Journal of
Nonlinear
Science

Check for
updates

Spreading Equilibria Under Mildly Singular Potentials: Pancakes Versus Droplets

Riccardo Durastanti[1] · Lorenzo Giacomelli[2]

Abstract

We study global minimizers of a functional modeling the free energy of thin liquid layers over a solid substrate under the combined effect of surface, gravitational, and intermolecular potentials. When the latter ones have a mild repulsive singularity at short ranges, global minimizers are compactly supported and display a microscopic contact angle of $\pi/2$. Depending on the form of the potential, the macroscopic shape can either be droplet-like or pancake-like, with a transition profile between the two at zero spreading coefficient for purely repulsive potentials. These results generalize, complete, and give mathematical rigor to de Gennes' formal discussion of spreading equilibria. Uniqueness and non-uniqueness phenomena are also discussed.

Communicated by Osmany Porto de Oliveira.

✉ Lorenzo Giacomelli
    lorenzo.giacomelli@uniroma1.it

   Riccardo Durastanti
   riccardo.durastanti@unina.it

[1]   Department of Mathematics and Applications "Renato Caccioppoli", University of Naples "Federico II", Via Cintia, Monte S. Angelo, 80126 Naples, Italy

[2]   SBAI Department, Sapienza University of Rome, Via Antonio Scarpa 16, 00161 Rome, Italy

⌖ Springer

1 Introduction and Results

1.1 The Problem

We consider a class of singular energy functionals of the form

$$E[u] = \int_{\mathbb{R}^N} \left(\tfrac{1}{2}|\nabla u|^2 + Q(u) \right) \mathrm{d}x, \tag{1.1}$$

where the potential $Q(u)$ satisfies the following structural assumptions:

$$\begin{cases} Q \in C((0, +\infty)), \ Q \equiv 0 \text{ in } (-\infty, 0], \ \inf Q > -\infty, \\ Q(u) \sim Au^{1-m} \ \text{ as } u \to 0^+, \ m > 1, \ A > 0. \end{cases} \tag{1.2}$$

In view of (1.2), (1.1) may be seen as a generalized and singular version of the Alt–Caffarelli or Alt–Phillips functionals (Alt and Caffarelli 1981; Alt and Phillips 1986). When modeling the height $u(x)$ of a liquid film over a solid substrate in lubrication approximation, γE represents the free energy of the liquid, with γ the liquid-vapor surface tension. In this case, the potential Q usually combines the effects of inter-molecular, gravitational, and surface forces:

$$Q(u) = (P(u) + G(u) - S)\chi_{\{u>0\}}$$

(de Gennes 1985; Oron et al. 1997) The function P is an inter-molecular potential, singular at $u = 0$ and decaying at $u = +\infty$; the function G is a gravitational potential; S is the non-dimensionalized spreading coefficient:

$$S = \frac{\text{spreading coefficient}}{\gamma} = \frac{\gamma_{\mathrm{SG}} - \gamma_{\mathrm{SL}} - \gamma}{\gamma},$$

where γ_{SG} and γ_{SL} are the solid–gas and solid–liquid tensions, respectively. There is, however, a caveat to be made at this point.

In thermodynamic equilibrium of the solid with the vapor phase (the so-called "moist" case, which concerns for instance a surface which has been pre-exposed to vapor), γ_{SG} is usually denoted by γ_{SV}, and its value can never exceed $\gamma_{\mathrm{SL}} + \gamma$. Indeed, otherwise the free energy of a solid/vapor interface could be lowered by inserting a liquid film in between: the equilibrium solid/vapor interface would then comprise such film, leading to $\gamma_{\mathrm{SV}} = \gamma_{\mathrm{SL}} + \gamma$. Therefore, $S \leq 0$ in the "moist" case. On the other hand, when the solid and the gaseous phase are not in equilibrium (the so-called "dry" case), there is no constraint on the sign of S.

The cases $S < 0$, resp. $S \geq 0$, are commonly referred to as *partial wetting*, resp. *complete wetting*: indeed, when $Q \equiv -S$, the global minimizer's support is compact if $-S > 0$, whereas if $-S \leq 0$ the global minimizer does not exist and the final spreading equilibrium is a zero-thickness unbounded film [see, e.g., Maggi (2012, §19.4)], where the complete form of the surface energy is considered instead of its lubrication approximation).

We are interested in nonnegative *global* minimizers (hereafter simply called *minimizers*) of E under the constraint of fixed mass; that is, in the set

$$\mathcal{D} = \mathcal{D}_M = \left\{ u \in H^1(\mathbb{R}^N) : u \geq 0, \int u = M \right\} \tag{1.3}$$

(we shall omit the subscript M when unnecessary).

1.2 The Potential Q

When gravity is not taken into account, Q is characterized by

$$Q(u) = \begin{cases} Au^{1-m}(1 + o(1)) & \text{as } u \to 0^+ \\ -S - Bu^{1-n}(1 + o(1)) & \text{as } u \to +\infty \end{cases} \tag{1.4}$$

with $A > 0$, $B \in \mathbb{R}$, and $m, n > 1$. Since $A > 0$, the singularity of Q at $u = 0$ disfavors small heights of the droplet and corresponds to short-range repulsive forces. When the strength of the singularity is sufficiently high, namely when $m \geq 3$, the very existence of a minimizer is precluded, since $E[u] \equiv +\infty$ for any $u \in \mathcal{D}$ (see Lemma 2.10 below). However, this is not the case when the singularity is milder ($m < 3$), which is the focus of this manuscript.

At long ranges, $B < 0$ corresponds to considering the effect of repulsive forces only [cf. the discussion in de Gennes (1985, II.D.1) and references therein], whereas $B > 0$ corresponds to considering short-range repulsive, long-range attractive, forces [cf. Oron et al. (1997, II.E and references therein)]. We anticipate that the long-range decay exponent n is not essential: it enters the analysis only in critical cases.

Though our results cover a wide range of potentials, it will be convenient to introduce a few prototypical cases (Fig. 1A). The first one, Q_a, is repulsive-attractive for $B > 0$ and purely repulsive for $B \leq 0$:

$$Q_a(u) = Au^{1-m} - Bu^{1-n} - S \quad \text{for } u > 0, \\ B \in \mathbb{R}, \quad 1 < n < m. \tag{1.5}$$

For purely repulsive potentials ($B < 0$), a long-range decay exponent n larger than the short-range growth exponent m is often considered. A prototype which is suited to this situation is
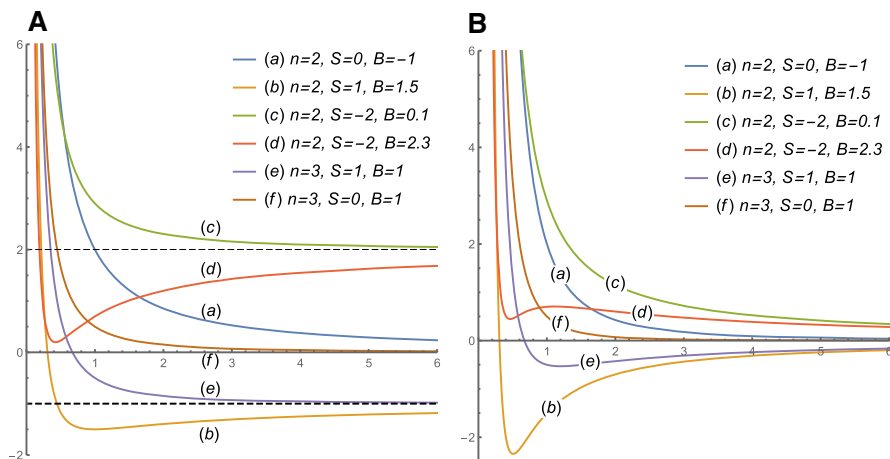
$$Q_b(u) = \frac{A|B|u}{|B|u^m + Au^n} - S \quad \text{for } u > 0, \quad B < 0, \quad m < n, \tag{1.6a}$$

for which we only consider the convex case, corresponding to the constraint

$$1 + 2m + m^2 + 2n - 6mn + n^2 \leq 0. \tag{1.6b}$$

Finally, when gravity is taken into account, the potential G has to be added:

$$Q_{a,g}(u) = Q_a(u) + \tfrac{1}{2}Du^2, \quad Q_{b,g}(u) = Q_b(u) + \tfrac{1}{2}Du^2. \tag{1.7}$$

Fig. 1 **A** On the left, prototypes of Q with $A = 1$ and $m = 5/2$: from (a) to (d), Q_a with $n = 2$; in (e) and (f), Q_b with $n = 3$. **B** On the right, the corresponding functions $R(u) = Q(u)/u$: R' has no zeros in (a), (c), and (f), one zero in (b) and (e), two zeroes in (d); $e_* < +\infty$ in (b) and (e)

1.3 The Framework

An enormous amount of work has been done on the fundamentals of wetting phenomena, from different perspectives: referenced discussions may be found in de Gennes (1985), Finn (1986), Oron et al. (1997), Bonn et al. (2009), Maggi (2012) and Snoeijer and Andreotti (2013). Concerning the analysis of energy functionals E of the form (1.1) *with a singular potential Q*, the focus has mainly been on two aspects.

- *Positive minimizers with $Q \equiv +\infty$ for $u \leq 0$ and/or $m \geq 3$*. In this case, short range repulsion is so strong that compactly supported minimizers do not exist, and energy minimization forces the creation of a tiny liquid layer fully separating gas and solid. In this framework, interesting qualitative properties of minimizers, such as (in)stability of the flat film, bifurcation, concentration, and asymptotic scaling laws with respect to the potential's parameters, have been successfully investigated, also in relation to dynamic phenomena such as coarsening and dewetting; see Becker et al. (2003), Bertozzi et al. (2001), Glasner and Witelski (2003), Chen and Jiang (2012), Chen et al. (2020), Jiang (2011), Laugesen and Pugh (2000a, b), Laugesen and Pugh (2002a, b), Liu and Witelski (2020), Otto et al. (2006), Glasner et al. (2009), the references therein, and Witelski (2020) for a recent overview.
- *Potentials with $A < 0$*. In this case, minimizers and critical points also have a rich structure: we refer to Jiang and Ni (2007) and again to Laugesen and Pugh (2000a, b, 2002a, b) for a thorough study, including classification, stability, and other qualitative properties.

On the other hand, in the case of mildly singular potentials,

$$Q(u) \equiv 0 \text{ for } u \leq 0 \text{ and } 1 < m < 3, \tag{1.8}$$

the minimization problem (1.1)–(1.3) does not seem to have been explored so far. We are only aware of two very recent and interesting works (De Silva and Savin 2022a, b), where the model case $Q(u) = u^{1-m}\chi_{\{u>0\}}$ is considered on a bounded domain Ω with Dirichlet boundary condition (and no mass constraint). There, existence and regularity of minimizers is discussed, together with the regularity of the free boundary $\partial\{u > 0\} \cap \Omega$ and the Γ-limit as $m \to 3^-$.

The case (1.8) is the focus of the present manuscript. Given the vastity of the potentials which have been introduced and considered through the years, we prefer to study generic potentials rather than concentrating on model cases.

1.4 Existence and Basic Properties of Minimizers

Solely under (1.2) and (1.8), the existence of a minimizer of E in $\mathcal{D}$ is guaranteed by standard direct methods and symmetry arguments (see Theorem 2.1). The assumption $m < 3$ is crucial, since $E[u] \equiv +\infty$ on $\mathcal{D}$ if $m \geq 3$ (see Lemma 2.10). It turns out that the minimizer we obtain is:

(a) compactly supported;
(b) radially symmetric (up to a translation of x);
(c-) non-increasing along radii.

In the rest of this introduction, we assume in addition that $Q \in C^1((0, +\infty))$. If either $N = 1$, or if $Q'(u)$ satisfies a very mild additional condition for $u \gg 1$ (see (3.2) below), then (a), (b), and (c-) in fact hold for *any* minimizer; in addition, *any* minimizer is (see Theorem 3.1):

(c) strictly decreasing along radii;
(d) a smooth solution to the Euler–Lagrange equation for some $\lambda \in \mathbb{R}$:

$$- \Delta u + Q'(u) = \lambda \quad \text{in } \{u > 0\}. \tag{1.9}$$

1.5 The One-Dimensional Case

For $N = 1$, we are able to obtain much more detailed information, such as uniqueness and asymptotic results, which are discussed in the next paragraphs. The key to both of them is the identification of λ, which we prove via a combination of ODE and variational arguments (Theorem 4.5):

$$\lambda = R(u(0)), \quad \text{where} \quad R(u) := \frac{Q(u)}{u}. \tag{1.10}$$

Not surprisingly, the function R plays a crucial role in the analysis. First of all, it follows from (1.10) that a constant function $u_s \in (0, +\infty)$ is a stationary solution to (1.9) if and only if $Q'(u_s) = \lambda = R(u_s) = Q(u_s)/u_s$; since $u^2 R'(u) = u Q'(u) - Q(u)$, in fact

u_s is a stationary solution to (1.9) if and only if $R'(u_s) = 0$.

We assume, as in the model cases (1.5), (1.6), and (1.7), that these stationary solutions do not accumulate at 0 or $+\infty$:

$$\delta \in (0, 1) \text{ exists such that } R' \neq 0 \; in \; (0, \delta) \cup (\delta^{-1}, +\infty).$$

Of crucial importance is the smallest among the absolute minimum points of R, provided they exist:

$$e_* := \begin{cases} +\infty & \text{if } \nexists \min R \\ \min R^{-1}(\min R) & \text{otherwise.} \end{cases} \tag{1.11}$$

In the model cases, e_* coincides with the unique global minimum point of R, whenever such point exists (Fig. 1B).

1.6 Uniqueness

As is often the case, uniqueness is related to convexity. If Q is convex in $(0, e_*)$, by comparison arguments we show that the minimizer is unique (see Theorem 4.13). In terms of the model cases, this translates into (see Sect. 8):

- uniqueness for Q_a if $B \leq 0$, or if $B > 0$ and $-S \leq 0$, or if $-S \geq 0$ and $B \geq c_1(A, S)$, where

$$c_1(A, S) := (m - 1) \left(\frac{A}{n-1} \right)^{\frac{n-1}{m-1}} \left(\frac{-S}{m-n} \right)^{\frac{m-n}{m-1}};$$

- uniqueness for $Q_{a,g}$ if $-S \leq 0$ or if $B \leq c_3(A, D)$, where

$$c_3(A, D) := \frac{m+1}{n(n-1)} \left(\frac{Am(m-1)}{n+1} \right)^{\frac{n+1}{m+1}} \left(\frac{D}{m-n} \right)^{\frac{m-n}{m+1}};$$

- uniqueness for Q_b and $Q_{b,g}$.

Interestingly, however, potentials Q exist such that the minimizer is *not* unique for at least one value of the mass M. Generally speaking, this occurs when R is not injective in $(0, e_*)$ (see Theorem 7.4): this is the case, for instance, in model Q_a with $-S > 0$ and $c_2(A, S) \leq B < c_1(A, S)$, where (see Sect. 8)

$$c_2(A, S) := \frac{m-1}{n} \left(\frac{Am}{n-1} \right)^{\frac{n-1}{m-1}} \left(\frac{-S}{m-n} \right)^{\frac{m-n}{m-1}}.$$

1.7 Micro–Macro Relations and the Regime $M \gg 1$

When continuum models are considered, wetting phenomena are characterized by the presence of two interfaces of codimension-one (liquid–solid and liquid–gas) and an (unknown) *contact line*, i.e., a codimension-two interface where the solid, the liquid,
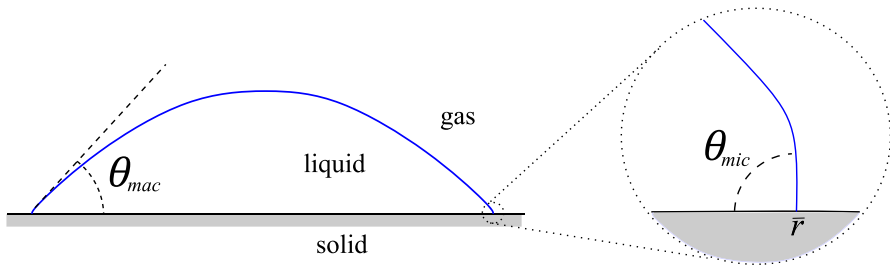
Fig. 2 A prototypical droplet ($N = 1$). On the left, the macroscopic profile and the macroscopic contact angle θ_{mac}; on the right, a zoom into the contact-line $\bar{r}$ and the microscopic contact angle θ_{mic}. All minimizers of E in $\mathcal{D}$ have $\theta_{\mathrm{mic}} = \pi/2$

and the surrounding gas or vapor meet (Fig. 2). Among the main topics of interest to the physics and applied math communities, are the modeling of the "microscopic" physics near these interfaces—e.g., in terms of intermolecular potentials, substrate's corrugation and, in a dynamic context, slippage, contact-line free boundary conditions, and rheological properties—and the analysis of how such microscopic laws affect the "macroscopic" behavior of droplets. See the reviews de Gennes (1985), Oron et al. (1997), Bonn et al. (2009), Snoeijer and Andreotti (2013) and also Feldman and Kim (2018), Flitton and King (2004), Ren et al. (2010) for referenced discussions.

In this context, of particular importance are the *microscopic contact angle* θ_{mic}, identified with the arctangent of the droplet's slope *at* the contact line, and (various notions of) *macroscopic, or effective, or apparent contact angle* θ_{mac}: generally speaking, this is the arctangent of the slope, near the contact line, of the profile that the droplet assumes in the bulk of the wetted region, see Fig. 2.

For droplet's dynamics, after the pioneering works (Huh and Scriven 1971; Dussan and Davis 1974; Voinov 1976; Tanner 1979), the relation of θ_{mac} and macroscopic profile with θ_{mic}, microscopic modeling, and speed of the contact line has been extensively studied via both formal asymptotic methods [see, e.g., Cox (1986), Hocking (1983), Hocking (1992), Haley and Miksis (1991), Bertsch et al. (2000), Eggers and Stone (2004), Ansini and Giacomelli (2002), Chiricotto and Giacomelli (2013) and the references therein] and rigorous arguments (Giacomelli and Otto 2002; Giacomelli et al. 2016; Delgadino and Mellet 2021), especially in the case $\theta_{\mathrm{mic}} = 0$. More details may be found, e.g., in Eggers and Stone (2004) and Bonn et al. (2009, §C).

In the framework of this paper, which is concerned with statics, the "microscopic" physics are encoded in the intermolecular part P of the potential Q. In order to associate to P a microscopic length-scale ε, for a given reference potential P_0 one could set

$$P(u) = P_0\left(\tfrac{u}{\varepsilon}\right) \quad \text{with} \ \ \varepsilon \ll 1. \tag{1.12}$$

Then, the macroscopic profile of minimizers could be identified by taking the limit as $\varepsilon \to 0$. However, due to the lack of scaling invariance of E for general Q, it is more convenient to look at the limit as $M \to +\infty$. The two regimes are equivalent when E has a scaling invariance. This is the case, for instance, when P has the form (1.12)

and $G = 0$ (no gravity): indeed, with the scaling $x = \varepsilon \hat{x}$, $u = \varepsilon \hat{u}$, one obtains

$$E[u] = \int_{\mathbb{R}} \left(\tfrac{1}{2}u_x^2 + P_0\left(\tfrac{u}{\varepsilon}\right) - S \right) \mathrm{d}x = \varepsilon \int_{\mathbb{R}} \left(\tfrac{1}{2}\hat{u}_{\hat{x}}^2 + P_0(\hat{u}) - S \right) \mathrm{d}\hat{x}$$

with

$$\int_{\mathbb{R}} \hat{u} \mathrm{d}\hat{x} = \varepsilon^{-2} \int_{\mathbb{R}} u \mathrm{d}x \gg 1.$$

Hence, we will analyze the limit $M \to +\infty$: the goal is to identify a macroscopic profile, whence a macroscopic contact angle (if it exists), as the limit of (suitably rescaled) minimizers u_M of E in $\mathcal{D}_M$.

1.8 Microscopic Behavior

The microscopic behavior of minimizers of E in $\mathcal{D}$ is universally determined by the short-range form of the potential. Indeed, we show in Theorem 4.7 that

$$u(x) \sim \left(\frac{A(m+1)^2}{2} \right)^{\frac{1}{m+1}} (\bar{r} - x)^{\frac{2}{m+1}} \quad \text{as } x \to \bar{r}^-,$$

where $\bar{r}$ denotes the right boundary of the minimizer's support. This shows that mildly singular potentials produce steady states with $\theta_{\mathrm{mic}} = \pi/2$ (Fig. 2).

1.9 Macroscopic Behavior: Pancakes Versus Droplets

Let u_M be a minimizer of E in $\mathcal{D}_M$. By translation invariance, we may assume that $\operatorname{supp} u_M = [-\bar{r}_M, \bar{r}_M]$ and that the maximal height is $u_{0M} = u_M(0)$. The behavior of u_M for $M \gg 1$ is essentially influenced by two quantities: the constant e_* defined in (1.11), which is always finite in presence of gravity (i.e., $D > 0$), and the non-dimensionalized spreading coefficient S, which for a generic potential Q is defined by

$$(-\infty, +\infty) \ni -S := \lim_{u \to +\infty} Q(u) \quad \text{when the limit exists and is finite.}$$

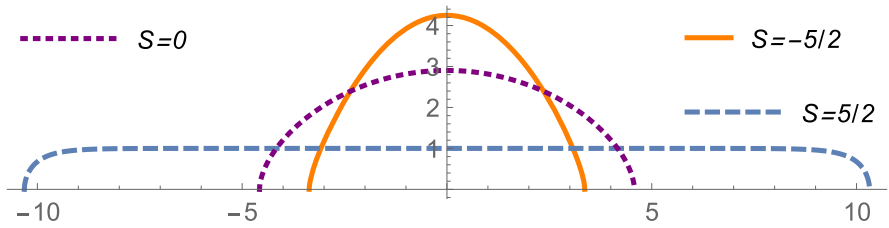We will prove in Sect. 5 that there are two generic behaviors of u_M as $M \to +\infty$.

- **Droplet:** as $M \to +\infty$,

$$u_{0M}^4 \sim \frac{9|S|}{32}M^2, \quad \bar{r}_M^4 \sim \frac{9}{8|S|}M^2, \quad u_{0M}^{-1}u_M(\bar{r}_M y) \sim 1 - y^2. \tag{1.13}$$

- **Pancake:** as $M \to +\infty$,

$$u_{0M} \sim e_*, \quad \bar{r}_M \sim \frac{1}{2e_*}M, \quad u_M(\bar{r}_M y) \sim e_*. \tag{1.14}$$

Fig. 3 In the case Q_a with $A = 1$, $B = 0$, $m = 5/2$, and $M = 20$, the unique minimizer of E for different values of S. Minimizers are obtained as numerical solutions of (1.9)–(1.10) with $u'(0) = 0$, tuning $u(0)$ so that $M = 20$ (with a tolerance of 10^{-2})

If $S = 0$ and $e_* < +\infty$, the behavior is of pancake type. On the other hand, if $S = 0$ and Q is strictly decreasing (whence $e_* = +\infty$), we also find a third, intermediate behavior, which connects droplets to pancakes (cf. Remark 5.14) through the decay exponent of $Q(u)$ as $u \to +\infty$.

- **Transition profile**: with $S = 0$, $Q' < 0$, and $Q(u) \sim Ku^{1-p}$ as $u \to +\infty$,

$$u_{0M}^{p+3} \sim \frac{pK}{2c_p^2 f_p(0)^2} M^2, \quad \bar{r}_M^{p+3} \sim \frac{f_p(0)^2}{2^{p+2} pK c_p^{p+1}} M^{p+1},$$

$$u_{0M}^{-1} u_M(\bar{r}_M y) \sim f_p^{-1}(f_p(0)|y|), \tag{1.15}$$

where

$$f_p(w) := \int_w^1 \frac{\sqrt{p}\,\tilde{w}^{\frac{p-1}{2}}\,\mathrm{d}\tilde{w}}{\sqrt{1 - \tilde{w}^p}} \quad \text{and} \quad c_p := \int_0^1 f_p^{-1}(f_p(0)y)\mathrm{d}y.$$

In Fig. 3, we report numerical solutions to the minimization problem in a prototypical case in which uniqueness holds. Table 1 summarizes the main assumptions which lead to each of these behaviors, together with the corresponding model cases and with references to the corresponding results.

1.10 Profiles of Minimizers: Macroscopic Contact Angles and Thickness

Combining the information in Paragraphs 1.8 and 1.9, we can characterize minimizers as follows.

