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On a geometric combination of functions related to Prékopa–Leindler inequality

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Abstract

We introduce a new operation between nonnegative integrable functions on $\mathbb{R}^n$, that we call *geometric combination*; it is obtained via a mass transportation approach, playing with inverse distribution functions. The main feature of this operation is that the Lebesgue integral of the geometric combination equals the geometric mean of the two separate integrals; as a natural consequence, we derive a new functional inequality of Prékopa–Leindler type. When applied to the characteristic functions of two measurable sets, their geometric combination provides a set whose volume equals the geometric mean of the two separate volumes. In the framework of convex bodies, by comparing the geometric combination with the 0-sum, we get an alternative proof of the log-Brunn–Minkowski inequality for unconditional convex bodies and for convex bodies with n symmetries.

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1 | INTRODUCTION

The classical Prékopa–Leindler inequality [29, 32–34] states that, given two functions $f, g \in L^1(\mathbb{R}^n; \mathbb{R}_+)$ and a parameter $\lambda \in [0, 1]$, for any measurable function $h : \mathbb{R}^n \to \mathbb{R}_+$ which satisfies

$$h((1-\lambda)x+\lambda y) \geqslant f(x)^{1-\lambda} g(y)^{\lambda} \qquad \forall x, y \in \mathbb{R}^n\,, \tag{1}$$

it holds that

$$\int_{\mathbb{R}^n} h \geqslant \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda} \tag{2}$$

(see also [5, 9, 15, 24, 30, 40] for several extensions and applications). Inequality (2) is commonly considered as a "functional form" of the Brunn–Minkowski inequality for the n-dimensional volume. Indeed, when f and g are the characteristic functions of two measurable sets K and L in $\mathbb{R}^n$, the smallest function h satisfying inequality (1) (named λ-supremal convolution of f and g) agrees with the characteristic function of the set $(1-\lambda)K+\lambda L$. Thus, (2) yields the multiplicative form of Brunn–Minkowski inequality

$$|(1-\lambda)K+\lambda L| \geqslant |K|^{1-\lambda}|L|^{\lambda}\,, \tag{3}$$

which is easily seen to be equivalent to the additive form, namely, to the $(1/n)$-concavity of volume under Minkowski addition.

It is well known that equality in (2) occurs if and only if $f(x) = g(x+b)$ for a log-concave function g and a constant vector b (see [21]), and in (3) if and only if K and L are homothetic convex bodies [38].

When these conditions are far from being satisfied, estimates (2) and (3) may be very rough. This is one of the motivations for the investigation on one hand of quantitative versions of the inequalities [3, 4, 12, 16, 23, 41], and on the other hand of possible different operations between functions (in the analytic framework) or sets (in the geometric one), still allowing to bound from below, respectively, the Lebesgue integral or the volume by the geometric mean of the corresponding quantities.

In the functional setting, a new inequality of Prékopa–Leindler type has been recently obtained in [1]: it is based on the idea of replacing the usual supremal convolution by a kind of geometric supremal convolution, and still implies the Brunn–Minkowski inequality.

In the geometric setting, an inequality closely related to the classical Brunn–Minkowski inequality (3), which is actually widely open and is among the most relevant questions under study in Convex Geometry, is the log-Brunn–Minkowski inequality: stated within the class of centrally symmetric convex sets, the conjecture reads

$$|(1-\lambda)\cdot K +_0 \lambda \cdot L| \geqslant |K|^{1-\lambda}|L|^{\lambda}\,, \tag{4}$$

where, denoting by h_K and h_L the support functions of K and L,

$$(1-\lambda)\cdot K +_0 \lambda \cdot L := \left\{ x \in \mathbb{R}^n \ : \ x\cdot\xi \leqslant h_K(\xi)^{1-\lambda} h_L(\xi)^{\lambda} \quad \forall \xi \in S^{n-1} \right\}.$$

Since the 0-Minkowski combination $(1-\lambda)\cdot K +_0 \lambda \cdot L$ is contained into $(1-\lambda)K+\lambda L$, inequality (4) is clearly a strengthening of (3) (whose relevance is also related to the uniqueness of the convex body with a prescribed cone volume measure, see [14]). Up to now, the log-Brunn–Minkowski conjecture has been proved just in some special cases, including: planar bodies [14], unconditional bodies [10, 19, 37], bodies with symmetries [13], and complex bodies [35]; local versions have been studied in [17, 18, 27].

To the best of our knowledge, no functional version of (4) has been proposed up to now. There is just a related functional inequality, namely, the so-called multiplicative Prékopa–Leindler inequality for functions on $\mathbb{R}^n_+ := (\mathbb{R}_+)^n$, which can be easily deduced from the classical one via an exponential change of variables: given $f, g \in L^1(\mathbb{R}^n_+; \mathbb{R}_+)$ and a parameter $\lambda \in (0,1)$, for any measurable function $h : \mathbb{R}^n_+ \to \mathbb{R}_+$, which satisfies

$$h\big(x_1^{1-\lambda} y_1^{\lambda}, \dots, x_n^{1-\lambda} y_n^{\lambda}\big) \geqslant f(x)^{1-\lambda} g(y)^{\lambda} \qquad \forall x, y \in \mathbb{R}^n_+ \,, \tag{5}$$

it holds that

$$\int_{\mathbb{R}^n_+} h \geqslant \left(\int_{\mathbb{R}^n_+} f \right)^{1-\lambda} \left(\int_{\mathbb{R}^n_+} g \right)^{\lambda} \tag{6}$$

(see [2, 11, 39]). When f and g are the characteristic functions of two unconditional sets K and L, the smallest function h satisfying (5) is the characteristic function of the product body

$$K^{1-\lambda} \cdot L^{\lambda} := \Big\{ z \in \mathbb{R}^n \ : \ \forall i = 1, \dots, n, \ |z_i| = |x_i|^{1-\lambda} |y_i|^{\lambda} \text{ for some } x \in K \,, \ y \in L \Big\} .$$

Hence (6) gives $|K^{1-\lambda} \cdot L^{\lambda}| \geqslant |K|^{1-\lambda} |L|^{\lambda}$, and the log-Brunn–Minkowski inequality readily follows since the product body is contained into the 0-sum (see [37, Lemma 4.1]).

In this paper, we present a different construction: it stems in the functional analytic setting, where it yields a new inequality of Prékopa–Leindler type, and reflects in the geometric one, where it yields a new concept of geometric mean of sets.

Given two nonnegative integrable functions f and g on $\mathbb{R}^n$ and a parameter $\lambda \in [0,1]$, we introduce a new function $f \star_\lambda g$, that we call *geometric combination of f and g* (in proportion λ), whose Lebesgue integral is equal to the geometric mean of the integrals of f and g. As a straightforward natural consequence, in order to have (2), it is sufficient that h is minorated almost everywhere by $f \star_\lambda g$.

The reason why we can handle functions defined on the whole space $\mathbb{R}^n$, and not merely on $\mathbb{R}^n_+$ as in the multiplicative Prékopa–Leindler inequality, is that the construction of $f \star_\lambda g$ does not involve the geometric mean of the variable's components appearing in (5), but rather the geometric mean of intrinsically positive quantities associated with f and g. In one space dimension, such positive quantities are precisely the absolutely continuous parts of the derivatives of the inverse distribution functions of f and g; in higher dimensions, the same procedure can be iterated by arguing along a prescribed family of linearly independent directions. In fact, the proof strategy is essentially one-dimensional, and is of mass transportation type: it incorporates the use of distribution functions originally due to Barthe (see [28, Theorem 2.13]) with the construction of the Knothe map ([26], see also [38, p. 372]).

Moving attention from functions to sets, when f and g are the characteristic functions of two measurable sets K and L, their geometric combination agrees with the characteristic function of a measurable set, denoted by $K \star_\lambda L$, such that

$$|K \star_\lambda L| = |K|^{1-\lambda} |L|^{\lambda} \,. \tag{7}$$

To the best of our knowledge, this way of "geometrically combining" two sets so that the equality (7) holds, is completely new. Actually, several attempts exist in the literature to define some notion of geometric mean of sets, in particular, of convex bodies. Besides the 0-sum mentioned above, let us recall the dual version of the 0-sum considered by Saroglou [36], the classical notion of complex interpolation studied in Banach geometry [8], the Partial Differential Equation approach introduced by Cordero-Erausquin and Klartag [20], and the nonstandard construction proposed by V. Milman and Rotem [31]. A more detailed description about each of these constructions, along with additional references, can be found in [31].

While in all these cases a major concern is getting a volume estimate for the geometric mean body, from this point of view, the behavior of our geometric combination is of striking simplicity, as the equality (7) holds. Thus, it is natural to wonder about possible relationships with the log-sum, and particularly with the log-Brunn–Minkowski conjecture.

In this direction, we are able to show that when K and L are unconditional convex bodies, their geometric combination $K \star_\lambda L$ with respect to the coordinate axes contains their 0-combination. The same inclusion occurs for the class of convex bodies with n symmetries considered in [6, 7, 13], provided that one works with the natural family of directions suggested by the shapes of K and L. Thus, both for unconditional bodies and for bodies with n symmetries, we obtain an alternative proof of the log-Brunn–Minkowski inequality.

Getting farther-reaching implications of our approach in the log-Brunn–Minkowski conjecture for wider classes of convex bodies remains an intriguing open question: though, in principle, we are not fatally limited to deal with sets with special symmetries, in order to handle arbitrary sets, the main difficulty seems to understand how to choose the directions and the primitives involved in our construction.

The paper is organized as follows. In Section 2, we introduce the geometric combination of functions and prove its integral property, for the sake of clarity first in one dimension (see Theorem 8) and then in n-dimensions (see Theorem 18). In Section 3, we turn attention to the geometric combination of centrally symmetric convex sets, and we show that this new operation seems to satisfy some good properties: in Section 3.1, we establish the convexity preserving property (in dimension $n = 1, 2$), and we exhibit some explicit examples of geometric combinations; in Sections 3.2 and 3.3, we deal, respectively, with unconditional convex bodies and convex bodies with n symmetries and we show that, in such classes, the comparison with the 0-sum yields, via Theorem 18, an alternative proof of the log-Brunn–Minkowski inequality. Finally, in Section 4, we give a short list of related open problems.

2 | GEOMETRIC COMBINATION OF FUNCTIONS

2.1 | The one-dimensional case

Definition 1. Let f be a nonnegative, integrable function of one real variable, with strictly positive integral. The *inverse distribution function (shortly i.d. function) of f* is the generalized inverse of the absolutely continuous nondecreasing function

$$F(x) := \frac{1}{\int_{\mathbb{R}} f} \int_{-\infty}^{x} f(t)\, dt\,, \qquad x \in \mathbb{R}\,, \tag{8}$$

namely,

$$u(t) := \inf \left\{ s \in \mathbb{R} \ : \ \int_{-\infty}^{s} f(x)\, dx > t \int_{\mathbb{R}} f \right\}, \qquad t \in (0, 1)\,.$$

Lemma 2. *Let f and F be as in the above definition, and let u be the i.d. function of f. Then:*

(i) *u is finite valued, right continuous, and strictly increasing (possibly unbounded) in (0,1);*
(ii) *$F(u(t)) = t$ for every $t \in (0,1)$;*
(iii) *for $\mathcal{L}^1$-a.e. $t \in (0,1)$, u is differentiable at t, F is differentiable at $u(t)$, and it holds that $F'(u(t))\,u'(t) = 1$, or equivalently*

$$f(u(t))u'(t) = \int_{\mathbb{R}} f(x)\,dx \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,1)\,. \tag{9}$$

In particular, we have that $f(u(t)) > 0$ and $u'(t) > 0$ for $\mathcal{L}^1$-a.e. $t \in (0,1)$.

