as

$$T^{\mathcal{B}} := \{\, (i, j) \mid i, j \in N, B_j \in \hat{C}^L(B_i \setminus \{i\}) \,\}.$$

According to Lemma 3.2 in Herings et al. [6] the above collection of admissible coalitions $\mathcal{B}$ has the following property.

Lemma 3.3 ([6])**.** *For a graph (N, L), let $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be a collection of admissible coalitions. Then, $(N, T^{\mathcal{B}})$ is a rooted spanning tree.*

We denote the collection of the rooted spanning trees of Definition 3.2 by $\mathcal{B}^{\mathrm{ADM}}$. Herings et al. [6] defined their average tree solution with respect to the rooted spanning tree of $\mathcal{B}^{\mathrm{ADM}}$.

Definition 3.4. For a graph game (N, v, L), the marginal contribution vector $\boldsymbol{y}^T \in \mathbb{R}^n$ corresponding to $T \in \mathcal{B}^{\mathrm{ADM}}$ is the vector of payoffs given by

$$y_i^T = v(B_i) - \sum_{K \in \hat{C}(B_i \setminus \{i\})} v(K), \quad i \in N. \tag{3.1}$$

According to Herings et al. [6], the average tree solution is defined as follows.

Definition 3.5 (Herings et al. [6])**.** On the class of arbitrary graph games (N, v, L), the average tree solution is defined by

$$\bar{\boldsymbol{y}} = \frac{1}{|\mathcal{B}^{\mathrm{ADM}}|} \sum_{T \in \mathcal{B}^{\mathrm{ADM}}} \boldsymbol{y}^T. \tag{3.2}$$

3.3. **Depth First Search Tree.** Now we introduce a tree growing algorithm, called *Depth First Search*(DFS), and define the average tree solution based on the collection of rooted spanning trees constructed by DFS. We prove that the set $\mathcal{B}^{\mathrm{DFS}}$ of thus obtained rooted spanning trees is always a subset of $\mathcal{B}^{\mathrm{ADM}}$.

The psudocode of DFS is presented as follows.

Algorithm 1 Depth First Search

Input: a connected graph $G = (N, L)$
Output: a spanning tree of G with predecessor function p, and two time functions d and f

$k \leftarrow 0$, $S \leftarrow \emptyset$
choose any node r (as root)
$k \leftarrow k + 1$
colour r black
set $d(r) := k$
add r to S
while S is nonempty **do**
  consider the top node i of S
  $k \leftarrow k + 1$
  if i has an uncoloured neighbour j **then**
    colour j black
    set $p(j) := x$ and $d(j) := k$
    add j to the top of S
  else
    set $f(i) := k$
    remove i from S
  end if
end while
return (p, d, f)

The DFS procedure computes the predecessor function p and two time functions, the discovery time function d and the finishing time function f. The function p forms the subgraph (N, T), where

$$T = \{\, (u, v) \mid u = p(v), v \in N \setminus \{r\} \,\}.$$

Since it is assumed that (N, L) is connected, the resulting graph (N, T) is a rooted spanning tree, called a *depth first search tree* (DFS tree for short). Several properties of DFS tree are given here.

Lemma 3.6. *For a graph (N, L), let $T \in \mathcal{B}^{\mathrm{DFS}}$ and $S \in C^L(N)$. Let v_1 be the first node discovered by DFS in S. Then it follows that*

$$des^T(v) \subseteq des^T(v_1) \quad \text{for all } v \in S \setminus \{v_1\}. \tag{3.3}$$

Proof. It holds from the construction of DFS tree. □

Theorem 3.7. *For a graph (N, L), let $\mathcal{B}^{\mathrm{DFS}}$ be the collection of rooted spanning trees of (N, L) constructed by DFS. Let $\mathcal{B}^{\mathrm{ADM}}$ be the collection of rooted spanning trees of Definition* 3.2. *Then it* *follows that*

$$\mathcal{B}^{\mathrm{DFS}} \subseteq \mathcal{B}^{\mathrm{ADM}}.$$

Proof. Consider any $T \in \mathcal{B}^{\mathrm{DFS}}$. We will prove that the collection of descendants for each node $i \in N$, $B = \{des^T(1), des^T(2), \ldots, des^T(n)\}$, satisfies the conditions (i) and (ii) of Definition 3.1.

From the definition of a descendant, $i \in des^T(i)$ for all $i \in N$. Since (N, T) is a rooted spanning tree there exists a root r such that $des^T(r) = N$. Hence, condition (i) is satisfied. Condition (ii) also holds since for all $i \in N$ and $K \in \hat{C}^L(des^T(i) \setminus \{i\})$, there exists a node $j \in suc^T(i)$ such that $K = des^T(j)$ and $\{i, j\} \in L$. Thus concludes that $B = \{des^T(1), des^T(2), \ldots, des^T(n)\}$ is a collection of admissible coalitions.

Now, let $T' := \{\, (i, j) \mid i, j \in N, des^T(j) \in \hat{C}^L(des^T(i) \setminus \{i\}) \,\}$. Then it follows that $T = T'$. Therefore, $T \in \mathcal{B}^{\mathrm{ADM}}$. $\square$

The average tree solution over a set of DFS trees is defined as follows.

Definition 3.8. For a graph game (N, v, L), the marginal contribution vector $\boldsymbol{x}^T \in \mathbb{R}^n$ corresponding to $T \in \mathcal{B}^{\mathrm{DFS}}$ is the vector of payoffs given by

$$x_i^T = v(des^T(i)) - \sum_{K \in \hat{C}(des^T(i) \setminus \{i\})} v(K), \quad i \in N. \tag{3.4}$$

Definition 3.9 (Average Tree Solution)**.** On the class of arbitrary graph games (N, v, L), the average tree solution constructed by DFS is given by

$$\bar{\boldsymbol{x}} = \frac{1}{|\mathcal{B}^{\mathrm{DFS}}|} \sum_{T \in \mathcal{B}^{\mathrm{DFS}}} \boldsymbol{x}^T. \tag{3.5}$$

To distinguish from the average tree solution of Definition 3.5, the average tree solution constructed by DFS is denoted by $\bar{\boldsymbol{x}}$.