- **Droplet:** we have

$$u(x) \sim \begin{cases} \frac{\sqrt{|S|}}{\bar{r}\sqrt{2}}(\bar{r}^2 - x^2) & \text{if } \delta \lesssim \bar{r} - x \leq \bar{r} \text{ (macroscopic profile)}, \\ \left(\frac{A(m+1)^2}{2}\right)^{\frac{1}{m+1}}(\bar{r}-x)^{\frac{2}{m+1}} & \text{if } 0 \leq \bar{r} - x \lesssim \delta \text{ (microscopic profile)}, \end{cases} \tag{1.16}$$

(see Fig. 4), where

$$\bar{r}^4 \sim \frac{9M^2}{8|S|}, \quad \delta^{m-1} \sim (2|S|)^{-\frac{m+1}{2}} \frac{A(m+1)^2}{2}.$$

Table 1 Synopsis of the main results

Main assumptions	Asymptotics as $M \to +\infty$ (Theorem 5.8, Theorem 5.12)	Uniqueness (Theorem 4.13, Theorem 7.4)	Model cases [cf. (1.5), (1.6), (1.7)] (Sect. 8)	Uniqueness in model cases (Sect. 8)
• $e_* < +\infty$	Pancake	Yes if $Q'' \geq 0$ in $(0, e_*)$	• $Q_a, -S < 0$	• Yes
			• $Q_a, -S = 0, B > 0$	• Yes
			• $Q_a, -S > 0, B \geq c_1(A, S)$	• Yes
			• $Q_b, -S < 0$	• Yes
			• $Q_{b,g}$	• Yes
			• $Q_{a,g}$	• Yes if $-S \leq 0$ or $B \leq c_3(A, D)$
• $e_* = +\infty$	Droplet	Yes if $Q'' \geq 0$	• $Q_a, -S > 0,\ \ c_2(A, S) \leq B < c_1(A, S)$	• No
• $-S > 0$			• $Q_a, -S > 0\ \ 0 < B < c_2(A, S)$,	• Not known
			• $Q_a, -S > 0, B \leq 0$	• Yes
			• $Q_b, -S > 0$	• Yes
• $e_* = +\infty$	Transition profile	Yes if $Q'' \geq 0$	• $Q_a, -S = 0, B \leq 0$	• Yes
• $-S = 0$			• $Q_b, -S = 0$	• Yes
• $Q' < 0$ in $(0, +\infty)$				

In this case, it is natural to define the *macroscopic contact angle* θ_{mac} as the arctangent of the slope of the macroscopic profile at the boundary of its support:

$$\tan \theta_{\mathrm{mac}} = \frac{\sqrt{|S|}}{\bar{r}\sqrt{2}} \frac{\mathrm{d}}{\mathrm{d}x}(\bar{r}^2 - x^2)\big|_{x=-\bar{r}} = \sqrt{2|S|}. \qquad (1.17)$$

This analysis also identifies the *transitional thickness* as the height $\sqrt{2|S|}\delta$ at which the crossover takes place:

$$\sqrt{2|S|}\delta \sim \left(\frac{A(m+1)^2}{4|S|}\right)^{\frac{1}{m-1}}. \qquad (1.18)$$

- **Pancake:** when $e_* < +\infty$, we have

$$u(x) \sim \begin{cases} e_* & \text{if } \delta \lesssim \bar{r} - x \leq \bar{r} \text{ (macroscopic profile)}, \\ \left(\frac{A(m+1)^2}{2}\right)^{\frac{1}{m+1}} (\bar{r} - x)^{\frac{2}{m+1}} & \text{if } 0 \leq \bar{r} - x \lesssim \delta \text{ (microscopic profile)}, \end{cases}$$

where

$$\bar{r} \sim \frac{M}{2e_*}, \quad \delta^2 \sim e_*^{m+1}\left(\frac{A(m+1)^2}{2}\right)^{-1}.$$
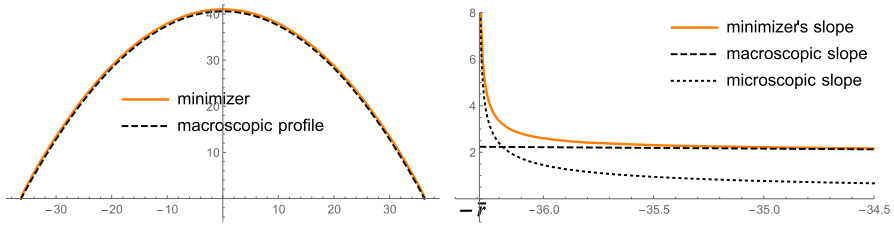
Fig. 4 As in Fig. 3, $Q = Q_a$ with $A = 1$, $B = 0$, $m = 5/2$; here $S = -5/2$. Left: the minimizer of E (solid) with $M = 2000$—obtained as in Fig. 3 with a tolerance of 10^{-1}—and its macroscopic profile (dashed), as defined in (1.16). Right: the slope of the same functions near the contact line $-\bar{r}$, together with the slope of the microscopic profile (dotted), as defined in (1.16)

The pancake's thickness $e_* < +\infty$ is defined in (1.11): it satisfies $R'(e_*) = 0$, that is,

$$Q(e_*) = e_* Q'(e_*). \tag{1.19}$$

- **Transition profile:** the behavior of f_p and f_p^{-1} is detailed in Remark 5.13. From there, we see that the droplet has three regimes:

$$u(x) \sim \begin{cases} \dfrac{u_0}{\bar{r}^2}\left(\bar{r}^2 - \dfrac{f_p^2(0)}{4}x^2\right) & \text{if } \bar{r} - x \approx \bar{r} \quad \text{(macroscopic profile)}, \\[2ex] \dfrac{u_0}{\bar{r}^{\frac{2}{p+1}}}\left(\dfrac{(p+1)f_p(0)}{2\sqrt{p}}(\bar{r}-x)\right)^{\frac{2}{p+1}} & \text{if } 1 \ll \bar{r} - x \ll \bar{r} \text{ (intermediate profile)}, \\[2ex] \left(\dfrac{A(m+1)^2}{2}\right)^{\frac{1}{m+1}}(\bar{r}-x)^{\frac{2}{m+1}} & \text{if } 0 \leq \bar{r} - x \ll 1 \text{ (microscopic profile)}, \end{cases}$$

where $\bar{r} = \bar{r}_M$ and $u_0 = u_{0M}$ are as in (1.15).

1.11 Repulsive Potentials: Comparison with de Gennes' Results

In part II.D of his milestone review (de Gennes 1985), where final spreading equilibria are discussed, de Gennes considers two model cases. The first one ("van der Waals forces"), on which we focus, is of the generic form (1.4) with $m = 3$ and $n = 4$, which corresponds to Q_b with $m = 3$ and $n = 4$. Now, we know from Lemma 2.10 that minimizers of E in $\mathcal{D}$ do not exist if $m \geq 3$. However, de Gennes confines his analysis to scales not below 30Å, where "a continuum picture is still applicable". In any event, our results show that, replacing $m = 3$ by a generic exponent $m \in (1, 3)$, most of his formal predictions can be rigorously justified down to $u = 0$. To proceed further, we distinguish three cases.

Partial wetting ($-S > 0$ in Q_b). When $-S > 0$, the macroscopic shape is of droplet type [see (1.13)]. Our results confirm, in the case of negligible gravitational effects, both the relation between S and the macroscopic contact angle and, in the limiting case $m = 3$, the estimate for the transitional thickness [compare (1.17) and (1.18) with de Gennes (1985, (2.54)) and the discussion below it].

Limiting case ($-S = 0$ in Q_b). In the limiting case $-S = 0$, the macroscopic shape is given by f_p^{-1} [see (1.15)]. In particular, in the limiting case $m = 3$ and for $p = 4$, we recover the same scaling exponents for the microscopic and intermediate regimes in de Gennes (1985, (2.55)–(2.56)),

$$
u(x) \sim
\begin{cases}
\dfrac{u_0}{\bar{r}^2}\left(\bar{r}^2 - \dfrac{f_p^2(0)}{4}x^2\right) & \text{if } \bar{r} - x \approx \bar{r} \quad\quad\ \text{(macroscopic profile)}, \\[2ex]
\dfrac{u_0}{\bar{r}^{2/5}}\left(\dfrac{5f_p(0)}{4}(\bar{r}-x)\right)^{2/5} & \text{if } 1 \ll \bar{r} - x \ll \bar{r} \ \text{(intermediate profile)}, \\[2ex]
(8A)^{1/4}(\bar{r}-x)^{1/2} & \text{if } 0 \le \bar{r} - x \ll 1 \ \text{(microscopic profile)},
\end{cases}
$$

the only difference being in the multiplicative constants, which turn out to depend on $f_p(0)$ and are therefore expressed in terms of Γ functions.

"Dry" complete wetting ($-S < 0$ in Q_b, and $Q_{b,g}$). In de Gennes (1985), only the case $Q_{b,g}$ (with gravity) with $-S < 0$ is discussed. However, we see from Table 1 that the same qualitative result (pancake shape) holds for two other cases which do not seem to have been discussed there:

- model Q_b (without gravity) when $-S < 0$;
- model $Q_{b,g}$ with $-S \ge 0$.

The characterization of e_* in (1.19) coincides with that in de Gennes (1985, (2.63)) whenever e_* is uniquely defined. In particular, one easily checks that if D is relatively small and S is relatively large, namely

$$
S^{n-m} \gg A^{n-1}|B|^{1-m} \quad \text{and} \quad D^{n-m} \ll A^{n+1}B^{-m-1},
$$

then

$$
uQ'(u) - Q(u) \sim -Amu^{1-m} + S \quad \text{for } u \ll \left(\frac{|B|}{A}\right)^{\frac{1}{n-m}}, \quad \text{hence} \quad e_* \sim \left(\frac{mA}{S}\right)^{\frac{1}{m-1}},
$$

which coincides with de Gennes (1985, (2.72)) in the critical case $m = 3$. However, the reader can easily realize that there are various other possibilities, depending on the relation between the four parameters S, A, B, D.

Remark 1.1 As Table 1 shows, the above conclusions hold not only for model Q_b, but also for model Q_a when $B \le 0$ (which, in this case, is also purely repulsive). In addition, the qualitative aspects of our results remain true for the second model potential considered by de Gennes ("double-layer forces"):

$$
Q(u) =
\begin{cases}
A|\log u|(1 + o(1)) & \text{as } u \to 0^+ \\
-S - Be^{-u}(1 + o(1)) & \text{as } u \to +\infty.
\end{cases}
$$

However, quantitative information need be modified in this case, taking into account that a log singularity of the potential corresponds to the limiting case "$m = 1$." We refrain from doing that for the sake of brevity.

Fig. 5 Model Q_a: D = droplet, P = pancake, ! = uniqueness
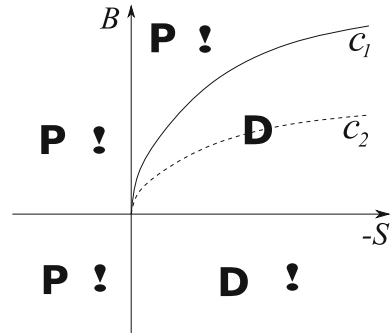
1.12 Repulsive/Attractive Potentials

As we mentioned in Sect. 1.3, potentials which are short-range repulsive and long-range attractive, such as model case Q_a with $B > 0$, have been widely discussed in the thin-film literature, using various forms of them, especially in order to model and analyze coarsening dynamics and dewetting phenomena; however, qualitative studies of mild singularities ($Q(u) \equiv 0$ for $u \leq 0$ and $1 < m < 3$) seem to have been missing so far.

We focus on Q_a with $B > 0$. The different possible behaviors are summarized in Fig. 5, where $B \leq 0$ is also shown for completeness.

If $-S \leq 0$ (complete wetting), a unique minimizer exists with pancake shape. However, as opposed to purely repulsive potentials, a unique pancake-shaped minimizer may exist in the partial wetting regime ($-S > 0$), too, provided B is sufficiently large. In addition, as we mentioned already in Sect. 1.6, droplet-shaped minimizers can fail to be unique for moderate values of B.

1.13 Open Questions

We conclude by discussing what we believe to be the main open questions in the problem we discussed.

Uniqueness. Uniqueness is left open only in some cases (see Table 1). In particular, we expect non-uniqueness phenomena to occur whenever Q is not convex in $(0, e_*)$, without the additional assumption that R is not injective (Theorem 7.4); e.g., model Q_a with $-S > 0$ and $0 < B < c_2(A, S)$.

Higher dimension. Our qualitative study is one-dimensional. In higher dimensions, it is still possible to characterize the eigenvalue λ. However, the relation between u and λ becomes nonlocal, involving integrals of functions of u rather than $u(0)$ alone (cf. 1.10). In addition, the Euler–Lagrange equation (in radial variable) becomes non-autonomous. The combination of these two features so far prevented us from developing an analogous qualitative study for $N > 1$.

Critical points and their stability. This manuscript is concerned with global minimizers of E in $\mathcal{D}$. However, we expect that E also has critical points in $\mathcal{D}$, consisting of two or more radially decreasing solutions to (1.9)–(1.10), suitably translated so to

have disjoint positivity sets, with a possibly different λ for each of them. It would be very interesting to prove that such configurations are indeed critical points of E in $\mathcal{D}$ and to study their (in)stability in either variational and/or dynamical sense, see, e.g., Burchard et al. (2012), Cheung and Chou (2010, 2017), Chou and Zhang (2012), Kang et al. (2016), Laugesen and Pugh (2002a) and Nicolaou (2018).

Full curvature problem. The gradient part of the functional in (1.1) may be derived from Stokes system on the basis of the main assumption in lubrication approximation: the vertical length scale is much smaller than the horizontal one (Giacomelli and Otto 2001, 2003). If not for the full Stokes system, it would be interesting to perform an analogous study at least for the functional E with $\frac{1}{2}|\nabla u|^2$ replaced by $\sqrt{1 + |\nabla u|^2}$; in other words, the full curvature effect is retained, though the droplet is yet assumed to be a subgraph. In this case, we are only aware of the studies in Novick-Cohen (1992), Novick-Cohen (1993), Minkov and Novick-Cohen (2001), and Minkov and Novick-Cohen (2006), which concern existence and uniqueness for convex potentials in the one-dimensional case.

Dynamics. Since the nineties (Bernis and Friedman (1990)), a lot of work has been done on existence (Grün 1995; Beretta et al. 1995; Bertozzi and Pugh 1996; Dal Passo et al. 1998; Grün 2004) and qualitative properties (such as finite speed of propagation, waiting time, long-time behavior) (Bernis 1996a, b; Bertsch et al. 1998; Dal Passo et al. 2001; Giacomelli and Grün 2006; Fischer 2014, 2016) of the spreading dynamics associated to E, as modeled by *thin-film equations*, which in one space dimension formally read as

$$u_t + \left(f(u)(u_{xx} - Q'(u))_x \right)_x = 0 \quad \text{on } \{u > 0\} \tag{1.20}$$

with f depending on the slip condition adopted at the liquid-solid interface ($f(u) = u^3 + bu^2$, $b > 0$, for Navier slip). When the potential is sufficiently singular ($Q(u) \equiv +\infty$ for $u \leq 0$ and/or $m \geq 3$), existence and uniqueness are rather simple, since the datum is to be positive and the solution will as well (Grün and Rumpf 2001; Bertozzi et al. 2001). On the other hand, for mildly or non-singular potentials, (1.20) turns into a genuine free boundary problem. Concerning weak solutions, most efforts in its study concentrated on existence and qualitative properties of "zero contact-angle" solutions, which satisfy $u_x = 0$ at $\partial\{u > 0\}$ for a.e. $t > 0$. We mention in particular (Bertozzi and Pugh 1994; Dal Passo et al. 2001; Novick-Cohen and Shishkov 2010), where the case of power-law potentials is discussed. However, these zero contact-angle solutions have the property of converging to their mean for the Neumann problem on a bounded domain (Bertozzi and Pugh 1994), regardless of their initial mass. It would be very interesting to see whether different classes of solutions to (1.20) exist, which instead satisfy a right-angle condition at $\partial\{u > 0\}$ and converge to a stable critical point of E for long times. Such achievements would be analogous to the ones regarding weak solutions with $Q \equiv 0$ and finite nonzero microscopic contact-angle (Otto 1998; Bertsch et al. 2005; Mellet 2015; Chiricotto and Giacomelli 2017). First results in this direction are contained in Durastanti and Giacomelli (2022): there, formal arguments support the existence of generic (both advancing and receding) traveling wave solutions of (1.20) for any speed and any $m \in (1, 3)$, with a contact angle of $\pi/2$ at $\partial\{u > 0\}$.

Notably, such waves exist even *without* slip conditions (i.e., for $b = 0$): hence mildly singular potentials may be seen as an alternative solution to the contact-line paradox.

  More recently, a well-posedness theory of "classical" solutions has been developed for both zero (Giacomelli et al. 2008, 2014; Gnann 2015, 2016; Gnann and Petrache 2018; Seis 2018) and fixed nonzero (Knüpfer 2011, 2015; Knüpfer and Masmoudi 2013, 2015) contact-angle. As a further step, it would also be interesting to develop a theory of "classical" solutions for the singular potentials Q addressed here. Another interesting question concerns, in the pancake case with $e_* \ll 1$, intermediate scaling laws for macroscopic droplets spreading over a microscopic pancake, in the spirit of Giacomelli and Otto (2002).

1.14 Notations

We define $\mathbb{R}_+ := (0, +\infty)$. By $C^\alpha_{[loc]}(\Omega)$ we mean the space of [locally] Hölder continuous functions with exponent α in $\Omega \subset \mathbb{R}^N$. The subscript c denotes spaces of functions with compact support. We denote by $|\Omega|$ the Lebesgue measure of a Lebesgue measurable subset Ω of $\mathbb{R}^N$. We denote by $B_R(x)$ the ball of radius R and center x in $\mathbb{R}^N$ and by ω_{N-1} the $(N-1)$-dimensional measure of the unit sphere $\mathbb{S}^{N-1} = \partial B_1(0)$. The Sobolev conjugate exponent of 2 is denoted by $2^* = \frac{2N}{N-2}$. For a measurable function f, we define

$$f_+ := \max(f, 0), \quad f_- := -\min(f, 0), \quad \operatorname{supp} f := \overline{\{x \in \operatorname{dom} f : f(x) \neq 0\}}.$$

We omit the domain of integration when it coincides with $\mathbb{R}^N$, and (when no ambiguity occurs) we also omit the differential dx when x is (a rescaling of) the spatial independent variable. If not otherwise specified, we will denote by C several constants whose value may change from line to line. These values will only depend on the data (for instance, C may depend on N).

  We say that a function is *radially strictly decreasing*, resp. *radially non-increasing* (with respect to $x_0 \in \mathbb{R}^N$), if it is radially symmetric (with respect to $x_0 \in \mathbb{R}^N$) and the corresponding radial function is strictly decreasing, resp. non-increasing.

2 Existence of a Minimizer

Note that (1.2) implies that

$$s_1 := \sup\{s \in \mathbb{R} : Q > 0 \text{ in } (0, s)\} \in (0, +\infty]. \tag{2.1}$$

In this section, we discuss existence and basic properties of minimizers.

Theorem 2.1 (Existence and basic properties of minimizers) *Assume* (1.2)*, $m < 3$, and $M > 0$. Then, there exists a minimizer u of E in $\mathcal{D}$. Moreover, u is radially non-increasing w.r.to a certain $x_0 \in \mathbb{R}^N$, u is compactly supported, and $u \in C^{1/2}_{loc}(\mathbb{R}^N \setminus \{x_0\})$.*

  We divide the proof into lemmas.

Lemma 2.2 *There exists $u \in \mathcal{D}$ such that $E[u] < +\infty$.*

Proof We define $u : \mathbb{R}^N \to [0, +\infty)$ as

$$u(x) := d(1 - |x|^2)_+^\alpha, \qquad \alpha \in \left(\tfrac{1}{2}, \tfrac{1}{m-1}\right),$$

where d is chosen so that $\int u = M$. Note that the range of α is not empty since $1 < m < 3$. Straightforward computations show that

$$\int |\nabla u|^2 = 4\alpha^2 d^2 \int (1 - |x|^2)_+^{2(\alpha-1)} |x|^2$$

$$= 4\alpha^2 d^2 \omega_{N-1} \int_0^1 (1 - r^2)^{2(\alpha-1)} r^{N+1} \mathrm{d}r \overset{\alpha > \frac{1}{2}}{<} +\infty.$$

Since $\operatorname{supp} u = B_1(0)$, we deduce that $u \in \mathcal{D}$. Moreover $Q(0) = 0$, so that

$$\int Q(u) = \int_{B_1(0)} Q(u) \overset{(1.2)}{\leq} C \int_{B_1(0)} \left(1 + \frac{1}{(1 - |x|^2)^{\alpha(m-1)}}\right)$$

$$= C\omega_{N-1} \int_0^1 \left(1 + \frac{1}{(1 - r^2)^{\alpha(m-1)}}\right) r^{N-1} \mathrm{d}r \overset{\alpha < \frac{1}{m-1}}{<} +\infty,$$

hence $E[u] < +\infty$. □

Remark 2.3 The previous lemma is false if $m \geq 3$ (see Lemma 2.10).

Lemma 2.4 *There exists $C \geq 0$ such that*

$$\int Q(u) \geq -C \tag{2.2}$$

for any $u \in \mathcal{D}$. In particular, E is bounded from below in $\mathcal{D}$.

Proof The lower bound of E is immediate from (2.2). Recall that $Q > 0$ in $(0, s_1)$, with s_1 defined in (2.1). If $s_1 = +\infty$ then Q is nonnegative and (2.2) is obvious with $C = 0$. Otherwise, we have

$$\int Q(u) \geq \int_{\{u \geq s_1\}} Q(u) \overset{(1.2)}{\geq} \inf Q \int_{\{u \geq s_1\}} 1 \geq \inf Q \int \frac{u}{s_1} \overset{(1.3)}{\geq} \frac{M}{s_1} (\inf Q)$$

for any u in $\mathcal{D}$, whence (2.2) with $C = -\frac{M}{s_1}(\inf Q)$. □

Lemma 2.5 *There exists a minimizer u of E in $\mathcal{D}$. Moreover, u is radially non-increasing.*

Proof It follows from Lemmas 2.2 and 2.4 that there exists a minimizing sequence $\{\tilde{u}_k\}$ in $\mathcal{D}$ such that

$$\lim_{k \to +\infty} E[\tilde{u}_k] = \inf_{\mathcal{D}} E \in \mathbb{R}. \tag{2.3}$$

In particular, for all $k \in \mathbb{N}$, we have that

$$-C \overset{(2.2)}{\leq} \int Q(\tilde{u}_k) \leq E[\tilde{u}_k] \overset{(2.3)}{\leq} C \tag{2.4}$$

and

$$\frac{1}{2} \int |\nabla \tilde{u}_k|^2 \overset{(1.1)}{=} E[\tilde{u}_k] - \int Q(\tilde{u}_k) \overset{(2.4)}{\leq} C. \tag{2.5}$$

In addition, by Nash inequality (cf. Carlen and Loss 1993),

$$\int \tilde{u}_k^2 \leq C \left(\int \tilde{u}_k \right)^{\frac{4}{N+2}} \left(\int |\nabla \tilde{u}_k|^2 \right)^{\frac{N}{N+2}} \overset{(2.5)}{\leq} C M^{\frac{4}{N+2}}. \tag{2.6}$$

For $k \in \mathbb{N}$, let u_k be the Schwarz symmetrization of $\tilde{u}_k$. Then, (Kesavan 2006, §1.3 and 1.4)

$$u_k \geq 0, \quad u_k(x) = v_k(|x|) \text{ with } v_k \text{ non-increasing in } (0, +\infty), \quad \int u_k = M. \tag{2.7}$$

Applying the Pólya–Szegő inequality (Kesavan 2006, Theorem 2.3.1 and Remark 2.3.5) and since the Schwarz symmetrization preserves the L^2-norm (Kesavan 2006, §1.3 and 1.4), we have that

$$\int |\nabla u_k|^2 \leq \int |\nabla \tilde{u}_k|^2 \quad \text{and} \quad \int u_k^2 = \int \tilde{u}_k^2. \tag{2.8}$$

In order to show that $Q(\tilde{u}_k)$ is uniformly bounded in $L^1(\mathbb{R}^N)$, we estimate

$$\int Q_-(\tilde{u}_k) \overset{(1.2),(2.1)}{\leq} -\inf Q \int_{\{\tilde{u}_k \geq s_1\}} 1 \leq -\inf Q \int \frac{\tilde{u}_k}{s_1} \overset{\tilde{u}_k \in \mathcal{D}}{=} -\frac{M}{s_1}(\inf Q) \tag{2.9}$$

and

$$\int Q_+(\tilde{u}_k) = \int Q(\tilde{u}_k) + \int Q_-(\tilde{u}_k) \overset{(2.4),(2.9)}{\leq} C. \tag{2.10}$$

Combining (2.9) and (2.10), we conclude that

$$\int |Q(\tilde{u}_k)| \le C \quad \text{for all } k \in \mathbb{N}. \tag{2.11}$$

Since $Q(\tilde{u}_k) \in L^1(\mathbb{R}^N)$, we may apply the results in Kesavan (2006, Section 1.3):

$$\int Q(u_k) = \int Q(\tilde{u}_k) \quad \text{and} \quad \int |Q(u_k)| = \int |Q(\tilde{u}_k)| \overset{(2.11)}{\le} C$$
$$\text{for every } k \in \mathbb{N}. \tag{2.12}$$

In particular,

$$\|u_k\|_{H^1(\mathbb{R}^N)} \overset{(2.8)}{\le} \|\tilde{u}_k\|_{H^1(\mathbb{R}^N)} \overset{(2.5),(2.6)}{\le} C$$
$$\text{and} \quad E[u_k] \overset{(2.8),(2.12)}{\le} E[\tilde{u}_k] \tag{2.13}$$

for all k in $\mathbb{N}$. This implies that $\{u_k\}$ is another minimizing sequence in $\mathcal{D}$. In view of (2.13), there exists a nonnegative, radially non-increasing function $u \in H^1(\mathbb{R}^N)$ such that

$$u_k \rightharpoonup u \text{ in } H^1(\mathbb{R}^N),$$
$$u_k \to u \text{ a.e. in } \mathbb{R}^N \text{ and in } L^p(B_R(0)) \text{ for any } R > 0 \text{ and any } 1 \le p < 2^*. \tag{2.14}$$

We now show that $\int u = M$, hence $u \in \mathcal{D}$. For any $R \ge 1$, we have

$$M \ge \int_{B_R(0) \backslash B_{R/2}(0)} u_k \overset{(2.7)}{=} \omega_{N-1} \int_{R/2}^R r^{N-1} v_k(r) \mathrm{d}r \overset{(2.7)}{\ge} \omega_{N-1} \left(\frac{R}{2}\right)^N v_k(R),$$

therefore

$$v_k(R) \le C R^{-N} \quad \text{for all } R \ge 1 \text{ and all } k \in \mathbb{N}. \tag{2.15}$$

Since $\int u_k = M$ for all k, we have

$$M = \lim_{k \to \infty} \int u_k = \lim_{k \to \infty} \int_{B_R(0)} u_k + \lim_{k \to \infty} \int_{\mathbb{R}^N \backslash B_R(0)} u_k$$
$$\overset{(2.14)}{=} \int_{B_R(0)} u + \lim_{k \to \infty} \int_{\mathbb{R}^N \backslash B_R(0)} u_k. \tag{2.16}$$

In order to estimate the second integral on the right-hand side of (2.16), we note that, in view of (1.2) and (2.15),

$$Q(u_k) \ge \frac{A}{2} u_k^{1-m} \chi_{\{u_k > 0\}} \text{ in } \mathbb{R}^N \backslash B_R(0) \text{ for } R \gg 1. \tag{2.17}$$

Therefore,

$$
\frac{1}{\omega_{N-1}} \int_{\mathbb{R}^N \setminus B_R(0)} u_k \overset{(2.7),(2.15)}{\leq} C R^{-N} \int_R^{+\infty} \chi_{\{v_k>0\}} r^{N-1} \mathrm{d}r
$$

$$
\overset{(2.7)}{\leq} C R^{-N} (v_k(R))^{m-1} \int_R^{+\infty} (v_k(r))^{1-m} \chi_{\{v_k>0\}} r^{N-1} \mathrm{d}r
$$

$$
\overset{(2.17),(2.15)}{\leq} C R^{-Nm} \int_R^{+\infty} Q(v_k) r^{N-1} \mathrm{d}r
$$

$$
\overset{(2.7)}{\leq} C R^{-Nm} \int |Q(u_k)| \overset{(2.12)}{\leq} C R^{-Nm}. \tag{2.18}
$$

Passing to the limit in (2.16) as $R \to +\infty$ using (2.18) and Beppo Levi's theorem, we conclude that $\int u = M$, hence $u \in \mathcal{D}$.