Proof. Statements (i)–(ii) follow from the basic properties of generalized inverse functions, see, for instance, [22, Proposition 1]. The proof of statement (iii) can be achieved as follows. Since F is (absolutely) continuous and nondecreasing, for every $t \in (0,1)$, the level set $\{F = t\}$ is a closed interval, and the family $\{t_j\} \subset (0,1)$ of levels such that $I_j := \{F = t_j\}$ has positive length is at most countable. We can decompose $\mathbb{R}$ as the disjoint union $\mathbb{R} = (\bigcup_j I_j) \cup L \cup N$, where L is the set of Lebesgue points of f in $\mathbb{R} \setminus \bigcup_j I_j$. Since $|F(N)| = 0$ and $|F(\bigcup_j I_j)| = 0$, we have that $F(L)$ has full measure in $(0,1)$; then, letting $M \subset F(L)$ be the set of points of differentiability of u in $F(L)$, also M has full measure in (0,1). If $t \in M$, then u is differentiable at t and $u(t)$ is a Lebesgue point of f, so that F is differentiable at $u(t)$; the relation $F'(u(t))\,u'(t) = 1$ is now obtained from (ii), and (9) follows as a direct consequence of the above analysis. □

Remark 3. We warn that, in general, the distributional derivative of an i.d. function may contain jump and/or Cantor parts as in the two examples hereafter.

(i) Let

$$f(x) = (1/2)\chi_{[0,1]\cup[2,3]}(x)\,, \tag{10}$$

where χ_A denotes the characteristic function of a set A. The i.d. function $u(t)$ equals $2t$ for $t \in [0,1/2)$ and $2t+1$ for $t \in [1/2,1]$.

(ii) Let

$$f(x) = F'(x)\,, \text{ with } F(x) = \begin{cases} 0 & \text{if } x \leqslant 0\,, \\ u^{-1}(x) & \text{if } 0 < x < 2\,, \\ 1 & \text{if } x \geqslant 2\,, \end{cases} \tag{11}$$

where u^{-1} is the inverse of the function $u : (0,1) \to (0,2)$, $u(t) := t + C(t)$, C being the Cantor function. By construction, the i.d. function of f is precisely the function $u(t)$.

Definition 4. Let u and v be the i.d. functions of two nonnegative, integrable functions with strictly positive integrals on $\mathbb{R}$, and let $\lambda \in [0,1]$. We call a *geometric primitive of* (u,v) *in proportion* λ any primitive of $(u')^{1-\lambda}(v')^{\lambda}$, that is, any function of the form

$$w_\lambda(t) = \int_{1/2}^{t} u'(s)^{1-\lambda} v'(s)^{\lambda}\,ds + c \qquad \text{with } c \in \mathbb{R}\,.$$

In case $c = 0$, we shall refer to the geometric primitive w_λ as the *standard* one.

Remark 5. The above definition is well posed and it holds that

(i) $w_\lambda \in AC_{loc}(0,1)$ with $w'_\lambda(t) > 0$ for $\mathcal{L}^1$-a.e. $t \in (0,1)$;
(ii) w_λ admits a classical inverse, defined on the interval $\operatorname{ran} w_\lambda := \{w_\lambda(t) : t \in (0,1)\}$.

Indeed, from Lemma 2, we have that u', v' are strictly positive $\mathcal{L}^1$-a.e. and belong to $L^1_{loc}(0,1)$. Hence, as a consequence of Hölder's inequality, also $(u')^{1-\lambda}(v')^\lambda$ is in $L^1_{loc}(0,1)$. This yields claim (i), which, in turn, implies (ii).

Definition 6. Let $f, g : \mathbb{R} \to \mathbb{R}$ be nonnegative, integrable functions having strictly positive integrals, with i.d. functions u, v, respectively. Let $\lambda \in [0,1]$, and let w_λ be a geometric primitive of (u, v) in proportion λ. We call the function

$$f \star_\lambda g(x) := \begin{cases} f(u(t))^{1-\lambda} g(v(t))^\lambda & \text{if } x = w_\lambda(t),\ t \in (0,1)\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{12}$$

a *geometric combination of* (f, g) *in proportion* λ.

In case the geometric primitive w_λ in (12) is chosen as standard one, we shall refer to $f \star_\lambda g$ as to the *standard* geometric combination of (f, g) in proportion λ.

Remark 7. Thanks to Remark 5, the function $f \star_\lambda g$ is well defined and, denoting by $\operatorname{ran} w_\lambda$ the image of w_λ, it can be equivalently written as

$$f \star_\lambda g(x) = \begin{cases} f(u(w_\lambda^{-1}(x)))^{1-\lambda} g(v(w_\lambda^{-1}(x)))^\lambda & \text{if } x \in \operatorname{ran} w_\lambda\,, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Clearly, the above expression identifies uniquely $f \star_\lambda g$ up to a translation in the variable x, depending on the choice of the geometric primitive w_λ.

Theorem 8. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be nonnegative, integrable functions having strictly positive integrals, let $\lambda \in [0,1]$, and let $f \star_\lambda g$ be a geometric combination of (f, g) in proportion λ according to Definition 6. Then $f \star_\lambda g$ is measurable and satisfies*

$$\int_{\mathbb{R}} f \star_\lambda g(x)\, dx = \left(\int_{\mathbb{R}} f(x)\, dx\right)^{1-\lambda} \left(\int_{\mathbb{R}} g(x)\, dx\right)^{\lambda}.$$

Proof. In view of (13), to show that $f \star_\lambda g$ is measurable, since both f and g are measurable, it is enough to show the following Lusin property (see, e.g., [25]): if N is a set of measure zero, each of the two sets $(u \circ w_\lambda^{-1})^{-1}(N)$ and $(v \circ w_\lambda^{-1})^{-1}(N)$ has measure zero. Focusing for instance on the first one, we have

$$(u \circ w_\lambda^{-1})^{-1}(N) = w_\lambda \circ u^{-1}(N) = w_\lambda \circ F(N)$$

with F as in (8). Since $w_\lambda \in AC_{loc}(0,1)$ and $F \in AC(\mathbb{R})$, $w_\lambda \circ F(N)$ has measure zero.

We can now use the change of variable $x = w_\lambda(t)$ (see [42, Theorem 13.32]) to write

$$\int_{\mathbb{R}} f \star_\lambda g(x)\,dx = \int_{\operatorname{ran} w_\lambda} f \star_\lambda g(x)\,dx = \int_0^1 f \star_\lambda g(w_\lambda(t)) w_\lambda'(t)\,dt$$
$$= \int_0^1 f(u(t))^{1-\lambda} g(v(t))^\lambda u'(t)^{1-\lambda} v'(t)^\lambda\,dt\,.$$

Finally, using (9), we obtain

$$\int_0^1 f(u(t))^{1-\lambda} g(v(t))^\lambda u'(t)^{1-\lambda} v'(t)^\lambda\,dt = \left(\int_{\mathbb{R}} f(x)\,dx\right)^{1-\lambda} \left(\int_{\mathbb{R}} g(x)\,dx\right)^{\lambda}. \qquad \square$$

Remark 9. An immediate consequence of Theorem 8 is the following Prékopa–Leindler-type result: under the assumptions of Theorem 8 on f and g, if h is any integrable function such that

$$h(x) \geqslant f \star_\lambda g(x) \qquad \text{for a.e. } x \in \mathbb{R}\,, \tag{14}$$

we have

$$\int_{\mathbb{R}} h(x)\,dx \geqslant \left(\int_{\mathbb{R}} f(x)\,dx\right)^{1-\lambda} \left(\int_{\mathbb{R}} g(x)\,dx\right)^{\lambda}, \tag{15}$$

and equality holds in (15) if and only if equality holds in (14).

Remark 10. Notice that, in general, $f \star_0 g$ and $f \star_1 g$ do not coincide necessarily with (a translation of) f and g, respectively. Indeed, for $\lambda = 0$ and $\lambda = 1$, a geometric primitive w_λ of their i.d. functions (u, v) does not necessarily coincide, up to a translation, with u and v, respectively. For instance, referring to the examples given in Remark 3: if f is given by (10) (and g is arbitrary), we have $w_0(t) = 2(t - \frac{1}{2})$, and hence, $f \star_0 g = (1/2)\chi_{[0,2]}$; if f is given by (11) (and g is arbitrary), we have $w_0(t) = t$ and hence $f \star_0 g = (\int_{\mathbb{R}} f)\chi_{[0,1]}$.

When the i.d. functions of f, g enjoy suitable regularity assumptions (which as we shall see occurs for the characteristic functions of convex bodies, see Remark 26), the geometric combination $f \star_\lambda g$ provides a continuous interpolation between f and g.

Proposition 11. *Let f and g satisfy the same assumptions of Theorem 8. If in addition their i.d. functions belong to $AC_{loc}(0,1)$, we have that:*

(i) *up to translations, $f \star_0 g = f$ and $f \star_1 g = g$;*
(ii) *the map $[0,1] \ni \lambda \mapsto f \star_\lambda g$ is continuous in $L^1(\mathbb{R})$.*

Proof. Let u, v be the i.d. functions of f, g, respectively. By assumption $u, v \in AC_{\text{loc}}(0,1)$, and thus, statement (i) immediately follows from the equalities $w_0 = u$ and $w_1 = v$.

Let us prove statement (ii). Let $(\lambda_j) \subset [0,1]$ be a sequence converging to $\overline{\lambda} \in [0,1]$, and let us define

$$\psi_j(t) := f(u(t))^{1-\lambda_j} g(v(t))^{\lambda_j}\,, \quad \psi(t) := f(u(t))^{1-\overline{\lambda}} g(v(t))^{\overline{\lambda}}\,, \qquad t \in (0,1),$$

$f \star_{\lambda_j} g = \psi_j \circ w_{\lambda_j}^{-1}$ on $I_j := \operatorname{ran} w_{\lambda_j}$, $f \star_{\overline{\lambda}} g = \psi \circ w_{\overline{\lambda}}^{-1}$ on $I := \operatorname{ran} w_{\overline{\lambda}}$, all functions vanishing otherwise.

We claim that:

(a) $\lim_j f \star_{\lambda_j} g(x) = f \star_\lambda g(x)$, for a.e. $x \in I$,

(b) $\lim_{j\to+\infty} \int_I f \star_{\lambda_j} g = \int_I f \star_{\bar\lambda} g, \qquad \lim_{j\to+\infty} \int_{\mathbb{R}\setminus I} f \star_{\lambda_j} g = 0.$

Before proving this claim, let us show how the convergence of $f \star_{\lambda_j} g$ to $f \star_{\bar\lambda} g$ in $L^1(\mathbb{R})$ follows from (a) and (b).