4. Average Tree Solutions and the Core

This section studies conditions for graph games such that average tree solutions belong to the core. We first introduce link-convexity given by Herings et al. [6] and present an alternative condition for arbitrary graph games to make the average tree solution be in the core. Next we give the class of graph games such that link-convexity ensures that the average tree solution is an element of the core.

4.1. **Link-convexity.**

Definition 4.1 (Link-convexity)**.** (N, v, L) is link-convex if

$$v(S) + v(T) \leq v(S \cup T) + \sum_{K \in \hat{C}^L(S \cap T)} v(K) \tag{4.1}$$

for all $S, T \in C^L(N)$ that satisfy

(LC1) $S \setminus T \in C^L(N)$ and $T \setminus S \in C^L(N)$

(LC2) $(S \setminus T) \cup (T \setminus S) \in C^L(N)$

(LC3) $N \setminus S \in C^L(N)$ or $N \setminus T \in C^L(N)$.

It was shown in Herings et al. [6] that for games on a complete graph link-convexity and convexity coincides with each other and that for games on a cycle-free graph link-convexity is even weaker than superadditivity.

Concerning arbitrary graph games satisfying link-convexity the following is claimed in Herings et al. [6].

Claim 4.2 (Herings et al. [6])**.** *Let (N, v, L) be a link-convex game. Then, the average tree solution $\bar{\boldsymbol{y}}$ is an element of the core.*

Herings et al. [6] proved Claim 4.2 by showing that all the marginal contribution vectors of Definition 3.4 are in the core of the game, i.e.,

$$\boldsymbol{y}^T \in \text{Core}(N, v, L) \quad \text{for all } T \in \mathcal{B}^{\text{ADM}}. \tag{4.2}$$

From the above, $\bar{\boldsymbol{y}} \in \text{Core}(N, v, L)$ holds since the core is a convex set.

We give here an example of link-convex game and its marginal contribution vectors $\boldsymbol{y}^T \in \mathbb{R}^n$.

Example 4.3. Consider the graph game with $N := \{1, 2, 3, 4, 5\}$ and $L := \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$ in Figure 1. The characteristic function values are given by

$$\begin{array}{cccc}
v(\{1\}) = 0 & v(\{1,4\}) = 1 & v(\{1,2,4\}) = 1 & v(\{1,2,3,4\}) = 10 \\
v(\{2\}) = 0 & v(\{2,3\}) = 6 & v(\{1,3,4\}) = 2 & v(\{1,2,3,5\}) = 7 \\
v(\{3\}) = 0 & v(\{3,4\}) = 1 & v(\{1,4,5\}) = 7 & v(\{1,2,4,5\}) = 7 \\
v(\{4\}) = 0 & v(\{3,5\}) = 6 & v(\{2,3,4\}) = 9 & v(\{1,3,4,5\}) = 13 \\
v(\{5\}) = 0 & v(\{4,5\}) = 1 & v(\{2,3,5\}) = 6 & v(\{2,3,4,5\}) = 9 \\
v(\{1,2\}) = 1 & v(\{1,2,3\}) = 6 & v(\{3,4,5\}) = 7 & v(\{1,2,3,4,5\}) = 19
\end{array}$$

$$\begin{array}{cccc}
v(\{1,3\}) = 0 & v(\{1,5\}) = 0 & v(\{1,2,4\}) = 0 & v(\{1,2,5\}) = 0 \\
v(\{2,4\}) = 0 & v(\{2,5\}) = 0 & v(\{1,3,5\}) = 0 & v(\{2,4,5\}) = 0
\end{array}$$

It is a routine to see that this graph game is link-convex.

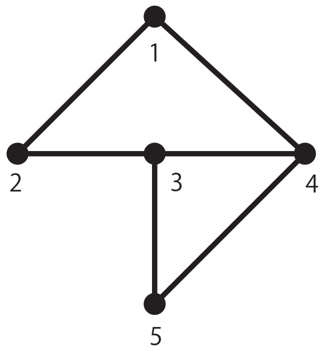


FIGURE 1. counter graph

Consider a tree $T_1 := \{(1,2),(2,3),(3,5),(5,4)\} \in \mathcal{B}^{\mathrm{ADM}}$ in Figure 2. We obtain the corresponding marginal contribution vector as follows.

$$\begin{aligned}
y_1^{T_1} &= v(\{1,2,3,4,5\}) - v(\{2,3,4,5\}) = 19 - 9 = 10 \\
y_2^{T_1} &= v(\{2,3,4,5\}) - v(\{3,4,5\}) = 9 - 7 = 2 \\
y_3^{T_1} &= v(\{3,4,5\}) - v(\{4,5\}) = 7 - 1 = 6 \\
y_4^{T_1} &= v(\{4\}) = 0 \\
y_5^{T_1} &= v(\{4,5\}) - v(\{4\}) = 1 - 0 = 1.
\end{aligned}$$

Then we have

$$y^{T_1}(\{2,3,4\}) = \ y_2^{T_1} + y_3^{T_1} + y_4^{T_1} = 2 + 6 + 0 = 8 < v(\{2,3,4\}) = 9.$$

It follows that

$$\boldsymbol{y}^{T_1} \notin \mathrm{Core}(N, v, L).$$

Similarly let $T_2 := \{(2,1),(1,4),(4,5),(5,3)\} \in \mathcal{B}^{\mathrm{ADM}}$ in Figure 3 and the corresponding marginal contribution vector is as follows.

$$\begin{aligned}
y_1^{T_2} &= v(\{1,3,4,5\}) - v(\{3,4,5\}) = 13 - 7 = 6 \\
y_2^{T_2} &= v(\{1,2,3,4,5\}) - v(\{1,3,4,5\}) = 19 - 13 = 6 \\
y_3^{T_2} &= v(\{3\}) = 0 \\
y_4^{T_2} &= v(\{3,4,5\}) - v(\{3,5\}) = 7 - 6 = 1 \\
y_5^{T_2} &= v(\{3,5\}) - v(\{3\}) = 6 - 0 = 6.
\end{aligned}$$

Then we have

$$y^{T_2}(\{2,3,4\}) = \ y_2^{T_2} + y_3^{T_2} + y_4^{T_2} = 6 + 0 + 1 = 7 < v(\{2,3,4\}) = 9,$$

hence

$$\boldsymbol{y}^{T_2} \notin \mathrm{Core}(N, v, L).$$

These vectors are counter-examples to the statement (4.2).

