

Eigenvalues upper bounds for the magnetic Schrödinger operator

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Abstract

We study the eigenvalues $\lambda_k(H_{A,q})$ of the magnetic Schrödinger operator $H_{A,q}$ associated with a magnetic potential A and a scalar potential q , on a compact Riemannian manifold M , with Neumann boundary conditions if $\partial M \neq \emptyset$. We obtain various bounds on $\lambda_k(H_{A,q})$. Besides the dimension and the volume of the manifold, the geometric quantity which plays an important role in these estimates is the first eigenvalue $\lambda''_{1,1}(M)$ of the Hodge-de Rham Laplacian acting on co-exact 1-forms. In the 2-dimensional case, $\lambda''_{1,1}(M)$ is nothing but the first positive eigenvalue of the Laplacian acting on functions. As for the dependence of the bounds on the potentials, it brings into play the mean value of the scalar potential q , the L^2 -norm of the magnetic field $B = dA$, and the distance, taken in L^2 , between the harmonic component of A and the subspace of all closed 1-forms whose cohomology class is integral (that is, having integral flux around any loop). In particular, this distance is zero when the first cohomology group $H^1(M, \mathbf{R})$ is trivial.

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1 Introduction

Let (M, g) be a compact Riemannian manifold with smooth boundary ∂M , if non empty. Consider the trivial complex line bundle $M \times \mathbf{C}$ over M ; its space of sections can be identified with $C^\infty(M, \mathbf{C})$, the space of smooth complex valued functions on M . Given a smooth real 1-form A on M we define a connection ∇^A on $C^\infty(M, \mathbf{C})$ as follows:

$$\nabla_X^A u = \nabla_X u - iA(X)u \quad (1)$$

for all vector fields X on M and for all $u \in C^\infty(M, \mathbf{C})$ (here ∇ denotes the Levi-Civita connection of (M, g)). The operator

$$\Delta_A = (\nabla^A)^* \nabla^A \quad (2)$$

is called the *magnetic Laplacian* associated to the magnetic potential A , and the smooth two form

$$B = dA$$

*Our colleague and friend Ahmad El Soufi passed away on December 29, 2016

is the associated *magnetic field*. In this paper, we are interested in magnetic Schrödinger operators of the form

$$H_{A,q} = \Delta_A + q$$

where q is a real valued continuous function on M . If $A = 0$, Δ_A is simply the usual Laplacian Δ on M . Note that we have

$$\Delta_A u = \Delta u + 2i\langle A, du \rangle + (|A|^2 + i \operatorname{div} A) u \quad (3)$$

where $\operatorname{div} A$ (often denoted also δA) is the co-differential of A .

If the boundary of M is non empty, we will consider Neumann magnetic conditions, that is:

$$\nabla_N^A u = 0 \quad \text{on} \quad \partial M, \quad (4)$$

where N denotes the inner unit normal. Then, it is well-known that $H_{A,q}$ is self-adjoint, and admits a discrete spectrum

$$\lambda_1(H_{A,q}) \leq \lambda_2(H_{A,q}) \leq \dots \rightarrow \infty.$$

Estimates of eigenvalues of such operators have received a great attention in the last decades, especially in the case where the underlying manifold is a bounded Euclidean domain with Dirichlet boundary conditions (see for instance [1, 4, 15, 27]) or with Neumann boundary conditions (see [3, 5, 9, 16, 17, 24, 25, 31]).

In this paper, we first give *upper bounds* for the spectrum of $H_{A,q}$ in terms of the harmonic part of the potential A , the magnetic field B , the integral $\int_M q$ and the geometry of M . These estimates are compatible with the Weyl law, and they are deduced from the fact that we have the relation (see (16) for a proof)

$$\lambda_k(H_{A,q}) \leq \lambda_k(H_{0,|A|^2+q}) = \lambda_k(\Delta + |A|^2 + q) \quad (5)$$

where $|A|$ denote the pointwise norm of A . We will also focus on the two first eigenvalues $\lambda_1(H_{A,q})$ and $\lambda_2(H_{A,q})$, where we can get more precise results.

In Theorems 2, 5 and 6, we observe that the geometry of the underling manifold (M, g) appears through the first nonzero eigenvalue $\lambda''_{1,1}$ of the Hodge-De Rham Laplacian Δ_{HR} acting on coexact 1-forms (with absolute condition when $\partial\Omega$ is not empty). This lead us to collect in sections 2.5 and 3 a lot of already known results where we have an explicit control of $\lambda''_{1,1}(M, g)$.

A last point, is that, when A is closed (i.e. zero magnetic field), we can establish a *sharp* upper bound for the first eigenvalue $\lambda_1(H_{A,q})$ in term of the distance of A to the an integral lattice of harmonic 1-forms (see Theorem 3) and we can discuss the equality case (see Theorems 3 and 4).

In the rest of the introduction we will recall some known facts and discuss the main results.

1.1 Preliminary facts and notation

First, we recall the absolute boundary conditions for a differential p -form ω . A form ω is said to be *tangential* if $i_N A = 0$ on ∂M , where N denote the exterior normal vector to the boundary; then, ω satisfies the *absolute boundary conditions* if ω and $d\omega$ are both tangential. We denote $\lambda_{1,p}$ the first eigenvalue of the Hodge Laplacian on p -forms (with absolute boundary conditions if ∂M non empty), and by $\lambda''_{1,p}$ (resp. $\lambda'_{1,p}$) the first eigenvalue when restricted to co-exact (resp. exact) p -forms. It follows that

$$\lambda_{1,p} \leq \min\{\lambda'_{1,p}, \lambda''_{1,p}\}$$

and as $\lambda''_{1,p} = \lambda'_{1,p+1}$ (by differentiating eigenfunctions) we see

$$\lambda''_{1,p} \geq \max\{\lambda_{1,p}, \lambda_{1,p+1}\}.$$

In particular,

$$\lambda''_{1,1} \geq \max\{\lambda_{1,1}, \lambda_{1,2}\}.$$

We recall now the variational definition of the spectrum. Let M be a compact manifold. If the boundary is non empty, we assume for $u \in C^\infty(M, \mathbf{C})$ the magnetic Neumann conditions, as in (4). Then one verifies that