Specifically, from (a), the first equality in (b) and [42, Theorem 16.28], it follows that $f \star_{\lambda_j} g$ converges to $f \star_{\bar\lambda} g$ in $L^1(I)$. On the other hand, from the second equality in (b), it follows that $f \star_{\lambda_j} g$ converges to $0 = f \star_{\bar\lambda} g$ in $L^1(\mathbb{R} \setminus I)$.

Proof of (a). As a first step, let us show that if $x \in I$, then $x \in I_j$ for j large enough. Specifically, it is enough to observe that, for every $\lambda \in [0,1]$,

$$\operatorname{ran} w_\lambda = (-a_\lambda, b_\lambda), \quad \text{with} \quad a_\lambda := \int_0^{1/2} (u')^{1-\lambda}(v')^\lambda, \quad b_\lambda := \int_{1/2}^1 (u')^{1-\lambda}(v')^\lambda,$$

and that, by Fatou's lemma,

$$a_{\bar\lambda} \leqslant \liminf_{j\to+\infty} a_{\lambda_j}, \qquad b_{\bar\lambda} \leqslant \liminf_{j\to+\infty} b_{\lambda_j}.$$

Since, for a.e. $t \in (0,1)$, we have that $\lim_j \psi_j(t) = \psi(t)$, (a) will follow if we show that, for every $x \in I$, $\lim_j w_{\lambda_j}^{-1}(x) = w_{\bar\lambda}^{-1}(x)$. Given $x \in I$, for j large enough let $t_j := w_{\lambda_j}^{-1}(x)$. Let (t_{j_k}) be a subsequence converging to some $\tau \in [0,1]$.

If $\tau \in (0,1)$, it holds that

$$x = w_{\lambda_{j_k}}(t_{j_k}) = \int_{1/2}^{t_{j_k}} (u')^{1-\lambda_{j_k}}(v')^{\lambda_{j_k}} \longrightarrow \int_{1/2}^{\tau} (u')^{1-\bar\lambda}(v')^{\bar\lambda} = w_{\bar\lambda}(\tau),$$

where the convergence holds since $0 \leqslant (u')^{1-\lambda_{j_k}}(v')^{\lambda_{j_k}} \leqslant u' + v' \in L^1_{\text{loc}}(0,1)$. Hence, $x = w_{\bar\lambda}(\tau)$, that is, $\tau = w_{\bar\lambda}^{-1}(x)$.

Assume now that $\tau = 1$. The sequence $q_k := (u')^{1-\lambda_{j_k}}(v')^{\lambda_{j_k}} \chi_{[1/2, t_{j_k}]}$ converges a.e. in $[1/2, 1]$ to $q := (u')^{1-\bar\lambda}(v')^{\bar\lambda}$, hence, by Fatou's Lemma we have that

$$b_{\bar\lambda} = \int_{1/2}^1 q \leqslant \liminf_{k\to+\infty} \int_{1/2}^1 q_k = x,$$

in contradiction with the assumption that x belongs to the open interval $I = (-a_{\bar\lambda}, b_{\bar\lambda})$.

A similar argument shows that $\tau \neq 0$.

We have thus shown that every converging subsequence of $w_{\lambda_j}^{-1}(x)$ converges to $w_{\bar\lambda}^{-1}(x)$; hence, the whole sequence converges to $w_{\bar\lambda}^{-1}(x)$.

Proof of (b). Since $\int_\mathbb{R} f$ and $\int_\mathbb{R} g$ are strictly positive, by Theorem 8, it holds that

$$\int_\mathbb{R} f \star_{\lambda_j} g = \left(\int_\mathbb{R} f\right)^{1-\lambda_j} \left(\int_\mathbb{R} g\right)^{\lambda_j} \xrightarrow{j\to+\infty} \left(\int_\mathbb{R} f\right)^{1-\bar\lambda} \left(\int_\mathbb{R} g\right)^{\bar\lambda} = \int_\mathbb{R} f \star_{\bar\lambda} g. \tag{16}$$

Hence, it is enough to prove only the second equality in (b). From (16), (a) and Fatou's lemma, we have that

$$0 \leqslant \limsup_{j\to+\infty} \int_{\mathbb{R}\setminus I} f \star_{\lambda_j} g = \limsup_{j\to+\infty} \left(\int_{\mathbb{R}} f \star_{\lambda_j} g - \int_I f \star_{\lambda_j} g \right) \leqslant \int_{\mathbb{R}} f \star_{\overline{\lambda}} g - \int_I f \star_{\overline{\lambda}} g = 0,$$

so that the second equality in (b) follows. □

2.2 | The n-dimensional case

Let $(z_1, \dots, z_n)$ be a family of linearly independent vectors in $\mathbb{R}^n$ (a typical choice will be the canonical basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$, cf. Section 3). For simplicity of notation, we do not indicate the dependence of our construction on the family $(z_1, \dots, z_n)$, as it will remain fixed throughout this section.

Definition 12. Given a nonnegative integrable function f on $\mathbb{R}^n$ with strictly positive integral, we call $(z_1, \dots, z_n)$*-inverse distribution field (shortly i.d. field) of* f the vector field

$$U(t) = u_1(t_1)z_1 + u_2(t_1, t_2)z_2 + u_3(t_1, t_2, t_3)z_3 + \cdots + u_n(t_1, t_2, \dots, t_n)z_n\,,$$

defined for $\mathcal{L}^n$-a.e. $t = (t_1, \dots, t_n) \in (0, 1)^n$ as follows:

- $t_1 \mapsto u_1(t_1)$ is the i.d. function of the map

$$x_1 \mapsto \int_{\mathbb{R}^{n-1}} f(x_1z_1 + x_2z_2 + \cdots + x_nz_n)\, dx_2 \dots dx_n\,;$$

- for $\mathcal{L}^1$-a.e. $t_1 \in (0, 1)$, $t_2 \mapsto u_2(t_1, t_2)$ is the i.d. function of the map

$$x_2 \mapsto \int_{\mathbb{R}^{n-2}} f(u_1(t_1)z_1 + x_2z_2 + \cdots + x_nz_n)\, dx_3 \dots\, dx_n\,;$$

- ...
- for $\mathcal{L}^{n-1}$-a.e. $(t_1, \dots, t_{n-1}) \in (0, 1)^{n-1}$, $t_n \mapsto u_n(t_1, \dots, t_n)$ is the i.d. function of the map

$$x_n \mapsto f(u_1(t_1)z_1 + u_2(t_1, t_2)z_2 + \cdots + u_{n-1}(t_1, t_2, \dots, t_{n-1})z_{n-1} + x_nz_n)\,.$$

Remark 13. The above definition is well posed and leads to a n-dimensional analog of the identity (9). Indeed:

- the i.d. function $t_1 \mapsto u_1(t_1)$ is well defined (because by assumption f has strictly positive integral), and for $\mathcal{L}^1$-a.e. $t_1 \in (0, 1)$, it satisfies the equality

$$\begin{aligned} &u_1'(t_1) \int_{\mathbb{R}^{n-1}} f(u_1(t_1)z_1 + x_2z_2 + \cdots + x_nz_n)\, dx_2 \dots dx_n \\ &\quad = \int_{\mathbb{R}^n} f(x_1z_1 + \cdots + x_nz_n)\, dx_1 \dots dx_n \quad \left(= \frac{1}{|z_1 \wedge \cdots \wedge z_n|} \int_{\mathbb{R}^n} f \right); \end{aligned}$$

- for $\mathcal{L}^1$-a.e. $t_1 \in (0,1)$, the i.d. function $t_2 \mapsto u_1(t_1, t_2)$ is well defined (because by the previous item the integral at the r.h.s. of the equality below is strictly positive), and for $\mathcal{L}^1$-a.e. $t_2 \in (0,1)$, it satisfies

$$\frac{\partial u_2}{\partial t_2}(t_1,t_2)\int_{\mathbb{R}^{n-2}} f(u_1(t_1)z_1 + u_2(t_1,t_2)z_2 + \cdots + x_n z_n)\,dx_3 \dots dx_n$$
$$= \int_{\mathbb{R}^{n-1}} f(u_1(t_1)z_1 + x_2 z_2 + \cdots + x_n z_n)\,dx_2 \dots dx_n$$

- ...
- for $\mathcal{L}^{n-1}$-a.e. $(t_1, \dots, t_{n-1}) \in (0,1)^{n-1}$, the i.d. function $t_n \mapsto u_n(t_1, \dots, t_{n-1}, t_n)$ is well defined (because by the previous items, the integral at the r.h.s. of the equality below is strictly positive), and for $\mathcal{L}^1$-a.e. $t_n \in (0,1)$, it satisfies:

$$\frac{\partial u_n}{\partial t_n}(t_1,t_2,\dots,t_n) f(u_1(t_1)z_1 + u_2(t_1,t_2)z_2 + \cdots + u_n(t_1,t_2,\dots,t_n)z_n)$$
$$= \int_{\mathbb{R}} f(u_1(t_1)z_1 + u_2(t_1,t_2)z_2 + \cdots + x_n z_n)\,dx_n\,.$$

Multiplying side by side the above equalities, we get by analogy with (9) the identity

$$f(U(t))\prod_{i=1}^{n}\frac{\partial u_i}{\partial t_i} = \frac{1}{|z_1 \wedge \cdots \wedge z_n|}\int_{\mathbb{R}^n} f \qquad \text{for } \mathcal{L}^n\text{-a.e. } t \in (0,1)^n\,. \tag{17}$$

Definition 14. Let U and V be the $(z_1, \dots, z_n)$-i.d. fields of two nonnegative integrable function f and g with strictly positive integrals on $\mathbb{R}^n$. We call $(z_1, \dots, z_n)$-*geometric potential of* (U,V) *in proportion* λ any vector field

$$W_\lambda(t) = w_{\lambda,1}(t_1)z_1 + w_{\lambda,2}(t_1,t_2)z_2 + \cdots + w_{\lambda,n}(t_1, \dots t_n)z_n$$

such that:

- $t_1 \mapsto w_{\lambda,1}(t_1)$ is a geometric primitive in proportion λ of $t_1 \mapsto u_1(t_1)$, $t_1 \mapsto v_1(t_1)$;
- for $\mathcal{L}^1$-a.e. $t_1 \in (0,1)$, $t_2 \mapsto w_{\lambda,2}(t_1,t_2)$ is a geometric primitive in proportion λ of $t_2 \mapsto u_2(t_1,t_2)$, $t_2 \mapsto v_2(t_1,t_2)$;
- ...
- for $\mathcal{L}^{n-1}$-a.e. $(t_1, \dots, t_{n-1}) \in (0,1)^{n-1}$, $t_n \mapsto w_{\lambda,n}(t_1, \dots, t_n)$ is a geometric primitive in proportion λ of $t_n \mapsto u_n(t_1, \dots, t_n)$, $t_n \mapsto v_n(t_1, \dots, t_n)$.