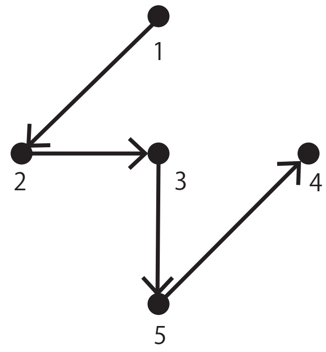


FIGURE 2. T_1

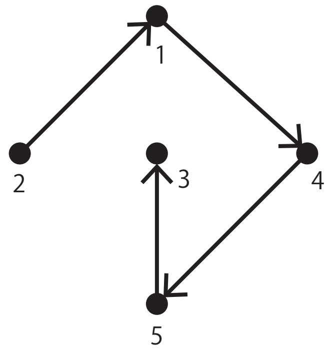


FIGURE 3. T_2

4.2. **Revised link-convexity.** The next condition should replace to link-convexity for the average tree solution to lie in the core.

Definition 4.4 (Revised link-convexity)**.** (N, v, L) is revised link-convex if

$$v(S) + v(T) \leq v(S \cup T) + \sum_{K \in \hat{C}^L(S \cap T)} v(K) \tag{4.3}$$

for all $S, T \in C^L(N)$ that satisfy

(RL1) $S \setminus T \in C^L(N)$ or $T \setminus S \in C^L(N)$

(RL2) $S \cup T \in C^L(N)$

(RL3) $N \setminus S \in C^L(N)$ or $N \setminus T \in C^L(N)$.

If $S, T \in C^L(N)$ satisfy (LC1) and (LC2), these two sets also satisfy (RL2). Additionally, it is clear that (RL1) is a weaker condition of (LC1) and (RL3) coincides with (LC3). Thus in general revised link-convexity is a stronger condition than link-convexity.

Theorem 4.5. *Let (N, v, L) be a revised link-convex game. Then, the average tree solution $\bar{\boldsymbol{x}}$ is an element of the core.*

Proof. Since the core is a convex set, it suffices to prove that for every $T \in \mathcal{B}^{\mathrm{DFS}}$ the marginal vector $\boldsymbol{x}^T$ with respect to tree $T \in \mathcal{B}^{\mathrm{DFS}}$ is an element of the core, i.e.,

$$\boldsymbol{x}^T \in \mathrm{Core}(N, v, L) \quad \text{for all } T \in \mathcal{B}^{\mathrm{DFS}}.$$

Take any tree $T \in \mathcal{B}^{\mathrm{DFS}}$ and let $\boldsymbol{x}^T$ be the corresponding marginal contribution vector. We will show that

$$x^T(S) \geq v(S) \quad \text{for all } S \in C^L(N),$$

from which it follows that $\boldsymbol{x}^T \in \mathrm{Core}(N, v, L)$. Take any $S \in C^L(N)$. The subgraph $(S, T(S))$ has components $S_1, S_2, \ldots, S_m$, which are all rooted trees with roots $r_1, r_2, \ldots, r_m$. Let $r_1, r_2, \ldots, r_m$ be indexed such that

$$m_1 < m_2 \Rightarrow des(r_{m_1}) \subseteq des(r_{m_2}) \text{ or } des(r_{m_1}) \cap des(r_{m_2}) = \emptyset. \tag{4.4}$$

For $k = 0, \ldots, m$, let $D^k := des(r_1) \cup des(r_2) \cup \cdots \cup des(r_k)$, with the convention that $D^0 = \emptyset$. For $k = 0, \ldots, m$, those successors of S_k in the tree T that lie outside S are denoted by $\delta(S_k) := \{\, i \mid (j, i) \in T, j \in S_k, i \notin S_k \} = \{i_1, i_2, \ldots, i_l\,\}$. We write $R := \{r_1, r_2, \ldots, r_m\}$ and $I := \bigcup_{k=1}^m \delta(S_k)$. For a node $i \in I$ we define $\Delta(i) := \{\, r \in R \mid des(r) \subseteq des(i)\,\}$, $\Delta^*(i) := \{\, r \in R \mid des(r) \subseteq des(i), \nexists r' \in R \setminus \{r\} \text{ s.t. } des(r) \subseteq des(r') \subseteq des(i)\,\}$. Note that $\bigcup_{r \in \Delta(i)} des(r) = \bigcup_{r \in \Delta^*(i)} des(r)$.

For $i \in N$ we simply denote $D_i := des(i)$. Consider some $k \in \{1, 2, \ldots, m\}$ and suppose $\delta(S_k) \neq \emptyset$. Take any $i_h \in \delta(S_k)$ and the following two sets

$$\begin{aligned} U &:= S \cup D^{k-1} \cup D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_{h-1}} \\ W &:= D_{i_h}. \end{aligned}$$

Then $U, W \in C^L(N)$ and satisfy the following three conditions of Definition 4.4.

(RL1) $W \setminus U = D_{i_h} \setminus (\bigcup_{r \in \Delta(i_h)} D_r) = D_{i_h} \setminus (\bigcup_{r \in \Delta^*(i_h)} D_r) \in C^L(N)$
(RL2) $U \cup W = S \cup D^{k-1} \cup D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_{h-1}} \cup D_{i_h} \in C^L(N)$
(RL3) $N \setminus W = N \setminus D_{i_h} \in C^L(N)$.