Now we prove that u is a minimizer of E in $\mathcal{D}$. The passage to the limit in the Dirichlet energy is straightforward by lower semi-continuity:

$$
\int |\nabla u|^2 \leq \liminf_{k \to \infty} \int |\nabla u_k|^2. \tag{2.19}
$$

Let us focus on the potential energy. Let $0 < \delta < s_1$ be such that $|\{u = \delta\}| = 0$ (note that this is the case for a.e. $\delta > 0$, see Lemma A.1). Choosing $R \gg \delta^{-1/N}$ and using (2.15), we deduce that $u_k \leq C R^{-N} < \delta$ in $\mathbb{R}^N \setminus B_R(0)$ for all k sufficiently large. Therefore,

$$
\int Q(u_k) \overset{\delta < s_1}{\geq} \int Q(u_k) \chi_{\{u_k > \delta\}}
$$

$$
= \int_{B_R(0)} Q_+(u_k) \chi_{\{u_k > \delta\}} - \int_{B_R(0)} Q_-(u_k) \chi_{\{u_k > \delta\}}. \tag{2.20}
$$

It follows from (2.14) that $u_k \to u$ a.e. in $\mathbb{R}^N$. Hence, since $|\{u = \delta\}| = 0$, $\chi_{\{u_k > \delta\}} \to \chi_{\{u > \delta\}}$ a.e. Moreover, by (1.2), $Q_- \leq C$. Hence by Fatou lemma, applied to the first term on the right-hand side of (2.20), and Lebesgue theorem, applied to the second term on the right-hand side of (2.20), we obtain

$$
\liminf_{k \to \infty} \int Q(u_k) \geq \int_{B_R(0)} Q(u) \chi_{\{u > \delta\}}. \tag{2.21}
$$

Since $u < \delta$ in $\mathbb{R}^N \setminus B_R(0)$, the right-hand side of (2.21) coincides with $\int Q(u) \chi_{\{u > \delta\}}$. Therefore,

$$
\liminf_{k \to \infty} \int Q(u_k) \geq \int Q(u) \chi_{\{u > \delta\}} = \int Q(u) \chi_{\{\delta < u < s_1\}} + \int Q(u) \chi_{\{u \geq s_1\}}.
$$

By Lemma A.1, $|\{u = \delta\}| = 0$ for a.e. $\delta > 0$; by Lemma A.2, $\chi_{\{u>\delta\}} \to \chi_{\{u>0\}}$ a.e. in $\mathbb{R}^N$ as $\delta \to 0$. Therefore, by Beppo Levi's theorem, we obtain

$$\liminf_{k\to\infty} \int Q(u_k) \geq \lim_{\delta\to 0} \int Q(u)\chi_{\{\delta<u<s_1\}} + \int Q(u)\chi_{\{u\geq s_1\}} = \int Q(u). \quad (2.22)$$

Since $\{u_k\}$ is a minimizing sequence, it follows from (2.19) and (2.22) that $E[u] \leq \liminf_{k\to+\infty} E[u_k] = \inf_{\mathcal{D}} E$; hence, u is a minimizer of E in $\mathcal{D}$. $\qquad\square$

Remark 2.6 If $N = 1$, the compact embedding $H^1(\mathbb{R}) \Subset C^{\frac{1}{2}}(\mathbb{R})$ implies that any minimizer belongs to $C^{\frac{1}{2}}(\mathbb{R})$.

Remark 2.7 Any minimizer u of E in $\mathcal{D}$ satisfies $Q(u) \in L^1(\mathbb{R}^N)$. Indeed, with the same argument just used in the proof of Lemma 2.5 [cf. (2.9)–(2.11)], we have

$$\int Q_-(u) \overset{(1.2),(2.1),u\in\mathcal{D}}{\leq} -\frac{M}{s_1}(\inf Q) \quad \text{and}$$

$$\int Q_+(u) = \int Q(u) + \int Q_-(u) \overset{E[u]<+\infty}{\leq} C.$$

The next Lemma implies that the minimizer given by Lemma 2.5 has compact support.

Lemma 2.8 *Any radially non-increasing function $u : \mathbb{R}^N \to [0, +\infty)$ such that*

$$u \in L^1(\mathbb{R}^N) \quad \text{and} \quad \int Q(u) < +\infty \quad (2.23)$$

has compact support.

Proof We can assume without loss of generality that u is radially symmetric with respect to $x_0 = 0$ in $\mathbb{R}^N$: $u(x) = v(|x|)$ with v non-increasing in $(0, +\infty)$. Since $u \in L^1(\Omega)$, arguing as in the proof of (2.15) we see that $v(R) \leq CR^{-N}$ for all $R \geq 1$. Consequently, arguing as in (2.18) we obtain for $R \gg 1$

$$\int_R^{+\infty} \chi_{\{v>0\}} r^{N-1} dr \leq CR^{-N(m-1)} \int_{\mathbb{R}^N\backslash B_R(0)} Q(u). \quad (2.24)$$

Since

$$\int_{\mathbb{R}^N\backslash B_R(0)} Q(u) = \int Q(u) - \int_{B_R(0)} Q(u) \leq \int Q(u) + \omega_{N-1}\frac{R^N}{N}\max\{0, -\inf Q\},$$

it follows from (2.24) that

$$\int_R^{+\infty} \chi_{\{v>0\}} r^{N-1} dr \leq CR^{-N(m-1)}\left(\int Q(u) + R^N \max\{0, -\inf Q\}\right) \overset{(2.23),(1.2)}{<} +\infty.$$

Since v is non-increasing, this implies that $v(r) = 0$ for all r sufficiently large and completes the proof. $\qquad \square$

Now we recall a simple property of radially symmetric functions in $H^1(\mathbb{R}^N)$.

Lemma 2.9 *If $u \in H^1(\mathbb{R}^N)$ is radially symmetric w.r.to x_0, then $u \in C_{loc}^{\frac{1}{2}}(\mathbb{R}^N \setminus \{x_0\})$.*

Proof We may assume w.l.o.g. that $x_0 = 0$. Let $\delta > 0$. Since u is radially symmetric, there exists a function $v : (0, +\infty) \to \mathbb{R}$ such that $u(x) = v(|x|)$. Since $u \in H^1(\mathbb{R}^N)$, there exists v', the distributional derivative of v, such that $\nabla u(x) = v'(|x|)x/|x|$. Then

$$C \geq \int_{\mathbb{R}^N \setminus B_\delta(0)} |\nabla u|^2 = \omega_{N-1} \int_\delta^{+\infty} |v'(r)|^2 r^{N-1} dr \geq \omega_{N-1} \delta^{N-1} \int_\delta^{+\infty} |v'(r)|^2 dr.$$

This implies that v' belongs to $L^2((\delta, +\infty))$. Using the same argument, we get $v \in L^2((\delta, +\infty))$. Therefore, $v \in H^1((\delta, +\infty)) \Subset C^{\frac{1}{2}}((\delta, +\infty))$, by the Sobolev embedding theorem. Hence,

$$|u(x_1) - u(x_2)| = |v(|x_1|) - v(|x_2|)| \leq C_\delta ||x_1| - |x_2||^{1/2} \leq C_\delta |x_1 - x_2|^{1/2}$$

for all $x_1, x_2 \in \mathbb{R}^N \setminus B_\delta(0)$. Since δ is arbitrary, the proof is complete. $\qquad \square$

Collecting Lemmas 2.5, 2.8, and 2.9, we obtain Theorem 2.1.

We conclude the section by proving the result anticipated in Remark 2.3.

Lemma 2.10 *If $m \geq 3$, then $\int Q(u) = +\infty$ for all $u \in \mathcal{D}$. Consequently, $E[u] = +\infty$ for all $u \in \mathcal{D}$.*

Remark 2.11 Lemma 2.10 is related to Theorem 2 of Lazer and McKenna (1991) [see also Yijing and Duanzhi (2014)]: there, in the model case $Q(s) = As^{1-m}$, it is proved that a solution to the Euler–Lagrange equation associated to E [cf. (3.3) below] belongs to H^1 if and only if $m < 3$.

Proof Assume by contradiction that $\tilde{u} \in \mathcal{D}$ exists such that $\int Q(\tilde{u}) < +\infty$. Let u be the Schwarz symmetrization of $\tilde{u}$. Then

$$u \geq 0 \quad \text{and} \quad u(x) = v(|x|) \text{ with } v \text{ non-increasing in } (0, +\infty). \qquad (2.25)$$

Arguing as in the first part of the proof of Lemma 2.5, we deduce that

$$u \in \mathcal{D} \quad \text{and} \quad \int Q(u) = \int Q(\tilde{u}) < +\infty. \qquad (2.26)$$

It follows from Lemma 2.8 and (2.25) that $\bar{r}$ exists such that $\text{supp } v = [0, \bar{r}]$. Furthermore, by Lemma 2.9, $C > 0$ exists such that

$$v(r) = v(r) - v(\bar{r}) \leq C(\bar{r} - r)^{\frac{1}{2}} \qquad \text{for all } \bar{r}/2 \leq r \leq \bar{r}. \qquad (2.27)$$

This implies that there exists $R \in [\bar{r}/2, \bar{r})$ such that

$$Q(v) \overset{(1.2)}{\geq} \frac{A}{2} v^{1-m} \overset{(2.27)}{\geq} C(\bar{r} - r)^{\frac{1-m}{2}} \quad \text{for all } v \in [R, \bar{r}). \tag{2.28}$$

Therefore

$$\int Q(\tilde{u}) \overset{(2.26)}{=} \int Q(u) \overset{(2.25)}{=} \omega_{N-1} \int_0^{\bar{r}} Q(v) r^{N-1} \mathrm{d}r$$

$$\overset{(1.2),(2.28)}{\geq} \frac{\omega_{N-1}}{N} R^N \inf Q + C\omega_{N-1} \int_R^{\bar{r}} (\bar{r} - r)^{\frac{1-m}{2}} \mathrm{d}r \overset{m \geq 3}{=} +\infty,$$

and we have obtained a contradiction. □

3 The Euler–Lagrange Equation

In this section, we assume that

$$m < 3 \quad \text{and} \quad Q \in C^1((0, +\infty)) \text{ is such that (1.2) holds.} \tag{3.1}$$

In higher dimension, we will also need additional information on the behavior of $Q'(s)$ for large s (more precisely on Q'_- and Q'_+, the negative and positive parts of Q'):

$$Q'_-(s) \leq C \text{ for } s \gg 1 \quad \text{and} \quad Q'_+(s) \leq Cs^q \text{ for } s \gg 1 \quad \text{if } N \geq 2 \tag{3.2}$$

for some $C > 0$, with $q < +\infty$ if $N = 2$ and $q \leq 2^*$ if $N \geq 3$. We will show:

Theorem 3.1 *Assume (3.1); if $N \geq 2$, assume in addition (3.2). Then, any minimizer of E in $\mathcal{D}$ has compact support, is radially strictly decreasing w.r.to some $x_0 \in \mathbb{R}^N$ in $\{u > 0\}$, and is a classical solution to*

$$-\Delta u + Q'(u) = \lambda \quad \text{in } \{u > 0\}, \tag{3.3}$$

for some $\lambda \in \mathbb{R}$, in the sense that $u \in C^2(\{u > 0\}) \cap C^{\frac{1}{2}}(\mathbb{R}^N)$ with $\nabla u(x_0) = \mathbf{0}$.

Remark 3.2 Starting from the pioneering works (Crandall et al. 1977; Lazer and McKenna 1991), an enormous interest has been given to the homogeneous Dirichlet problem for elliptic equations with a source term which is singular with respect to u [see, e.g., the Introduction of Hernández et al. (2007)]. Of course, in this case both the domain and λ are fixed. Concerning the case $Q'(u) \approx -u^{-m}$ with $1 < m < 3$, we refer in particular to Giacomoni and Saoudi (2009), Oliva and Petitta (2018), Godoy and Guerin (2018), Casado-Díaz and Murat (2021), De Silva and Savin (2022a) and Durastanti and Oliva (2022).

We begin by proving that (3.3) is satisfied by any radially non-increasing minimizer.

Proposition 3.3 *Assume* (3.1)*; if $N \geq 2$, assume in addition* (3.2)*. Then, any radially non-increasing minimizer u of E in $\mathcal{D}$ is a distributional solution to* (3.3) *for some $\lambda \in \mathbb{R}$.*

Proof Let x_0 be the symmetry center of u. It follows from Lemmas 2.8 and 2.9 that $u \in C(\mathbb{R}^N \setminus \{x_0\})$ and that $\operatorname{supp} u = \overline{B_{\bar{r}}(x_0)}$ for some $\bar{r} \in (0, +\infty)$. Let $\varphi \in \mathcal{I}$, where

$$\mathcal{I} = \left\{ \varphi \in C_c^1(\mathbb{R}^N) : \operatorname{supp} \varphi \Subset B_{\bar{r}}(x_0), \int \varphi = 0 \right\}.$$

Since $u \in C(\mathbb{R}^N \setminus \{x_0\})$ and u is radially non-increasing, $\varepsilon > 0$ (depending on φ) exists such that $u \geq \varepsilon$ in $\operatorname{supp} \varphi$. Choosing $|t| < \frac{\varepsilon}{\|\varphi\|_\infty}$, we have that

$$u + t\varphi \in \mathcal{D} \quad \text{and} \quad \chi_{\{u+t\varphi>0\}} = \chi_{\{u>0\}} = B_{\bar{r}}(x_0)$$
$$\text{for all } |t| < \frac{\varepsilon}{\|\varphi\|_\infty}. \tag{3.4}$$

It follows that

$$\frac{E[u + t\varphi] - E[u]}{t} \overset{(1.1),(3.4)}{=} \int \nabla u \cdot \nabla \varphi + \frac{t}{2} \int |\nabla \varphi|^2 + \int \frac{Q(u + t\varphi) - Q(u)}{t} \chi_{\{u>0\}}. \tag{3.5}$$

Passing to the limit as $t \to 0$ on the first two terms on the right-hand side of (3.5) is trivial. For the third one, we will apply Lebesgue theorem. Firstly, since $Q \in C^1((0, +\infty))$ and in view of (3.4), we have

$$\frac{Q(u + t\varphi) - Q(u)}{t} \to Q'(u)\varphi \quad \text{pointwise in } B_{\bar{r}}(x_0) \setminus \{x_0\} \quad \text{as } t \to 0. \tag{3.6}$$

Next, we work out the L^1-estimate. Since $u \geq \varepsilon$ in $\operatorname{supp} \varphi$, we have

$$\frac{Q(u + t\varphi) - Q(u)}{t} \equiv 0 \quad \text{in } \{u < \varepsilon\} \quad \text{for all } |t| < \frac{\varepsilon}{\|\varphi\|_\infty}. \tag{3.7}$$

On $\{u \geq \varepsilon\} \setminus \{x_0\}$, we write

$$\frac{Q(u + t\varphi) - Q(u)}{t} = \frac{1}{t} \int_0^t Q'(u + \tau\varphi)\varphi \, d\tau. \tag{3.8}$$

We note that $u + t\varphi \geq \varepsilon/2$ in $\{u \geq \varepsilon\}$ for any $|t| < \frac{\varepsilon}{2\|\varphi\|_\infty}$. Therefore, if $N = 1$, the boundedness of u guaranteed by Remark (2.6) implies that

$$\left| \frac{Q(u+t\varphi)-Q(u)}{t} \right| \overset{(3.8)}{\leq} \|\varphi\|_\infty \|Q'\|_{L^\infty([\frac{\varepsilon}{2}, \|u\|_\infty])}$$
$$\text{in } \{u \geq \varepsilon\} \, \forall |t| < \frac{\varepsilon}{2\|\varphi\|_\infty} \quad \text{if } N = 1. \tag{3.9}$$

If $N \geq 2$, since $u + t\varphi \geq \varepsilon/2$ and $Q \in C^1((0, +\infty))$ satisfies (3.2), a constant C (depending on ε) exists such that $|Q'(s)| \leq C(1+s^q)$ for any $s \in [\frac{\varepsilon}{2}, +\infty)$. Therefore, by the Sobolev embedding theorem,

$$
\left| \frac{Q(u+t\varphi) - Q(u)}{t} \right| \overset{(3.8)}{\leq} C\|\varphi\|_\infty \left(1 + (u + |t|\varphi)^q \right)
$$
$$
\leq C\|\varphi\|_\infty \left(1 + u^q \right) \in L^1(\{u \geq \varepsilon\}) \quad \text{for any } |t| < \frac{\varepsilon}{2\|\varphi\|_\infty} \quad \text{if } N \geq 2.
$$
(3.10)

In view of (3.6), (3.7), and (3.9)–(3.10), an application of Lebesgue theorem yields $\frac{Q(u+t\varphi)-Q(u)}{t}\chi_{\{u>0\}} \to Q'(u)\varphi$ in $L^1(B_{\bar{r}}(x_0))$ as $t \to 0$. Hence, passing to the limit as $t \to 0$ in (3.5), since u is a minimizer of E in $\mathcal{D}$, we obtain

$$
\int_{B_{\bar{r}}(x_0)} \left(\nabla u \cdot \nabla \varphi + Q'(u)\varphi \right) = 0 \quad \text{for all } \varphi \in \mathcal{I}. \tag{3.11}
$$

To conclude, fix $\phi \in C_c^1(B_{\bar{r}}(x_0))$ such that $\phi \geq 0$ and $\int \phi \neq 0$. For any $\psi \in C_c^1(B_{\bar{r}}(x_0))$, we have

$$
\varphi = \psi - \left(\frac{\int \psi}{\int \phi} \right) \phi \in \mathcal{I}.
$$

It follows from (3.11) that

$$
\int_{B_{\bar{r}}(x_0)} \left(\nabla u \cdot \nabla \psi + Q'(u)\psi \right) = \lambda \int_{B_{\bar{r}}(x_0)} \psi \quad \text{for all } \psi \in C_c^1(B_{\bar{r}}(x_0)), \tag{3.12}
$$

where

$$
\lambda = \frac{1}{\int \phi} \int_{B_{\bar{r}}(x_0)} \left(\nabla u \cdot \nabla \phi + Q'(u)\phi \right).
$$

Note that $\lambda \in \mathbb{R}$. Indeed, since $\phi \in C_c^1(B_{\bar{r}}(x_0))$, $\varepsilon > 0$ exists such that $u \geq \varepsilon$ in $\operatorname{supp} \phi$. If $N \geq 2$, it follows from (3.2) and the Sobolev embedding theorem that $Q'(u)\phi \in L^1(B_{\bar{r}}(x_0))$ [cf. (3.10)]. If $N = 1$, the boundedness of u implies that $Q'(u)\phi \in L^\infty(B_{\bar{r}}(x_0))$ [cf. (3.9)]. Thus, the arbitrariness of ψ in (3.12) implies the result. □

Next, we give a boundedness result for solutions to (3.3) if $N \geq 2$.

Lemma 3.4 *Let $N \geq 2$. Assume (3.1) and (3.2). Let $u \in H_0^1(B_{\bar{r}}(x_0))$ be a radially non-increasing (with respect to $x_0 \in \mathbb{R}^N$) distributional solution to (3.3) with $\operatorname{supp} u = \overline{B_{\bar{r}}(x_0)}$. Then $u \in L^\infty(B_{\bar{r}}(x_0))$.*

Proof Since u is a distributional solution to (3.3) with supp $u = \overline{B_{\bar{r}}(x_0)}$, we have

$$\int_{B_{\bar{r}}(x_0)} \nabla u \cdot \nabla \varphi + \int_{B_{\bar{r}}(x_0)} Q'(u)\varphi = \lambda \int_{B_{\bar{r}}(x_0)} \varphi \quad \forall \varphi \in C_c^1(B_{\bar{r}}(x_0)).$$

Hence, taking a nonnegative φ and dropping the nonnegative term concerning Q'_+, we obtain

$$\int_{B_{\bar{r}}(x_0)} \nabla u \cdot \nabla \varphi \le \int_{B_{\bar{r}}(x_0)} \left(\lambda + Q'_-(u)\right)\varphi \quad \forall \, 0 \le \varphi \in C_c^1(B_{\bar{r}}(x_0)). \quad (3.13)$$

By Lemma 2.9, u is continuous outside x_0, which, since u is radially non-increasing, implies that u is bounded away from zero in any compact subset of $B_{\bar{r}}(x_0)$. This fact, together with (3.2), allows us to extend the class of test functions in (3.13) to nonnegative functions in $H_c^1(B_{\bar{r}}(x_0))$.