$$\int_M (H_{A,q}u)\bar{u}v_g = \int_M (|\nabla^A u|^2 + q|u|^2)v_g,$$

and the associated quadratic form is then

$$Q_{A,q}(u) = \int_M (|\nabla^A u|^2 + q|u|^2)v_g.$$

We also introduce the Rayleigh quotient of a smooth function $u \neq 0$, defined by

$$R_{A,q}(u) = \frac{Q_{A,q}(u)}{\|u\|^2} \tag{6}$$

The spectrum of $H_{A,q}$ admits the usual variational characterization:

$$\lambda_1(H_{A,q}) = \min \left\{ R_{A,q}(u) \mid u \in C^1(M, \mathbf{C})/\{\mathbf{0}\} \right\} \tag{7}$$

and

$$\lambda_k(H_{A,q}) = \min_{E_k} \max \left\{ R_{A,q}(u) : u \in E_k/\{\mathbf{0}\} \right\} \tag{8}$$

where E_k runs through the set of all k -dimensional vector subspaces of $C^1(M, \mathbf{C})$.

The following proposition recalls some well-known facts. If c is a closed curve (a loop), the quantity

$$\Phi_c^A = \frac{1}{2\pi} \oint_c A \quad (9)$$

is called the *flux* of A across c . We will not specify the orientation of the loop, so that the flux will only be defined up to sign. This will not affect any of the statements, definitions or results which we will prove in this paper.

Proposition 1. *1. The spectrum of $H_{A,q}$ is equal to the spectrum of $H_{A+d\phi,q}$ for all smooth real valued functions ϕ ; in particular, when A is exact, the spectrum of $H_{A,q}$ reduces to that of the classical Schrödinger operator with potential q acting on functions (with Neumann boundary conditions if ∂M is not empty).*

2. Let A be 1-form on M . Then, there exists a smooth real valued function ϕ on M such that the 1-form $\tilde{A} = A + d\phi$ is co-closed and tangential, that is:

$$\delta\tilde{A} = 0, \quad i_N\tilde{A} = 0. \quad (10)$$

3. Set:

$$\text{Har}_1(M) = \left\{ h \in \Lambda^1(M) : dh = \delta h = 0 \text{ on } M, \quad i_N h = 0 \text{ on } \partial M \right\}.$$

Assume that the 1-form A is co-closed and tangential. Then A can be decomposed

$$A = \delta\psi + h, \quad (11)$$

where ψ is a smooth tangential 2-form and $h \in \text{Har}_1(M)$. Note that the vector space $\text{Har}_1(M)$ is isomorphic to the first de Rham absolute cohomology space $H^1(M, \mathbf{R})$.

4. We have $\lambda_1(H_{A,0}) \doteq \lambda_1(\Delta_A) = 0$ if and only if A is closed (i.e. $B = 0$) and the cohomology class of A is an integer (that is $\Phi_c^A \in \mathbf{Z}$ for any loop c in M).

Assertion (1) expresses the well-known *Gauge invariance* of the spectrum. Thanks to Assertion (2), in the study of the spectrum of the magnetic Laplacian, we can always assume that the potential A is co-closed and tangential.

Proof. 1. This comes from the fact that

$$\Delta_A e^{-i\phi} = e^{-i\phi} \Delta_{A+d\phi} \quad (12)$$

hence Δ_A and $\Delta_{A+d\phi}$ are unitarily equivalent.

2. Observe that the problem:

$$\begin{cases} \Delta\phi = -\delta A & \text{on } M, \\ \frac{\partial\phi}{\partial N} = -A(N) & \text{on } \partial M \end{cases}$$

has a unique solution (modulo a multiplicative constant). It is immediate to verify that $\tilde{A} = A + d\phi$ is indeed co-closed and tangential.

3. We apply the Hodge decomposition to the 1-form A (see [30], Thm. 2.4.2), and get:

$$A = df + \delta\psi + h, \quad (13)$$

where f is a function which is zero on the boundary, ψ is a tangential 2-form and h is a 1-form satisfying $dh = \delta h = 0$ (in particular, h is harmonic). Now, as $\delta A = 0$ we obtain $\delta df = 0$ hence f is a harmonic function; since f is zero on the boundary, we get $f = 0$ also on M and we can write

$$A = \delta\psi + h. \quad (14)$$

Now, since both A and $\delta\psi$ are tangential, also h will be tangential.

4. This result was proved by Shigekawa [31] for closed manifolds; for Neumann boundary conditions see also [24]. □

- *In the sequel, when we write the decomposition $A = \delta\psi + h$, it will be implicitly supposed that ψ is a tangential 2-form and h is a 1-form satisfying $dh = \delta h = 0$ and $i_N h = 0$.*

From definition (1) we see

$$|\nabla^A u|^2 = |du|^2 + |A|^2 |u|^2 + 2\text{Im}\langle A, \bar{u} du \rangle. \quad (15)$$

Since A is real, it is clear that if $u \in C^\infty(M, \mathbf{R})$ is a real valued function, then $\text{Im}\langle A, \bar{u} du \rangle = 0$ and, then,

$$R_{A,q}(u) = \frac{\int_M (|du|^2 + (|A|^2 + q)|u|^2) v_g}{\int_M |u|^2 v_g}.$$

Since $C^2(M, \mathbf{R})$ is a subspace of $C^2(M, \mathbf{C})$, it follows that the eigenvalues of $H_{A,q}$ are dominated by those of the scalar Schrödinger operator $H_{0,|A|^2+q} = \Delta + |A|^2 + q$, that is

$$\lambda_k(H_{A,q}) \leq \lambda_k(H_{0,|A|^2+q}) = \lambda_k(\Delta + |A|^2 + q). \quad (16)$$

For the first eigenvalue of $H_{A,q}$, one also has a lower estimate by the first eigenvalue of the scalar Schrödinger operator $H_{0,q} = \Delta + q$; in other words:

$$\lambda_1(H_{A,q}) \geq \lambda_1(H_{0,q}). \quad (17)$$

This property can be seen as an immediate consequence of the so-called diamagnetic inequality (see for instance Theorem 2.1.1 in [18]).