In case all the geometric primitives $w_{\lambda,i}$ are the standard ones, namely, when

$$w_1(1/2) = 0\,, \quad w_2(t_1, 1/2) = 0\,, \quad \dots, \quad w_n(t_1, \dots, t_{n-1}, 1/2) = 0\,,$$

we shall refer to the geometric potential W_λ as to the *standard* one.

Remark 15. By analogy with one-dimensional case, we have that

(i) for $\mathcal{L}^{i-1}$-a.e. $(t_1, \dots, t_{i-1}) \in (0,1)^{i-1}$, the maps $t_i \mapsto w_{\lambda,i}(t_1, \dots, t_{i-1}, t_i)$ are in $AC_{loc}(0,1)$, with

$$\frac{\partial w_{\lambda,i}}{\partial t_i} = \left(\frac{\partial u_i}{\partial t_i}\right)^{1-\lambda} \left(\frac{\partial v_i}{\partial t_i}\right)^{\lambda} > 0 \qquad \text{for } \mathcal{L}^1\text{-a.e. } t_i \in (0,1)\,; \tag{18}$$

(ii) W_λ admits a classical inverse defined on $\operatorname{ran} W_\lambda := \{W_\lambda(t) : t \in (0,1)^n\}$.

Definition 16. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be nonnegative, integrable functions having strictly positive integrals, with $(z_1, \dots, z_n)$-i.d. fields U, V, respectively. If W_λ is a $(z_1, \dots, z_n)$-geometric potential of (U, V) in proportion λ, we call the function

$$f \star_\lambda g(x) := \begin{cases} f(U(t))^{1-\lambda} g(V(t))^{\lambda} & \text{if } x = W_\lambda(t)\,,\ t \in (0,1)^n\,, \\ 0 & \text{otherwise,} \end{cases} \tag{19}$$

a $(z_1, \dots, z_n)$-geometric combination of (f, g) in proportion λ.

In case the geometric potential W_λ in (19) is chosen as the standard one, we shall refer to $f \star_\lambda g$ as to the *standard* geometric combination of (f, g) in proportion λ.

Remark 17. Again, by analogy with the one-dimensional case, denoting by $\operatorname{ran} W_\lambda$ the image of W_λ, we have

$$f \star_\lambda g(x) = \begin{cases} f(U(W_\lambda^{-1}(x)))^{1-\lambda} g(V(W_\lambda^{-1}(x)))^{\lambda} & \text{if } x \in \operatorname{ran} W_\lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Notice that, in the present n-dimensional setting, the function W_λ is identified up to an additive constant in the first component, up to a function of x_1 in the second component, up to a function of x_2 in the third one, and so on, up to a function of $(x_1, \dots, x_{n-1})$ in the last component.

Theorem 18. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be nonnegative, integrable functions having strictly positive integrals. Let $\lambda \in [0,1]$, and let $f \star_\lambda g$ be a $(z_1, \dots, z_n)$-geometric combination of (f, g) in proportion λ according to Definition 16. Then $f \star_\lambda g$ is measurable and satisfies*

$$\int_{\mathbb{R}^n} f \star_\lambda g(x)\, dx = \left(\int_{\mathbb{R}^n} f(x)\, dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)\, dx\right)^{\lambda}.$$

Proof. To show that $f \star_\lambda g$ is measurable, thanks to (20) it is enough to show that if N is Lebesgue negligible, each of the two sets $(U \circ W_\lambda^{-1})^{-1}(N)$ and $(V \circ W_\lambda^{-1})^{-1}(N)$ has measure zero. Considering, for instance, the set $(U \circ W_\lambda^{-1})^{-1}(N)$, we can write it as $W_\lambda \circ U^{-1}(N)$. Thus, we see that it is Lebesgue negligible because U is the inverse distribution field of the integrable function f, and the components of W_λ have the property that the maps $t_i \mapsto w_{\lambda,i}$ are locally absolutely continuous (see Remark 15).

Let us compute the integral of $f \star_\lambda g$. Let us define the function $h(x) := f \star_\lambda g(x_1 z_1 + \cdots + x_n z_n)$, $x \in \mathbb{R}^n$, so that

$$\int_{\mathbb{R}^n} f \star_\lambda g(x)\, dx = |z_1 \wedge \cdots \wedge z_n| \int_{\mathbb{R}^n} h(x)\, dx\,. \tag{21}$$

In order to compute the integral of h, we perform n one-dimensional changes of variable. Proceeding as in the proof of Theorem 8, and recalling that, by Definition 16, h vanishes outside the set $\{(w_{\lambda,1}(t_1), \dots, w_{\lambda,n}(t_1, \dots, t_n)) : t_i \in (0,1)\}$, we first use the change of variable $x_1 = w_{\lambda,1}(t_1)$ (see [42, Theorem 13.32]), obtaining

$$\int_{\mathbb{R}^n} h(x)\,dx = \int_{(0,1)} \left(\int_{\mathbb{R}^{n-1}} h(w_{\lambda,1}(t_1), x_2, \dots, x_n)\,dx_2 \dots dx_n \right) w'_{\lambda,1}(t_1) \Big)\,dt_1\,.$$

Next, in the inner integral, we proceed with the change of variable $x_2 = w_{\lambda,2}(t_1, t_2)$, $t_2 \in (0,1)$. Finally, after the last change of variable $x_n = w_{\lambda,n}(t_1, \dots, t_n)$, $t_n \in (0,1)$, using the family of equalities (18), and recalling the Definition 16 of $f \star_\lambda g$, we end up with

$$\begin{aligned}
\int_{\mathbb{R}^n} h(x)\,dx &= \int_{(0,1)^n} h(w_{\lambda,1}(t_1), \dots, w_{\lambda,n}(t_1, \dots, t_n)) \prod_{i=1}^n \frac{\partial w_{\lambda,i}}{\partial t_i}\,dt \\
&= \int_{(0,1)^n} h(w_{\lambda,1}(t_1), \dots, w_{\lambda,n}(t_1, \dots, t_n)) \prod_{i=1}^n \left(\frac{\partial u_i}{\partial t_i}\right)^{1-\lambda} \prod_{i=1}^n \left(\frac{\partial v_i}{\partial t_i}\right)^{\lambda} dt \\
&= \int_{(0,1)^n} f(U(t))^{1-\lambda} g(V(t))^{\lambda} \prod_{i=1}^n \left(\frac{\partial u_i}{\partial t_i}\right)^{1-\lambda} \prod_{i=1}^n \left(\frac{\partial v_i}{\partial t_i}\right)^{\lambda} dt\,.
\end{aligned}$$

Finally, from (21) and in view of (17), we conclude that

$$\int_{\mathbb{R}^n} f \star_\lambda g(x)\,dx = \left(\int_{\mathbb{R}^n} f(x)\,dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)\,dx \right)^{\lambda}. \qquad \square$$

3 | GEOMETRIC COMBINATION OF CONVEX BODIES

In this section, we focus attention on the geometric combination between the characteristic functions of two nondegenerate, centrally symmetric, convex bodies K and L in $\mathbb{R}^n$.

Below, we write for simplicity $K \star_\lambda L$ in place of $\chi_K \star_\lambda \chi_L$; moreover, we refer to the i.d. function (or field) of the characteristic function χ_K as the i.d. function (or field) of K. Since we are dealing with nondegenerate convex bodies, for simplicity of notation in the proofs, sometimes we identify a convex body with its interior.

3.1 | Convexity preserving for $n = 1, 2$

When K is a symmetric interval, $K = [-a, a]$, its i.d. function is given by

$$u(t) = a(2t-1) \qquad \forall t \in (0,1)\,. \tag{22}$$

It is elementary to deduce the behavior of the geometric combination for intervals.

Proposition 19. *Let K and L be symmetric intervals, and $\lambda \in [0,1]$. Then the standard geometric combination $K \star_\lambda L$ is itself a symmetric interval, precisely*

$$K \star_\lambda L = (1-\lambda)\cdot K +_0 \lambda \cdot L\,.$$

In particular, the equality $|K \star_\lambda L| = |K|^{1-\lambda}|L|^\lambda$ given by Theorem 8 is equivalent to the one-dimensional log-Brunn–Minkowski equality.

Proof. If u and v are the i.d. functions of $K=[-a,a]$ and $L=[-b,b]$, by (22) the standard geometric primitive of (u,v) in proportion λ is given by

$$w_\lambda(t) = a^{1-\lambda}b^\lambda(2t-1)\,.$$

Hence, the body $K \star_\lambda L$, which is the image of w_λ, is given by

$$K \star_\lambda L = \{w_\lambda(t) \ :\ t\in(0,1)\} = \left(-a^{1-\lambda}b^\lambda, a^{1-\lambda}b^\lambda\right).$$

On the other hand, since the support functions of K and L are given, respectively, by $h_K(\xi) = a|\xi|$ and $h_L(\xi) = b|\xi|$, the log-Brunn–Minkowski sum $(1-\lambda)K +_0 \lambda L$ is the interval with support function $h_K(\xi)^{1-\lambda}h_L(\xi) = a^{1-\lambda}b^\lambda|\xi|$, namely, we have as well

$$(1-\lambda)\cdot K +_0 \lambda\cdot L = \left(-a^{1-\lambda}b^\lambda, a^{1-\lambda}b^\lambda\right). \qquad \square$$

When K is a centrally symmetric convex body in $\mathbb{R}^2$, setting

$$K_{x_1} := \left\{x_2e_2 \ :\ x_1e_1 + x_2e_2 \in K\right\},$$

the components (u_1, u_2) of the (e_1,e_2)-i.d. field of K satisfy

- $t_1 \mapsto u_1(t_1)$ is the i.d. function of the map $x_1 \mapsto \mathcal{H}^1(K_{x_1})$, hence

$$u_1'(t_1) = \frac{|K|}{\mathcal{H}^1(K_{u_1(t_1)})} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t_1\in(0,1)\,;$$

- for $\mathcal{L}^1$-a.e. $t_1 \in (0,1)$, $t_2 \mapsto u_2(t_1,t_2)$ is the i.d. function of the map $x_2 \mapsto \chi_{K_{u_1(t_1)}}(x_2)$; hence,

$$\frac{\partial u_2}{\partial t_2}(t_1,t_2) = \mathcal{H}^1(K_{u_1(t_1)}) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t_2\in(0,1)\,.$$

As a consequence, we prove below that the standard geometric combination of centrally symmetric planar convex bodies preserves convexity (and always produces an unconditional set).

Proposition 20. *Let K and L be centrally symmetric convex bodies in $\mathbb{R}^2$, and let $\lambda \in (0,1)$. Then the standard (e_1,e_2)-geometric combination $K \star_\lambda L$ is an unconditional convex body.*

Proof. For brevity, we are going to denote by M the standard (e_1,e_2)-geometric combination $K \star_\lambda L$ (where λ is fixed in $[0,1]$).