Let assume that $\sup u > 1$ (otherwise there is nothing to prove). For $k \ge 1$, let $G_k(u) = (u - k)_+$. Note that $G_k(u) \in H_0^1(B_{\bar{r}}(x_0))$ and supp $G_k(u) = \{u \ge k\} \subseteq \{u \ge 1\} \Subset B_{\bar{r}}(x_0)$ (again using continuity and monotonicity of u). Hence, we can take $G_k(u)$ as test function in (3.13), obtaining

$$\int_{B_{\bar{r}}(x_0)} |\nabla G_k(u)|^2 \le \left(\lambda + \sup_{[1,+\infty)} Q'_-\right) \int_{B_{\bar{r}}(x_0)} G_k(u) \overset{(3.2)}{\le} C \int_{B_{\bar{r}}(x_0)} G_k(u). \quad (3.14)$$

Now fix

$$2 < q < \begin{cases} +\infty & \text{if } N = 2, \\ 2^* & \text{if } N \ge 3. \end{cases}$$

By the Sobolev embedding theorem, applied to the left-hand side of (3.14), and Hölder inequality, applied to the right-hand side of (3.14), we have

$$\left(\int_{B_{\bar{r}}(x_0)} |G_k(u)|^q\right)^{\frac{2}{q}} \le C \int_{B_{\bar{r}}(x_0)} G_k(u) \le C |\{u \ge k\}|^{\frac{q-1}{q}} \left(\int_{B_{\bar{r}}(x_0)} G_k(u)^q\right)^{\frac{1}{q}},$$

for a constant C depending on N and $\bar{r}$. Hence, for $h > k$,

$$(h-k)^q |\{u \ge h\}| \le \int_{\{u \ge h\}} |G_k(u)|^q \le \int_{\{u \ge k\}} |G_k(u)|^q \le C |\{u \ge k\}|^{q-1}. \quad (3.15)$$

Starting from inequality (3.15) and applying Lemma 4.1 of Stampacchia (1965), it is standard to conclude that $u \in L^\infty(B_{\bar{r}}(x_0))$. □

The facts that u is bounded and solves the Euler–Lagrange equation yield regularity of u:

Corollary 3.5 (Regularity of radially non-increasing minimizers) *Assume* (3.1)*; if* $N \geq 2$*, assume in addition* (3.2)*. Let* $u \in \mathcal{D}$ *be a radially non-increasing (w.r.to some* $x_0 \in \mathbb{R}^N$*) minimizer of* E *in* $\mathcal{D}$*. Then,* $u \in C^2(\{u > 0\}) \cap C^{\frac{1}{2}}(\mathbb{R}^N)$ *and* $\nabla u(x_0) = \mathbf{0}$*.*

Proof By Proposition 3.3, u is a distributional solution to (3.3). Moreover, by Lemma 2.8, $\operatorname{supp} u = \overline{B_{\bar{r}}(x_0)}$ for some $\bar{r} > 0$. It follows from Remark 2.6 (for $N = 1$) and Lemma 3.4 (for $N > 1$) that $u \in L^\infty(B_{\bar{r}}(x_0))$. Since $Q \in C^1((0, +\infty))$, Q' is locally bounded in $(0, +\infty)$: hence $-\Delta u$ is locally bounded in $\{u > 0\}$. This implies that u is locally Hölder continuous in $\{u > 0\}$ [Theorem 8.22 of Gilbarg and Trudinger (2001)]. Then, using once again that u solves (3.3) and that $Q' \in C((0, +\infty))$, we deduce that $u \in C^2(\{u > 0\})$. Finally, Hölder continuity in $\mathbb{R}^N$ follows from Lemma 2.9 and $\nabla u(x_0) = \mathbf{0}$ follows from regularity and symmetry. □

Remark 3.6 Under the assumptions of Corollary 3.5, if Q is more regular, say $Q \in C^k((0, +\infty))$, then by a bootstrap argument we obtain $u \in C^{k+1}(\{u > 0\})$.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1 Let $\tilde{u}$ be a minimizer of E in $\mathcal{D}$ and let u be the Schwarz symmetrization of $\tilde{u}$. Arguing as in the proof of Lemma 2.5, we deduce that

$$\int |\nabla u|^2 \leq \int |\nabla \tilde{u}|^2, \quad \int Q(u) = \int Q(\tilde{u}), \quad \text{and} \quad \int u = M, \qquad (3.16)$$

that is, $u \in \mathcal{D}$ and $E[u] \leq E[\tilde{u}]$. Therefore, $E[u] = E[\tilde{u}]$, which together with (3.16) implies that

$$\int |\nabla u|^2 = \int |\nabla \tilde{u}|^2. \qquad (3.17)$$

Since u is a radially non-increasing minimizer, by Lemma 2.8 it has compact support. Let v be the non-increasing function defined by $u(x) = v(r)$ with $r = |x - x_0|$ and let $\operatorname{supp} v = [0, \bar{r}]$. By Proposition 3.3 and Corollary 3.5, v is a classical solution of the following one-dimensional problem:

$$\begin{cases} -\dfrac{d^2v}{dr^2} - \dfrac{N-1}{r}\dfrac{dv}{dr} + Q'(v) = \lambda & \text{in } (0, \bar{r}), \\ v(0) = u(x_0), \ v(\bar{r}) = 0, \ \dfrac{dv}{dr}(0) = 0, \\ \displaystyle\int r^{N-1}v(r)dr = \omega_{N-1}^{-1}M. \end{cases}$$

We claim that

$$\frac{dv}{dr} < 0 \quad \text{in } (0, \bar{r}). \qquad (3.18)$$

We argue by contradiction assuming that there exists a first $r_0 \in (0, \bar{r})$ such that $\frac{dv}{dr}(r_0) = 0$. Since $\frac{dv}{dr} \leq 0$, $\frac{d^2v}{dr^2}(r_0) = 0$: but then $Q'(v(r_0)) = \lambda$, whence $v(r) \equiv v(r_0) > 0$ for all $r \geq r_0$, in contradiction with $v(\bar{r}) = 0$.

In view of (3.17) and (3.18), it follows from Kesavan (2006, Theorem 2.3.3) or Ferone and Volpicelli (2004, Theorem 1) that $\tilde{u} = u$ (up to a translation of the center of symmetry). This, together with Proposition 3.3 and Corollary 3.5, completes the proof of Theorem 3.1. $\qquad\square$

4 The One-Dimensional Case

The rest of the manuscript is concerned with the case $N = 1$. In the next statement, we summarize, for $N = 1$, the results contained in Sect. 3:

Corollary 4.1 *Assume* (3.1) *and* $N = 1$. *Any minimizer* u *of* E *in* $\mathcal{D}$ *is even with respect to some* $x_0 \in \mathbb{R}$, *which up to a translation we may assume to be zero:* $x_0 = 0$. *Moreover, supp* $u = [-\bar{r}, \bar{r}]$ *for some* $\bar{r} \in (0, +\infty)$, $u' < 0$ *in* $(0, \bar{r})$, $u \in C^2((-\bar{r}, \bar{r})) \cap C([-\bar{r}, \bar{r}])$, *and* $\lambda > 0$ *exists such that* u *is a classical solution to*

$$\begin{cases} -u'' + Q'(u) = \lambda & in\ \{u > 0\} = (-\bar{r}, \bar{r}), \\ u'(0) = 0, \ u(\pm\bar{r}) = 0. \end{cases} \tag{4.1}$$

In the rest of the manuscript, we will always assume (up to a translation) that the symmetry point of a minimizer is located at $x_0 = 0$. In one space dimension, we will be able to obtain uniqueness (up to a translation) and qualitative properties of minimizers of E in $\mathcal{D}$. The key additional information is a characterization of the eigenvalue λ, which we now discuss under a mild additional information on the behavior of $Q'(s)$ for $s \ll 1$. We assume that $C > 0$ exists such that

$$(3.1)\ \text{holds} \quad \text{and} \quad |s\,Q'(s)| \le C Q(s)\ \text{for}\ s \ll 1. \tag{4.2}$$

Remark 4.2 For $s \ll 1$, $(4.2)_2$ essentially rules out exponential growth of $Q(s)$, which is however already excluded by (1.2), as well as too wild oscillations of $Q'(s)$, such as $Q(s) = As^{1-m}(1 + s \sin s^{-2})$ for $s \ll 1$. It is obviously satisfied by all model cases.

4.1 The Identification of the Eigenvalue λ

The identification of λ is based on three identities:

Lemma 4.3 *Assume* (4.2) *and* $N = 1$. *Let* u *be a minimizer of* E *in* $\mathcal{D}$. *Then, a constant* $K \in \mathbb{R}$ *exists such that*

$$\frac{1}{2}u'^2 - Q(u) + \lambda u = K \quad in\ \{u > 0\}. \tag{4.3}$$

In addition

$$\int_{\{u>0\}} [u'^2 + Q'(u)u - \lambda u] = 0 \tag{4.4}$$

and

$$\frac{3}{2}\int_{\{u>0\}}|u'|^2 - \int_{\{u>0\}} Q(u) + \int_{\{u>0\}} Q'(u)u = 0. \tag{4.5}$$

Remark 4.4 The function $uQ'(u)\chi_{\{u>0\}}$ in (4.4) and (4.5) belongs to $L^1(\mathbb{R})$ since $Q(u) \in L^1(\mathbb{R})$ (cf. Remark 2.7) and (4.2) holds.

Proof In order to prove (4.3), it suffices to multiply the equation in (4.1) by $-u'$:

$$\frac{1}{2}[u'^2]' - Q'(u)u' + \lambda u' = \left[\frac{1}{2}u'^2 - Q(u) + \lambda u\right]' = 0 \quad \text{in } \{u > 0\} = (-\bar{r}, \bar{r}),$$

hence (4.3) holds. We now prove (4.4). As $x \to \bar{r}^-$, we have

$$(u')^2 \overset{(4.3)}{=} 2\left(Q(u) - \lambda u + K\right) \sim 2Q(u), \quad \text{that is,} \quad u' \sim -\sqrt{2Q(u)} \overset{(1.2)}{\sim} -\sqrt{2A}u^{\frac{1-m}{2}}.$$

Therefore,

$$uu' \sim -\sqrt{2A}u^{\frac{3-m}{2}} \to 0 \quad \text{as } x \to \bar{r}^-. \tag{4.6}$$

Multiplying (4.1) by u and integrating over $(-\bar{r}, \bar{r})$, we have

$$\int_{(-\bar{r},\bar{r})}[-u''u + Q'(u)u - \lambda u] = 0.$$

Integrating by parts the first term and using (4.6), we obtain (4.4).

We now prove (4.5). For $\alpha > 0$, we consider the mass-preserving rescaling $u_\alpha(\hat{x}) = \alpha u(\alpha\hat{x}) \in \mathcal{D}$. Performing the change of variable $x = \alpha\hat{x}$, we obtain

$$E[u_\alpha] = \int \frac{1}{2}\left|\frac{du_\alpha(\hat{x})}{d\hat{x}}\right|^2 d\hat{x} + \int Q(u_\alpha(\hat{x}))d\hat{x} = \alpha^3 \int \frac{1}{2}|u'|^2 dx + \alpha^{-1}\int Q(\alpha u)dx.$$

We show that $E[u_\alpha]$ is differentiable with respect to α and

$$\frac{d}{d\alpha}E[u_\alpha] = \frac{3}{2}\alpha^2 \int |u'|^2 - \alpha^{-2}\int Q(\alpha u) + \alpha^{-1}\int Q'(\alpha u)u$$
$$\text{for all } \alpha > 0. \tag{4.7}$$

The only non-trivial limit is

$$\lim_{t\to 0}\int \frac{Q((\alpha + t)u) - Q(\alpha u)}{t} = \int Q'(\alpha u)u, \tag{4.8}$$

for which we apply Lebesgue theorem. Fix $\alpha > 0$. First of all, it follows from (3.1) that $\frac{Q((\alpha+t)u)-Q(\alpha u)}{t} \to Q'(\alpha u)u$ pointwise in $(-\bar{r}, \bar{r})$ as $t \to 0$. For the L^1-bound, we write

$$\left| \frac{Q((\alpha+t)u) - Q(\alpha u)}{t} \right| \leq \frac{1}{|t|} \int_0^{|t|} |Q'((\alpha+\tau)u)| u \, \mathrm{d}\tau. \tag{4.9}$$

Hereafter in the proof, C denotes a generic constant which may depend on α, but not on t and τ. Take $|t| < \frac{\alpha}{2}$, so that $\frac{\alpha}{2} < \alpha + \tau < \frac{3\alpha}{2}$ for $|\tau| < |t|$. In view of (1.2) and (4.2), $\delta > 0$ exists such that

$$\tfrac{A}{2} s^{1-m} \leq Q(s) \leq 2A s^{1-m} \quad \text{and} \quad s|Q'(s)| \leq C Q(s) \quad \text{for all } s \in (0, \delta). \tag{4.10}$$

Using $(4.10)_1$ twice, we see that

$$Q((\alpha+\tau)u) \leq C((\alpha+\tau)u)^{1-m} \leq C u^{1-m} \leq C Q(u)$$
$$\text{if } u < \delta_0 := \min\{\frac{2\delta}{3\alpha}, \delta\}. \tag{4.11}$$

Therefore,

$$|Q'((\alpha+\tau)u)|u \begin{cases} = \dfrac{|Q'((\alpha+\tau)u)|}{\alpha+\tau}(\alpha+\tau)u \overset{(4.10)_2,(4.11)}{\leq} C Q(u) & \text{if } u < \delta_0 \\ \leq u\|Q'\|_{L^\infty([\frac{\alpha\delta_0}{2}, \frac{3}{2}\alpha u(0)])} & \text{if } u \geq \delta_0 \end{cases}$$

and the L^1-bound follows from (4.9) since $Q(u) \in L^1(\mathbb{R})$ (cf. Remark 2.7). Thus, (4.8), whence (4.7), hold. Since $u = u_1$ is a minimizer of E in $\mathcal{D}$, it follows that $\frac{\mathrm{d}}{\mathrm{d}\alpha} E[u_\alpha]|_{\alpha=1} = 0$: hence (4.7) coincides with (4.5). $\qquad \square$

Now we are ready to characterize λ.

Theorem 4.5 *Assume* (4.2) *and* $N = 1$. *Let* u *be a minimizer of* E *in* $\mathcal{D}$. *Then,* $K = 0$ *in* (4.3), *that is*

$$\tfrac{1}{2}u'^2 - Q(u) + \lambda u = 0 \quad \text{in } \{u > 0\}, \tag{4.12}$$

and

$$\lambda = \frac{Q(u(0))}{u(0)}. \tag{4.13}$$

Proof By Lemma 4.3, u satisfies (4.3), (4.4), and (4.5). Integrating (4.3) over $\{u > 0\}$ and adding (4.4), we obtain

$$\frac{3}{2} \int_{\{u>0\}} |u'|^2 - \int_{\{u>0\}} Q(u) + \int_{\{u>0\}} Q'(u)u = \int_{\{u>0\}} K. \tag{4.14}$$

Subtracting (4.14) from (4.5), we have

$$\int_{\{u>0\}} K = 0,$$

hence $K = 0$. Evaluating (4.12) in $x = 0$ and recalling that $u'(0) = 0$, we deduce
(4.13).   □

4.2 Admissible Maximal Heights and the First Integral

In view of (4.13), it is convenient to introduce the function

$$R(s) := \frac{Q(s)}{s} \overset{(3.1)}{\in} C^1((0, +\infty)), \quad R'(s) = \frac{1}{s}\left(Q'(s) - R(s)\right), \quad s > 0.$$
(4.15)

In view of (4.1) and (4.13), any minimizer of E in $\mathcal{D}$ is a solution to

$$\begin{cases} -u'' + Q'(u) = R(u_0) & \text{in } \{u > 0\}, \\ u(0) = u_0 > 0, \ u'(0) = 0 \end{cases} \qquad (P_{u_0})$$

whose solutions we now discuss. First of all, any solution to (P_{u_0}) is even and

$$u''(0) \overset{(P_{u_0})}{=} Q'(u_0) - R(u_0) \overset{(4.15)}{=} u_0 R'(u_0). \tag{4.16}$$

In addition, multiplying (P_{u_0}) by u' and integrating from $x = 0$, we obtain

$$\tfrac{1}{2}(u')^2 = Q(u) - R(u_0)u \overset{(4.15)}{=} u(R(u) - R(u_0)) \quad \text{in } \{u > 0\}. \tag{4.17}$$

In the next lemma, we give a necessary and sufficient condition on u_0 for a solution
of (P_{u_0}) to have compact support (supp $u = [-\bar{r}, \bar{r}]$) and negative derivative in $(0, \bar{r})$.
This will identify an *admissible set* $\mathcal{A}$ to which the maximal height of minimizers
must belong. We also list a few properties of such solutions, which will be used in the
sequel.

Lemma 4.6 *Assume* (4.2) *and* $N = 1$. *Let* u *be a solution of* (P_{u_0}). *Then, the following
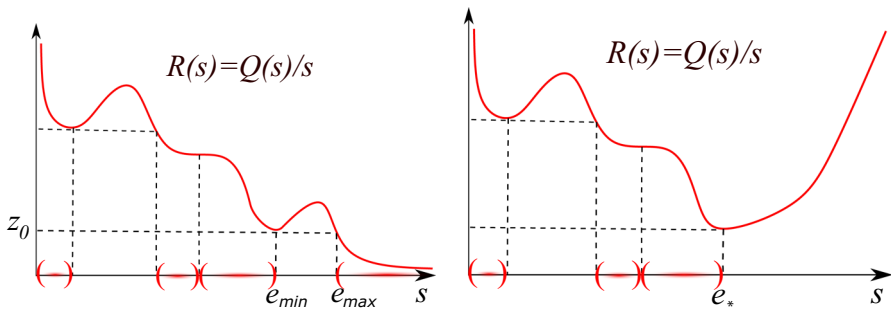are equivalent:*

(a) *supp* $u = [-\bar{r}, \bar{r}]$ *for some* $\bar{r} \in (0, +\infty)$ *and* $u' < 0$ *in* $(0, \bar{r})$;
(b) $u_0 \in \mathcal{A}$, *where* $\mathcal{A} \subseteq (0, +\infty)$ *is the open set defined by Fig.* 6

$$\mathcal{A} := \left\{ s \in (0, +\infty) : \ R'(s) < 0 \text{ and } R(t) > R(s) \text{ for all } t < s \right\}. \tag{4.18}$$

In particular, $\max u \in \mathcal{A}$ *for any minimizer* u *of* E *in* $\mathcal{D}$. *If* (a) *or* (b) *hold, then*

$$u' = -\sqrt{2u(R(u) - R(u_0))} \quad \text{in } [0, \bar{r}), \tag{4.19}$$

$$u_0 - u(x) = \int_0^x \sqrt{2u(R(u) - R(u_0))} \quad \text{for all } x \in [0, \bar{r}), \tag{4.20}$$

Fig. 6 The set $\mathcal{A}$ in Lemma 4.6 in two cases: $e_* = +\infty$ and $z_0 < +\infty$ [cf. (4.27) and (4.28)] and $e_* < +\infty$. The numbers e_*, e_{min} and e_{max} are also visualized [cf. (4.27) and (4.29)]

and for all $x \in [0, \bar{r})$ it holds that

$$Z(u(x)) = x, \quad where \ \ Z(u)$$
$$:= \frac{1}{\sqrt{2}} \int_u^{u_0} (s(R(s) - R(u_0)))^{-\frac{1}{2}} \, ds \ \ for \, u \in [0, u_0]. \tag{4.21}$$

Note that Z is well defined since $R'(u_0) \neq 0$.

Proof Proof of $(a) \implies (b)$. Since u is strictly decreasing in $(0, \bar{r})$, $R'(u_0) \leq 0$ (by (4.16)). If $R'(u_0) = 0$ we would have $u \equiv u_0$, in contradiction with $u' < 0$. Therefore, $R'(u_0) < 0$. In view of (4.17), $R(u(x)) > R(u_0)$ for any $x \in (0, \bar{r})$. Since u is continuous and $u(\bar{r}) = 0$, we deduce that $R(t) > R(u_0)$ for every $t \in (0, u_0)$, hence $u_0 \in \mathcal{A}$.

Proof of $(b) \implies (a)$. We now assume that $u_0 \in \mathcal{A}$, which implies $R'(u_0) < 0$. Because of (4.16), u is strictly decreasing in a right neighborhood of $x = 0$. Assume by contradiction that $x > 0$ exists such that $u'(x) = 0$ and $u' < 0$ in $(0, x)$. In particular, $u(x) < u_0$ and, by (4.17), $R(u(x)) = R(u_0)$. This contradicts the definition of $\mathcal{A}$. Therefore, $u' < 0$ as long as u is defined, and (4.19) follows from (4.17). Integrating (4.19) with respect to x we obtain (4.20). Integrating (4.19) with respect to u, we obtain (4.21). In particular,

$$\sqrt{2}x = \int_{u(x)}^{u_0} (s(R(s) - R(u_0)))^{-\frac{1}{2}} \, ds \leq \int_0^{u_0} (s(R(s) - R(u_0)))^{-\frac{1}{2}} \, ds.$$

Since $R'(u_0) \neq 0$ and $s(R(s) - R(u_0)) \sim Q(s) \to +\infty$ as $s \to 0^+$, the right-hand side is finite: therefore, u has compact support and the proof is complete. □

4.3 Asymptotics Near the Interface

Now we investigate the asymptotic behavior near $\partial\{u > 0\} = \{-\bar{r}, \bar{r}\}$ of solutions to (P_{u_0}) with $u_0 \in \mathcal{A}$; in particular, for minimizers of E. Since any solution of (P_{u_0}) is even, it suffices to study the behavior of u as $x \to \bar{r}^-$.

Theorem 4.7 *Assume* (4.2) *and* $N = 1$. *Let* u *be a solution of* (P_{u_0}) *with* $u_0 \in \mathcal{A}$. *Then*

$$u(x) \sim \left(\frac{A(m+1)^2}{2} \right)^{\frac{1}{m+1}} (\bar{r} - x)^{\frac{2}{m+1}} \quad as\ x \to \bar{r}^-, \tag{4.22}$$

$$u'(x) \sim -\frac{2}{m+1} \left(\frac{A(m+1)^2}{2} \right)^{\frac{1}{m+1}} (\bar{r} - x)^{\frac{2}{m+1}-1} \quad as\ x \to \bar{r}^-. \tag{4.23}$$

In particular, $u \in H_0^1((-\bar{r}, \bar{r}))$.

Proof By Lemma 4.6, (4.21) holds. Since both Z and u are strictly decreasing, whence invertible, $x \to \bar{r}^-$ if and only if $u \to 0^+$. Therefore,

$$\lim_{x \to \bar{r}^-} \frac{\bar{r} - x}{u^\alpha(x)} \stackrel{(4.21)}{=} \lim_{u \to 0^+} \frac{\bar{r} - Z(u)}{u^\alpha} = \lim_{u \to 0^+} \frac{\frac{1}{\sqrt{2}} (Q(u) - R(u_0)u)^{-1/2}}{\alpha u^{\alpha-1}}$$
$$\stackrel{(1.2)}{=} \lim_{u \to 0^+} \frac{1}{\sqrt{2A}\alpha u^{\alpha-1+\frac{1-m}{2}}}, \tag{4.24}$$

where in the second equality of (4.24) we used L'Hôpital's rule. Choosing $\alpha = \frac{m+1}{2}$ we obtain (4.22). Consequently, we obtain (4.23):

$$u'(x) \stackrel{(4.19)}{\sim} -\sqrt{2Q(u)} \stackrel{(1.2)}{\sim} -\sqrt{2A}u(x)^{\frac{1-m}{2}} \stackrel{(4.22)}{\sim} -\sqrt{2A} \left(\sqrt{\frac{A(m+1)^2}{2}} \right)^{\frac{1-m}{m+1}} (\bar{r} - x)^{\frac{1-m}{m+1}}$$

$$= -\frac{2}{m+1} \left(\frac{A(m+1)^2}{2} \right)^{\frac{1}{m+1}} (\bar{r} - x)^{\frac{2}{m+1}-1} \quad as\ x \to \bar{r}^-.$$

Finally, since u is even, $u \in C^2((-\bar{r}, \bar{r}))$, and (4.23) holds, $C > 0$ exists such that

$$\int_{-\bar{r}}^{\bar{r}} (u')^2 \le C \int_0^{\bar{r}} (\bar{r} - x)^{\frac{2(1-m)}{m+1}} \stackrel{m<3}{<} +\infty.$$

$\square$

4.4 Bounds on the Maximal Height

To avoid pathological situations, we assume in what follows that $\delta > 0$ exists such that

$$R' \neq 0 \text{ in } (0, \delta) \cup (\delta^{-1}, +\infty). \tag{4.25}$$

In particular, the limit of R as $s \to +\infty$ exists, and it follows from $\inf Q > -\infty$ that

$$\lim_{s \to +\infty} R(s) = \lim_{s \to +\infty} \frac{Q(s)}{s} \ge 0. \tag{4.26}$$

We let e_* be the smallest among the absolute minimum points of R, provided they exist:

$$e_* := \begin{cases} +\infty & \text{if } \nexists \min R, \\ \min R^{-1}(\min R) & \text{otherwise.} \end{cases} \qquad (4.27)$$

Remark 4.8 If $e_* = +\infty$, then $\inf_{(0,+\infty)} R = \lim_{s \to +\infty} R(s)$. In particular, in view of (4.26), $Q > 0$ in $(0, +\infty)$. Note that in this case $Q(s) \lesssim s$ for $s \gg 1$. Moreover, if $R(+\infty) = 0$ the converse holds true: $Q > 0$ in $(0, +\infty)$ implies $e_* = +\infty$.