1.2 Statement of results

Before stating the results, let us define a distance associated to the 1-form A which will play an important role in our estimates (see (18) below). Let $\mathcal{L}_{\mathbf{Z}}$ be the lattice in $\text{Har}_1(M) \sim H^1(M, \mathbf{R})$ formed by the integral harmonic 1-forms (those having integral flux around any loop). Given $A \in \text{Har}_1(M)$, we define its distance to the lattice $\mathcal{L}_{\mathbf{Z}}$ by the formula:

$$d(A, \mathcal{L}_{\mathbf{Z}})^2 = \min \left\{ \|\omega - A\|^2, \omega \in \mathcal{L}_{\mathbf{Z}} \right\}, \quad (18)$$

where $\|\cdot\|$ denotes the L^2 -norm of forms in M . Of course, when $H^1(M, \mathbf{R}) = 0$ any harmonic 1-form is zero and we set $d(A, \mathcal{L}_{\mathbf{Z}}) = 0$.

Theorem 2. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a compact Riemannian manifold (M, g) of dimension n , where $A = \delta\psi + h$ is a potential as in (11). One has, under Neumann boundary conditions if $\partial M \neq \emptyset$:*

1.

$$\lambda_1(H_{A,q}) \leq \Gamma(M, A, q) := \frac{1}{|M|} \left(d(h, \mathcal{L}_{\mathbf{Z}})^2 + \frac{\|B\|^2}{\lambda''_{1,1}(M)} + \int_M qv_g \right) \quad (19)$$

where $|M|$ denotes the volume of M and $\lambda''_{1,1}(M)$ is the first eigenvalue of the Hodge-De Rham Laplacian Δ_{HR} acting on co-exact 1-forms (with absolute boundary condition if $\partial M \neq \emptyset$).

2. *If the first absolute De Rham cohomology group vanishes : $H^1(M, \mathbf{R}) = 0$, then*

$$\lambda_1(H_{A,q}) \leq \frac{1}{|M|} \left(\frac{\|B\|^2}{\lambda''_{1,1}(M)} + \int_M qv_g \right) \quad (20)$$

with equality if and only if $\Delta_{HR}(\delta\psi) = \lambda''_{1,1}(\delta\psi)$ and $|\delta\psi|^2 + q$ is constant, equal to $\lambda_1(H_{A,q})$.

The case when the potential A is closed (that is $B = 0$) is of special interest. We have

Theorem 3. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a compact Riemannian manifold (M, g) of dimension n , where the potential A is closed, so that we can write $A = h$ as in (11). One has under Neumann boundary conditions if $\partial M \neq \emptyset$:*

$$\lambda_1(H_{A,q}) \leq \frac{d(h, \mathcal{L}_{\mathbf{Z}})^2 + \int_M qv_g}{|M|}. \quad (21)$$

In case of equality in (21), there exists an integer harmonic form $\omega \in \mathcal{L}_{\mathbf{Z}}$ such that $|A - \omega|^2 + q$ is constant. In particular, if the potential q is constant, (M, g) carries a harmonic 1-form of constant length.

Sometimes we can characterize equality. Precisely:

Theorem 4. 1. When (M, g) is a flat torus, we have equality in (21) if and only if the potential q is constant.

2. When M is a two-dimensional torus (that is, a genus one surface) and q is constant with have equality in (21) if and only if (M, g) is a flat torus.

In section 2 we will give applications of Theorem 2 for manifolds for which we have a good control of $\lambda'_{1,1}(M, g)$. First of all, using the Bochner formula, we show such control for closed manifolds with Ricci curvature bounded below by a positive constant; when the boundary is not empty, we have to impose that it is convex. Then, we extend such lower bound also when the inner curvature is not everywhere positive; for example, for convex domains in \mathbf{R}^n , and for hypersurfaces of manifolds with curvature operator with arbitrary sign, provided that the extrinsic curvatures are large enough. The general principle is that one still has a positive lower bound for $\lambda'_{1,1}$ if the positivity of the principal curvatures of the boundary compensate, in some sense, for the negativity of the inner curvature.

Thanks to the Li-Yau *conformal volume* $V_c(M)$ associated to the Riemannian manifold (M, g) , which depends only on the conformal class of g (see [10] for a definition and detail about it), and the results in [11], it is also possible to get an upper bound for the second eigenvalue.

Theorem 5. Let $H_{A,q}$ be a magnetic Schrödinger operator on a compact Riemannian manifold (M, g) of dimension n , where $A = \delta\psi + h$ is a potential as in (11). One has (under Neumann boundary conditions if $\partial M \neq \emptyset$):

$$\lambda_2(H_{A,q}) \leq n \frac{V_c(M)}{|M|} + \Gamma(M, A, q) \quad (22)$$

with $\Gamma(M, A, q)$ as in (19).

In section 3, we will give applications of Theorem 5 in specific situations.

We then state an upper bound valid for all the eigenvalues.

Theorem 6. Let $H_{A,q}$ be a magnetic Schrödinger operator on a closed Riemannian manifold (M, g) of dimension n , where $A = \delta\psi + h$ is a potential as in (11).

1. There exists a constant $c([g])$ depending on the conformal class of g such that

$$\lambda_k(H_{A,q}) \leq \Gamma(M, A, q) + c([g]) \left(\frac{k}{|M|} \right)^{2/n}. \quad (23)$$

2. If (M^n, g) has a Ricci curvature bound $\text{Ric}(M, g) \geq -a^2(n-1)$ and if $|A|^2 + q \geq 0$ (in particular, if $q \geq 0$), there exist positive constants c_1, c_2, c_3 depending only on the dimension n of M such that

$$\lambda_k(H_{A,q}) \leq c_1 \Gamma(M, A, q) + c_2 a^2 + c_3 \left(\frac{k}{|M|} \right)^{2/n}, \quad (24)$$

with $\Gamma(M, A, q)$ as in (19).