To see that M is unconditional, recall that M is the image of the standard geometric potential W_λ of the i.d. fields (U, V) of K and L. In view of the above expressions of $u_1'(t_1)$ and $\frac{\partial u_2}{\partial t_2}$ (and of their analog for the derivatives of the components of V), we have

$$W_\lambda(t_1, t_2) = \left(\int_{\frac{1}{2}}^{t_1} \left(\frac{|K|}{\gamma_K(s)} \right)^{1-\lambda} \left(\frac{|L|}{\gamma_L(s)} \right)^{\lambda} ds \,,\ \gamma_K(t_1)^{1-\lambda} \gamma_L(t_1)^{\lambda} \left(t_2 - \frac{1}{2} \right) \right) \quad \forall (t_1, t_2) \in (0,1)^2 \,,$$

where we have set for brevity

$$\gamma_K(t_1) := \mathcal{H}^1(K_{u_1(t_1)}) \,, \quad \gamma_L(t_1) := \mathcal{H}^1(L_{v_1(t_1)}) \qquad \forall t_1 \in (0,1) \,.$$

It readily follows that the components of W_λ satisfy, for every $\delta \in (0, 1/2)$,

$$\begin{aligned} w_{\lambda,1}(\tfrac{1}{2} + \delta) &= -w_{\lambda,1}(\tfrac{1}{2} - \delta) \,, \\ w_{\lambda,2}(t_1, \tfrac{1}{2} + \delta) &= -w_{\lambda,1}(t_1, \tfrac{1}{2} - \delta) \quad \forall t_1 \in (0,1) \,, \end{aligned}$$

which shows that M is unconditional.

We now turn attention to convexity. We claim that the convexity of a generic centrally symmetric set K is related to the concavity of the corresponding function γ_K^2 as follows:

$$K \text{ convex } \Rightarrow \gamma_K^2 \text{ concave,} \tag{23}$$

$$K \text{ unconditional and } \gamma_K^2 \text{ concave } \Rightarrow K \text{ convex.} \tag{24}$$

Specifically, setting $\psi_K(x_1) := \mathcal{H}^1(K_{x_1})$, we have that: if K is convex, the map ψ_K is concave; vice versa, if ψ_K is concave and K is symmetric about the x_1-axis (and hence unconditional since it is assumed to be centrally symmetric), then K is convex.

In view of this observation, to obtain (23)–(24), it is enough to show that the concavity of ψ_K is equivalent to the concavity of γ_K^2. From the definition of the distribution function u_1, we have that

$$\psi_K(u_1(s)) u_1'(s) = |K| \,, \qquad \text{for a.e. } s \in (0,1) \,.$$

Since $\gamma_K(s) = \psi_K(u_1(s))$, squaring both sides and differentiating with respect to s gives

$$(\gamma_K^2)'(s) = 2\psi_K(u_1)\psi_K'(u_1)u_1' = 2|K|\psi_K'(u_1),$$

which shows that $(\gamma_K^2)'$ is nondecreasing if and only if ψ_K' is (recall indeed that u_1 is increasing).

Now, thanks to (23)–(24), and since we have already proved that M is unconditional, in order to prove that M is convex, we are reduced to show that

$$\gamma_K^2 \text{ and } \gamma_L^2 \text{ concave } \Longrightarrow \gamma_M^2 \text{ concave} \,. \tag{25}$$

We observe that, by the definition of M, it holds that $\gamma_M = \gamma_K^{1-\lambda} \gamma_L^{\lambda}$. Then the validity of the implication (25) follows from the fact that the geometric mean of two nonnegative concave function is still concave. For the sake of completeness, we enclose the elementary proof. Denoting by φ and

ψ the two functions, we have

$$\begin{aligned}
&\varphi^{1-\lambda}((1-\theta)s+\theta t)\psi^{\lambda}((1-\theta)s+\theta t)\\
&\qquad \geqslant ((1-\theta)\varphi(s)+\theta\varphi(t))^{1-\lambda}((1-\theta)\psi(s)+\theta\psi(t))^{\lambda}\\
&\qquad \geqslant (1-\theta)\varphi(s)^{1-\lambda}\psi(s)^{\lambda}+\theta\varphi(t)^{1-\lambda}\psi(t)^{\lambda}\,,
\end{aligned}$$

where in the last line, we have exploited the inequality $(x_1+y_1)^{1-\lambda}(x_2+y_2)^{\lambda} \geqslant x_1^{1-\lambda}x_2^{\lambda}+y_1^{1-\lambda}y_2^{\lambda}$, holding for nonnegative numbers x_1, x_2, y_1, y_2 (as it follows by adding the a.m.–g.m. inequality applied separately to the pair $(\frac{x_1}{x_1+y_1}, \frac{x_2}{x_2+y_2})$ and to the pair $(\frac{y_1}{x_1+y_1}, \frac{y_2}{x_2+y_2})$). □

Below we give some explicit examples of geometric combinations in dimension 2.

Example 21 (Geometric combination of two rectangles). If $K = Q(a_1, a_2)$ is the centrally symmetric rectangle with vertices at $(\pm a_1, \pm a_2)$, with $a_i > 0$, we have $\mathcal{H}^1(K_{u_1(s)}) = 2a_2$, so that the i.d. field of K is given by

$$U(t_1,t_2) = (a_1(2t_1-1)\,,\ a_2(2t_2-1)) \qquad \forall (t_1,t_2)\in(0,1)^2\,.$$

If we take another rectangle $L = Q(b_1, b_2)$ with i.d. field V, the standard (e_1, e_2)-geometric potential of (U, V) in proportion λ has components

$$W_\lambda(t_1,t_2) = \left(a_1^{1-\lambda}b_1^{\lambda}(2t_1-1)\,,\ a_2^{1-\lambda}b_2^{\lambda}(2t_2-1)\right), \qquad \forall (t_1,t_2)\in(0,1)^2\,.$$

Hence,

$$Q(a_1,a_2)\star_\lambda Q(b_1,b_2) = Q(a_1^{1-\lambda}b_1^{\lambda}, a_2^{1-\lambda}b_2^{\lambda})\,.$$

Example 22 (Geometric combination of two parallelograms with parallel sides). By arguing as in the previous example, it is immediate to obtain that the (z_1, z_2)-geometric combination in proportion λ of two parallelograms with sides parallel to the vectors (z_1, z_2), of lengths $(2\alpha_1, 2\alpha_2)$ and $(2\beta_1, \beta_2)$, respectively, is still a parallelograms with sides parallel to (z_1, z_2), of lengths $(2\alpha_1^{1-\lambda}\beta_1^{\lambda}, 2\alpha_2^{1-\lambda}\beta_2^{\lambda})$.

Example 23. Let us consider again two parallelograms with parallel sides, and let us determine their (e_1, e_2)-geometric combination, for example, in proportion $\frac{1}{2}$. Let us denote by $P(a_1, a_2, b_1)$ the centrally symmetric parallelogram with two vertices at the points (a_1, a_2) and $(b_1, -a_2)$, with $a_1 > b_1 > 0$, $a_2 > 0$. Let us determine the standard (e_1, e_2)-geometric combination of $K := P(3, 2, 1)$ and $L := P(4, 1, 3)$. Setting for brevity $P := P(a_1, a_2, b_1)$, some elementary computations give

$$\mathcal{H}^1(P_{u_1(s)}) = \begin{cases} 2a_2 & \text{if } \frac{1}{2}\leqslant s\leqslant \frac{1}{2}+\frac{b_1}{a_1+b_1}\\ \frac{2a_2}{a_1-b_1}\left((a_1-b_1)^2+2(b_1^2-a_1^2)(s-\frac{1}{2}-\frac{b_1}{a_1+b_1})\right)^{\frac{1}{2}} & \text{if } \frac{1}{2}+\frac{b_1}{a_1+b_1}\leqslant s\leqslant 1\,.\end{cases}$$

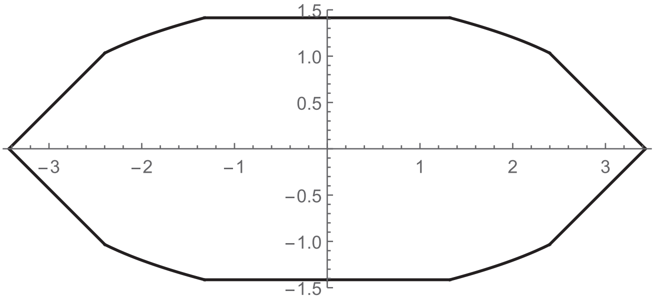

FIGURE 1 The geometric combination of the two parallelograms in Example 23.

Thus, denoting by $U = (u_1, u_2)$, $V = (v_1, v_2)$ the i.d. fields of K and L, we have

$$\gamma_K(s) := \mathcal{H}^1(K_{u_1(s)}) = \begin{cases} 4 & \text{if } \frac{1}{2} \leqslant s \leqslant \frac{3}{4} \\ 8\sqrt{1-s} & \text{if } \frac{3}{4} \leqslant s \leqslant 1 \,, \end{cases}$$

$$\gamma_L(s) := \mathcal{H}^1(L_{v_1(s)}) = \begin{cases} 2 & \text{if } \frac{1}{2} \leqslant s \leqslant \frac{13}{14} \\ 2\sqrt{14}\sqrt{1-s} & \text{if } \frac{13}{14} \leqslant s \leqslant 1 \,. \end{cases}$$

Let us then compute the (e_1, e_2)-geometric potential of (U, V) in proportion $\frac{1}{2}$, denoted as usual by $W_{\frac{1}{2}}(t_1, t_2)$. By Proposition 20, we already know that M is unconditional, so we can restrict ourselves to determine one quarter of the boundary, which is found by computing $W_{\frac{1}{2}}(t_1, 1)$, for $t_1 \in [\frac{1}{2}, 1]$. From the above expressions of γ_K and γ_L, we find that it is given, for any $t_1 \in [\frac{1}{2}, 1]$, by

$$\begin{aligned} W_{\frac{1}{2}}(t_1, 1) &= \left(\int_{\frac{1}{2}}^{t_1} \left(\frac{|K||L|}{\gamma_L(s)\gamma_K(s)} \right)^{\frac{1}{2}} ds \,,\ \frac{1}{2}(\gamma_K(t_1)\gamma_L(t_1))^{\frac{1}{2}} \right) \\ &= \left(4\sqrt{14} \int_{\frac{1}{2}}^{t_1} \frac{1}{\psi(s)}\, ds \,,\ \frac{1}{2}\psi(t_1) \right), \end{aligned}$$

where

$$\psi(s) := (\gamma_K(s)\gamma_L(s))^{\frac{1}{2}} = \begin{cases} 2\sqrt{2} & \text{if } \frac{1}{2} \leqslant s \leqslant \frac{3}{4} \\ 4(1-s)^{\frac{1}{4}} & \text{if } \frac{3}{4} \leqslant s \leqslant \frac{13}{14} \\ 4(14)^{\frac{1}{4}}(1-s)^{\frac{1}{2}} & \text{if } \frac{13}{14} \leqslant s \leqslant 1 \,. \end{cases}$$

Figure 1 represents the boundary of the geometric combination $K \star_{\frac{1}{2}} L$, which is obtained in the first quadrant by plotting the curve $t_1 \mapsto W_{\frac{1}{2}}(t_1, 1)$ over $[\frac{1}{2}, 1]$ (composed by two line segments and a positively curved arc), and in the other quadrants by reflection.