Remark 4.9 In view of (4.26), $Q \not\equiv 0$ in $(0, +\infty)$ –that is, $R \not\equiv 0$ in $(0, +\infty)$—implies that $e_* < +\infty$.

If $e_* = +\infty$, we introduce the smallest level at which R ceases to be injective:

$$z_0 := \sup\{z > 0 : \#\{R^{-1}(z')\} \leq 1 \text{ for every } z' \in (0, z)\} \quad \text{if } e_* = +\infty. \quad (4.28)$$

Thanks to (4.25) and Remark 4.8, the set is non-empty and $z_0 \in (0, +\infty]$. If $z_0 = +\infty$, then R is strictly decreasing in $(0, +\infty)$. If instead $z_0 < +\infty$, we define (cf. Fig. 6)

$$e_{max} := \max\{s > 0 : R(s) = z_0\} \quad \text{and} \quad e_{min} := \min\{s > 0 : R(s) = z_0\}. \qquad (4.29)$$

It follows from (4.25) that $0 < e_{min} < e_{max} < +\infty$.

In the following lemma, we give a characterization of e_* in terms of $\mathcal{A}$ and some restrictions on the maximal height.

Lemma 4.10 *Assume $N = 1$, (4.2) and (4.25). Then $e_* = \sup \mathcal{A}$, hence $\mathcal{A} \subseteq (0, e_*)$. In addition, if $e_* = +\infty$ and $z_0 < +\infty$ [cf. (4.28)], then $\mathcal{A} \subseteq (0, e_{min}) \cup (e_{max}, +\infty)$ [cf. (4.29)].*

Proof It is obvious that $\sup \mathcal{A} \leq e_*$. If by contradiction $\sup \mathcal{A} < e_*$, then for all $s \in (\sup \mathcal{A}, e_*)$ we would have either $R'(s) \geq 0$ or $R(t) \leq R(s)$ for some $t < s$, in contradiction with the definition of e_*. Therefore, $\sup \mathcal{A} = e_*$.

If $e_* = +\infty$ and $z_0 < +\infty$, let $u_0 \in \mathcal{A}$. By its definition, $R(e_{min})$ is a global minimum for R in $(0, e_{max}]$, which implies that $R'(e_{min}) = 0$ and $R(s) \geq R(e_{min})$ for $s \in [e_{min}, e_{max}]$. The former implies that $u_0 \neq e_{min}$ and the latter implies that $u_0 \notin (e_{min}, e_{max}]$. $\qquad \square$

4.5 Uniqueness

We will now prove comparison and uniqueness results for minimizers of E in $\mathcal{D}_M$ under the following additional assumption on Q:

$$Q'(s) \text{ is non-decreasing for } s \in (0, e_*). \qquad (4.30)$$

In Sect. 7, we will provide examples of non-uniqueness for a large class of potentials which do not satisfy (4.30).

Lemma 4.11 *Assume $N = 1$, (4.2), (4.25), and (4.30). Let u_1 and u_2 be minimizers of E in $\mathcal{D}_{M_1}$, resp. $\mathcal{D}_{M_2}$, both symmetric with respect to $x_0 = 0$. Then, the following are equivalent:*

(i) $M_1 < M_2$;
(ii) $u_1(0) =: u_{01} < u_{02} := u_2(0)$;
(iii) $u_1 < u_2$ *in* $\operatorname{supp} u_1$.

Remark 4.12 If $e_* = +\infty$, (4.30) can be attained only if $z_0 = +\infty$.

Proof The proof of $(iii) \Rightarrow (i)$ is obvious.

Proof of $(ii) \Rightarrow (iii)$. It follows from Lemma 4.6 that $u_{0i} \in \mathcal{A}$. Since u_i are strictly decreasing, it follows from Lemma 4.10 that $u_1, u_2 < e_*$. Subtracting the corresponding equations, we obtain

$$-(u_1 - u_2)'' = R(u_{01}) - R(u_{02}) - (Q'(u_1) - Q'(u_2)) \quad \text{in } \{u_1 > 0\} \cap \{u_2 > 0\}.$$

Since $u_{02} \in \mathcal{A}$, $R(u_{02}) < R(t)$ for all $t \in (0, u_{02})$: in particular, $R(u_{02}) < R(u_{01})$. As long as $u_1 \leq u_2$, by (4.30), we have $Q'(u_1) \leq Q'(u_2)$. Hence, $(u_1 - u_2)'' < 0$ as long as $u_1 \leq u_2$. Integrating twice using $u_1'(0) = u_2'(0) = 0$, we deduce that $(u_1 - u_2) \leq (u_1 - u_2)(0) < 0$ as long as $u_1 \leq u_2$, whence $u_1 < u_2$ in $\operatorname{supp} u_1$.

Proof of $(i) \Rightarrow (ii)$. First we note that $u_{01} \neq u_{02}$: otherwise, u_1 and u_2 would solve the same equation (P_{u_0}), whence $u_1 = u_2$ by Picard–Lindelöf theorem, in contradiction with $M_1 \neq M_2$. On the other hand, using $(ii) \Rightarrow (iii)$, $u_{01} > u_{02}$ implies $u_1 > u_2$ in $\operatorname{supp} u_2$, in contradiction with $M_1 < M_2$. □

As a by-product of Lemma 4.11, we obtain the uniqueness result:

Theorem 4.13 *Assume $N = 1$, (4.2), (4.25), and (4.30). Then for any $M > 0$, there exists at most one minimizer of E in $\mathcal{D}_M$ (up to translation).*

Proof Let u_1 and u_2 be two minimizers of E in $\mathcal{D}_M$ (both of them symmetric with respect to $x_0 = 0$). Since u_1 and u_2 have the same mass, it follows from Lemma 4.11 that $u_1(0) = u_2(0)$. This implies $u_1 = u_2$ by Picard–Lindelöf theorem. □

5 Pancakes Versus Droplets

We assume throughout the section that

$$N = 1, \text{ (4.2), and (4.25) hold.} \tag{5.1}$$

For $s \in \mathcal{A}$, we consider the solution u_s to (P_s) with $\operatorname{supp} u_s = [-\bar{r}_s, \bar{r}_s]$ (cf. Lemma 4.6) and we define

$$\mu : \mathcal{A} \to (0, +\infty), \quad \mu(s) := 2 \int_0^{\bar{r}_s} u_s. \tag{5.2}$$

We claim that

$$\mu \in C(\mathcal{A}). \tag{5.3}$$

Let $u_0 \in \mathcal{A}$. Since $\mathcal{A}$ is open, $s \in \mathcal{A}$ in a neighborhood of u_0. By Lemma 4.6, we know that u_s is even, has compact support, say $[-\bar{r}_s, \bar{r}_s]$, and is strictly decreasing in $(0, \bar{r}_s)$. By Lemma A.3, $u_s \to u_{u_0}$ in $C^2_{loc}((-\bar{r}_{u_0}, \bar{r}_{u_0}))$ and $\bar{r}_s \to \bar{r}_{u_0}$ as $s \to u_0$, hence a.e. in $(-\bar{r}_{u_0}, \bar{r}_{u_0})$. Therefore, by dominated convergence, $\mu(s) \to \mu(u_0)$.

We will also need the following a-priori estimate, in the spirit of Theorem 4.7.

Lemma 5.1 *Assume* (5.1). *A constant* $C > 0$ *exists such that for all* $K \geq 1$ *there exists* $\delta_K > 0$ *such that*

$$u(x) \geq C(\bar{r} - x)^{\frac{2}{m+1}} \quad \text{for all } x \in (\bar{r} - \delta_K, \bar{r}) \tag{5.4}$$

for all $u_0 \in \mathcal{A} \cap [K^{-1}, K]$ *and for all* $u_0 \in \mathcal{A} \cap [K^{-1}, +\infty)$ *if* $R(+\infty) < +\infty$, *where* u *is the solution to* (P_{u_0}) *and* $[-\bar{r}, \bar{r}] = supp\, u$.

Proof For u_0 as in the statement, the properties of Q imply that $\varepsilon_K < K^{-1}$ exist such that

$$Q(s) - R(u_0)s \leq 2As^{1-m} \quad \text{for all } s \in (0, \varepsilon_K), \tag{5.5}$$

$$Q(s) \geq \tfrac{A}{2}s^{1-m} \quad \text{for all } s \in (0, \varepsilon_K), \tag{5.6}$$

$$R(s) \geq 2|R(u_0)| \quad \text{for all } s \in (0, \varepsilon_K). \tag{5.7}$$

We preliminarily work out an upper bound on u. It follows from (4.19) that

$$-u' = \sqrt{2(Q(u) - R(u_0)u)} \overset{(5.5)}{\leq} Cu^{\frac{1-m}{2}} \quad \text{as long as } u < \varepsilon_K.$$

Integrating it in $(x, \bar{r})$, we see that

$$u(x) \leq C(\bar{r} - x)^{\frac{2}{m+1}} \quad \text{as long as } u < \varepsilon_K.$$

Choosing δ_K such that $C\delta_K^{\frac{2}{m+1}} = \varepsilon_K/2$, we conclude that $u(x) < \varepsilon_K/2$ for all $x \in (\bar{r} - \delta_K, \bar{r})$. Now we can work out the lower bound:

$$-u' \overset{(4.19)}{=} \sqrt{2u(R(u) - R(u_0))} \overset{(5.7)}{\geq} \sqrt{Q(u)} \overset{(5.6)}{\geq} Cu^{\frac{1-m}{2}} \quad \text{in } (\bar{r} - \delta_K, \bar{r}). \tag{5.8}$$

Integrating (5.8) in $(x, \bar{r})$, $x \in (\bar{r} - \delta_K, \bar{r})$, we deduce

$$C(\bar{r} - x) \overset{(5.8)}{\leq} -\int_x^{\bar{r}} u^{\frac{m-1}{2}} u' = \tfrac{2}{m+1} u^{\frac{m+1}{2}} \quad \text{for all } x \in (\bar{r} - \delta_K, \bar{r}),$$

whence (5.4). □

5.1 The Droplet Case

In this subsection, we prove the following result.

Lemma 5.2 (the droplet case) *Assume* (5.1), $e_* = +\infty$,

$$Q(s) \to -S \in (0, +\infty), \quad \text{and} \quad sQ'(s) \to 0 \text{ as } s \to +\infty. \tag{5.9}$$

If $\mathcal{A} \ni u_{0k} \to +\infty$ as $k \to +\infty$ for a sequence, then $\mu(u_{0k}) \to +\infty$,

$$u_{0k}^4 \sim \frac{9|S|}{32}\mu^2(u_{0k}), \quad \bar{r}_k^4 \sim \frac{9}{8|S|}\mu^2(u_{0k}) \quad \text{as } k \to +\infty \tag{5.10}$$

and

$$w_k(y) = u_{0k}^{-1}u_k(\bar{r}_k y) \to 1 - y^2 \quad \text{in } C_{loc}^2((-1,1)) \quad \text{as } k \to +\infty, \tag{5.11}$$

where u_k is the solution to $(P_{u_{0k}})$.

Proof We recall that $e_* = +\infty$ implies $Q > 0$ (thus $R > 0$) in $(0, +\infty)$ (Remark 4.8), and we note for later reference that only $-S \geq 0$ is used in Step 1 of this proof.

Step 1. We show that $C \geq 1$ exists such that

$$\bar{r}_k \to +\infty \quad \text{and} \quad \frac{u_{0k}}{\bar{r}_k} \leq C \quad \text{as } k \to +\infty. \tag{5.12}$$

Let $\bar{s}$ be such that $Q(s) \leq |S| + 1$ in $(\bar{s}, +\infty)$, and let $x_k = u_k^{-1}(\bar{s})$ (which is well defined for $k \gg 1$ since $u_{0k} \to +\infty$). Since $R > 0$ we obtain (5.12):

$$+\infty \leftarrow u_{0k} - \bar{s} \overset{(4.20)}{=} \int_0^{x_k} \sqrt{2(Q(u_k) - R(u_{0k})u_k)} \leq \sqrt{2(|S|+1)}x_k \leq \sqrt{2(|S|+1)}\bar{r}_k.$$

Step 2. Note that $-S > 0$ and $Q > 0$ in $(0, +\infty)$ imply that $\inf_{(0,+\infty)} Q > 0$. Let $w_k(y) = u_{0k}^{-1}u_k(\bar{r}_k y)$. It follows from (4.19) that

$$-w_k' = \sqrt{2}\frac{\bar{r}_k}{u_{0k}}\sqrt{Q(u_{0k}w_k) - Q(u_{0k})w_k} \quad \text{in } [0,1), \tag{5.13}$$

that is,

$$1 = \sqrt{2}\frac{\bar{r}_k}{u_{0k}}\int_0^1 \sqrt{Q(u_{0k}w_k(y)) - Q(u_{0k})w_k(y)}\,dy. \tag{5.14}$$

We claim that a constant $C \geq 1$ exists such that

$$\frac{u_{0k}}{\bar{r}_k} \geq C^{-1}. \tag{5.15}$$

Assume by contradiction that $\frac{u_{0k}}{\bar{r}_k} \to 0$ as $k \to +\infty$ for a subsequence (not relabeled). Then, (5.14) yields

$$Q(u_{0k}w_k(y)) - Q(u_{0k})w_k(y) \to 0 \quad \text{for a.e. } y \in (0, 1) \text{ as } k \to +\infty. \quad (5.16)$$

Take one of such y's and any convergent subsequence $w_k(y) \to c \in [0, 1]$. If $c = 0$, we would have $Q(u_{0k})w_k(y) \to |S| \cdot 0$, whence by (5.16) $Q(u_{0k}w_k(y)) \to 0$, as $k \to +\infty$, which is impossible since $\inf\limits_{(0,+\infty)} Q > 0$. Therefore $c > 0$, and (5.16) implies $|S| - |S|c = 0$, i.e., $c = 1$. Thus $w_k(y) \to 1$ for a.e. $y \in (0, 1)$. Now, take $0 < b < 1$. Integrating (5.13) in $(0, b)$, we obtain

$$\frac{1}{\sqrt{2}} \int_{w_k(b)}^{1} \frac{\mathrm{d}w}{\sqrt{Q(u_{0k}w) - Q(u_{0k})w}} = \frac{\bar{r}_k}{u_{0k}}b \to +\infty \quad \text{as } k \to +\infty. \quad (5.17)$$

On the other hand, by Lagrange theorem, for all $w \in [w_k(b), 1]$ we have

$$Q(u_{0k}w) - Q(u_{0k})w = \left(Q(u_{0k}) - u_{0k}Q'(u_{0k}\eta_{k,w})\right)(1 - w)$$
$$\text{for some } \eta_{k,w} \in [w_k(b), 1].$$

Noting that $\eta_{k,w} \to 1$ and, by (5.9), that $u_{0k}Q'(u_{0k}\eta_{k,w}) = \frac{1}{\eta_{k,w}}u_{0k}\eta_{k,w}Q'(u_{0k}\eta_{k,w}) \to 0$ uniformly with respect to $w \in [w_k(b), 1]$ as $k \to +\infty$, since $Q(s) \to |S| > 0$ as $s \to +\infty$ we obtain

$$Q(u_{0k}w) - Q(u_{0k})w \geq \frac{|S|}{2}(1 - w) \quad \text{for all } w \in [w_k(b), 1] \text{ and all } k \gg 1,$$
$$(5.18)$$

hence

$$\frac{1}{\sqrt{2}} \int_{w_k(b)}^{1} \frac{\mathrm{d}w}{\sqrt{Q(u_{0k}w) - Q(u_{0k})w}} \leq \frac{1}{\sqrt{|S|}} \int_{w_k(b)}^{1} \frac{\mathrm{d}w}{\sqrt{1 - w}} < +\infty,$$

in contradiction with (5.17). Hence, (5.15) holds.

Step 3. We now prove that, as $k \to +\infty$,

$$\left(\frac{u_{0k}}{\bar{r}_k}\right)^2 \to \frac{|S|}{2} \quad \text{and} \quad w_k(y) \to 1 - y^2 \quad \text{in } C_{loc}^2((-1, 1)). \quad (5.19)$$

Take any subsequence $k \to +\infty$ (not relabeled) such that $\frac{u_{0k}}{\bar{r}_k} \to \beta$. By (5.12) and (5.15), $\beta \in (0, +\infty)$. Since w_k is strictly decreasing, $y_k := w_k^{-1}$ is well defined in $[0, 1]$. Integrating (5.13) in $(0, y_k(w))$ with $w > 0$, we obtain

$$y_k(w) = \frac{1}{\sqrt{2}} \frac{u_{0k}}{\bar{r}_k} \int_{w}^{1} \frac{\mathrm{d}\tilde{w}}{\sqrt{Q(u_{0k}\tilde{w}) - Q(u_{0k})\tilde{w}}}$$

$$\sim \frac{\beta}{\sqrt{2}} \int_w^1 \frac{\mathrm{d}\tilde{w}}{\sqrt{Q(u_{0k}\tilde{w}) - Q(u_{0k})\tilde{w}}}. \tag{5.20}$$

Since $u_{0k} \to +\infty$, by (5.9) we deduce that $Q(u_{0k}\tilde{w}) - Q(u_{0k})\tilde{w} \to |S|(1 - \tilde{w})$ a.e. in $(w, 1)$. Now, repeating the same argument used to obtain (5.18) we get

$$\frac{1}{\sqrt{Q(u_{0k}\tilde{w}) - Q(u_{0k})\tilde{w}}} \le \frac{\sqrt{2}}{\sqrt{|S|(1 - \tilde{w})}} \in L^1((w, 1)) \quad \text{for all } k \gg 1.$$

Thus, we can apply Lebesgue theorem to (5.20), obtaining

$$\lim_{k \to +\infty} y_k(w) = y(w) := \frac{\beta}{\sqrt{2}} \int_w^1 \frac{\mathrm{d}\tilde{w}}{\sqrt{|S|(1 - \tilde{w})}}$$

$$= \frac{\sqrt{2}\beta}{\sqrt{|S|}} \sqrt{1 - w} \in \left[0, \frac{\sqrt{2}\beta}{\sqrt{|S|}}\right) \tag{5.21}$$

for every $w \in (0, 1]$. By construction, $\frac{\sqrt{2}\beta}{\sqrt{|S|}} \le 1$. Since y_k and y are strictly decreasing, (5.21) is equivalent to

$$\lim_{k \to +\infty} w_k(y) = w(y) := 1 - \frac{|S|}{2\beta^2} y^2 \quad \text{for all } y \in \left[0, \frac{\sqrt{2}\beta}{\sqrt{|S|}}\right). \tag{5.22}$$

Assume by contradiction that $\frac{\sqrt{2}\beta}{\sqrt{|S|}} < 1$ and let $y \in \left(\frac{\sqrt{2}\beta}{\sqrt{|S|}}, 1\right)$. Then, by (5.22), $w_k(y) \to 0$ as $k \to +\infty$, which implies for $k \gg 1$

$$-w_k'(y) \overset{(5.13)}{=} \sqrt{2} \frac{\bar{r}_k}{u_{0k}} \sqrt{Q(u_{0k}w_k(y)) - Q(u_{0k})w_k(y)}$$

$$\ge \frac{1}{\beta} \inf_{(0,+\infty)} \sqrt{Q} > 0. \tag{5.23}$$

Therefore,

$$w_k(y) = -\int_y^1 w_k'(z)\mathrm{d}z \overset{(5.23)}{\ge} \frac{1}{\beta} \inf_{(0,+\infty)} \sqrt{Q} > 0 \quad \text{for } k \gg 1,$$

a contradiction. Hence, $\frac{\sqrt{2}\beta}{\sqrt{|S|}} = 1$ and (5.19) follows.

Conclusion. As $k \to +\infty$, we have

$$\frac{\mu(u_{0k})}{2} = \int_0^{\bar{r}_k} u_k \mathrm{d}x = u_{0k}\bar{r}_k \int_0^1 w_k \mathrm{d}y \sim u_{0k}\bar{r}_k \int_0^1 (1 - y^2)\mathrm{d}y = \frac{2}{3} u_{0k}\bar{r}_k. \tag{5.24}$$

In particular, $\mu(u_{0k}) \to +\infty$. Combined with (5.19), (5.24) leads after straightforward computations to (5.10), and (5.11) follows from (5.19). □

Lemma 5.3 (the droplet's energy) *Under the assumptions of Lemma* 5.2,

$$E[u_k] \lesssim \sqrt{\mu(u_{0k})} \quad for\, k \gg 1.$$

Proof We let $u = u_k$ and $\bar{r} = \bar{r}_k$ for notational convenience. Recalling that Q, thus R, is positive in $(0, +\infty)$ (Remark 4.8), we note that

$$E[u] = 2 \int_0^{\bar{r}} \left(\tfrac{1}{2}(u')^2 + Q(u) \right) \stackrel{(4.17)}{=} 2 \int_0^{\bar{r}} (2Q(u) - R(u_{0k})u)$$

$$\lesssim \int_0^{\bar{r}} Q(u). \tag{5.25}$$

Also, note that $Q > 0$, $-S > 0$, and (1.2) imply that

$$1 + u^{1-m} \lesssim Q(u) \lesssim 1 + u^{1-m} \quad \text{for all } u > 0. \tag{5.26}$$

In particular,

$$E[u] \stackrel{(5.25)}{\lesssim} \int_0^{\bar{r}} Q(u) \stackrel{(5.26)}{\lesssim} \bar{r} + \int_0^{\bar{r}} u^{1-m}. \tag{5.27}$$

Note that (5.9) implies that $R(+\infty) = 0$. Therefore, we may apply Lemma 5.1 with $K = 1$, leading to

$$u(x) \gtrsim (\bar{r} - x)^{\frac{2}{m+1}} \quad \text{for } x \in (\bar{r} - \delta_K, \bar{r}), \tag{5.28}$$

which by monotonicity implies that

$$u(x) \geq u(\bar{r} - \delta_K) \stackrel{(5.28)}{\sim} 1 \quad \text{for } x \in (0, \bar{r} - \delta_K). \tag{5.29}$$

Therefore, since $\frac{2(1-m)}{m+1} + 1 = \frac{3-m}{m+1} > 0$,

$$E[u] \stackrel{(5.27)}{\lesssim} \bar{r} + \int_0^{\bar{r}-\delta_K} u^{1-m} + \int_{\bar{r}-\delta_K}^{\bar{r}} u^{1-m} \stackrel{(5.28),(5.29)}{\lesssim} \bar{r} + \int_{\bar{r}-\delta_K}^{\bar{r}} (\bar{r} - x)^{\frac{2(1-m)}{m+1}} \lesssim \bar{r},$$

hence the result follows from $(5.10)_2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

5.2 The Pancake Case

Lemma 5.4 (The pancake case) *Assume* (5.1) *and* $e_* < +\infty$. *If* $\mathcal{A} \ni u_{0k} \to e_*(> 0)$ *as* $k \to +\infty$ *for a sequence, then* $\mu(u_{0k}) \to +\infty$ *as* $k \to +\infty$ *and*

$$\left. \begin{array}{l} \dfrac{2}{\mu(u_{0k})}\bar{r}_k \to \dfrac{1}{e_*} \quad and \\ v_k(y) = u_k(\bar{r}_k y) \to e_* \quad in\ C_{loc}^2((-1,1)) \end{array} \right\} \quad as\, k \to +\infty, \tag{5.30}$$

where u_k is the solution to $(P_{u_{0k}})$.