Now it follows from revised link-convexity that for $i_h \in \{i_1, i_2, \ldots, i_l\}$,

$$\begin{aligned} &v(S \cup D^{k-1} \cup (D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_{h-1}})) + v(D_{i_h}) \\ &\quad \leq v(S \cup D^{k-1} \cup D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_{h-1}} \cup D_{i_h}) + \sum_{r \in \Delta^*(i_h)} v(D_r) \end{aligned}$$

By repeated application of this argument, it follows that

$$v(S \cup D^{k-1}) + \sum_{i \in \delta(S_k)} v(D_i) \leq v(S \cup D^k) + \sum_{i \in \delta(S_k)} \sum_{r \in \Delta^*(i)} v(D_r). \tag{4.5}$$

Notice that this formula (4.5) is also valid if $\delta(S_k) = \emptyset$, since $S \cup D^{k-1} = S \cup S_k \cup D^{k-1} = S \cup D_{r_k} \cup D^{k-1} = S \cup D^k$. By repeated application of the last inequality (4.5), we see that

$$v(S) + \sum_{k=1}^m \sum_{i \in \delta(S_k)} v(D_i) \leq v(S \cup D^m) + \sum_{k=1}^m \sum_{i \in \delta(S_k)} \sum_{r \in \Delta^*(i)} v(D_r). \tag{4.6}$$

Recall that r_m is the first node discovered by Depth First Search in S. From lemma 3.6, it follows that

$$D_r \subseteq D_{r_m} \quad \text{for all } r \in R \setminus \{r_m\}.$$

Hence, every D_{r_k} for $k = 1, 2, \dots m-1$ appears exactly once in the right-hand side, i.e.,

$$\sum_{k=1}^{m} \sum_{i \in \delta(S_k)} \sum_{r \in \Delta^*(i)} v(D_r) = \sum_{k=1}^{m-1} v(D_{r_k}).$$

Since $v(S \cup D^m) = v(D_{r_m})$, we obtain

$$v(S \cup D^m) + \sum_{k=1}^{m} \sum_{i \in \delta(S_k)} \sum_{r \in \Delta^*(i)} v(D_r) = \sum_{k=1}^{m} v(D_{r_k}).$$

Therefore,

$$\begin{aligned}
(4.8) \Leftrightarrow \quad & v(S) + \sum_{k=1}^{m} \sum_{i \in \delta(S_k)} v(D_i) \le \sum_{k=1}^{m} v(D_{r_k}) \\
\Leftrightarrow \quad & v(S) \le \sum_{k=1}^{m} (v(D_{r_k}) - \sum_{i \in \delta(S_k)} v(D_i)) = \sum_{k=1}^{m} x^T(S_k) = x^T(S).
\end{aligned}$$

□

Corollary 4.6. *Let (N, v^L) be a convex game. Then, the average tree solution $\bar{\boldsymbol{x}}$ is an element of the core.*

Proof. Let $S, T \in C^L(N)$ satisfy the conditions (RL1)∼(RL3) of revised link-convexity. Convexity of (N, v^L) implies that

$$\begin{aligned}
& v^L(S) + v^L(T) \le v^L(S \cup T) + v^L(S \cap T) \\
\Leftrightarrow \quad & v(S) + v(T) \le v(S \cup T) + \sum_{K \in \hat{C}^L(S \cap T)} v(K).
\end{aligned}$$

Thus the game (N, v, L) is revised link-convex. It immediately follows from Theorem 4.5 that the average tree solution is an element of the core. □

4.3. **Cycle-complete graph.** As we have seen in the previous section, for arbitrary graph games link-convexity is not a sufficient condition to make all the marginal contribution vectors lie in the core. In this section we consider games on the class of cycle-complete graphs, which includes the class of cycle-free and complete graphs. It will be proved that the marginal contribution vectors belong to the core for arbitrary cycle-complete graph games satisfying link-convexity.

4.3.1. *Convexity on cycle-complete graph games.* Van den Nouweland and Borm [7] presented that convexity of (N, v) is a necessary and sufficient condition for convexity of (N, v^L). We prove that convexity of (N, v, L) is a necessary and sufficient condition for convexity of (N, v^L) when the underlying graph is cycle-complete, following the proof given by Van den Nouweland and Borm [7].

Theorem 4.7. *Let (N, L) be a cycle-complete graph. (N, v, L) is a convex game if and only if (N, v^L) is convex.*

Proof. Suppose that (N, v, L) is convex. Let $S, T \in C^L(N)$ be such that $S \cup T \in C^L(N)$ and $S \cap T \in C^L(N) \cup \{\emptyset\}$. Then, from convexity of (N, v, L), it holds that

$$v^L(S) + v^L(T) \leq v^L(S \cup T) + v^L(S \cap T)$$
$$\Leftrightarrow v(S) + v(T) \leq v(S \cup T) + v(S \cap T).$$

For the converse part, suppose (N, v^L) is convex. Let $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$. It suffices to show that $v^L(S \cup \{i\}) - v^L(S) \leq v^L(T \cup \{i\}) - v^L(T)$, i.e.,

$$\sum_{K \in \hat{C}^L(S \cup \{i\})} v(K) - \sum_{K \in \hat{C}^L(S)} v(K) \leq \sum_{K \in \hat{C}^L(T \cup \{i\})} v(K) - \sum_{K \in \hat{C}^L(T)} v(K). \tag{4.7}$$