These result will be deduced from inequality (16) and from estimates for the Schrödinger Laplacian derived in [23] and [20]. Note that if $A = 0$, we recover the result of [6] for the usual Laplacian.

In the specific situation of an Euclidean domain, we get other estimates in Theorem 18 using Riesz means, as a corollary of Inequality (16) and of [13].

2 Upper bounds for the first eigenvalue of $H_{A,q}$

2.1 Proof of Theorem 2.

We recall that $A = \delta\psi + h$ denotes the potential, ψ is a smooth tangential 2-form, $h \in \text{Har}_1(\Omega)$, $\lambda''_{1,1}(\Omega)$ denotes the first eigenvalue of the Laplacian acting on co-exact 1-forms, $B = dA$ is the curvature of the potential A , and $\mathcal{L}_{\mathbf{Z}}$ denotes the integral lattice of $H^1(M)$ formed by the integer harmonic 1-forms $\text{Har}_1(M)$.

Let $\omega \in \mathcal{L}_{\mathbf{Z}}$. Fix a base point x_0 and define, for $x \in \Omega$:

$$\phi(x) \doteq \int_{x_0}^x \omega, \quad (25)$$

where on the right we mean integration of ω along any path joining x_0 with x . As ω is closed, $\phi(x)$ does not depend on the choice of two homotopic paths and since the flux of A across each c_j is an integer, $\phi(x)$ is multivalued and defined up to $2\pi\mathbf{Z}$. This implies that the function $u(x) = e^{i\phi(x)}$ is well defined. As $d\phi = \omega$ we see that $du = iu\omega$ and therefore

$$\nabla^A u = du - iuh - iu\delta\psi = iu(\omega - h - \delta\psi).$$

Since $|u| = 1$, we obtain:

$$|\nabla^A u|^2 = |\omega - h - \delta\psi|^2.$$

We use $u(x)$ as test-function for the first eigenvalue of Δ_A . Then, for each $\omega \in \mathcal{L}_{\mathbf{Z}}$, we have the relation

$$\lambda_1(H_{A,q}) \leq \frac{\int_M |\nabla^A u|^2 v_g + \int_M |u|^2 q v_g}{\int_M |u|^2 v_g} = \frac{\|\omega - h - \delta\psi\|^2}{|M|} + \frac{\int_M q v_g}{|M|} \quad (26)$$

As $\omega - h$ is harmonic, it is L^2 -orthogonal to $\delta\psi$ and we get

$$\lambda_1(H_{A,q}) \leq \frac{\|\omega - h\|^2 + \|\delta\psi\|^2 + \int_M q v_g}{|M|} \quad (27)$$

Now observe that, since $\delta\psi$ is coexact and tangential, one has by the variational characterization of the eigenvalue $\lambda''_{1,1}(M, g)$:

$$\frac{\int_M |d\delta\psi|^2 v_g}{\int_M |\delta\psi|^2 v_g} \geq \lambda''_{1,1}(M, g)$$

As $d\delta\psi = B$, we have

$$\int_M |\delta\psi|^2 v_g \leq \frac{1}{\lambda''_{1,1}(M, g)} \|B\|^2. \quad (28)$$

Taking the infimum on the right-hand side of (27) over all $\omega \in \mathcal{L}_{\mathbf{Z}}$ we obtain, taking into account (28):

$$\lambda_1(H_{A,q}) \leq \frac{d(h, \mathcal{L}_{\mathbf{Z}})^2}{|M|} + \frac{\|B\|^2}{\lambda''_{1,1}(M)|M|} + \frac{1}{|M|} \int_M q v_g$$

as asserted.

When $H^1(M, \mathbf{R}) = 0$, we have immediately the relation

$$\lambda_1(H_{A,q}) \leq \frac{1}{|M|} \left(\frac{\|B\|^2}{\lambda''_{1,1}(M, g)} + \int_M q v_g \right).$$

In case of equality, we must have equality in all the step of the proof: in particular, we must have

$$\frac{\int_M |d\delta\psi|^2 v_g}{\int_M |\delta\psi|^2 v_g} = \lambda''_{1,1}(M, g)$$

which means that $\lambda''_{1,1}$ is an eigenvalue for the eigenfunction $\delta\psi$. For $u = 1$, Equation (3) becomes

$$\Delta_A u = |\delta\psi|^2$$

and the equation $H_{A,q}u = \lambda_1(H_{A,q})u$ becomes

$$|\delta\psi|^2 + q = \lambda_1(H_{A,q}).$$

as asserted.

2.2 Proof of Theorem 3

1. Inequality (21) is an immediate consequence of Inequality (19).
2. In order to investigate the equality case, we will derive the inequality using a different approach. Let $\omega \in \mathcal{L}_{\mathbf{Z}}$ and $u = e^{i\phi}$ the associated function on M as defined in (25). Recall that $|\cdot|$ denotes the pointwise norm, thus defining a smooth function on M .

First we observe that

$$\Delta_A u = |A - \omega|^2 u. \quad (29)$$

In fact recall that, as $\delta A = 0$:

$$\Delta_A u = \Delta u + |A|^2 u + 2i\langle du, A \rangle.$$

As $du = iu\omega$ one gets:

$$\Delta u = \delta du = \delta(iu\omega) = i(-\langle du, \omega \rangle + u\delta\omega) = -i\langle du, \omega \rangle = |\omega|^2 u$$

and (29) follows after an easy computation. In turn, one has:

$$H_{A,q} u = |A - \omega|^2 u + qu.$$

Using u as a test-function, and recalling that $|u|^2 = 1$, we have

$$\lambda_1(H_{A,q}) \int_M |u|^2 v_g \leq \int_M \langle H_{A,q} u, u \rangle v_g = \|\omega - A\|^2 + \int_M q v_g. \quad (30)$$

In particular, if we choose ω so that $d(\omega, A)^2 = d(A, \mathcal{L}_{\mathbf{Z}})^2$ we recover inequality (21). But now, if equality holds, we see that u must be an eigenfunction for $\lambda_1(H_{A,q})$, that is

$$\lambda_1(H_{A,q}) u = H_{A,q} u = \Delta_A u + qu = (|A - \omega|^2 + q)u.$$

So, we deduce that $|A - \omega|^2 + q = \lambda_1(H_{A,q})$ as asserted. In particular, if q is constant, $|A - \omega|$ is constant, and (M, g) carries a harmonic 1-form of constant length.