Proof Since $e_* < +\infty$ is a stationary solution of (P_{e_*}) (recall (4.16) and that $R'(e_*) = 0$), by continuous dependence [Theorem 8.40 of Kelley and Peterson (2010)] $u_k \to e_*$ in $C^2_{loc}(\mathbb{R})$ and $\bar{r}_k \to +\infty$ as $k \to +\infty$. Since $u_{0k} \in \mathcal{A}$, Lemma 4.10 implies that $u_{0k} < e_*$ for all k. Let $v_k(y) = u_k(\bar{r}_k y)$. It follows from (4.19) that

$$-v'_k = \sqrt{2}\bar{r}_k \sqrt{Q(v_k) - R(u_{0k})v_k} \quad \text{in } [0, 1).$$

Integrating, we obtain

$$e_* \leftarrow u_{0k} = \sqrt{2}\bar{r}_k \int_0^1 \sqrt{Q(v_k) - R(u_{0k})v_k} \, dy \quad \text{as } k \to +\infty.$$

Since $\bar{r}_k \to +\infty$ as $k \to +\infty$, we deduce that

$$0 = \lim_{k \to +\infty} (Q(v_k) - R(u_{0k})v_k) \quad \text{a.e. in } [0, 1). \tag{5.31}$$

Take any such y, and take any subsequence (not relabeled) such that $v_k(y) \to \overline{v} \in [0, e_*]$ as $k \to +\infty$ (recall that $v_k(y) \leq u_{0k} \to e_*$). If $\overline{v} = 0$ we would have $0 = +\infty$ in (5.31). Therefore, $\overline{v} > 0$ and $Q(\overline{v}) = R(e_*)\overline{v}$, that is, $R(e_*) = R(\overline{v})$. By the definition of e_* [cf. (4.27)], $R(s) > R(e_*)$ for $s \in (0, e_*)$: therefore $\overline{v} = e_*$. The arbitrariness of the subsequence and of y (in this order) implies that $v_k(y) \to e_*$ a.e. in $(0, 1)$. By dominated convergence we obtain $(5.30)_1$:

$$\frac{\mu(u_{0k})}{\bar{r}_k} = \frac{1}{\bar{r}_k} \int_{-\bar{r}_k}^{\bar{r}_k} u_k(x) dx = \int_{-1}^1 v_k(y) dy \to \int_{-1}^1 e_* = 2e_*.$$

Finally, fix $\overline{y} < 1$ such that $v_k(\overline{y})$ converges. We have

$$0 = e_* - e_* \leftarrow u_{0k} - v_k(\overline{y}) = \omega_k := \sqrt{2}\bar{r}_k \int_0^{\overline{y}} \sqrt{Q(v_k) - R(u_{0k})v_k} \, dy$$

with $\omega_k \to 0$ as $k \to +\infty$. Hence, $|v_k(y_1) - v_k(y_2)| \leq \omega_k$ for all $y_1, y_2 \in [0, \overline{y}]$. Therefore, $v_k \to e_*$ in $C_{loc}([0, 1))$, and the conclusion follows from standard ODE theory. $\square$

Lemma 5.5 (The pancake's energy) *Under the assumptions of Lemma 5.4,*

$$\frac{E[u_k]}{\mu(u_{0k})} \to R(e_*) \quad \text{as } k \to +\infty.$$

Proof With $v_k(y) = u_k(\bar{r}_k y)$, we write

$$\frac{E[u_k]}{\mu(u_{0k})} = \frac{2}{\mu(u_{0k})} \int_0^{\bar{r}_k} \left(\frac{1}{2}(u'_k)^2 + Q(u_k)\right) \overset{(4.17)}{=} \frac{2}{\mu(u_{0k})} \int_0^{\bar{r}_k} (2Q(u_k) - R(u_{0k})u_k)$$

$$\overset{[x=\bar{r}_k y]}{=} \frac{2\bar{r}_k}{\mu(u_{0k})} \int_0^1 (2Q(v_k) - R(u_{0k})v_k)\, dy.$$

By (5.30), $\frac{2\bar{r}_k}{\mu(u_{0k})} \to \frac{1}{e_*}$ and $R(u_{0k})v_k \to R(e_*)e_* = Q(e_*)$ in $L^1((0,1))$ as $k \to +\infty$. Hence, it suffices to prove that $Q(v_k) \to Q(e_*)$ in $L^1((0,1))$. Applying Lemma 5.1 with $K = e_* + 1$, we have $u_k(x) \geq C(\bar{r}_k - x)^{\frac{2}{m+1}}$ for all $x \in (\bar{r}_k - \delta_K, \bar{r}_k)$, which in terms of v_k means that

$$v_k(y) \geq C\bar{r}_k^{\frac{2}{m+1}}(1-y)^{\frac{2}{m+1}} \quad \text{for all } \frac{\bar{r}_k - \delta_K}{\bar{r}_k} < y < 1.$$

Therefore, by monotonicity of v_k,

$$|Q(v_k)| \overset{(1.2)}{\leq} C\left(1 + \bar{r}_k^{\frac{2(1-m)}{m+1}}(1-y)^{\frac{2(1-m)}{m+1}}\right) \overset{\bar{r}_k>1, m>1}{\leq} C\left(1 + (1-y)^{\frac{2(1-m)}{m+1}}\right). \tag{5.32}$$

Since $\frac{2(1-m)}{m+1} > -1$ if $m < 3$, the right-hand side of (5.32) belongs to $L^1((0,1))$: therefore, an application of Lebesgue theorem yields the result. $\qquad\square$

5.3 Conclusion

Let u_M be a minimizer of E in $\mathcal{D}_M$, supp $u_M = [-\bar{r}_M, \bar{r}_M]$, and $u_{0M} = u_M(0)$. We are interested in the behavior of u_M as $M \to +\infty$. We preliminarily estimate the energy of any sequence of minimizers whose maximal height remains finite.

Lemma 5.6 *Assume* (5.1). *If* $\mathcal{A} \ni u_{0M} \to \alpha \in [0, +\infty)$ *for a sequence* $M \to +\infty$ *(not labeled), then the corresponding solutions* u_M *to* $(P_{u_{0M}})$ *are such that:*

(i) supp $u_M = [-\bar{r}_M, \bar{r}_M]$, $\bar{r}_M < +\infty$, *with* $\bar{r}_M \to +\infty$ *as* $M \to +\infty$;
(ii) $\liminf\limits_{M \to +\infty} \dfrac{1}{M} E[u_M] \geq R(\alpha)$ $(R(0) = +\infty)$.

Proof *(i)* follows from Corollary 4.1 and mass constraint. For *(ii)*, since $u_{0M} \in \mathcal{A}$, we have $R(u_M) > R(u_{0M})$ in $(0, \bar{r}_M)$. Therefore, as $M \to +\infty$,

$$E[u_M] = 2\int_0^{\bar{r}_M} \left(\tfrac{1}{2}(u_M')^2 + Q(u_M)\right) \overset{(4.17)}{=} 2\int_0^{\bar{r}_M} (2Q(u_M) - R(u_{0M})u_M)$$

$$= 2\int_0^{\bar{r}_M} u_M(2R(u_M) - R(u_{0M})) \geq R(u_{0M})M.$$

$\qquad\square$

To avoid pathological situations, we assume a minimal monotonicity property on R:

$$R' < 0 \text{ in a left neighborhood of } e_*. \tag{5.33}$$

Note that (5.33) is already included in (4.25) if $e_* = +\infty$. Assumption (5.33) suffices to infer the following.

Lemma 5.7 *Assume* (5.1) *and* (5.33). *Then, a left neighborhood of* e_* *is contained in* $\mathcal{A}$.

Proof If $e_* = +\infty$, then inf $R = R(+\infty)$ (cf. Remark 4.8): together with (5.33), this immediately implies the conclusion. If $e_* < +\infty$, let $\delta > 0$ such that $R' < 0$ in $(e_* - \delta, e_*)$. If e_* is the first local minimum point of R, R is non-increasing in $(0, e_*)$ and there is nothing to prove. Otherwise, let $R_0 := \min_{(0, e_* - \delta]} R > R(e_*)$ and take $\tilde{\delta} \leq \delta$ such that $R(e_* - \tilde{\delta}) < R_0$. Then $(e_* - \tilde{\delta}, e_*) \subseteq \mathcal{A}$. $\qquad\square$

We are now ready to conclude the analysis of generic macroscopic shapes of minimizers.

Theorem 5.8 *Assume* (5.1) *and* (5.33). *Let* u_M *be a minimizer of* E *in* $\mathcal{D}_M$. *Let* supp $u_M = [-\bar{r}_M, \bar{r}_M]$ *and* $u_{0M} = u_M(0)$.

- *(droplet) If* $e_* = +\infty$ *and* (5.9) *holds, then*

$$
\left.
\begin{aligned}
u_{0M}^4 &\sim \frac{9|S|}{32} M^2, \quad \bar{r}_M^4 \sim \frac{9}{8|S|} M^2, \quad and \\
u_{0M}^{-1} u_M(\bar{r}_M y) &\to 1 - y^2 \quad in \ C_{loc}^2((-1, 1))
\end{aligned}
\right\} \quad as \ M \to +\infty.
$$

- *(pancake) If* $e_* < +\infty$, *then*

$$
\left.
\begin{aligned}
u_{0M} &\sim e_*, \quad \bar{r}_M \sim \frac{1}{2e_*} M, \quad and \\
u_M(\bar{r}_M y) &\to e_* \quad in \ C_{loc}^2((-1, 1))
\end{aligned}
\right\} \quad as \ M \to +\infty.
$$

Proof The droplet's case. By Lemma 5.7, $a > 0$ exists such that $(a, +\infty) \subset \mathcal{A}$. By (5.3), $\mu \in C((a, +\infty))$. Hence Lemma 5.2 is applicable and yields $\mu(s) \to +\infty$ as $s \to +\infty$. Therefore, for any sequence $M_k \to +\infty$ there exists a sequence $\mathcal{A} \ni u_{0k} \to +\infty$ such that $\mu(u_{0k}) = M_k$. Moreover, by Theorem 4.7, u_k (the solution to $(P_{u_{0k}})$) belongs to $\mathcal{D}_{M_k}$. It follows from Lemma 5.3 that u_k is such that $E[u_k] \lesssim \sqrt{M_k}$.

To conclude, assume by contradiction that u_{0M} does not converge to $+\infty$. Then, a subsequence M_k exists such that $u_{0M_k} \to \alpha \in [0, +\infty)$. By Lemma 4.6, $u_{0M_k} \in \mathcal{A}$ for all k. By Lemma 5.6, since $R > 0$ in $(0, +\infty)$ (cf. Remark 4.8), we have $E[u_{M_k}] \gtrsim M_k$ as $k \to +\infty$. Hence, for k sufficiently large we would have $E[u_k] < E[u_{M_k}]$, in contradiction with the definition of u_M. Thus $u_{0M} \to +\infty$ as $M \to +\infty$, and the result follows from Lemma 5.2.

The pancake's case. By Lemma 4.10, $\mathcal{A} \subseteq (0, e_*)$. By Lemma 5.7, $I = (e_* - \delta, e_*) \subseteq \mathcal{A}$ for some $\delta \leq e_*$. By (5.3), $\mu \in C(I)$. By Lemma 5.4, $\mu(s) \to +\infty$ as $s \to e_*^-$. Therefore, for any sequence $M_k \to +\infty$ there exists a sequence $\mathcal{A} \ni u_{0k} \to e_*^-$ such that $\mu(u_{0k}) = M_k$. By Theorem 4.7 the solution u_k to $(P_{u_{0k}})$ belongs to $\mathcal{D}_{M_k}$ and, by Lemma 5.5, u_k is such that $\frac{E[u_k]}{M_k} \to R(e_*)$.

Assume by contradiction that u_{0M} does not converge to e_*. Then a subsequence M_k exists such that $u_{0M_k} \to \alpha \in [0, e_*)$. By Lemma 4.6, $u_{0M_k} \in \mathcal{A}$ for all k. By

Lemma 5.6, we have $\liminf\limits_{M_k \to +\infty} \frac{E[u_{M_k}]}{M_k} \geq R(\alpha)$. Since $R(e_*) < R(\alpha)$ [cf. (4.27)], for k sufficiently large we would have $E[u_{M_k}] > E[u_k]$, in contradiction with the definition of u_M. Hence, u_{0M} converges to e_* as $M \to +\infty$, and the conclusion follows from Lemma 5.4. □

5.4 Transition Profiles

This is a limiting case: the same assumptions leading to Lemma 5.2 are assumed to hold (in particular, $e_* = +\infty$), except that in this case $S = 0$. In addition, we need to assume monotonicity of Q:

$$Q'(s) < 0 \quad \text{for all } s \in (0, +\infty). \tag{5.34}$$

Lemma 5.9 *Assume (5.1), (5.34), and $S = 0$. Let u_M be a minimizer of E in $\mathcal{D}_M$ and let $u_{0M} = u_M(0)$. Then, $u_{0M} \to +\infty$ as $M \to +\infty$.*

Remark 5.10 Note that (5.34) and $S = 0$ already imply $Q > 0$ and $R > 0$ in $(0, +\infty)$. Thus, $R'(s) = s^{-1}Q'(s) - s^{-2}Q(s) < 0$ in $(0, +\infty)$, which implies $\mathcal{A} = (0, +\infty)$ and $e_* = +\infty$. Also, note that the case $S = 0$ and $e_* = +\infty$ remains open if (5.34) does not hold; however, this never happens in model cases.

Proof Let $\operatorname{supp} u_M = [-\bar{r}_M, \bar{r}_M]$. We recall that u_M solves $(P_{u_{0M}})$. Let u_{0M} be any sequence (not relabeled) such that $u_{0M} \to \alpha \in [0, +\infty]$ as $M \to +\infty$. We will exclude that $\alpha = 0$ and that $\alpha \in (0, +\infty)$, which implies the statement.

Assume that $\alpha = 0$. Then, $R(u_{0M}) \to +\infty$. By Lemma 5.6 (i), $(0, \bar{r}_M) \supset (0, 1)$ for $M \gg 1$. Since $u_M \leq u_{0M}$ in $(0, \bar{r}_M)$, by (5.34) we have

$$u'_M(y) = u'_M(0) + \int_0^y u''_M \overset{(P_{u_{0M}})}{=} \int_0^y \left(Q'(u_M) - R(u_{0M})\right) \leq -R(u_{0M})y$$

for all $y \in (0, 1)$, that is,

$$0 < u_M(x) = u_{0M} + \int_0^x u'_M \leq u_{0M} - R(u_{0M})\int_0^x y\,\mathrm{d}y = u_{0M} - \tfrac{1}{2}R(u_{0M})x^2 \to -\infty$$

for all $x \in (0, 1)$, a contradiction.

Assume that $\alpha \in (0, +\infty)$. By continuous dependence [Theorem 8.40 of Kelley and Peterson (2010)], $u_M \to u$ in $C^2_{loc}(\{u > 0\})$, where u is the solution of (P_α). By Remark 5.10 we have $\alpha \in \mathcal{A}$, hence it follows from Lemma 4.6 that u has compact support and, therefore, finite mass $\mu(\alpha)$. Since $\mu \in C(\mathcal{A})$, we have $M = \mu(u_{0M}) \to \mu(\alpha)$ which contradicts $M \to +\infty$. □

In this limiting case, the macroscopic shape turns out to depend on the behavior of Q and its derivatives as $s \to +\infty$: we assume that

$$\begin{aligned} &K > 0 \text{ and } p > 1 \text{ exist such that} \\ &Q(s) \sim Ks^{1-p} \quad \text{and} \quad Q'(s) \sim -(p-1)Ks^{-p} \quad \text{as } s \to +\infty. \end{aligned} \tag{5.35}$$

We note for later reference that

$$- s^2 R'(s) = G(s) := Q(s) - s Q'(s) \sim pK s^{1-p} \quad \text{as } s \to +\infty. \quad (5.36)$$

Remark 5.11 In the model cases Q_a and Q_b with $S = 0$, we have $K = |B| > 0$ and $p = n$ for Q_b, $K = -B > 0$ and $p = n$ for Q_a with $B < 0$, and $K = A > 0$ and $p = m$ for Q_a with $B = 0$ [cf. (1.5) and (1.6)].

Theorem 5.12 (transition profiles) *Assume* (5.1), (5.34), *and* (5.35). *Then, as* $M \to +\infty$,

$$u_{0M}^{p+3} \sim \frac{pK}{2c_p^2 f_p(0)^2} M^2, \quad \bar{r}_M^{p+3} \sim \frac{f_p(0)^2}{2^{p+2} pK c_p^{p+1}} M^{p+1},$$

$$thus \quad \frac{u_{0M}^{p+1}}{\bar{r}_M^2} \sim \frac{2pK}{f_p(0)^2}, \quad (5.37)$$

and

$$w_M(y) = u_{0M}^{-1} u_M(\bar{r}_M y) \to f_p^{-1}(f_p(0)|y|) \quad in \ C_{loc}^2((-1, 1)), \quad (5.38)$$

where

$$f_p(w) := \int_w^1 \frac{\sqrt{p}\, \tilde{w}^{\frac{p-1}{2}}\, \mathrm{d}\tilde{w}}{\sqrt{1 - \tilde{w}^p}} \quad for \ w \in [0, 1], \quad (5.39)$$

and $c_p = \int_0^1 f_p^{-1}(f_p(0)y)\mathrm{d}y$.

Proof We let $f = f_p$ for notational convenience. Thanks to Lemma 5.9, $u_{0M} \to +\infty$ as $M \to +\infty$. As we noticed there, Step 1 in the proof of Lemma 5.2 holds for $-S \geq 0$: therefore

$$\bar{r}_M \to +\infty \quad \text{as } M \to +\infty.$$

Let $w_M(y) = u_{0M}^{-1} u_M(\bar{r}_M y)$. It follows from (4.19) that

$$- \frac{w_M'}{\sqrt{2}\sqrt{Q(u_{0M} w_M) - Q(u_{0M}) w_M}} = \frac{\bar{r}_M}{u_{0M}} \quad \text{in } [0, 1). \quad (5.40)$$

We claim that $C \geq 1$ exists such that

$$C^{-1} \leq \frac{\bar{r}_M^2 G(u_{0M})}{u_{0M}^2} \overset{(5.36)}{\sim} \frac{pK \bar{r}_M^2}{u_{0M}^{p+1}} \leq C \quad \text{for all } M \gg 1. \quad (5.41)$$

Integrating (5.40) in (0, 1) and multiplying by $\sqrt{G(u_{0M})}$, we obtain

$$\frac{\bar{r}_M \sqrt{G(u_{0M})}}{u_{0M}} = \frac{1}{\sqrt{2}} \int_0^1 \frac{\sqrt{G(u_{0M})}dw}{\sqrt{Q(u_{0M}w) - Q(u_{0M})w}}. \tag{5.42}$$

Since $u_{0M} \to +\infty$, we have

$$\int_{1/2}^1 \frac{\sqrt{G(u_{0M})}dw}{\sqrt{Q(u_{0M}w) - Q(u_{0M})w}} \overset{(5.35),(5.36),(5.39)}{\sim} f(1/2), \tag{5.43}$$

which, by (5.42), already proves the lower bound in (5.41). For the upper bound, we note that, since $Q > 0$ in $(0, +\infty)$ (cf. Remark 5.10),

$$\frac{d}{dw}(Q(u_{0M}w) - Q(u_{0M})w)$$
$$= u_{0M}Q'(u_{0M}w) - Q(u_{0M}) < u_{0M}Q'(u_{0M}w) \overset{(5.34)}{<} 0. \tag{5.44}$$

Therefore, again since $u_{0M} \to +\infty$,

$$\int_0^{\frac{1}{2}} \frac{\sqrt{G(u_{0M})}dw}{\sqrt{Q(u_{0M}w) - Q(u_{0M})w}} \overset{(5.44)}{\leq} \frac{1}{2} \frac{\sqrt{G(u_{0M})}}{\sqrt{Q\left(\frac{u_{0M}}{2}\right) - \frac{1}{2}Q(u_{0M})}}$$
$$\overset{(5.35),(5.36)}{\sim} \frac{1}{\sqrt{2}}\sqrt{\frac{p}{2^p - 1}}$$

as $M \to +\infty$, which together with (5.43) yields the upper bound in (5.41).

Thanks to (5.41), any sequence $M \to +\infty$ has a subsequence (not relabeled) such that

$$\frac{\bar{r}_M \sqrt{G(u_{0M})}}{u_{0M}} \sim \frac{1}{\ell} > 0 \qquad \text{as } M \to +\infty \tag{5.45}$$

for some $\ell > 0$. We will show that $\ell = \sqrt{2}/f(0)$: together with the arbitrariness of the (sub)sequence and with (5.36), this will prove (5.37)$_3$. Since w_M is strictly decreasing in (0, 1), $y_M = w_M^{-1}$ is well defined. Fix $0 < w \leq 1$. Multiplying (5.40) by $\sqrt{G(u_{0M})}$ and integrating in $(0, y_M(w))$, since $u_{0M} \to +\infty$ we obtain

$$f(w) \overset{(5.39)}{=} \int_w^1 \frac{\sqrt{p}\tilde{w}^{\frac{p-1}{2}}d\tilde{w}}{\sqrt{1-\tilde{w}^p}} \overset{(5.35)}{\sim} \int_w^1 \frac{\sqrt{G(u_{0M})}d\tilde{w}}{\sqrt{Q(u_{0M}\tilde{w}) - Q(u_{0M})\tilde{w}}}$$
$$= \frac{\bar{r}_M \sqrt{2G(u_{0M})}}{u_{0M}} y_M(w) \overset{(5.45)}{\sim} \frac{\sqrt{2}}{\ell} y_M(w) \quad \text{as } M \to +\infty. \tag{5.46}$$

It follows from (5.46) that, for every $0 < w \leq 1$,

$$y_M(w) \to y(w) := \frac{\ell}{\sqrt{2}} f(w) \in \left[0, \frac{\ell f(0)}{\sqrt{2}}\right) \quad \text{as } M \to +\infty. \tag{5.47}$$

By construction, $\frac{\ell f(0)}{\sqrt{2}} \leq 1$. Since w_M and f are strictly decreasing, (5.47) is equivalent to

$$w_M(y) \to w(y) := f^{-1}\left(\frac{\sqrt{2}}{\ell}y\right) \quad \text{as } M \to +\infty \quad \text{for all } y \in \left[0, \frac{\ell f(0)}{\sqrt{2}}\right).$$
(5.48)

We now prove that $\frac{\ell f(0)}{\sqrt{2}} = 1$. It follows from (5.40) that for all $M \gg 1$

$$-w'_M(y) \quad = \quad \frac{\bar{r}_M\sqrt{G(u_{0M})}}{u_{0M}}\sqrt{\frac{2(Q(u_{0M}w_M(y)) - Q(u_{0M})w_M(y))}{G(u_{0M})}}$$

$$\overset{(5.45),(5.36)}{\geq} \ell^{-1}\sqrt{\frac{Q(u_{0M}w_M(y)) - Q(u_{0M})w_M(y)}{pKu_{0M}^{1-p}}}.$$
(5.49)

By (5.34) and since $w_M(y) \leq 1$, we have for all $M \gg 1$

$$Q(u_{0M}w_M(y)) - Q(u_{0M})w_M(y) \geq Q(u_{0M})(1 - w_M(y)) \overset{(5.35)}{\geq} \frac{K}{2}u_{0M}^{1-p}(1 - w_M(y)),$$

which inserted in (5.49) implies

$$-w'_M(y) \geq \frac{\ell^{-1}}{\sqrt{2p}}\sqrt{1 - w_M(y)} \quad \text{for every } y \in [0, 1).$$
(5.50)

Assume by contradiction that $\frac{\ell f(0)}{\sqrt{2}} < 1$. Then, by (5.48), $w_M(y) \to 0$ for every $y \in \left(\frac{\ell f(0)}{\sqrt{2}}, 1\right]$. Hence, using (5.50), we deduce for all $M \gg 1$ that $-w'_M(y) \geq \frac{\ell^{-1}}{2\sqrt{p}}$ for every $y \in \left(\frac{\ell f(0)}{\sqrt{2}}, 1\right]$. It follows that

$$w_M(y) = -\int_y^1 w'_M(z)\,\mathrm{d}z \geq \frac{\ell^{-1}}{2\sqrt{p}}(1 - y) \quad \text{for all } y \in \left(\frac{\ell f(0)}{\sqrt{2}}, 1\right], \ M \gg 1,$$

a contradiction. Therefore, $\ell = \sqrt{2}/f(0)$.

The pointwise convergence in (5.38) now follows from (5.48) recalling that w_M is an even function. To prove the uniform convergence, we note that w_M solves

$$\begin{cases} -\dfrac{u_{0M}^2}{\bar{r}_M^2 G(u_{0M})}w''_M = \dfrac{Q(u_{0M}) - u_{0M}Q'(u_{0M}w_M)}{G(u_{0M})} \\ w_M(0) = 1, \ w'_M(0) = 0. \end{cases}$$

In view of (5.45) and (5.35), as $M \to +\infty$, w_M converges in $C^2_{loc}(\{w > 0\})$ to the unique solution of

$$\begin{cases} -\frac{2}{f(0)^2} w'' = \dfrac{w^p + p - 1}{p w^p} \\ w(0) = 1, \ w'(0) = 0, \end{cases}$$

which coincides with $f_p^{-1}(f_p(0)|y|)$ for $|y| < 1$. In particular, as $M \to +\infty$ we have

$$\frac{M}{2} = \int_0^{\bar{r}_M} u_M \mathrm{d}x = u_{0M} \bar{r}_M \int_0^1 w_M(y) \mathrm{d}y \overset{(5.38)}{\sim} u_{0M} \bar{r}_M \int_0^1 f_p^{-1}(f_p(0)y) \mathrm{d}y$$
$$= c_p u_{0M} \bar{r}_M. \tag{5.51}$$

Combining (5.37)$_3$ and (5.51), after straightforward computations we obtain (5.37)$_1$ and (5.37)$_2$. □

Remark 5.13 The function f_p behaves as

$$f_p(w) \sim \begin{cases} f_p(0) - \frac{2\sqrt{p}}{p+1} w^{\frac{p+1}{2}} & \text{as } w \to 0^+ \\ 2\sqrt{1-w} & \text{as } w \to 1^-. \end{cases}$$

Hence,

$$w_p(y) := f_p^{-1}(f_p(0)y) \sim \begin{cases} \left(\frac{p+1}{2\sqrt{p}} f_p(0)(1-y) \right)^{\frac{2}{p+1}} & \text{as } y \to 1^- \\ 1 - \frac{f_p(0)^2}{4} y^2 & \text{as } y \to 0^+. \end{cases}$$

Remark 5.14 The previous remark implies that the critical case is a transition between droplets and pancakes:

(i) for $p = 1$, we have $f_1(w) = 2\sqrt{1-w}$, $w_1(y) = 1 - y^2$, and $c_1 = 2/3$, so that

$$u_{0M}^4 \sim \frac{9K}{32} M^2 \quad \text{and} \quad \bar{r}_M^4 \sim \frac{9}{8K} M^2 \quad \text{as } M \to +\infty \text{ and } p \to 1,$$

thus recovering the droplet case with $K = -S > 0$ in the limit as $p \to 1$;

(ii) as $p \to +\infty$, we have $w_p(y) \to 1$, $c_p \to 1$, and

$$u_{0M} \sim 1 \quad \text{and} \quad \bar{r}_M \sim \frac{1}{2} M \quad \text{as } M \to +\infty \text{ and } p \to +\infty,$$

thus recovering the pancake case with $e_* = 1$ in the limit as $p \to +\infty$.