Now, let $\mathcal{E}$ denote the set of connected components containing at least one node j with $\{i, j\} \in L$, i.e.,

$$\mathcal{E} := \{\ E \in \hat{C}^L(S) \mid \exists j \in E \text{ s.t. } \{i, j\} \in L\ \},$$

and let $E_i := \{i\} \cup \bigcup_{E \in \mathcal{E}} E$. By definition, we have $E_i \in \hat{C}^L(S \cup \{i\})$, and for every $E \in \hat{C}^L(S \cup \{i\})$ such that $E \neq E_i$ it holds that $E \in C^L(S)$. As a consequence, we obtain

$$\sum_{E \in \hat{C}^L(S \cup \{i\})} v(E) - \sum_{E \in \hat{C}^L(S)} v(E) = v(\{i\} \cup (\bigcup_{E \in \mathcal{E}} E)) - \sum_{E \in \mathcal{E}} v(E).$$

Similarly, we have

$$\sum_{F \in \hat{C}^L(T \cup \{i\})} v(F) - \sum_{F \in \hat{C}^L(T)} v(F) = v(\{i\} \cup (\bigcup_{F \in \mathcal{F}} F)) - \sum_{F \in \mathcal{F}} v(F),$$

where $\mathcal{F} := \{\ F \in \hat{C}^L(T) \mid \exists j \in F \text{ s.t. } \{i, j\} \in L\ \}$. Hence, (4.7) is equivalent to

$$v(\{i\} \cup (\bigcup_{E \in \mathcal{E}} E)) - \sum_{E \in \mathcal{E}} v(E) \leq v(\{i\} \cup (\bigcup_{F \in \mathcal{F}} F)) - \sum_{F \in \mathcal{F}} v(F).$$

Next we will consider the relationship between the set $\mathcal{E}$ and $\mathcal{F}$. Since $S \subseteq T$, there exists a unique $F \in \mathcal{F}$ with $E \subseteq F$, for all $E \in \mathcal{E}$. Here, we will show that for each $F \in \mathcal{F}$,

$$\exists! E \in \mathcal{E} \text{ s.t. } E \subseteq F, \text{ or } \nexists E \in \mathcal{E} \text{ s.t. } E \subseteq F.$$

Assume that there are $E^1, E^2 \in \mathcal{E}(E^1 \neq E^2)$ and $F' \in \mathcal{F}$ such that $E^1 \subseteq F'$ and $E^2 \subseteq F'$. Let $j_1 \in E^1$ and $j_2 \in E^2$ be such that $\{i, j_1\} \in L$ and $\{i, j_2\} \in L$. Note that $\{j_1, j_2\} \notin L$ since E^1 and E^2 are connected components of S, respectively. Since $j_1, j_2 \in F' \in \hat{C}^L(T)$, there exists a path $j_1 \sim_{F'} j_2$. Since $i \notin T$, there is a cycle from i to i over j_1 and j_2 in (N, L). However, since a graph (N, L) is cycle-complete this should imply that $\{j_1, j_2\} \in L$, which leads to a contradiction. Hence we can number the elements of $\mathcal{E}$ and $\mathcal{F}$ as follows:

$$\mathcal{E} = \{E_1, E_2, \ldots, E_s\} \text{ and } \mathcal{F} = \{F_1, F_2, \ldots, F_t\},$$

where $t \geq s$ and $E_k \subseteq F_k$ for all $k \in \{1, 2, ..., s\}$.

Now, we can use the properties of the game (N, v, L). For all $k = s+1, s+2, \ldots, t$, $F_k \in C^L(N)$, $\{i\} \cup (\bigcup_{h=1}^{s} F_h) \in C^L(N)$ and $\{i\} \cup (\bigcup_{h=1}^{t} F_k) \in C^L(N)$. Moreover, (N, v, L) is superadditive when (N, v, L) is convex. Superadditivity of the game (N, v, L) implies

$$v(\{i\} \cup (\bigcup_{F \in \mathcal{F}} F)) \geq v(\{i\} \cup (\bigcup_{h=1}^{s} F_h)) + \sum_{h=s+1}^{t} v(F_h). \tag{4.8}$$

For all $k = 1, 2, \ldots, s$,

$$\begin{aligned}
&F_k \in C^L(N),\\
&\{i\} \cup (\bigcup_{h=k+1}^{s} F_h) \cup (\bigcup_{h=1}^{k} E_h) \in C^L(N),\\
&F_k \cap (\{i\} \cup (\bigcup_{h=k+1}^{s} F_h) \cup (\bigcup_{h=1}^{k} E_h)) = E_k \in C^L(N) \text{ and}\\
&F_k \cup (\{i\} \cup (\bigcup_{h=k+1}^{s} F_h) \cup (\bigcup_{h=1}^{k} E_h)) = \{i\} \cup (\bigcup_{h=k}^{s} F_h) \cup (\bigcup_{h=1}^{k-1} E_h) \in C^L(N).
\end{aligned}$$

Convexity of the game (N, v, L) implies

$$\begin{gathered}
v(\{i\} \cup (\bigcup_{h=1}^{s} F_h)) - v(F_1) \geq v(\{i\} \cup (\bigcup_{h=2}^{s} F_h) \cup E_1) - v(E_1)\\
\vdots\\
v(\{i\} \cup (\bigcup_{h=k}^{s} F_h) \cup (\bigcup_{h=1}^{k-1} E_h)) - v(F_k) \geq v(\{i\} \cup (\bigcup_{h=k+1}^{s} F_h) \cup (\bigcup_{h=1}^{k} E_h)) - v(E_k)\\
\vdots\\
v(\{i\} \cup F_s \cup (\bigcup_{h=1}^{s-1} E_h)) - v(F_s) \geq v(\{i\} \cup (\bigcup_{h=1}^{s} E_h)) - v(E_s).
\end{gathered}$$

Adding all these s inequalities, we obtain

$$v(\{i\} \cup (\bigcup_{h=1}^{s} F_h)) - \sum_{h=1}^{s} v(F_h) \geq v(\{i\} \cup (\bigcup_{E \in \mathcal{E}} E)) - \sum_{E \in \mathcal{E}} v(E). \tag{4.9}$$

Now, (4.8) and (4.9) readily imply (4.7). □

Theorem 4.8. *Let (N, L) be a cycle-complete graph and let (N, v, L) be a convex game. Then (N, v, L) is link-convex.*