Example 24 (Geometric combination of two rhombi). If $K = R(a_1, a_2)$ is the centrally symmetric rhombus with one vertex at $(a_1, 0)$ and another one at $(0, a_2)$, with $a_i > 0$, we have $\mathcal{H}^1(K_{u_1(s)}) =$

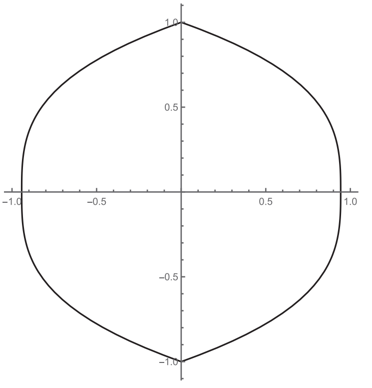

FIGURE 2 The geometric combination of the square and the rhombus in Example 25.

$2a_2\sqrt{2s}$, so that the i.d. field of K is given by

$$U(t_1,t_2)=\left(a_1\left(\sqrt{2t_1}-1\right),\ a_2\sqrt{2t_1}(2t_2-1)\right)\qquad\forall(t_1,t_2)\in(0,1)^2\,.$$

If we take another rhombus $L=R(b_1,b_2)$ with i.d. field V, and we choose, for instance, $\lambda=\frac{1}{2}$, the standard (e_1,e_2)-geometric potential of (U,V) in proportion $\frac{1}{2}$ has components

$$W_{1/2}(t_1,t_2)=\left(\sqrt{a_1b_1}(\sqrt{2t_1}-1),\ \sqrt{b_1b_2}\sqrt{2t_1}(2t_2-1)\right),\qquad\forall(t_1,t_2)\in(0,1)^2\,.$$

Hence,

$$R(a_1,a_2)\star_{\frac{1}{2}}R(b_1,b_2)=R(\sqrt{a_1a_2},\sqrt{b_1b_2})\,.$$

Example 25 (Geometric combination of a rectangle and a rhombus). Using the same notation as in Examples 21 and 24, if $K=Q(a_1,a_2)$ and $L=R(b_1,b_2)$, the standard (e_1,e_2)-geometric potential in proportion $1/2$ has components

$$W_{1/2}(t_1,t_2)=\left(\frac{4}{3}2^{\frac{1}{4}}\sqrt{a_1b_1}\left(t_1^{\frac{3}{4}}-\left(\frac{1}{2}\right)^{\frac{3}{4}}\right),2^{\frac{1}{4}}\sqrt{a_2b_2t_1^{\frac{1}{4}}}\,(2t_2-1)\right)\qquad\forall(t_1,t_2)\in(0,1)^2\,.$$

Taking, for example, $a_1=a_2=b_1=b_2=1$, one quarter of the boundary of $Q(1,1)\star_{\frac{1}{2}}R(1,1)$ is the curve

$$2^{\frac{1}{4}}\left(\frac{4}{3}\left(t_1^{\frac{3}{4}}-\left(\frac{1}{2}\right)^{\frac{3}{4}}\right),-t_1^{\frac{1}{4}}\right),\qquad t_1\in\left[0,\frac{1}{2}\right],$$

namely, the graph of the function

$$x(y)=-\frac{2\sqrt{2}}{3}(y^3+1)\,,\qquad y\in[-1,0]\,,$$

see Figure 2.

3.2 Unconditional convex bodies

When K is an unconditional convex body in $\mathbb{R}^n$, setting

$$K_{x_1,\dots,x_i} := \left\{(x_{i+1}e_{i+1} + \cdots + x_n e_n) \;:\; (x_1 e_1 + \cdots + x_i e_i + x_{i+1}e_{i+1} + \cdots + x_n e_n) \in K\right\},$$

the components $(u_1, \dots, u_n)$ of the $(e_1, \dots, e_n)$-i.d. field of K satisfy

- $t_1 \mapsto u_1(t_1)$ is the i.d. function of the map $x_1 \mapsto \mathcal{H}^{n-1}(K_{x_1})$, hence

$$u_1'(t_1) = \frac{|K|}{\mathcal{H}^{n-1}(K_{u_1(t_1)})} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t_1 \in (0,1)\,;$$

- for $\mathcal{L}^1$-a.e. $t_1 \in (0,1)$, $t_2 \mapsto u_2(t_1,t_2)$ is the i.d. function of the map $x_2 \mapsto \mathcal{H}^{n-2}(K_{(u_1(t_1),x_2)})$; hence,

$$\frac{\partial u_2}{\partial t_2}(t_1,t_2) = \frac{\mathcal{H}^{n-1}(K_{u_1(t_1)})}{\mathcal{H}^{n-2}(K_{u_1(t_1),u_2(t_1,t_2)})} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t_2 \in (0,1)\,;$$

- …
- for $\mathcal{L}^{n-1}$-a.e. $(t_1, \dots, t_{n-1}) \in (0,1)^{n-1}$, $t_n \mapsto u_n(t_1, \dots, t_{n-1}, t_n)$ is the i.d. function of the map $x_n \mapsto \chi_K(u_1(t_1)e_1 + \cdots + u_{n-1}(t_{n-1})e_{n-1} + x_n e_n)$; hence,

$$\frac{\partial u_n}{\partial t_n}(t_1, \dots, t_{n-1}, t_n) = \mathcal{H}^1(K_{u_1(t_1),\dots,u_{n-1}(t_1,\dots,t_{n-1})}) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t_n \in (0,1)\,.$$

Remark 26. Given an unconditional convex body K, the maps

$$s \mapsto \mathcal{H}^{n-i}\left(K_{u_1(t_1),\dots,u_{i-1}(t_1,\dots,t_{i-1},s)}\right) \qquad i = 1, \dots, n\,,$$

are continuous and strictly positive on the interior of their support (which is an interval). Hence, from their definition above, the components $(u_1, \dots, u_n)$ of the $(e_1, \dots, e_n)$-i.d. field of K satisfy

$$t_i \mapsto u_i(t_1, \dots, t_i) \in \mathrm{Lip}_{\mathrm{loc}}(0,1) \qquad i = 1, \dots, n\,.$$

In view of Proposition 11, we infer that, given two unconditional convex body K and L, the map $\lambda \mapsto K \star_\lambda L$ is a continuous interpolation in L^1 from $K \star_0 L = K$ to $K \star_1 L = L$. In addition, for $n = 2$, by Proposition 20, this interpolation is made up of convex bodies.

Proposition 27. *Let K and L be unconditional convex bodies in $\mathbb{R}^n$, and let $\lambda \in [0,1]$. Then the standard $(e_1, \dots, e_n)$-geometric combination $K \star_\lambda L$ enjoys the following properties:*

(i) *it is an unconditional set;*
(ii) *it satisfies the inclusion*

$$K \star_\lambda L \subseteq (1-\lambda)\cdot K +_0 \lambda \cdot L\,. \tag{26}$$

In particular, the equality $|K \star_\lambda L| = |K|^{1-\lambda}|L|^\lambda$ given by Theorem 18 implies the log-Brunn–Minkowski inequality for unconditional convex bodies.

Proof.

(i) Let us check, for every fixed $i \in \{1, \dots, n\}$, the implication

$$(x_1, \dots, x_i, \dots, x_n) \in K \star_\lambda L \implies (x_1, \dots, -x_i, \dots, x_n) \in K \star_\lambda L\,. \tag{27}$$

By construction, $K \star_\lambda L$ agrees with the image of the standard geometric potential W_λ, that is,

$$K \star_\lambda L = \Big\{ \big(w_{\lambda,1}(t_1), w_{\lambda,2}(t_1,t_2), \dots, w_{\lambda,n}(t_1, \dots, t_n) \big) \ : \ t_i \in (0,1) \Big\}\,.$$

From the expressions of $\frac{\partial u_i}{\partial t_i}$ recalled at the beginning of Section 3.2, we see that, since K is unconditional, the components u_i of its i.d. field satisfy

$$\frac{\partial u_i}{\partial t_i}(t_1, \dots, t_{i-1}, s) = \frac{\partial u_i}{\partial t_i}(t_1, \dots, t_{i-1}, 1-s) \qquad \text{for } \mathcal{L}^1\text{-a.e. } s \in (0,1)\,, \tag{28}$$

and similarly for the components v_i of the i.d. field of L.

Then, for every $i = 1, \dots, n$, $\delta \in (0, \frac{1}{2})$ and $t_1, \dots, t_{i-1} \in (0,1)$, it holds that

$$\begin{aligned}
w_{\lambda,i}(t_1, \dots, t_{i-1}, \tfrac{1}{2} + \delta) &= \int_{1/2}^{\frac{1}{2}+\delta} \frac{\partial u_i}{\partial t_i}(t_1, \dots, t_{i-1}, s)^{1-\lambda} \frac{\partial v_i}{\partial t_i}(t_1, \dots, t_{i-1}, s)^{\lambda}\, ds \\
&= \int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \frac{\partial u_i}{\partial t_i}(t_1, \dots, t_{i-1}, 1-s)^{1-\lambda} \frac{\partial v_i}{\partial t_i}(t_1, \dots, t_{i-1}, 1-s)^{\lambda}\, ds \\
&= -\int_{\frac{1}{2}-\delta}^{\frac{1}{2}} \frac{\partial u_i}{\partial t_i}(t_1, \dots, t_{i-1}, s')^{1-\lambda} \frac{\partial v_i}{\partial t_i}(t_1, \dots, t_{i-1}, s')^{\lambda}\, ds' \\
&= -w_{\lambda,i}(t_1, \dots, t_{i-1}, \tfrac{1}{2} - \delta)\,,
\end{aligned}$$

where the first and fourth equalities hold by definition of $w_{\lambda,i}$, the third one by the change of variable $s' = 1 - s$, and the second one is due to the fact that the functions u_i (and v_i) satisfy (28). This shows the implication (27).

(ii) In order to show the inclusion (26), it is enough to prove that

$$K \star_\lambda L \subseteq K^{1-\lambda} \cdot L^{\lambda}\,, \tag{29}$$

where

$$K^{1-\lambda} \cdot L^{\lambda} := \Big\{ \big(\pm|x_1|^{1-\lambda}|y_1|^{\lambda}, \dots, \pm|x_n|^{1-\lambda}|y_n|^{\lambda} \big) \ : \ (x_1, \dots, x_n) \in K\,,\ (y_1, \dots, y_n) \in L \Big\}\,.$$

Indeed, (26) will follow from (29) because $K^{1-\lambda} \cdot L^{\lambda} \subseteq (1-\lambda) \cdot K +_0 \lambda \cdot L$ (see [37, Lemma 4.1]).