To see (ii), note that $f_p(0) \to 0$, $\sqrt{p} f_p(0) \to \pi$, and $\left(\frac{f_p(0)^2}{2^{p+2}p} \right)^{\frac{1}{p+3}} \to \frac{1}{2}$ as $p \to +\infty$. Since w_p is strictly decreasing, $w_p(0) = 1$ and $w_p(1) = 0$, y_p exists such that

$w_p(y_p) = 1 - \frac{f_p(0)^2}{4}$ for $p \gg 1$. Moreover, $p \to +\infty$ if and only if $w_p(y_p) \to 1$, hence we deduce that as $p \to +\infty$

$$f_p(0)y_p = f_p(w_p(y_p)) \sim 2\sqrt{1 - w_p(y_p)} = f_p(0),$$

which implies that $y_p \to 1$ as $p \to +\infty$. Therefore, $w_p \to 1$ and (ii) follows.

6 Asymptotic Behavior for Small Masses

As expected, minimizers tend uniformly to zero as $M \to 0$.

Lemma 6.1 *Assume* (5.1). *Let* u_M *be a minimizer of* E *in* $\mathcal{D}_M$. *Then,* $u_{0M} := u_M(0) \to 0$ *as* $M \to 0^+$.

Proof By Corollary 4.1, Theorem 4.5, and Lemma 4.6, $\operatorname{supp} u_M = [-\bar{r}_M, \bar{r}_M]$ with $r_M < +\infty$, $\{u_{0M}\} \subset \mathcal{A}$, and $\lambda_M = R(u_{0M})$. Assume by contradiction that $u_{0M} \to \alpha \in \overline{\mathcal{A}} \setminus \{0\}$ for a subsequence (not relabeled).

We first analyze the case $\alpha < +\infty$. Since $\lambda_M \to \lambda := R(\alpha)$, it follows from Lemma A.3 that $\bar{r}_M \to \bar{r} \in (0, +\infty]$ and $u_M \to u$ in $C_{loc}((-\bar{r}, \bar{r}))$ as $M \to 0^+$, where u is the solution to (P_α) and $\{u > 0\} = (-\bar{r}, \bar{r})$. This implies, applying Fatou's Lemma, that

$$0 = \lim_{M \to 0^+} M = \liminf_{M \to 0^+} \int_{-\bar{r}_M}^{\bar{r}_M} u_M \geq \int_{-\bar{r}}^{\bar{r}} u > 0,$$

a contradiction.

If instead $u_{0M} \to +\infty$, then $R > 0$ in $(0, +\infty)$: indeed, otherwise we would have $e_* < +\infty$ (Remark 4.9), which would imply $u_{0M} < e_*$ (Lemma 4.10), a contradiction. In addition, $Q(s) \leq cs$ for $s \geq 1$ for some $c > 0$ (otherwise, we would again have $e_* < +\infty$). Since $u_{0M} \to +\infty$ and u_M is strictly decreasing with compact support, $x_M \in (0, \bar{r}_M)$ exists such that $x_M = u_M^{-1}(1)$ for M sufficiently small. Hence,

$$u_{0M} - 1 = \int_0^{x_M} -u'(x) \overset{(4.19)}{=} \int_0^{x_M} \sqrt{2(Q(u_M(x)) - R(u_{0M})u_M(x))}$$
$$\overset{R>0}{\leq} \int_0^{x_M} \sqrt{2Q(u_M(x))} \overset{Q(s) \leq cs^2}{\leq} \sqrt{2c} \int_{-\bar{r}_M}^{\bar{r}_M} u_M(x) = \sqrt{2c}M.$$
$$(6.1)$$

Letting $M \to 0$ in (6.1), we obtain a contradiction.   □

7 Non-uniqueness of Minimizers

In this subsection, we prove non-uniqueness of minimizers of E in $\mathcal{D}_M$ (here we explicit the dependence of $\mathcal{D}$ on M) when the weighted potential R is such that

$$\mathcal{A} \neq (0, e_*). \tag{7.1}$$

Moreover, we need an additional property of Q, namely:

$$Q'(s) \sim -A(m-1)s^{-m} \quad \text{as } s \to 0^+. \tag{7.2}$$

Throughout the section, $\{u_{0k}\} \subset \mathcal{A}$ and u_k is the solution of $(P_{u_{0k}})$, defined in $\{u_k > 0\} = (-\bar{r}_k, \bar{r}_k)$. We let $E(u_{0k}) := E[u_k]$ and we recall that $\mu(u_{0k})$ denotes the mass of u_k [cf. (5.2)]. In what follows we study the behavior of μ and E on $\overline{\mathcal{A}}$. Let us start by showing what happens when u_{0k} tends to 0.

Lemma 7.1 *Assume* (5.1) *and* (7.2). *Let* $\{u_{0k}\} \subset \mathcal{A}$ *be such that* $u_{0k} \to 0$ *as* $k \to +\infty$. *Then,* $\mu(u_{0k}) \to 0$ *as* $k \to +\infty$.

Proof Let $\omega_k = \sqrt{A/u_{0k}^{m+1}}$. The function $w_k(y) = u_{0k}^{-1} u_k \left(\omega_k^{-1} y \right)$ is a solution to

$$\begin{cases} -w_k'' = f_k(w_k) := A^{-1}(u_{0k}^{m-1} Q(u_{0k}) - u_{0k}^m Q'(u_{0k} w_k)) & \text{in } (-\omega_k \bar{r}_k, \omega_k \bar{r}_k), \\ w_k(0) = 1, \ w_k'(0) = 0, \ w_k(\pm \omega_k \bar{r}_k) = 0. \end{cases}$$

By (1.2) and (7.2), $f_k(w) \to 1 + (m-1)w^{-m}$ in $C_{loc}((0, +\infty))$ as $k \to +\infty$ and

$$F_k(w) := \int_w^1 f_k(t) \mathrm{d}t = A^{-1} u_{0k}^{m-1} (Q(u_{0k} w) - Q(u_{0k}) w) \overset{(1.2)}{\sim} w^{1-m} - w \to +\infty$$

as $(k, w) \to (+\infty, 0^+)$. Applying Lemma A.3, we deduce that, as $k \to +\infty$, $w_k \to w$ in $C_{loc}^2(\{w > 0\})$, where w is the solution to

$$\begin{cases} -w'' = 1 + (m-1)w^{-m} & \text{in } \{w > 0\} = (-\bar{r}, \bar{r}), \\ w(0) = 1, \ w'(0) = 0, \ w(\bar{r}) = 0, \end{cases}$$

and $\omega_k \bar{r}_k \to \bar{r}$. Note that w is concave, hence $\bar{r} < +\infty$. Therefore, $\bar{r}_k \to 0$ as $k \to +\infty$, whence

$$\mu(u_{0k}) = 2 \int_0^{\bar{r}_k} u_k(x) \mathrm{d}x \leq 2\bar{r}_k u_{0k} \to 0 \quad \text{as } k \to +\infty.$$

$\square$

Now we show that E is continuous on $\mathcal{A}$.

Lemma 7.2 *Assume* (5.1). *Then,* $\mu \in C(\mathcal{A})$ *and* $E \in C(\mathcal{A})$.

Proof Continuity of μ follows from (5.3), hence we only show continuity of E. Let $\mathcal{A} \ni u_{0k} \to \alpha \in \mathcal{A}$ as $k \to +\infty$. Since $\alpha \in \mathcal{A}$, we have $\alpha > 0$. By Lemma A.3, $u_k \to u$, where u solves (P_α) in $C^2_{loc}((-\bar{r}, \bar{r}))$ with $\{u > 0\} = (-\bar{r}, \bar{r})$, and $\bar{r}_k \to \bar{r} > 0$ as $k \to +\infty$. Note that $\bar{r} < +\infty$ since $\alpha \in \mathcal{A}$ [cf. Lemma 4.6].

Let $\rho_k = \bar{r}_k/\bar{r}$ and $w_k(y) = u_k(\rho_k y)$, $y \in (-\bar{r}, \bar{r})$. Obviously $\rho_k \to 1$ and $w_k \to u$ locally uniformly, hence a.e., in $(-\bar{r}, \bar{r})$. We have

$$\frac{1}{2} E(u_{0k}) = \int_0^{\bar{r}_k} \left(\frac{1}{2}(u_k')^2 + Q(u_k) \right) \overset{(4.17)}{=} \int_0^{\bar{r}_k} (2Q(u_k) - R(u_{0k})u_k)$$
$$= \rho_k \int_0^{\bar{r}} (2Q(w_k) - R(u_{0k})w_k). \tag{7.3}$$

The properties of Q imply that

$$|Q(s)| \le C(1 + s^{1-m}) \quad \text{for all } s \in (0, \alpha + 1). \tag{7.4}$$

Applying Lemma 5.1 with $K = \alpha + 1$, we have $u_k(x) \ge C(\bar{r}_k - x)^{\frac{2}{m+1}}$ for all $x \in (\bar{r}_k - \delta_K, \bar{r}_k)$ and k sufficiently large. In terms of w_k, this means that

$$w_k(y) \ge C(\bar{r} - y)^{\frac{2}{m+1}} \quad \text{for all } y \in \left(\bar{r} - \frac{\delta_K}{2}, \bar{r}\right) \tag{7.5}$$

for k sufficiently large. It follows from the monotonicity of w_k and the locally uniform convergence $w_k \to u$ that

$$|Q(w_k)| \overset{(7.4),(7.5)}{\le} C\left(1 + (\bar{r} - y)^{\frac{2(1-m)}{m+1}}\right). \tag{7.6}$$

Since $\frac{2(1-m)}{m+1} > -1$ if $m < 3$, the right-hand side of (7.6) belongs to $L^1((0, \bar{r}))$: therefore, an application of Lebesgue theorem in (7.3) implies the result. $\qquad\square$

Finally, we study the behavior of μ on $\partial\mathcal{A} \setminus \{0\}$.

Lemma 7.3 *Assume* (5.1); *if* $e_* = +\infty$, *assume in addition* (5.9). *Let* $\{u_{0k}\} \subset \mathcal{A}$ *be such that* $u_{0k} \to \alpha \in \partial\mathcal{A} \setminus \{0\}$ *as* $k \to +\infty$, *where* $\partial\mathcal{A}$ *denote the boundary of* $\mathcal{A}$ *with respect to the topology of* $\mathbb{R}^*$. *Then,* $\mu(u_{0k}) \to +\infty$ *as* $k \to +\infty$.

Proof If $\alpha = +\infty$, it follows from Lemma 4.10 that $e_* = +\infty$. Thus, by Lemma 5.2, we have that $\mu(u_{0k}) \to +\infty$.

Now let $\alpha \in \partial\mathcal{A} \setminus \{0, +\infty\}$. By continuous dependence [Theorem 8.40 of Kelley and Peterson (2010)], $u_k \to u$ in $C^2_{loc}(\{u > 0\})$, where u solves (P_α) and $\{u > 0\} = (-\bar{r}, \bar{r})$ for $\bar{r} \in (0, +\infty]$. Since R is continuous in $(0, +\infty)$ and $\alpha \in \partial\mathcal{A} \setminus \{0, +\infty\}$, by definition of $\mathcal{A}$ at least one of the following holds:

(i) $R'(\alpha) = 0$. Then, $u \equiv \alpha$, hence (by Fatou's Lemma) $\liminf_{k \to +\infty} \mu(u_{0k}) \ge \int_{\mathbb{R}} \alpha = +\infty$;

(ii) $0 < t < \alpha$ exists such that $R'(t) = 0$ and $R(t) = R(\alpha)$. In this case, we claim that $u > t$ for every $x \in \mathbb{R}$. Indeed, if $x > 0$ exists such that $u(x) = t$, then

$$u''(x) \overset{(P_\alpha)}{=} Q'(u(x)) - R(\alpha) = Q'(t) - R(t) \overset{(4.15)}{=} t R'(t) = 0 \quad \text{and}$$
$$u'(x) \overset{(4.17)}{=} 0,$$

whence $u \equiv t$, in contradiction with $u(0) = \alpha > t$. It follows from Fatou's Lemma that $\liminf_{k \to +\infty} \mu(u_{0k}) \geq \int_{\mathbb{R}} t = +\infty$.

$\square$

Now we are ready to prove the non-uniqueness result.

Theorem 7.4 *Assume* (5.1), (5.33), (7.1), *and* (7.2); *if* $e_* = +\infty$, *assume in addition* (5.9). *Then,* $M > 0$ *exists such that* E *has at least two minimizers in* $\mathcal{D}_M$.

Proof We argue by contradiction assuming that for every $M > 0$ there exists a unique minimizer u_M of E in $\mathcal{D}_M$. By Lemma 4.6, we know that $u_{0M} := u_M(0) \in \mathcal{A}$. Then, the function $\mathcal{P} : (0, +\infty) \to \mathcal{A}$, $\mathcal{P}(M) := u_{0M}$, is well defined, and u_M is the unique solution to $(P_{u_{0M}})$.

We claim that $\mathcal{P}$ is continuous. Take any sequence $M_k \to M \in (0, +\infty)$ as $k \to +\infty$ and take any subsequence (not relabeled) such that $\mathcal{P}(M_k) \to \alpha \in \overline{\mathcal{A}}$. Since $\alpha \in \partial\mathcal{A}$ is excluded by Lemmas 7.1 and 7.3, in fact $\alpha \in \mathcal{A}$. We will show that $\alpha = \mathcal{P}(M)$, which implies continuity in view of the arbitrariness of the subsequence.

Let u be the solution of (P_α). By Lemma 7.2,

$$M_k = \mu(\mathcal{P}(M_k)) \to \mu(\alpha) = M \quad \text{and}$$
$$E(\mathcal{P}(M_k)) \to E(\alpha) = E[u] \quad \text{as } k \to +\infty. \tag{7.7}$$

Since $\mu(\alpha) = M$ and since by assumption u_M is unique, it suffices to prove that $E[u] \leq E[u_M]$. Indeed, then $u = u_M$ and thus $\alpha = \mathcal{P}(M)$, proving continuity of $\mathcal{P}$.

We set $\gamma_k = \frac{M_k}{M} \to 1$ as $k \to +\infty$. Since $\gamma_k u_M \in \mathcal{D}_{M_k}$ and u_{M_k} is a minimizer in $\mathcal{D}_{M_k}$, it holds that

$$\int \left(\frac{\gamma_k^2}{2}(u'_M)^2 + Q(\gamma_k u_M) \right) = E[\gamma_k u_M] \geq E[u_{M_k}] \quad \forall k > 0.$$

A straightforward application of dominated convergence theorem then yields $E[u] \leq E[u_M]$:

$$E[u_M] \longleftarrow E[\gamma_k u_M] \geq E[u_{M_k}] \overset{(7.7)}{\longrightarrow} E[u].$$

We are now ready to conclude. We know that

$$\mathcal{P}(M) \to 0 \text{ as } M \to 0 \qquad \text{(Lemma 6.1)},$$
$$\mathcal{P}(M) \to e_* \text{ as } M \to +\infty \text{ (Theorem 5.8)}.$$

These two information, together with the assumption $\mathcal{A} \neq (0, e_*)$, contradict the continuity of $\mathcal{P}$ in $(0, +\infty)$ and complete the proof.   □

8 Model Cases

Here, we take a closer look at the four model cases referred to in the Introduction. We recall that large-mass asymptotic results depend on two quantities: e_*, which is the smallest among the global minimum points of $R(s) = Q(s)/s$, if any, and $+\infty$ otherwise; and the sign of S. Uniqueness also depends on the sign of $Q''(s)$ for $s \in (0, e_*)$.

Throughout the section, $N = 1$ and u_M is a minimizer of E in $\mathcal{D}_M$. Any Q (and R) in this section obviously satisfies (5.1), (5.33), (7.2), (5.9) if $-S > 0$ and $D = 0$, and (5.34)–(5.35) if $-S = 0$, $B \leq 0$ and $D = 0$. Therefore, for the macroscopic shape we will only need to check whether $e_* < +\infty$ or not (cf. Theorem 5.8 and Theorem 5.12), and for uniqueness we will only need to discuss convexity of Q in $(0, e_*)$ (cf. Theorem 4.13).

Proposition 8.1 (model Q_a) *Let*

$$Q(s) = Q_a(s)$$
$$= \begin{cases} As^{1-m} - Bs^{1-n} - S & \text{for } s > 0, \quad A > 0, \ B \in \mathbb{R}, \ S \in \mathbb{R}, \ 1 < n < m < 3, \\ 0 & \text{for } s \leq 0. \end{cases}$$
(8.1)

(i) Uniqueness. u_M is:

– *unique for any $M \in (0, +\infty)$ if $B \leq 0$ or if $B > 0$ and $-S \leq 0$ or if $-S > 0$ and $B \geq c_1(A, S)$, where*

$$c_1(A, S) := (m-1) \left(\frac{A}{n-1} \right)^{\frac{n-1}{m-1}} \left(\frac{-S}{m-n} \right)^{\frac{m-n}{m-1}};$$
(8.2)

– *not unique for at least one $M \in (0, +\infty)$ if $-S > 0$ and $c_2(A, S) \leq B < c_1(A, S)$, where*

$$c_2(A, S) := \frac{m-1}{n} \left(\frac{Am}{n-1} \right)^{\frac{n-1}{m-1}} \left(\frac{-S}{m-n} \right)^{\frac{m-n}{m-1}}.$$

(ii) Macroscopic behavior. For $M \gg 1$, u_M is:

– *droplet-shaped (Theorem 5.8) if $-S > 0$ and $B < c_1(A, S)$;*
– *pancake-shaped (Theorem 5.8) if $-S < 0$, or if $-S = 0$ and $B > 0$, or if $-S > 0$ and $B \geq c_1(A, S)$;*
– *transition-shaped (Theorem 5.12) if $S = 0$ and $B \leq 0$.*

Uniqueness of minimizers remains open if $-S > 0$ and $0 < B < c_2(A, S)$. Figure 5 summarizes the results.

Proof We have

$$R(+\infty) = 0, \qquad G(s) := -s^2 R'(s) = Ams^{1-m} - Bns^{1-n} - S \quad \text{for } s > 0,$$

hence $G(0) = +\infty$ and $G(+\infty) = -S$. Simple computations show that G' has no zeroes if $B \leq 0$. In this case G is monotone decreasing in $(0, +\infty)$, which means that G, whence R', never changes sign if $-S \geq 0$, whereas it changes sign once if $-S < 0$. If instead $B > 0$, G' changes sign once, with its zero located at $g = \left(\frac{Am(m-1)}{Bn(n-1)} \right)^{\frac{1}{m-n}}$, and

$$g^{m-1} G(g) = Am - Bng^{m-n} - Sg^{m-1} = Am\frac{n-m}{n-1} - S \left(\frac{Am(m-1)}{Bn(n-1)} \right)^{\frac{m-1}{m-n}}.$$

Hence, if $B > 0$,

$$G(g) < 0 \quad \Leftrightarrow \quad -S \leq 0 \text{ or } -S > 0 \text{ and } -S \left(\frac{Am(m-1)}{Bn(n-1)} \right)^{\frac{m-1}{m-n}} < Am\frac{m-n}{n-1},$$

and the latter inequality is equivalent to $B > c_2(A, S)$. Summarizing:

$$R' \begin{cases} \text{changes sign once} & \text{if } -S < 0 \text{ or if } -S = 0 \text{ and } B > 0 \\ \leq 0 & \text{if } -S \geq 0 \text{ and } B \leq c_2(A, S) \\ \text{changes sign twice} & \text{if } -S > 0 \text{ and } B > c_2(A, S). \end{cases}$$

If R' never changes sign, then $e_* = +\infty$. If R' changes sign once, then $e_* < +\infty$ and $R(e_*) < 0 = R(+\infty)$. If R' changes sign twice, then

$$Q > 0 \text{ in } (0, +\infty) \quad \Longleftrightarrow \quad 0 < B < c_1(A, S) = nm^{\frac{1-n}{m-1}} c_2(A, S),$$

with $c_1(A, S)$ as in (8.2). Note that $c_1(A, S) > c_2(A, S)$ since $n < m$. Therefore, recalling Remarks 4.8 and 4.9, we deduce that $e_* < +\infty$ in the cases listed in the pancake case of Proposition 8.1, and $e_* = +\infty$ otherwise. Distinguishing between droplets and transition is obvious.

As to (i), we need to discuss the sign of $Q''(s) = Am(m-1)s^{-m-1} - Bn(n-1)s^{-n-1}$ in $(0, e_*)$. We distinguish various cases.

– If $B \leq 0$, then $Q'' \geq 0$ in $(0, +\infty)$.
– If $B > 0$ and $-S \leq 0$, then $e_* < +\infty$ and Q'' has one zero. Since $Q''(s) \to +\infty$ as $s \to 0^+$ and

$$Q''(e_*) = e_*^{-2}(m-1) \left(Ame_*^{1-m} - Bn\frac{n-1}{m-1} e_*^{1-n} \right)$$
$$\overset{R'(e_*)=0}{=} e_*^{-2}(m-1) \left(Bn\frac{m-n}{m-1} e_*^{1-n} + S \right) \geq 0,$$

$Q'' > 0$ in $(0, e_*)$.

- If $-S > 0$ and $B \geq c_1(A, S)$, we have seen that $e_* < +\infty$ and that R' changes sign twice. Since $G'(s) = -sQ''(s)$, the unique zero of Q'' is located at the above defined point $s_2 = g$. Above, we have also seen that $G(g) < 0$, hence $R'(g) > 0$. Since R' changes sign twice and recalling the definition of e_*, this implies that $e_* \leq g$. Therefore, $Q'' \geq 0$ in $(0, e_*)$.
- If $-S > 0$ and $c_2(A, S) \leq B < c_1(A, S)$, we have seen that $e_* = +\infty$ and that R' has at least one zero. Hence, $\mathcal{A} \neq (0, +\infty)$ and Theorem 7.4 applies.

$\square$

Proposition 8.2 (model Q_b) *Assume* (1.6b) *and let*

$$
Q(s) = Q_b(s) = \begin{cases} \dfrac{A|B|s}{|B|s^m + As^n} - S & \text{for } s > 0, \quad B < 0, \quad 1 < m < n, \ m < 3, \\ 0 & \text{for } s \leq 0. \end{cases}
$$

$$(8.3)$$

(i) *Uniqueness.* u_M *is unique for any* $M \in (0, +\infty)$.
(ii) *Macroscopic behavior. For* $M \gg 1$, u_M *is droplet-shaped if* $-S > 0$, *pancake-shaped if* $-S < 0$, *and transition-shaped if* $S = 0$ *(cf. Theorems* 5.8 *and* 5.12*).*

Proof Simple computations show that $Q' < 0$ in $(0, +\infty)$ and that $Q'' \geq 0$ in $(0, +\infty)$ if and only if (1.6b) holds. Hence (i) follows. As to (ii), we have

$$
G(s) = -s^2 R'(s) = \frac{A|B|s(|B|ms^m + Ans^n)}{(|B|s^m + As^n)^2} - S,
$$

which has no zeroes if $-S \geq 0$ (hence $e_* = +\infty$) and at least one zero if $-S < 0$. In the latter case, since $-S < 0$, Q is not always positive, hence, by Remark 4.9, $e_* < +\infty$. $\square$

Proposition 8.3 (model $Q_{a,g}$) *With* Q_a *as in* (8.1)*, let*

$$
Q(s) = Q_{a,g}(s) = Q_a(s) + \tfrac{1}{2}Ds^2 \quad \text{for } s > 0, \quad D > 0.
$$

(i) *Uniqueness.* u_M *is unique for any* $M \in (0, +\infty)$ *if* $-S \leq 0$ *or*

$$
B \leq \frac{m+1}{n(n-1)} \left(\frac{Am(m-1)}{n+1} \right)^{\frac{n+1}{m+1}} \left(\frac{D}{m-n} \right)^{\frac{m-n}{m+1}} =: c_3(A, D).
$$

(ii) *Macroscopic behavior. For* $M \gg 1$, u_M *is pancake-shaped (Theorem* 5.8*).*

In the case $-S > 0$ and $B > c_3(A, D)$, we do not know how to ascertain the sign of Q'' in $(0, e_*)$.