Proof. Let S, $T \in C^L(N)$ satisfy (LC1)∼(LC3) of Definition 4.1. When $S \cap T = \emptyset$, $S \cup T = (S \setminus T) \cup (T \setminus S) \in C^L(N)$. When $S \cap T \neq \emptyset$, clearly $S \cup T \in C^L(N)$ because $S, T \in C^L(N)$. Now, assume that $S \cap T \notin C^L(N)$, i.e.,

$$\exists i_1, i_2 \in S \cap T \text{ such that } \nexists \text{path } i_1 \sim_{S\cap T} i_2. \tag{4.10}$$

Since $i_1, i_2 \in S \in C^L(N)$,

$$\exists \text{path } i_1 \sim_S i_2$$

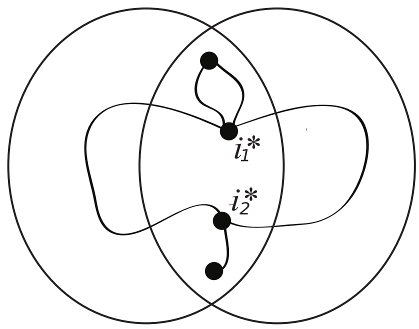
Let P^S be the shortest path among the above paths. Since $i_1, i_2 \in T \in C^L(N)$,

$$\exists \text{path } i_1 \sim_T i_2$$

Let P^T be the shortest path among the above paths. By assumption (4.10), P^S has at least one node in $S \setminus T$ and P^T has at least one node in $T \setminus S$. Thus P^S and P^T are different. Let $\rho(i_1, i_2)$ denote the sum of the lengths of P^S and P^T. Let $i_1^*, i_2^* \in S \cap T$ be such that

$$\rho(i_1^*, i_2^*) = \min\{\, \rho(i_1, i_2) \mid i_1, i_2 \in S \cap T, \nexists \text{ path } i_1 \sim_{S\cap T} i_2 \,\}.$$

For i_1^* and i_2^*, the corresponding P^S and P^T form a cycle. By the assumption that (N, L) is a cycle-complete graph, there exists a edge between any two nodes in the cycle. Thus, $\{i_1^*, i_2^*\} \in L$, which leads to a contradiction and we conclude that $S \cap T \in C^L(N)$. Convexity of the game (N, v, L) implies that $v(S) + v(T) \leq v(S \cup T) + v(S \cap T) = v(S \cup T) + \sum_{K \in \hat{C}^L(S\cap T)} v(K)$.



$\square$

Theorem 4.9. *Let (N, L) be a cycle-complete graph and let (N, v, L) be a link-convex game. Then, the average tree solution $\bar{\boldsymbol{x}}$ is an element of the core.*

Proof. We will prove this theorem in a similar way to the proof of Theorem 4.5.

Take any tree $T \in \mathcal{B}^{\mathrm{DFS}}$ and let $\boldsymbol{x}^T$ be the corresponding marginal vector. Take any $S \in C^L(N)$, and consider the subgraph $(S, T(S))$. It has components $S_1, S_2, \ldots, S_m$, which are all rooted trees with roots $r_1, r_2, \ldots, r_m$. Let $r_1, r_2, \ldots, r_m$ be indexed such that

$$m_1 < m_2 \Rightarrow des(r_{m_1}) \subseteq des(r_{m_2}) \text{ or } des(r_{m_1}) \cap des(r_{m_2}) = \emptyset. \tag{4.11}$$

For $k = 0, \dots, m$, let $D^k := des(r_1) \cup des(r_2) \cup \cdots \cup des(r_k)$, with the convextion that $D^0 = \emptyset$. For $k = 0, \dots, m$, those successors of S_k in the tree T that lie outside S are denoted by $\delta(S_k) := \{ i \mid (j,i) \in T, j \in S_k, i \notin S_k \} = \{i_1, i_2, \dots, i_l\}$. We write $R := \{r_1, r_2, \dots, r_m\}$ and $I := \bigcup_{k=1}^m \delta(S_k)$. For a node $i \in I$, we define $\Delta(i) := \{ r \in R \mid des(r) \subseteq des(i) \}$, $\Delta^*(i) := \{ r \in R \mid des(r) \subseteq des(i), \nexists r' \in R \setminus \{r\} \text{ s.t. } des(r) \subseteq des(r') \subseteq des(i) \}$. Note that $\bigcup_{r \in \Delta(i)} des(r) = \bigcup_{r \in \Delta^*(i)} des(r)$. For $i \in N$, we simply denote $D_i := des(i)$.

Consider some $k \in \{1, 2, \dots, m\}$ and suppose $\delta(S_k) \neq \emptyset$. Take any $i_h \in \delta(S_k)$ and consider the following two sets

$$U := S \cup D^{k-1} \cup D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_{h-1}}$$
$$W := D_{i_h}.$$

We will show that $U, W \in C^L(N)$ satisfy (LC1)$\sim$(LC3) of Definition 4.1. From the proof of Theorem 4.5, we obtain $W \setminus U, N \setminus W \in C^L(N)$.

Next we will prove that

$$U \setminus W \in C^L(N),\ (U \setminus W) \cup (W \setminus U) \in C^L(N).$$

First, we consider whether $U \setminus W$ is connected or not. Since $U \cap W = \bigcup_{r \in \Delta(i_h)} D_r$,

$$\begin{aligned}
U \setminus W &= U \setminus (U \cap W) \\
&= S \cup D^{k-1} \cup (D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_{h-1}}) \setminus \bigcup_{r \in \Delta(i_h)} D_r \\
&= \bigcup_{p \in \{1,2,\dots,m\}} S_p \cup \bigcup_{p \in \{1,2,\dots,k-1\}} D_{r_p} \cup (D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_{h-1}}) \setminus \bigcup_{r \in \Delta(i_h)} D_r \\
&= \bigcup_{\substack{p \in \{1,2,\dots,m\} \\ r_p \notin \Delta(i_h)}} S_p \cup \bigcup_{\substack{p \in \{1,2,\dots,k-1\} \\ r_p \notin \Delta(i_h)}} D_{r_p} \cup (D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_{h-1}}).
\end{aligned} \tag{4.12}$$