Since we now know that both sets $K \star_\lambda L$ and $K^{1-\lambda} \cdot L^{\lambda}$ are unconditional, we are reduced to show that

$$(K \star_\lambda L) \cap \mathbb{R}^n_+ \subseteq \big(K^{1-\lambda} \cdot L^{\lambda} \big) \cap \mathbb{R}^n_+\,.$$

Let us consider a generic element of $(K \star_\lambda L) \cap \mathbb{R}^n_+$, which will be of the form

$$W_\lambda(t) := \big(w_{\lambda,1}(t_1), w_{\lambda,2}(t_1,t_2), \dots, w_{\lambda,n}(t_1,\dots,t_n)\big) \tag{30}$$

for some $t = (t_1, \dots, t_n)$ with $t_i \in (1/2, 1)$ for every $i = 1, \dots, n$.

By applying Hölder's inequality with conjugate exponents $\frac{1}{1-\lambda}$ and $\frac{1}{\lambda}$, we see that

$$0 \leqslant w_{\lambda,i}(t_1,\dots,t_i) \leqslant u_i(t_1,\dots,t_i)^{1-\lambda} v_i(t_1,\dots,t_i)^{\lambda} \qquad \forall i = 1,\dots,n\,. \tag{31}$$

Since the vector

$$\big(u_1(t_1)^{1-\lambda}v_1(t_1)^\lambda, u_2(t_1,t_2)^{1-\lambda}v_2(t_1,t_2)^\lambda, \dots, u_n(t_1,\dots,t_n)^{1-\lambda}v_n(t_1,\dots,t_n)^\lambda\big)$$

belongs to $(K^{1-\lambda} \cdot L^\lambda) \cap \mathbb{R}^n_+$ and, since the sets K and L are convex (which implies $K^{1-\lambda} \cdot L^\lambda$ convex as well), for every $s = (s_1, \dots, s_n) \in (0,1)^n$ the vector

$$\big(s_1u_1(t_1)^{1-\lambda}v_1(t_1)^\lambda, s_2u_2(t_1,t_2)^{1-\lambda}v_2(t_1,t_2)^\lambda, \dots, s_nu_n(t_1,\dots,t_n)^{1-\lambda}v_n(t_1,\dots,t_n)^\lambda\big)$$

belongs to $(K^{1-\lambda} \cdot L^\lambda) \cap \mathbb{R}^n_+$. Therefore, the inequalities (31) imply that the vector $W_\lambda(t)$ in (30) belongs to $(K^{1-\lambda} \cdot L^\lambda) \cap \mathbb{R}^n_+$. □

3.3 | Convex bodies with n symmetries

The class of convex bodies with n symmetries has been considered in the literature on Convex Geometry, in particular, to give a partial answer to some long-standing open questions such as the Mahler conjecture [6, 7] and the log-Brunn–Minkowski conjecture [13].

Before giving the definition, let us recall first that a linear reflection is a map $A \in GL(n)$ which acts identically into some $(n-1)$-dimensional linear subspace H of $\mathbb{R}^n$, and there exists $u \in S^{n-1} \setminus H$ such that $A(u) = -u$. In particular, an orthogonal reflection is a linear reflection A which belongs to $O(n)$.

Now, given a family $A_1, \dots, A_n$ of linear reflections in $\mathbb{R}^n$, such that the corresponding hyperplanes $H_1, \dots, H_n$ satisfy $H_1 \cap \dots \cap H_n = \{0\}$, we set

$$\mathcal{S}ym(A_1,\dots,A_n) := \Big\{K \subseteq \mathbb{R}^n \ : \ A_iK = K \quad \forall i = 1,\dots,n\Big\}.$$

To deal with the operation of geometric combination in the class $\mathcal{S}ym(A_1, \dots, A_n)$, it is crucial to choose an appropriate family of directions. This requires to fix some background from [13].

- If $(A_1, \dots, A_n)$ are orthogonal reflections in $\mathbb{R}^n$, such that the corresponding hyperplanes $H_1, \dots, H_n$ satisfy $H_1 \cap \dots \cap H_n = \{0\}$, we denote by $C(A_1, \dots, A_n)$ an *n-dimensional simplicial convex cone* (namely, the positive hull of n linearly independent vectors) that is associated with the closure of the group generated by $(A_, \dots, A_n)$ as in [13, Proposition 1]. If C is the positive hull of $w_1, \dots, w_n$, the linear subspaces generated by $\{w_1, \dots, w_n\} \setminus w_i$, for $i = 1, \dots, n$, are called the *walls* of C.
- If $(A_1, \dots, A_n)$ are merely linear reflections in $\mathbb{R}^n$, such that the corresponding hyperplanes $H_1, \dots, H_n$ satisfy $H_1 \cap \dots \cap H_n = \{0\}$, there exists a map $\Psi \in GL(n)$ such that $A_i' := \Psi A_i \Psi^{-1}$ are orthogonal reflections through hyperplanes $H_1', \dots, H_n'$ satisfying $H_1' \cap \dots \cap H_n' = \{0\}$ (Ψ can

be found as a map which sends the Löwner ellipsoid of K into the unit ball, see the proof of [13, Theorem 2]).

We refer to [13] for more details, and we proceed to state the following.

Proposition 28. *Let K and L be convex bodies in the class $\mathcal{S}ym(A_1, \dots, A_n)$, and let $\lambda \in [0,1]$. Let $K \star_\lambda L$ be their $(z_1, \dots, z_n)$-geometric combination in proportion λ, where the family $(z_1, \dots, z_n)$ is chosen as follows:*

(a) *if A_i are orthogonal reflections, $(z_1, \dots, z_n)$ are the normals to the walls of the simplicial convex cone $C(A_1, \dots, A_n)$;*
(b) *if A_i are merely linear reflections, denoting by Ψ a map in $GL(n)$ such that $A'_i := \Psi A_i \Psi^{-1}$ are orthogonal reflections, $(z_1, \dots, z_n)$ are the image through Ψ^{-1} of the normals to the walls of the simplicial convex cone $C(A'_1, \dots, A'_n)$.*

Then $K \star_\lambda L$ enjoys the following properties:

(i) *it belongs to the class $\mathcal{S}ym(A_1, \dots, A_n)$;*
(ii) *it satisfies the inclusion*

$$K \star_\lambda L \subseteq (1-\lambda) \cdot K +_0 \lambda \cdot L \,. \tag{32}$$

In particular, the equality $|K \star_\lambda L| = |K|^{1-\lambda}|L|^\lambda$ given by Theorem 18 implies the log-Brunn–Minkowski inequality in the class $\mathcal{S}ym(A_1, \dots, A_n)$.

Proof. Let us prove the statement first in case A_i are orthogonal reflections and then in case they are merely linear reflections.

– *Case (a). (A_i orthogonal reflections).*

(i) In order to check that $K \star_\lambda L$ belongs to the class $\mathcal{S}ym(A_1, \dots, A_n)$, let us write the sets K, L and $K \star_\lambda L$ as

$$\begin{aligned}
K &= \Big\{ u_1(t_1) z_1 + u_2(t_1,t_2) z_2 + \cdots + u_n(t_1, \dots, t_n) z_n \ : \ t_i \in (0,1) \Big\}, \\
L &= \Big\{ v_1(t_1) z_1 + v_2(t_1,t_2) z_2 + \cdots + v_n(t_1, \dots, t_n) z_n \ : \ t_i \in (0,1) \Big\}, \\
K \star_\lambda L &= \Big\{ w_{\lambda,1}(t_1) z_1 + w_{\lambda,2}(t_1,t_2) z_2 + \cdots + w_{\lambda,n}(t_1, \dots, t_n) z_n \ : \ t_i \in (0,1) \Big\},
\end{aligned}$$

where $U = u_1 z_1 + \cdots + u_n z_n$, $V = v_1 z_1 + \cdots + v_n z_n$ are the $(z_1, \dots, z_n)$-inverse distribution fields of K and L, and $W_\lambda = w_{\lambda,1} z_1 + \cdots + w_{\lambda,n} z_n$ is the standard geometric potential of (U, V) in proportion λ.

The fact that K and L belong to $\mathcal{S}ym(A_1, \dots, A_n)$ can be expressed as the system of equalities

$$\begin{aligned}
u_i(t_1, \dots, t_{i-1}, s) &= -u_i(t_1, \dots, t_{i-1}, 1-s) \qquad \forall s \in (0,1)\,, \\
v_i(t_1, \dots, t_{i-1}, s) &= -v_i(t_1, \dots, t_{i-1}, 1-s) \qquad \forall s \in (0,1)\,.
\end{aligned} \tag{33}$$

Then, proceeding in the same way as in the first part of the proof of Proposition 27, we see that the functions $w_{\lambda,i}$ continue to satisfy the analogous equalities:

$$w_{\lambda,i}(t_1, \dots, t_{i-1}, s) = -w_{\lambda,i}(t_1, \dots, t_{i-1}, 1-s) \qquad \forall s \in (0,1)\,, \tag{34}$$

which implies that also $K \star_\lambda L$ belongs to $\mathcal{S}ym(A_1, \dots, A_n)$.

(ii) By statement (i) already proved, we know that $K \star_\lambda L$ belongs to $Sym(A_1, \dots, A_n)$. Also, since the 0-sum is linear covariant (i.e., $A_i((1-\lambda)\cdot K +_0 \lambda \cdot L)) = (1-\lambda)\cdot A_i(K) +_0 \lambda \cdot A_i(L))$, we have that $(1-\lambda)\cdot K +_0 \lambda\cdot L$ belongs to $Sym(A_1, \dots, A_n)$. Therefore, in order to prove the inclusion (32), we are reduced to show that, denoting for brevity by C the simplicial convex cone $C(A_1, \dots, A_n)$, it holds that

$$C \cap (K \star_\lambda L) \subseteq C \cap ((1-\lambda)\cdot K +_0 \lambda \cdot L)\,. \tag{35}$$

In turn, to have (35) it is enough to show that

$$\Phi(C \cap (K \star_\lambda L)) \subseteq \Phi(C \cap ((1-\lambda)\cdot K +_0 \lambda \cdot L))\,,$$

where Φ is a map in $GL(n)$ with $\Phi(z_i) = e_i$, so that Φ maps C into $\mathbb{R}^n_+$ (recall that $(z_1, \dots, z_n)$ denote the normals to the walls of C).

Since K and L are invariant under the closure of the group generated by $(A_1, \dots, A_n)$, by [13, Proposition 1 (v)], we know that the unconditional sets $\overline{K}$ and $\overline{L}$ defined by

$$\mathbb{R}^n_+ \cap \overline{K} := \Phi(C \cap K) \qquad \text{and} \qquad \mathbb{R}^n_+ \cap \overline{L} := \Phi(C \cap L)$$

are unconditional convex bodies.