2.3 Spectrum of flat tori

In order to prove Theorem 4, we investigate the spectrum of flat tori. Let Σ be a flat n -dimensional torus, quotient of \mathbf{R}^n by a lattice Γ . Recall that the dual lattice Γ^* is defined by

$$\Gamma^* = \{v : \langle v, w \rangle \in \mathbf{Z} \text{ for all } w \in \Gamma\}$$

On a flat torus any harmonic 1-form ξ is parallel, and then it has constant pointwise norm $|\xi|$. In particular

$$\|\xi\| = |M| |\xi|.$$

The lattice $\mathcal{L}_{\mathbf{Z}}$ is an additive subgroup of the vector space of harmonic (hence parallel) 1-forms. If ω is one such consider the associated dual parallel vector field, ω^\sharp . We remark that this induces an isomorphism of groups:

$$\mathcal{L}_{\mathbf{Z}} \cong 2\pi\Gamma^*. \quad (31)$$

To prove that, associate to each $X \in \Gamma$ the curve $c_X : [0, 1] \rightarrow \Sigma$ given by $c_X(t) = tX$. Note that c_X is a loop because Σ is Γ -invariant. The flux of ω across c_X is easily seen to be

$$\Phi_{c_X}^\omega = \frac{1}{2\pi}\omega(X) = \frac{1}{2\pi}\langle \omega^\sharp, X \rangle.$$

Hence any such flux is an integer if and only if $\langle \omega^\sharp, X \rangle \in 2\pi\mathbf{Z}$. This is true for all $X \in \Gamma$ iff $\omega^\sharp \in 2\pi\Gamma^*$, which proves (31).

Now if $\omega \in \mathcal{L}_{\mathbf{Z}}$, it is readily seen that the associated function u as in (25) is given by:

$$u(x) = e^{i\langle \omega^\sharp, x \rangle}$$

which is well-defined on $(M, g) = \mathbf{R}^n/\Gamma$. Hence, for each $\omega \in \mathcal{L}_{\mathbf{Z}}$, thanks to (29), we have:

$$\Delta_A u = |A - \omega|^2 u$$

and the constant $|A - \omega|^2$ is thus an eigenvalue of Δ_A associated to the eigenfunction u . Because of (31) the set

$$\{u(x) = e^{i\langle \omega^\sharp, x \rangle}, \omega \in \mathcal{L}_{\mathbf{Z}}\}$$

gives rise to a complete orthonormal basis of $L^2(M)$, hence we have found all the eigenvalues of Δ_A . In conclusion, we have the following fact.

Proposition 7. *Let Σ be a flat torus, quotient of \mathbf{R}^n by the lattice Γ and let Γ^* denote the lattice dual to Γ . Let A be a harmonic 1-form. Then the spectrum of the magnetic Laplacian with potential A , that is, the operator $\Delta_A = H_{A,0}$, is given by*

$$\{|A - \omega|^2 : \omega \in \mathcal{L}_{\mathbf{Z}} \cong 2\pi\Gamma^*\}$$

with associated eigenfunctions $\{u(x) = e^{i\langle \omega^\sharp, x \rangle}\}$. In particular

$$\lambda_1(\Delta_A) = \inf_{\omega \in \mathcal{L}_{\mathbf{Z}}} |A - \omega|^2.$$

2.4 Proof of Theorem 4

1. Now let M be a flat torus, $A = h$ a harmonic 1-form, and let ω_0 be an element in $\mathcal{L}_{\mathbf{Z}}$ such that

$$d(A, \mathcal{L}_{\mathbf{Z}})^2 = \|A - \omega_0\|^2 = |M||A - \omega_0|^2.$$

Inequality (21) takes the form:

$$\lambda_1(H_{A,q}) \leq |A - \omega_0|^2 + \frac{1}{|M|} \int_M qv_g$$

with equality if and only if the associated test-function $u(x) = e^{i\langle \omega_0^\sharp, x \rangle}$ (of constant modulus one) is an eigenfunction of $H_{A,q}$. As

$$H_{A,q}u = \Delta_A u + qu = (|A - \omega_0|^2 + q)u$$

we see indeed that we have equality in (21) if and only if

$$q = \frac{1}{|M|} \int_M qv_g$$

that is, iff q is constant.

2. Now assume that (M, g) is a genus one surface and q is constant. It remains to show that, if equality holds, M has to be flat.

Since q is constant, there exists a harmonic one form ξ with constant length by the second assertion of Theorem 3. We will apply Bochner formula to ξ . Let Δ_{HR} the Laplacian on 1-forms and ∇ the covariant derivative. Bochner's identity gives, for any 1-form α :

$$\langle \Delta_{HR}\alpha, \alpha \rangle = |\nabla\alpha|^2 + \frac{1}{2}\Delta|\alpha|^2 + \text{Ric}(\alpha, \alpha). \quad (32)$$

In dimension 2 one has $\text{Ric} = Kg$, where K is the Gaussian curvature. As ξ is harmonic and of constant pointwise norm, we get

$$0 = \int_M |\nabla\xi|^2 v_g + |\xi|^2 \int_M K v_g.$$

As M has genus one we see $\int_M K v_g = 0$; this means that ξ must actually be parallel. But then $\star\xi$ must also be parallel; by normalization, we have a global orthonormal basis $(\xi, \star\xi)$ of parallel one forms, which forces (M, g) to be flat.