By definition of $S_1, S_2, \dots, S_m$,

$$\begin{aligned}
&S_p \subseteq D_{r_p} \text{ and } D_{r_p} \in C^L(N) \quad \text{for all } p \in \{1, 2, \dots, m\} \\
&\Rightarrow S_p \subseteq D_{r_p} \text{ and } D_{r_p} \in C^L(N) \quad \text{for all } p \in \{1, 2, \dots, k-1\} (r_p \notin \Delta(i_h)).
\end{aligned} \tag{4.13}$$

Since $i_1, i_2, \cdots, i_{h-1} \in \delta(S_k)$,

$$S_k \cup D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_{h-1}} \in C^L(N). \tag{4.14}$$

In addition, $r_k \notin \Delta(i_h)$ implies

$$S_k \subseteq \bigcup_{\substack{p \in \{1,2,\dots,m\} \\ r_p \notin \Delta(i_h)}} S_p. \tag{4.15}$$

In order to prove $U \setminus W \in C^L(N)$, from (4.12)∼(4.15), it suffices to show that

$$\bigcup_{\substack{p\in\{1,2,\dots,m\}\\ r_p\notin\Delta(i_h)}} S_p \in C^L(N).$$

Let $\hat{S} := \bigcup_{\substack{p\in\{1,2,\dots,m\}\\ r_p\notin\Delta(i_h)}} S_p$. Since $S_1, S_2, \dots, S_m \in C^L(N)$, in order to prove $\hat{S} \in C^L(N)$, it suffices to show that

$$\text{for all } m_1, m_2 \in \{1, 2, \dots, m\}\ (r_{m_1}, r_{m_2} \notin \Delta(i_h), m_1 \neq m_2),$$
$$\text{there exists a path } i \sim_{\hat{S}} j \text{ for any } i \in S_{m_1}, j \in S_{m_2}.$$

Now consider two distinct indexes $m_1, m_2 \in \{1, \dots, m\}$ $(r_{m_1}, r_{m_2} \notin \Delta(i_h), m_1 \neq m_2)$. Without loss of generality we suppose that $m_2 > m_1$. From (4.11) we find that $des(r_{m_1})$ and $des(r_{m_2})$ satisfy

$$des(r_{m_1}) \subseteq des(r_{m_2}) \text{ or } des(r_{m_1}) \cap des(r_{m_2}) = \emptyset.$$

(I) When $des(r_{m_1}) \subseteq des(r_{m_2})$, consider the following two cases.

(i) Suppose $\nexists\ r \in R \setminus \{r_{m_1}, r_{m_2}\}$ such that $des(r_{m_1}) \subseteq des(r) \subseteq des(r_{m_2})$.

Take any $i_1 \in S_{m_1}, i_2 \in S_{m_2}$. Since T is a rooted spanning tree of the graph (N, L), $(N, L(T))$ is connected. Hence, there exists a path from $i_1 \in S_{m_1}$ to $i_2 \in S_{m_2}$ in $(N, L(T))$, that is,

$$\exists \text{path } i_1 \sim i_2.$$

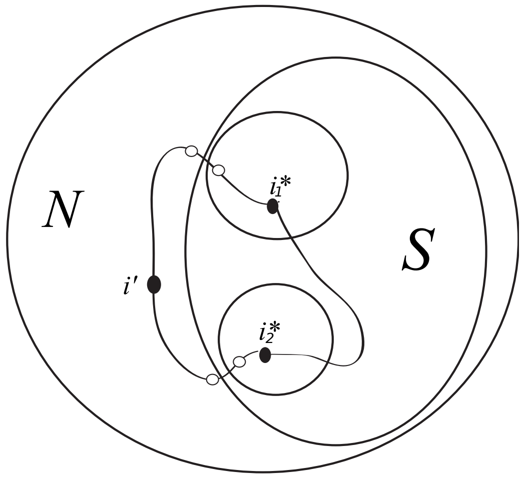
Let P^T be the shortest path among the above paths. By $i_1, i_2 \in S \in C^L(N)$ it holds that

$$\exists \text{path } i_1 \sim_S i_2$$

Let P^S be the shortest path among the above paths. Since S_{m_1} and S_{m_2} are components of $(S, T(S))$, P^T has at least one node that lies outside S. Thus, P^T is different from P^S. Let $\rho(i_1, i_2)$ denote the sum of the lengths of P^T and P^S determined by i_1 and i_2. Let $i_1^* \in S_{m_1}, i_2^* \in S_{m_2}$ be such that

$$\rho(i_1^*, i_2^*) = \min\{\, \rho(i_1, i_2) \mid i_1 \in S_{m_1}, i_2 \in S_{m_2} \,\}.$$

Meanwhile, by the assumption, $\nexists r \in R \setminus \{r_{m_1}, r_{m_2}\}$ such that $des(r_{m_1}) \subseteq des(r) \subseteq des(r_{m_2})$, P^T does not contain any nodes in S other than those in S_{m_1}, S_{m_2}. Therefore, the corresponding P^T and P^S for i_1^*, i_2^* form a cycle. By the assumption that (N, L) is a cycle-complete graph, there is a edge between any two nodes in the cycle. Thus, $\{i_1^*, i_2^*\} \in L$, which leads to $S_{m_1} \cup S_{m_2} \in C^L(N)$.