Moreover, from the proof of Theorem 8 and [13, Lemma 6 (ii)], we know that

$$\begin{aligned}\mathbb{R}^n_+ \cap \Big((1-\lambda)\cdot \overline{K} +_0 \lambda \cdot \overline{L}\Big) &\subseteq \Phi\big(\big\{x \in C \ : \ \langle x, u\rangle \leqslant h_K(u)^{1-\lambda} h_L(u)^\lambda \ \forall u \in C\big\}\big) \\ &= \Phi(C \cap ((1-\lambda)\cdot K +_0 \lambda \cdot L))\,.\end{aligned}$$

We are thus reduced to prove that

$$\Phi(C \cap (K \star_\lambda L)) \subseteq \mathbb{R}^n_+ \cap \Big((1-\lambda)\cdot \overline{K} +_0 \lambda \cdot \overline{L}\Big)\,. \tag{36}$$

The inclusion (36) will follow from Proposition 27 applied to the unconditional convex bodies $\overline{K}$ and $\overline{L}$, provided that we are able to show that

$$\Phi(C \cap (K \star_\lambda L)) = \mathbb{R}^n_+ \cap (\overline{K} \star_\lambda \overline{L})\,, \tag{37}$$

where $\overline{K} \star_\lambda \overline{L}$ is the $(e_1, \dots, e_n)$-geometric combination of $\overline{K}$ and $\overline{L}$.

We emphasize that the families of vectors with respect to which the two geometric combinations appearing in (37) are constructed are distinguished, and their indication is omitted just for notational simplicity. For the sake of clearness, let us repeat that $K \star_\lambda L$ is the $(z_1, \dots, z_n)$-geometric combination of K and L, with $(z_1, \dots, z_n)$ chosen as specified in the statement of Proposition 28, while $\overline{K} \star_\lambda \overline{L}$ is the $(e_1, \dots, e_n)$-geometric combination of the unconditional bodies $\overline{K}$ and $\overline{L}$.

Let us check the equality (37). By the system of equations (33)–(34), we see that

$$u_i(t_1, \dots, t_{i-1}, 1/2) = v_i(t_1, \dots, t_{i-1}, 1/2) = w_{\lambda,i}(t_1, \dots, t_{i-1}, 1/2) = 0 \quad \forall i = 1, \dots n\,.$$

Hence, the intersections of the walls of C with K, L, and $K \star_\lambda L$ are given, respectively, by

$$K \cap \Big\{t_i = \frac{1}{2}\Big\}, \quad L \cap \Big\{t_i = \frac{1}{2}\Big\}, \quad (K \star_\lambda L) \cap \Big\{t_i = \frac{1}{2}\Big\}\,.$$

Thus,

$$C \cap K = \Big\{ u_1(t_1) z_1 + u_2(t_1, t_2) z_2 + \cdots + u_n(t_1, \ldots, t_n) z_n \ : \ t_i \in (1/2, 1) \Big\},$$
$$C \cap L = \Big\{ v_1(t_1) z_1 + v_2(t_1, t_2) z_2 + \cdots + v_n(t_1, \ldots, t_n) z_n \ : \ t_i \in (1/2, 1) \Big\},$$
$$C \cap (K \star_\lambda L) = \Big\{ w_{\lambda,1}(t_1) z_1 + w_{\lambda,2}(t_1, t_2) z_2 + \cdots + w_{l,n}(t_1, \ldots, t_n) z_n \ : \ t_i \in (1/2, 1) \Big\}.$$

By applying the map Φ to the last equality above, we see that the set at the left-hand side of (37) satisfies

$$\Phi(C \cap (K \star_\lambda L)) = \Big\{ w_{\lambda,1}(t_1) e_1 + w_{\lambda,2}(t_1, t_2) e_2 + \cdots + w_{\lambda,n}(t_1, \ldots, t_n) e_n \ : \ t_i \in (1/2, 1) \Big\}. \tag{38}$$

On the other hand, we have

$$\begin{aligned} \mathbb{R}^n_+ \cap \overline{K} &= \Phi(C \cap K) = \Phi\Big(\Big\{ u_1(t_1) z_1 + u_2(t_1, t_2) z_2 + \cdots + u_n(t_1, \ldots, t_n) z_n \ : \ t_i \in (1/2, 1) \Big\} \Big) \\ &= \Big\{ u_1(t_1) e_1 + u_2(t_1, t_2) e_2 + \cdots + u_n(t_1, \ldots, t_n) e_n \ : \ t_i \in (1/2, 1) \Big\} \end{aligned}$$

and similarly for L. In view of the equalities (33), we infer that

$$\overline{K} = \Big\{ u_1(t_1) e_1 + u_2(t_1, t_2) e_2 + \cdots + u_n(t_1, \ldots, t_n) e_n \ : \ t_i \in (0, 1) \Big\},$$
$$\overline{L} = \Big\{ v_1(t_1) e_1 + v_2(t_1, t_2) e_2 + \cdots + v_n(t_1, \ldots, t_n) e_n \ : \ t_i \in (0, 1) \Big\}.$$

It follows straightforwardly that $\overline{U} := u_1(t_1) e_1 + u_2(t_1, t_2) e_2 + \cdots + u_n(t_1, \ldots, t_n) e_n$ and $\overline{V} := v_1(t_1) e_1 + v_2(t_1, t_2) e_2 + \cdots + v_n(t_1, \ldots, t_n) e_n$ are, respectively, the $(e_1, \ldots, e_n)$-i.d. fields of the unconditional bodies $\overline{K}$ and $\overline{L}$. Therefore,

$$\overline{K} \star_\lambda \overline{L} = \Big\{ w_{\lambda,1}(t_1) e_1 + w_{\lambda,2}(t_1, t_2) e_2 + \cdots + w_{\lambda,n}(t_1, \ldots, t_n) e_n \ : \ t_i \in (0, 1) \Big\}.$$

Hence, the set at the right-hand side of (37) satisfies

$$\mathbb{R}^n_+ \cap (\overline{K} \star_\lambda \overline{L}) = \Big\{ w_{\lambda,1}(t_1) e_1 + w_{\lambda,2}(t_1, t_2) e_2 + \cdots + w_{\lambda,n}(t_1, \ldots, t_n) e_n \ : \ t_i \in (1/2, 1) \Big\}. \tag{39}$$

By combining (38) and (39), equality (37) follows.

– *Case (b) (A_i linear reflections).* Let Ψ be a map in $GL(n)$ such that $A'_i := \Psi A_i \Psi^{-1}$ are orthogonal reflections. Let η_i be the normals to the walls of the simplicial convex cone $C(A'_1, \ldots, A'_n)$, so that $(z_1, \ldots, z_n) := \Psi^{-1}(\eta_1, \ldots, \eta_n)$. Statements (i)–(ii) readily follow from the corresponding items already proved in Case (a), provided we show that

$$\Psi(K \star_\lambda L) = \Psi(K) \star_\lambda \Psi(L), \tag{40}$$

where the geometric combination $K \star_\lambda L$ is made with respect to $(z_1, \ldots, z_n)$, and the geometric combination $\Psi(K) \star_\lambda \Psi(L)$ is made with respect to $(\eta_1, \ldots, \eta_n)$.

To prove (40), we start by writing $K \star_\lambda L$ as

$$K \star_\lambda L = \Big\{ w_{\lambda,1}(t_1) z_1 + w_{\lambda,2}(t_1, t_2) z_2 + \cdots + w_{\lambda,n}(t_1, \ldots, t_n) z_n \ : \ t_i \in (0, 1) \Big\},$$

where $w_{\lambda,i}$ are the components of the standard $(z_1, \dots, z_n)$-geometric potential of the $(z_1, \dots, z_n)$-i.d. fields of K and L.

Then

$$\Psi(K \star_\lambda L) = \left\{ w_{\lambda,1}(t_1)\eta_1 + w_{\lambda,2}(t_1, t_2)\eta_2 + \cdots + w_{\lambda,n}(t_1, \dots, t_n)\eta_n \ : \ t_i \in (0,1) \right\},$$

so that (40) holds true, provided that $w_{\lambda,i}$ are also the components of the standard $(\eta_1, \dots, \eta_n)$-geometric potential of the $(\eta_1, \dots, \eta_n)$-i.d. fields of $\Psi(K)$ and $\Psi(L)$. In turn, this is true, provided that the following implication holds: if $U(t) = u_1(t_1)z_1 + u_2(t_1, t_2)z_2 + \cdots + u_n(t_1, \dots, t_n)z_n$ is the $(z_1, \dots, z_n)$-i.d. field of K, then $\widetilde{U}(t) = u_1(t_1)\eta_1 + u_2(t_1, t_2)\eta_2 + \cdots + u_n(t_1, \dots, t_n)\eta_n$ is the $(\eta_1, \dots, \eta_n)$-i.d. field of $\Psi(K)$ (and similarly for L). Such implication follows immediately from Definition 12. □

4 | OPEN PROBLEMS

- *Problem 4.1: Continuous interpolations.* Given two functions $f, g : \mathbb{R}^n \to \mathbb{R}_+$, with strictly positive integrals and i.d. fields U, V, respectively, construct a family of fields $\{W_\lambda\}_{\lambda \in [0,1]}$ such that, if $f \star_\lambda g$ is defined according to (19), there holds as follows.
  - (i) For $\lambda = 0$ and $\lambda = 1$, we have $f \star_0 g = f$ and $f \star_1 g = g$.
  - (ii) The map $[0,1] \ni \lambda \mapsto f \star_\lambda g$ is continuous in $L^1(\mathbb{R}^n)$.
  - (iii) For every $\lambda \in [0,1]$, $\int_{\mathbb{R}^n} f \star_\lambda g(x)\,dx = (\int_{\mathbb{R}^n} f(x)\,dx)^{1-\lambda} (\int_{\mathbb{R}^n} g(x)\,dx)^\lambda$.

  In this respect, we have seen that, in dimension $n = 1$, taking w_λ equal to a geometric primitive of (u, v) in proportion λ, property (iii) is always satisfied by Theorem 8, whereas properties (i)–(ii) are satisfied in case u, v belong to $AC_{\mathrm{loc}}(0,1)$, but they are not in case the distributional derivatives of u or v contain jumps or Cantor parts (cf. Remark 10 and Proposition 11). Moreover, we recall that (i)–(iii) hold in any space dimension when f and g are the characteristic functions of two unconditional convex bodies, see Remark 26.
- *Problem 4.2: Convexity preserving of geometric combination in dimension $n > 2$.* Establish whether (at least under the additional assumption that K and L are unconditional), Proposition 20 continues to hold in dimension $n > 2$.
- *Problem 4.3: Comparison between geometric combination and 0-sum.* Establish whether (at least in dimension $n = 2$) the inclusion (26) in Proposition 27 continues to hold for arbitrary convex bodies (not necessarily unconditional), provided that the family of directions $(z_1, \dots, z_n)$ needed to construct $K \star_\lambda L$ is suitably chosen. An affirmative answer would imply the log-Brunn–Minkowski inequality (establishing the result if $n > 2$, and giving a new proof of it if $n = 2$).
- *Problem 4.4: Comparison between geometric combination and Minkowski-sum.* Establish whether the inclusion $K \star_\lambda L \subseteq (1 - \lambda)K + L$ holds for arbitrary convex bodies, provided that the family of directions $(z_1, \dots, z_n)$ needed to construct $K \star_\lambda L$ is suitably chosen. An affirmative answer would give another proof of the classical Brunn–Minkowski inequality.

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