2.5 A few consequences

We can now describe a few consequences of Theorem 2 in some specific situations where we are able to control the eigenvalue $\lambda''_{1,1}$ of the manifold M .

2.5.1 Positive Ricci curvature

When the Ricci curvature of M is positive (and ∂M is convex if nonempty), then $H^1(M, \mathbf{R}) = \{0\}$. This implies that the harmonic part h in the decomposition (14) of the potential

1-form A vanishes, so that $A = \delta\psi$ for a tangential two-form ψ . Moreover, the constant $\lambda''_{1,1}(M)$ can be controlled in terms of a lower bound of the Ricci curvature of M . Indeed, we have the

Lemma 8. *Let (M, g) be a compact Riemannian manifold whose Ricci curvature satisfies*

$$\text{Ric} \geq c g$$

for some positive c . When $\partial M \neq \emptyset$, assume furthermore that ∂M is convex (i.e. its shape operator S is nonnegative). One has

$$\lambda''_{1,1}(M) \geq 2c$$

Moreover, the equality holds if and only if every co-exact eigenform α associated with $\lambda''_{1,1}(M)$ is such that α^\sharp is a Killing vector field which satisfies $\text{Ric}(\alpha^\sharp) = c\alpha^\sharp$ and, when $\partial M \neq \emptyset$, $S(\alpha^\sharp) = 0$.

Here $S : T\partial M \rightarrow T\partial M$ is the shape operator of ∂M , defined as follows: if N is the inner unit normal vector to the boundary, and $X \in T\partial M$, then $S(X) = -\nabla_X N$.

Proof. We use again the Bochner identity (32):

$$\langle \Delta_{HR}\alpha, \alpha \rangle = |\nabla\alpha|^2 + \frac{1}{2}\Delta|\alpha|^2 + \text{Ric}(\alpha, \alpha).$$

On the other hand, we have the following general inequality (see [19], Lemma 6.8 p. 270)

$$|\nabla\alpha|^2 \geq \frac{1}{2}|d\alpha|^2 + \frac{1}{n}|\delta\alpha|^2 \geq \frac{1}{2}|d\alpha|^2 \quad (33)$$

in which the equality holds if and only if $\nabla\alpha$ is anti-symmetric, that is α^\sharp is a Killing vectorfield (see [2] Theorem 1.81, p. 40). When α is a co-exact eigenform associated to $\lambda''_{1,1}(M)$, we have

$$\lambda''_{1,1}(M)|\alpha|^2 = \langle \Delta_{HR}\alpha, \alpha \rangle \geq \frac{1}{2}|d\alpha|^2 + \frac{1}{2}\Delta|\alpha|^2 + c|\alpha|^2 \quad (34)$$

When M is closed, one has $\int_M \Delta|\alpha|^2 v_g = 0$ and $\int_M |d\alpha|^2 v_g = \lambda''_{1,1}(M) \int_M |\alpha|^2 v_g$ and the result follows from (34) after integration, that is

$$\lambda''_{1,1}(M) \int_M |\alpha|^2 v_g \geq \frac{1}{2} \int_M |d\alpha|^2 v_g + c \int_M |\alpha|^2 v_g = \left(\frac{\lambda''_{1,1}(M)}{2} + c \right) \int_M |\alpha|^2 v_g \quad (35)$$

which implies $\lambda''_{1,1}(M) \geq 2c$.

If $\partial M \neq \emptyset$, we observe that, since $i_N \alpha = 0$ and $i_N d\alpha = 0$, the vector field α^\sharp is tangent along the boundary and $0 = d\alpha(N, \alpha^\sharp) = \nabla_N \alpha(\alpha^\sharp) - \nabla_{\alpha^\sharp} \alpha(N) = \frac{1}{2} N \cdot |\alpha|^2 - \langle S(\alpha^\sharp), \alpha^\sharp \rangle$. Thus Green formula gives

$$\int_M \Delta |\alpha|^2 v_g = \int_{\partial M} N \cdot |\alpha|^2 v_g = 2 \int_{\partial M} \langle S(\alpha^\sharp), \alpha^\sharp \rangle v_g \geq 0$$

The rest of the proof is the same as above.

Let us discuss the equality case when ∂M is empty. Assume that a co-exact eigenform α satisfies : α^\sharp is a Killing vector field and $Ric(\alpha^\sharp, \alpha^\sharp) = c|\alpha|^2$. Under these conditions, the equality holds in the inequality (34) and, then, in (35), which implies $\lambda''_{1,1}(M) = 2c$.

Conversely, if $\lambda''_{1,1}(M) = 2c$, then, for any co-exact eigenform α , the equality holds in (35) and, then, in (34) and (33) which implies that $Ric(\alpha^\sharp, \alpha^\sharp) = c|\alpha|^2$ and that $D\alpha$ is anti-symmetric, that is α^\sharp is a Killing vector field.

When ∂M is not empty, the discussion of the equality case follows the same lines observing that since ∂M is convex, the equality $\langle S(\alpha^\sharp), \alpha^\sharp \rangle = 0$ occurs if and only if $S(\alpha^\sharp) = 0$. \square

An immediate consequence of Theorem 2 is the

Corollary 9. *Under the circumstances of Theorem 2 and the assumption that the Ricci curvature of M satisfies $Ric \geq c g$ for some positive c , and that the boundary ∂M is convex (if nonempty), one has*

$$\lambda_1(H_{A,q}) \leq \frac{1}{|M|} \left(\frac{\|B\|^2}{2c} + \int_M q v_g \right) \quad (36)$$

where $B = dA$ is the magnetic field. The equality holds in (36) if and only if A^\sharp is a Killing vector field with $Ric(A^\sharp) = cA^\sharp$, $|A^\sharp|^2 + q = \lambda_1(H_{A,q})$ and, when $\partial M \neq \emptyset$, $S(A^\sharp) = 0$.