Proof Since $R(0) = R(+\infty) = +\infty$, we always have $e_* < +\infty$ [cf. (4.27)], hence (ii) is trivial. As for uniqueness, we discuss the sign of $Q''(s) = Am(m-1)s^{-m-1} - Bn(n-1)s^{-n-1} + D$ in $(0, e_*)$. If $B \leq 0$, then $Q'' > 0$ in $(0, +\infty)$. Hence, we only

consider $B > 0$. It follows from the study of Q''' that $s_3 = \left(\frac{A(m+1)m(m-1)}{B(n+1)n(n-1)} \right)^{\frac{1}{m-n}}$ is the unique point of minimum of Q''. After straightforward computations we obtain that $Q''(s_3) \geq 0$ if and only if $B \leq c_3(A, D)$. In this case $Q'' \geq 0$ in $(0, +\infty)$. It remains to consider $B > c_3(A, D)$. If $-S \leq 0$, we take advantage of the results seen for Q_a. Indeed, in this case R'_a has a unique zero, located at e_a, with $R'_a \geq 0$ in $[e_a, +\infty)$ and $Q''_a \geq 0$ in $(0, e_a)$. Thus, $R' = R'_a + \frac{D}{2} > 0$ in $[e_a, +\infty)$. Since by definition $R'(e_*) = 0$, this implies that $e_* < e_a$. Since $Q'' = Q''_a + D \geq D > 0$ in $(0, e_a)$, we conclude that $Q'' > 0$ in $(0, e_*)$. □

Proposition 8.4 (model $Q_{b,g}$) *Assume* (1.6b). *With Q_b as in* (8.3), *let*

$$Q(s) = Q_{b,g}(s) = Q_b(s) + \tfrac{1}{2}Ds^2 \quad \text{for } s > 0, \quad D > 0.$$

(i) Uniqueness. u_M is unique for any $M \in (0, +\infty)$.
(ii) Macroscopic behavior. For $M \gg 1$, u_M is pancake-shaped (Theorem 5.8).

Proof Since $D > 0$, we have $R(0) = R(+\infty) = +\infty$, thus $e_* < +\infty$. Under assumption (1.6b), $Q''_b \geq 0$ in $(0, +\infty)$, thus $Q'' = Q''_b + D > 0$ in $(0, +\infty)$. □

Appendix

Lemma A.1 *Let u be a nonnegative measurable function. Then, the set $\mathcal{C} = \{\delta > 0 : |\{u = \delta\}| > 0\}$ is countable.*

Proof Let $\mu : [0, +\infty) \to [0, +\infty)$ be the distribution function of u, that is, $\mu(\delta) = |\{u > \delta\}|$. By definition, δ is a discontinuity point of μ if and only if $\delta \in \mathcal{C}$. Since μ is non-increasing (Kesavan 2006, Remark 1.1.1) and since a monotone function has at most a countable set of discontinuity points (Rudin 1976, Theorem 4.30), $\mathcal{C}$ is at most countable. □

Lemma A.2 *Let u be a nonnegative measurable function on $\mathbb{R}^N$. Then, $\chi_{\{u>\delta\}} \to \chi_{\{u>0\}}$ almost everywhere in $\mathbb{R}^N$ as $\delta \to 0^+$.*

Proof First we note that

$$\chi_{\{u>0\}} - \chi_{\{u>\delta\}} = \chi_{\{0<u\leq\delta\}}.$$

Therefore, the result is equivalent to proving that $\chi_{\{0<u\leq\delta\}} \to 0$ a.e. as $\delta \to 0$. Since $\{0 < u \leq \delta\}$ monotonically shrinks as δ decreases, we have

$$\lim_{\delta \to 0} |\{0 < u \leq \delta\}| = |\omega|, \quad \text{with } \omega = \bigcap_{\delta > 0} \{0 < u \leq \delta\}.$$

In order to show that $|\omega| = 0$, we note that, since $0 < u \leq \delta$ a.e. in ω for any $\delta > 0$,

$$\int_\omega u \leq \delta|\omega| \quad \forall \delta > 0 \quad \Rightarrow \quad \int_\omega u = 0.$$

Since $u > 0$ a.e. in ω, we conclude that $|\omega| = 0$. $\square$

We conclude with an ODE lemma.

Lemma A.3 *Let $D := (0, +\infty)$ and let f, $f_k : D \to \mathbb{R}$ be continuous functions such that $f_k \to f$ in $C_{loc}(D)$. Let $u_k \in C^2((-r, r)) \cap C^0([-r, r])$ be a solution of*

$$\begin{cases} -u_k'' = f_k(u_k) & \text{in } \{u_k > 0\} = (-r_k, r_k) \\ u_k(0) = u_{0k}, \ u_k'(0) = 0, \ u_k(\pm r_k) = 0, \end{cases} \quad \text{(A.1)}$$

non-increasing in $(0, r_k)$, and let $u \in C^2((-r, r)) \cap C^0([-r, r])$ be the unique solution of

$$\begin{cases} -u'' = f(u) & \text{in } \{u > 0\} = (-r, r) \\ u(0) = u_0, \ u'(0) = 0, \ u(\pm r) = 0 \ \text{if } r < +\infty. \end{cases}$$

Let $F_k(s) = \int_s^{u_{0k}} f_k(t)\mathrm{d}t$ be such that

$$\lim_{(k,s) \to (+\infty, 0^+)} F_k(s) = +\infty. \quad \text{(A.2)}$$

If $u_{0k} \to u_0 \in D$ as $k \to +\infty$, then $r_k \to r$ and $u_k \to u$ in $C_{loc}^2((-r, r))$.

Proof Thanks to Theorem 8.39 and Theorem 8.40 of Kelley and Peterson (2010), $u_k \to u$ in $C_{loc}^2((-r, r))$. It follows that for all $x \in (-r, r)$, $u_k(x)$ is positive for k sufficiently large; hence $r \leq \liminf_{k \to +\infty} r_k$. It remains to prove that $r \geq \limsup_{k \to +\infty} r_k =: R$. If $r = +\infty$, nothing is to be proved. If $r < +\infty$, assume by contradiction that $r < R$. Multiplying the equation in (A.1) by $-u_k'$, integrating in $(0, x)$ and using the initial conditions, we have

$$(u_k'(x))^2 = 2F_k(u_k(x)). \quad \text{(A.3)}$$

Since u is continuous at $x = r$ and $u(r) = 0$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $u(r - \delta) < \varepsilon/2$. By locally uniform convergence and recalling that u_k is non-increasing, $u_k < \varepsilon$ in $(r - \delta, R - \delta)$ for all k sufficiently large. Fix $x \in (r - \delta, R - \delta)$. Since u_k is non-increasing, it follows from (A.3) that $u_k'(x) = -\sqrt{2}\sqrt{F_k(u_k(x))}$. Hence, (A.2) implies that for every $M > 0$ we can choose ε sufficiently small and k sufficiently large such that

$$u_k'(y) < -M \quad \forall y \in (r - \delta, x), \quad \text{(A.4)}$$

whence we deduce that for every k sufficiently large

$$u_k(x) = \int_{r-\delta}^x u_k'(y)\mathrm{d}y + u_k(r - \delta) \overset{(A.4)}{<} -M(x - r + \delta) + \varepsilon.$$

Choosing M sufficiently large and recalling that u_k is nonnegative, we obtain a contradiction. $\qquad\square$

Data availability All data generated or analyzed during this study are included in this published article.

References

Alt, H.W., Caffarelli, L.A.: Existence and regularity for a minimum problem with free boundary. J. Reine Angew. Math. **325**, 105–144 (1981)

Alt, H.W., Phillips, D.: A free boundary problem for semilinear elliptic equations. J. Reine Angew. Math. **368**, 63–107 (1986)

Ansini, L., Giacomelli, L.: Shear-thinning liquid films: macroscopic and asymptotic behaviour by quasi-self-similar solutions. Nonlinearity **15**(6), 2147–2164 (2002)

Becker, J., Grün, G., Seemann, R., Mantz, H., Jacobs, K., Mecke, K., Blossey, R.: Complex dewetting scenarios captured by thin-film models. Nat. Mater. **2**(1), 59–63 (2003)

Beretta, E., Bertsch, M., Dal Passo, R.: Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation. Arch. Rational Mech. Anal. **129**(2), 175–200 (1995)

Bernis, F.: Finite speed of propagation and continuity of the interface for thin viscous flows. Adv. Differ. Equ. **1**(3), 337–368 (1996)

Bernis, F.: Finite speed of propagation for thin viscous flows when $2 \leq n < 3$. C. R. Acad. Sci. Paris Sér. I Math. **322**(12), 1169–1174 (1996)

Bernis, F., Friedman, A.: Higher order nonlinear degenerate parabolic equations. J. Differ. Equ. **83**(1), 179–206 (1990)

Bertozzi, A.L., Grün, G., Witelski, T.P.: Dewetting films: bifurcations and concentrations. Nonlinearity **14**(6), 1569–1592 (2001)

Bertozzi, A.L., Pugh, M.: The lubrication approximation for thin viscous films: the moving contact line with a "porous media" cut-off of van der Waals interactions. Nonlinearity **7**(6), 1535–1564 (1994)

Bertozzi, A.L., Pugh, M.: The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions. Commun. Pure Appl. Math. **49**(2), 85–123 (1996)

Bertsch, M., Dal Passo, R., Davis, S.H., Giacomelli, L.: Effective and microscopic contact angles in thin film dynamics. Eur. J. Appl. Math. **11**(2), 181–201 (2000)

Bertsch, M., Dal Passo, R., Garcke, H., Grün, G.: The thin viscous flow equation in higher space dimensions. Adv. Differ. Equ. **3**(3), 417–440 (1998)

Bertsch, M., Giacomelli, L., Karali, G.: Thin-film equations with "partial wetting" energy: existence of weak solutions. Phys. D **209**(1–4), 17–27 (2005)

Bonn, D., Eggers, J., Indekeu, J., Meunier, J.: Wetting and spreading. Rev. Mod. Phys. **81**(2), 739–805 (2009)

Burchard, A., Chugunova, M., Stephens, B.K.: Convergence to equilibrium for a thin-film equation on a cylindrical surface. Commun. Partial Differ. Equ. **37**(4), 585–609 (2012)

Carlen, E.A., Loss, M.: Sharp constant in Nash's inequality. Int. Math. Res. Not. **7**, 213–215 (1993)

Casado-Díaz, J., Murat, F.: Semilinear problems with right-hand sides singular at $u = 0$ which change sign. Ann. Inst. H. Poincaré Anal. Non Linéaire **38**(3), 877–909 (2021)

Chen, X., Jiang, H.: Singular limit of an energy minimizer arising from dewetting thin film model with van der Waal, Born repulsion and surface tension forces. Calc. Var. Partial Differ. Equ. **44**(1–2), 221–246 (2012)

Chen, X., Jiang, H., Liu, G.: Boundary spike of the singular limit of an energy minimizing problem. Discrete Contin. Dyn. Syst. **40**(6), 3253–3290 (2020)

Cheung, K.-L., Chou, K.-S.: On the stability of single and multiple droplets for equations of thin film type. Nonlinearity **23**(12), 3003–3028 (2010)

Cheung, K.-L., Chou, K.-S.: Energy stability of droplets and dry spots in a thin film model of hanging drops. Z. Angew. Math. Phys. **68**(5), 104, 21 (2017)

Chiricotto, M., Giacomelli, L.: Scaling laws for droplets spreading under contact-line friction. Commun. Math. Sci. **11**(2), 361–383 (2013)

Chiricotto, M., Giacomelli, L.: Weak solutions to thin-film equations with contact-line friction. Interfaces Free Bound. **19**(2), 243–271 (2017)

Chou, K.-S., Zhang, Z.: A mountain pass scenario and heteroclinic orbits for thin-film type equations. Nonlinearity **25**(12), 3343–3388 (2012)

Cox, R.: The dynamics of the spreading of liquids on a solid surface. Part 1. Viscous flow. J. Fluid Mech. **168**, 169–194 (1986)

Crandall, M.G., Rabinowitz, P.H., Tartar, L.: On a Dirichlet problem with a singular nonlinearity. Commun. Partial Differ. Equ. **2**(2), 193–222 (1977)

Dal Passo, R., Garcke, H., Grün, G.: On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions. SIAM J. Math. Anal. **29**(2), 321–342 (1998)

Dal Passo, R., Giacomelli, L., Grün, G.: A waiting time phenomenon for thin film equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **30**(2), 437–463 (2001)

Dal Passo, R., Giacomelli, L., Shishkov, A.: The thin film equation with nonlinear diffusion. Commun. Partial Differ. Equ. **26**(9–10), 1509–1557 (2001)

de Gennes, P.G.: Wetting: statics and dynamics. Rev. Mod. Phys. **57**(3, part 1), 827–863 (1985)

De Silva, D., Savin, O.: The Alt–Phillips functional for negative powers. Preprint arXiv:2203.07123v1 (2022)

De Silva, D., Savin, O.: Uniform density estimates and Γ-convergence for the Alt–Phillips functional of negative powers. Preprint arXiv:2205.08436v1 (2022)

Delgadino, M.G., Mellet, A.: On the relationship between the thin film equation and Tanner's law. Commun. Pure Appl. Math. **74**(3), 507–543 (2021)

Durastanti, R., Giacomelli, L.: Thin-film equations with mildly singular potentials: an alternative solution to the contact-line paradox. Preprint arXiv:2207.00700v1 (2022)

Durastanti, R., Oliva, F.: Comparison principle for elliptic equations with mixed singular nonlinearities. Potential Anal **57**, 83–100 (2022)

Dussan V., E., Davis, S.: On the motion of a fluid–fluid interface along a solid surface. J. Fluid Mech. **65**(1), 71–95 (1974)

Eggers, J., Stone, H.: Characteristic lengths at moving contact lines for a perfectly wetting fluid: the influence of speed on the dynamic contact angle. J. Fluid Mech. **505**, 309–321 (2004)

Feldman, W.M., Kim, I.C.: Liquid drops on a rough surface. Commun. Pure Appl. Math. **71**(12), 2429–2499 (2018)

Ferone, A., Volpicelli, R.: Convex rearrangement: equality cases in the Pólya–Szegö inequality. Calc. Var. Partial Differ. Equ. **21**(3), 259–272 (2004)

Finn, R.: Equilibrium Capillary Surfaces. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 284. Springer-Verlag, New York (1986)

Fischer, J.: Upper bounds on waiting times for the thin-film equation: the case of weak slippage. Arch. Ration. Mech. Anal. **211**(3), 771–818 (2014)

Fischer, J.: Behaviour of free boundaries in thin-film flow: the regime of strong slippage and the regime of very weak slippage. Ann. Inst. H. Poincaré Anal. Non Linéaire **33**(5), 1301–1327 (2016)

Flitton, J.C., King, J.R.: Surface-tension-driven dewetting of Newtonian and power-law fluids. J. Eng. Math. **50**(2–3), 241–266 (2004)

Giacomelli, L., Gnann, M.V., Knüpfer, H., Otto, F.: Well-posedness for the Navier-slip thin-film equation in the case of complete wetting. J. Differ. Equ. **257**(1), 15–81 (2014)

Giacomelli, L., Gnann, M.V., Otto, F.: Rigorous asymptotics of traveling-wave solutions to the thin-film equation and Tanner's law. Nonlinearity **29**(9), 2497–2536 (2016)

Giacomelli, L., Grün, G.: Lower bounds on waiting times for degenerate parabolic equations and systems. Interfaces Free Bound. **8**(1), 111–129 (2006)

Giacomelli, L., Knüpfer, H., Otto, F.: Smooth zero-contact-angle solutions to a thin-film equation around the steady state. J. Differ. Equ. **245**(6), 1454–1506 (2008)

Giacomelli, L., Otto, F.: Variational formulation for the lubrication approximation of the Hele–Shaw flow. Calc. Var. Partial Differ. Equ. **13**(3), 377–403 (2001)

Giacomelli, L., Otto, F.: Droplet spreading: intermediate scaling law by PDE methods. Commun. Pure Appl. Math. **55**(2), 217–254 (2002)

Giacomelli, L., Otto, F.: Rigorous lubrication approximation. Interfaces Free Bound. **5**(4), 483–529 (2003)

Giacomoni, J., Saoudi, K.: Multiplicity of positive solutions for a singular and critical problem. Nonlinear Anal. **71**(9), 4060–4077 (2009)

Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Classics in Mathematics, Springer-Verlag, Berlin (2001)

Glasner, K., Otto, F., Rump, T., Slepčev, D.: Ostwald ripening of droplets: the role of migration. Eur. J. Appl. Math. **20**(1), 1–67 (2009)

Glasner, K., Witelski, T.: Coarsening dynamics of dewetting films. Phys. Rev. E Stat. Nonlinear Soft Matter Phys. **67**(1 2), 163021–1630212 (2003)

Gnann, M.V.: Well-posedness and self-similar asymptotics for a thin-film equation. SIAM J. Math. Anal. **47**(4), 2868–2902 (2015)

Gnann, M.V.: On the regularity for the Navier-slip thin-film equation in the perfect wetting regime. Arch. Ration. Mech. Anal. **222**(3), 1285–1337 (2016)

Gnann, M.V., Petrache, M.: The Navier-slip thin-film equation for 3D fluid films: existence and uniqueness. J. Differ. Equ. **265**(11), 5832–5958 (2018)

Godoy, T., Guerin, A.: Existence of nonnegative solutions to singular elliptic problems, a variational approach. Discrete Contin. Dyn. Syst. **38**(3), 1505–1525 (2018)

Grün, G.: Degenerate parabolic differential equations of fourth order and a plasticity model with non-local hardening. Z. Anal. Anwendungen **14**(3), 541–574 (1995)

Grün, G.: Droplet spreading under weak slippage—existence for the Cauchy problem. Commun. Partial Differ. Equ. **29**(11–12), 1697–1744 (2004)

Grün, G., Rumpf, M.: Simulation of singularities and instabilities arising in thin film flow. Eur. J. Appl. Math. **12**(3), 293–320 (2001)

Haley, P., Miksis, M.: The effect of the contact line on droplet spreading. J. Fluid Mech. **223**, 57–81 (1991)

Hernández, J., Mancebo, F.J., Vega, J.M.: Positive solutions for singular nonlinear elliptic equations. Proc. R. Soc. Edinb. Sect. A **137**(1), 41–62 (2007)

Hocking, L.: The spreading of a thin drop by gravity and capillarity. Q. J. Mech. Appl. Math. **36**(1), 55–69 (1983)

Hocking, L.: Rival contact-angle models and the spreading of drops. J. Fluid Mech. **239**, 671–681 (1992)

Huh, C., Scriven, L.: Hydrodynamic model of steady movement of a solid/liquid/fluid contact line. J. Colloid Interface Sci. **35**(1), 85–101 (1971)

Jiang, H.: Energy minimizers of a thin film equation with Born repulsion force. Commun. Pure Appl. Anal. **10**(2), 803–815 (2011)

Jiang, H., Ni, W.-M.: On steady states of van der Waals force driven thin film equations. Eur. J. Appl. Math. **18**(2), 153–180 (2007)

Kang, D., Nadim, A., Chugunova, M.: Dynamics and equilibria of thin viscous coating films on a rotating sphere. J. Fluid Mech. **791**, 495–518 (2016)

Kelley, W.G., Peterson, A.C.: The Theory of Differential Equations. Universitext, 2nd edn. Springer, New York (2010)

Kesavan, S.: Symmetrization & Applications. Series in Analysis, vol. 31. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2006)

Knüpfer, H.: Well-posedness for the Navier slip thin-film equation in the case of partial wetting. Commun. Pure Appl. Math. **64**(9), 1263–1296 (2011)

Knüpfer, H.: Well-posedness for a class of thin-film equations with general mobility in the regime of partial wetting. Arch. Ration. Mech. Anal. **218**(2), 1083–1130 (2015)

Knüpfer, H., Masmoudi, N.: Well-posedness and uniform bounds for a nonlocal third order evolution operator on an infinite wedge. Commun. Math. Phys. **320**(2), 395–424 (2013)

Knüpfer, H., Masmoudi, N.: Darcy's flow with prescribed contact angle: well-posedness and lubrication approximation. Arch. Ration. Mech. Anal. **218**(2), 589–646 (2015)

Laugesen, R.S., Pugh, M.C.: Linear stability of steady states for thin film and Cahn–Hilliard type equations. Arch. Ration. Mech. Anal. **154**(1), 3–51 (2000)

Laugesen, R.S., Pugh, M.C.: Properties of steady states for thin film equations. Eur. J. Appl. Math. **11**(3), 293–351 (2000)

Laugesen, R.S., Pugh, M.C.: Energy levels of steady states for thin-film-type equations. J. Differ. Equ. **182**(2), 377–415 (2002)

Laugesen, R.S., Pugh, M.C.: Heteroclinic orbits, mobility parameters and stability for thin film type equations. Electron. J. Differ. Equ. **95**, 1–29 (2002)

Lazer, A.C., McKenna, P.J.: On a singular nonlinear elliptic boundary-value problem. Proc. Am. Math. Soc. **111**(3), 721–730 (1991)

Liu, W., Witelski, T.P.: Steady states of thin film droplets on chemically heterogeneous substrates. IMA J. Appl. Math. **85**(6), 980–1020 (2020)

Maggi, F.: Sets of Finite Perimeter and Geometric Variational Problems. Cambridge Studies in Advanced Mathematics, vol. 135. Cambridge University Press, Cambridge (2012)

Mellet, A.: The thin film equation with non-zero contact angle: a singular perturbation approach. Commun. Partial Differ. Equ. **40**(1), 1–39 (2015)

Minkov, E., Novick-Cohen, A.: Droplet profiles under the influence of van der Waals forces. Eur. J. Appl. Math. **12**(3), 367–393 (2001)

Minkov, E., Novick-Cohen, A.: Errata: "Droplet profiles under the influence of van der Waals forces" [European J. Appl. Math **12** (2001), no. 3, 367–393; mr1936203]. Eur. J. Appl. Math. **17**(1), 128 (2006)

Nicolaou, Z.G.: Stability and instability of axisymmetric droplets in thermocapillary-driven thin films. Nonlinearity **31**(3), 1009–1044 (2018)

Novick-Cohen, A.: On a minimization problem arising in wetting. SIAM J. Appl. Math. **52**(3), 593–613 (1992)

Novick-Cohen, A.: A singular minimization problem for droplet profiles. Eur. J. Appl. Math. **4**(4), 399–418 (1993)

Novick-Cohen, A., Shishkov, A.: The thin film equation with backwards second order diffusion. Interfaces Free Bound. **12**(4), 463–496 (2010)

Oliva, F., Petitta, F.: Finite and infinite energy solutions of singular elliptic problems: existence and uniqueness. J. Differ. Equ. **264**(1), 311–340 (2018)

Oron, A., Davis, S., Bankoff, S.: Long-scale evolution of thin liquid films. Rev. Mod. Phys. **69**(3), 931–980 (1997)

Otto, F.: Lubrication approximation with prescribed nonzero contact angle. Commun. Partial Differ. Equ. **23**(11–12), 2077–2164 (1998)

Otto, F., Rump, T., Slepčev, D.: Coarsening rates for a droplet model: rigorous upper bounds. SIAM J. Math. Anal. **38**(2), 503–529 (2006)

Ren, W., Hu, D., E, W.: Continuum models for the contact line problem. Phys. Fluids **22**(10), 102103 (2010)

Rudin, W.: Principles of Mathematical Analysis. International Series in Pure and Applied Mathematics, 3rd edn. McGraw-Hill Book Co., New York-Auckland-Düsseldorf (1976)

Seis, C.: The thin-film equation close to self-similarity. Anal. PDE **11**(5), 1303–1342 (2018)

Snoeijer, J., Andreotti, B.: Moving contact lines: scales, regimes, and dynamical transitions. Annu. Rev. Fluid Mech. **45**, 269–292 (2013)

Stampacchia, G.: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. Ann. Inst. Fourier (Grenoble) **15**(fasc. 1), 189–258 (1965)

Tanner, L.: The spreading of silicone oil drops on horizontal surfaces. J. Phys. D Appl. Phys. **12**(9), 1473–1484 (1979)

Voinov, O.: Hydrodynamics of wetting. Fluid Dyn. **11**(5), 714–721 (1976)

Witelski, T.P.: Nonlinear dynamics of dewetting thin films. AIMS Math. **5**(5), 4229–4259 (2020)

Yijing, S., Duanzhi, Z.: The role of the power 3 for elliptic equations with negative exponents. Calc. Var. Partial Differ. Equ. **49**(3–4), 909–922 (2014)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.