(ii) Suppose $\exists r \in R \setminus \{r_{m_1}, r_{m_2}\}$ such that $des(r_{m_1}) \subseteq des(r) \subseteq des(r_{m_2})$.
Let $R^* = \{\, r \in R \mid des(r_{m_1}) \subseteq des(r) \subseteq des(r_{m_2}) \,\}$. Assume that there is a node $r \in R^*$ such that $r \in \Delta(i_h)$, i.e., $des(r) \subseteq des(i_h)$. By $r_{m_1} \notin \Delta(i_h)$, it holds that

$$des(i_h) \cap des(r_{m_1}) = \emptyset \text{ or } des(i_h) \subsetneq des(r_{m_1}),$$

which implies

$$des(r) \cap des(r_{m_1}) = \emptyset \text{ or } des(r) \subsetneq des(r_{m_1}).$$

This contradicts the fact that $r \in R^*$. Hence, $r \notin \Delta(i_h)$ for all $r \in R^*$. Therefore,

$$S_p \subseteq \bigcup_{\substack{r_q \in R \\ r_q \notin \Delta(i_h)}} S_q \quad \text{for all } r_p \in R^*,$$

which implies

$$\bigcup_{r_p \in R^*} S_p \subseteq \bigcup_{\substack{r_q \in R \\ r_q \notin \Delta(i_h)}} S_q = \hat{S}. \tag{4.16}$$

Next, let $R^* := \{r_{\pi(1)}, r_{\pi(2)}, \ldots, r_{\pi(z)}\}(r_{m_1} = r_{\pi(1)}, r_{m_2} = r_{\pi(z)})$ and let $r_{\pi(1)}, r_{\pi(2)}, \ldots, r_{\pi(z)}$ be indexed such that for $q = 2, 3, \ldots, z$,

$$des(r_{\pi(q-1)}) \subseteq des(r_{\pi(q)}) \text{ and } \nexists r \in R \text{ s.t. } des(r_{\pi(q-1)}) \subseteq des(r) \subseteq des(r_{\pi(q)}).$$

From the result of (i),

$$S_{r_{\pi(q-1)}} \cup S_{r_{\pi(q)}} \in C^L(N) \quad \text{for } q = 2, 3, \ldots, z.$$

Hence,

$$\bigcup_{r_p \in R^*} S_p = \bigcup_{q=1}^{z} S_{r_{\pi(q)}} \in C^L(N),$$

and from (4.16) we obtain

$$\exists\ path\ i \sim_{\hat{S}} j \quad \text{for all } i \in S_{m_1},\ j \in S_{m_2}.$$

(II) When $des(r_{m_1}) \cap des(r_{m_2}) = \emptyset$, let $a \in N$ be the last common ancestor of r_{m_1} and r_{m_2} in (N, T). Let $r' \in R$ satisfy

$$|des(r')| = \min\{\, |des(r)| \mid r \in R, a \in des(r)\,\}.$$

Analogously, we obtain

$$\exists\ path\ i \sim_{\hat{S}} r' \quad \text{for all } i \in S_{m_1},$$
$$\exists\ path\ j \sim_{\hat{S}} r' \quad \text{for all } j \in S_{m_2}.$$

Thus,

$$\exists\ path\ i \sim_{\hat{S}} j \quad \text{for all } i \in S_{m_1},\ j \in S_{m_2}.$$

Therefore $\hat{S} \in C^L(N)$, i.e., $U \setminus W \in C^L(N)$.

Now we obtain $U \setminus W, W \setminus U \in C^L(N)$. From $i_h \in \delta(S_k)$, there exists a node $j \in S_k \subseteq U \setminus W$ such that $\{j, i_h\} \in L$. Moreover, $i_h \in W \setminus U$. Thus $(U \setminus W) \cup (W \setminus U) \in C^L(N)$.

Hence, U and W satisfy the following conditions of link-convexity.

(LC1) $U \setminus W \in C^L(N)$ and $W \setminus U \in C^L(N)$

(LC2) $(U \setminus W) \cup (W \setminus U) \in C^L(N)$

(LC3) $N \setminus W \in C^L(N)$.

Now link-convexity of the game implies that for all $i_h \in \{i_1, i_2, \ldots, i_l\}$,

$$\begin{aligned} &v(S \cup D^{k-1} \cup (D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_{h-1}})) + v(D_{i_h}) \\ &\quad \le v(S \cup D^{k-1} \cup D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_{h-1}} \cup D_{i_h}) + \sum_{r \in \Delta^*(i_h)} v(D_r). \end{aligned}$$

By repeated application of this argument, it follows that

$$v(S \cup D^{k-1}) + \sum_{i \in \delta(S_k)} v(D_i) \le v(S \cup D^k) + \sum_{i \in \delta(S_k)} \sum_{r \in \Delta^*(i)} v(D_r). \tag{4.17}$$

Notice that this formula (4.17) is also valid if $\delta(S_k) = \emptyset$, since $S \cup D^{k-1} = S \cup S_k \cup D^{k-1} = S \cup D_{r_k} \cup D^{k-1} = S \cup D^k$. Repeating the same argument in the proof of Theorem 4.5 completes the proof. □

Corollary 4.10. *Let (N, L) be a cycle-complete graph and let (N, v, L) be a convex game. Then, the average tree solution $\bar{\boldsymbol{x}}$ is an element of the core.*

Proof. The corollary follows immediately from Theorem 4.8 and 4.9. □

Corollary 4.11. *Let (N, L) be a cycle-complete graph and let (N, v) be a convex game. Then, the average tree solution* $\bar{\boldsymbol{x}}$ *is an element of the core.*

Proof. Convexity of the game (N, v) readily implies convexity of the game (N, v, L). Then, the corollary follows immediately from Corollary 4.10. □

5. Concluding remarks

In this paper we have discussed the relationship between the core and the average tree solution. We gave an alternative condition that should replace link-convexity for the average tree solution to be an element of the core of arbitrary graph games. For the class of games with a cycle-complete graph structure, we found that link-convexity guarantees that the average tree solution belongs to the core. In general, revised link-convexity is weaker than convexity and link-convexity is weaker than superadditivity for games with a cycle-free graph. Thus the average tree solution lies in the core for a class of games such that the previous solutions such as the Myerson value and the position value can be outside of the core. This result suggests that the average tree solution can be more stable allocation rule compared to the others.

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