Recall that Bochner vanishing Theorem tells us that a non Ricci-flat manifold M with non-negative Ricci curvature and mean-convex boundary if $\partial M \neq \emptyset$, satisfies $H^1(M, \mathbf{R}) = \{0\}$. On the other hand, Bochner's identity gives for any Killing vector field A , $\Delta_{HR} A = 2 Ric(A)$.

The inequality (36) improves by a factor 2 the estimate obtained by Cruzeiro, Malliavin and Taniguchi ([7], Theorem 1.1) for $\lambda_1(H_{A,0})$ for closed manifolds (be careful, the magnetic Laplacian defined in [7] coincides with $\frac{1}{2}\Delta_A$).

An important special case we want to emphasize is the following

Corollary 10. (i) *Let $H_{A,q}$ be a magnetic Schrödinger operator on the standard n -dimensional sphere \mathbb{S}^n . One has*

$$\lambda_1(H_{A,q}) \leq \frac{1}{\sigma_n} \left(\frac{\|B\|^2}{2(n-1)} + \int_{\mathbb{S}^n} q v_g \right) \quad (37)$$

where $\sigma_n = (n+1)\omega_{n+1}$ is the volume of \mathbb{S}^n and $B = dA$ is the magnetic field. The equality holds in (37) if and only if A^\sharp is a Killing vector field of \mathbb{S}^n and $|A^\sharp|^2 + q = \lambda_1(H_{A,q})$.

(ii) Let $H_{A,q}$ be a magnetic Schrödinger operator on a spherical cap $C_r(x_0)$ of radius $r \leq \frac{\pi}{2}$ centered at x_0 . One has

$$\lambda_1(H_{A,q}) \leq \frac{1}{v_n(r)} \left(\frac{\|B\|^2}{2(n-1)} + \int_{C_r} qv_g \right) \quad (38)$$

where $v_n(r) = \sigma_{n-1} \int_0^r (\sin t)^{n-1} dt$ is the volume of $C_r(x_0)$.

If $r < \frac{\pi}{2}$, then the equality holds if and only if $B = 0$ and q is constant. When $r = \frac{\pi}{2}$ (i.e. for a hemisphere), the equality holds in (38) if and only if A^\sharp is a Killing vector field which vanishes at x_0 and $|A^\sharp|^2 + q = \lambda_1(H_{A,q})$.

Indeed, a Killing vector field is tangent along $\partial C_r(x_0)$ if and only if it vanishes at x_0 . Since $\partial C_r(x_0)$ is totally umbilical, the condition $S(A^\sharp) = 0$ implies that $A^\sharp = 0$ unless $S = 0$ which only occurs when $r = \frac{\pi}{2}$.

In particular, for a magnetic Laplacian $\Delta_A = H_{A,0}$ on \mathbb{S}^n , the inequality (37) reads

$$\lambda_1(H_{A,0}) \leq \frac{1}{2(n-1)\sigma_n} \|B\|^2. \quad (39)$$

Notice that when n is even, there is no nonzero vector field of constant length on \mathbb{S}^n . Therefore, in the even dimensional case, the equality in (39) holds if and only if $A^\sharp = 0$ (or, equivalently, $B = 0$). If n is odd, then the equality in (39) implies that A^\sharp is proportional to the vector field $J(x) = (-x_2, x_1, \dots, -x_{n+1}, x_n)$ which is the only Killing vector field of constant length, up to a dilation (see [8]).

In dimension 2, the inequality (39) (i.e. $\lambda_1(H_{A,0}) \leq \frac{1}{8\pi} \|B\|^2$) improves the upper bound obtained by Besson, Colbois and Courtois in [3].

2.5.2 Closed hypersurfaces

We now assume that M is a closed, immersed hypersurface of a Riemannian manifold M' . At any point $x \in M$ denote the principal curvatures of M (eigenvalues of the shape operator) by $k_1(x), \dots, k_n(x)$. Let I_p denote the set of p -multi-indices

$$I_p = \{(j_1, \dots, j_p) : 1 \leq j_1 \leq \dots \leq j_p \leq n\},$$

and, for each $\alpha = (j_1, \dots, j_p) \in I_p$, consider the corresponding p -curvature

$$K_\alpha(x) = k_{j_1}(x) + \dots + k_{j_p}(x).$$

Set $\star\alpha = \{1, \dots, p\} \setminus \{j_1, \dots, j_p\}$, and moreover

$$\begin{cases} \beta_p(x) = \frac{1}{p(n-p)} \inf_{\alpha \in I_p} K_\alpha(x) K_{\star\alpha}(x) \\ \beta_p(\Sigma) = \inf_{x \in \Sigma} \beta_p(x) \end{cases}$$

We then have the following lower bound (see Theorem 7 in [29]):

Theorem 11. *Let M^n be a closed immersed hypersurface of the Riemannian manifold M^{n+1} having curvature operator bounded below by $\gamma_{M'} \in \mathbf{R}$. Then we have the following lower bound*

$$\lambda_{1,p}(M) \geq p(n-p+1)(\gamma_{M'} + \beta_p(M)).$$

Equality holds for geodesic spheres in constant curvature spaces. In particular

$$\lambda''_{1,1}(M) \geq \max\{2(n-1)(\gamma_{M'} + \beta_2(M)), n(\gamma_{M'} + \beta_1(M))\}.$$

We say that M is p -convex if all p -curvatures are non-negative; that is, $K_\alpha(x) \geq 0$ for all $\alpha \in I_p$ and for all $x \in M$. Clearly if M is p -convex then it is q -convex for all $q \geq p$. Then, 1-convexity is the usual convexity assumption and n -convex is equivalent to mean convexity.

Note that we could have a positive lower bound even when the curvature of M' is negative; it is enough to assume that the p -curvatures K_α are positive enough. For example, for a 2-convex hypersurface in hyperbolic space \mathbf{H}^{n+1} , (where $\gamma_{M'} = 1$) with 2-curvatures $K_\alpha(x)$ uniformly bounded below by $c > 2$, elementary algebra shows that $\beta_2(M) \geq c^2/4$ hence

$$\lambda''_{1,1} \geq 2(n-1)\left(\frac{c^2}{4} - 1\right) > 0$$

On the other hand, if M^n is a 2-convex hypersurface of the sphere \mathbf{S}^{n+1} then $\beta_2(M) \geq 0$ and therefore

$$\lambda''_{1,1} \geq 2(n-1).$$

We finally remark the following estimate by P. Guerini ([21]): if M is a convex hypersurface of \mathbf{R}^n then

$$\lambda_{1,p}(M) \geq \frac{p}{2e^3} \cdot \frac{1}{\text{diam}(M)^2}.$$

2.5.3 Convex domains in Euclidean space

Assume now that M is a convex domain in \mathbf{R}^n . Then we know from [28] that for all $p = 1, \dots, n$ one has $\lambda_{1,p} = \lambda'_{1,p}$; in particular :

$$\lambda''_{1,1} = \lambda_{1,2}.$$

Theorem 1.1 in [28] states that, for all $p = 1, \dots, n$:

$$\frac{a_{n,p}}{D_p^2} \leq \lambda_{1,p} \leq \frac{a'_{n,p}}{D_p^2}$$

for explicit constants $a_{n,p}, a'_{n,p}$. Here D_p is the p -th largest principal axis of the ellipsoid of maximal volume included in M , also called *John ellipsoid* of M . In particular,

$$\lambda''_{1,1} \geq \frac{4}{n^3 D_2^2}.$$

Accordingly we have an upper bound for the spectrum of the magnetic Laplacian:

$$\lambda_1(H_{A,0}) \leq \frac{1}{|M|} \left(\frac{\|B\|^2}{\lambda''_{1,1}(M)} + \int_M qv_g \right) \leq \frac{n^3 \|B\|^2 D_2^2}{4|M|} + \frac{1}{|M|} \int_M qv_g$$

For example, assume that $q = 0$ and

$$\frac{1}{|M|} \int_M \|B\|^2 \leq c.$$

We then see

$$\lambda_1(H_{A,0}) \leq \frac{cn^3}{4} D_2^2$$

2.5.4 Other estimates

We refer to [22] for a lower bound of $\lambda''_{1,1}$ of any compact manifold with boundary Ω , in terms of a lower bound $\gamma \in \mathbf{R}$ of the eigenvalues of the curvature operator of Ω , and the 2-curvatures of $\partial\Omega$: if the 2 curvatures are large enough, then the lower bound is positive (see Theorem 3.3 in [22]). We also remark that in certain cases it is possible to estimate from below the gap $\lambda_{1,p} - \lambda_{1,0}$ between the first eigenvalue for p -forms (absolute boundary conditions) and the first eigenvalue on functions (Neumann conditions). For example, for convex domains in \mathbf{S}^n one has, for $p = 2, \dots, \frac{n}{2}$:

$$\lambda_{1,p} \geq \lambda_{1,0} + (p-1)(n-p)$$

which reduces to an equality when Ω is the hemisphere. In particular,

$$\lambda''_{1,1} \geq \lambda_{1,0} + n - 2$$

which often improves the bound $\lambda''_{1,1} \geq 2(n-1)$ considered in Corollary 14 above : in fact, $\lambda_{1,0}$ is the first positive eigenvalue for the Neumann Laplacian acting on functions, which can be very large (for example, for small geodesic balls).

3 Upper bounds for the second eigenvalue of $H_{A,q}$

Let us first give the proof of Theorem 5: it is a consequence of the observation (16) and of a previous result of El Soufi and Ilias [11]. By (16), we have

$$\lambda_2(H_{A,q}) \leq \lambda_2(H(0, |A|^2 + q))$$

which corresponds to the usual Laplacian Δ on (M, g) with the potential $|A|^2 + q$, and $A = \delta\psi + h$ as in (11). By [11], for any scalar potential W on M one has

$$\lambda_2(\Delta + W) \leq n \left(\frac{V_c(M)}{|M|} \right)^{\frac{2}{n}} + \frac{1}{|M|} \int_M W v_g,$$

where $V_c(M)$ is the Li-Yau conformal volume of the Riemannian manifold M . In our situation, $W = |\delta\psi + h|^2 + q$ and we have already seen that

$$\frac{1}{|M|} \int_M (|\delta\psi + h|^2 + q) v_g \leq \Gamma(M, A, q) := \frac{1}{|M|} \left(d(h, \mathcal{L}_Z)^2 + \frac{\|B\|^2}{\lambda''_{1,1}(M)} + \int_M q v_g \right)$$

which allows to conclude.

As for the first eigenvalue, we have a lot of consequences of this result in specific situations. For example, the conformal volume of the sphere \mathbb{S}^n endowed with the conformal class of its standard metric g_s is equal to the volume $\sigma_n = |\mathbb{S}^n|_{g_s}$ of the standard metric. Hence, any domain $\Omega \subset \mathbb{S}^n$, endowed with a metric conformal to the standard one will satisfies $V_c(\Omega) \leq \sigma_n$.

Corollary 12. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a bounded domain $\Omega \subset \mathbf{R}^n$, endowed with a Riemannian metric g conformally equivalent to the Euclidean metric. One has, under Neumann boundary conditions,*

$$\lambda_2(H_{A,q}) \leq n \left(\frac{\sigma_n}{|\Omega|_g} \right)^{\frac{2}{n}} + \frac{1}{|\Omega|_g} \left(\frac{\|B\|^2}{\lambda''_{1,1}(\Omega)} + d(h, \mathcal{L}_Z)^2 + \int_\Omega q v_g \right). \quad (40)$$

This corollary applies of course when Ω is a domain of the Euclidean space, the hyperbolic space, and the sphere. Note that the equality holds in (40) when g is the spherical metric, $A = 0$, $q = 0$ and Ω is a ball whose Euclidean radius tends to infinity.

For a compact orientable surface M of genus γ , one has (see [26])

$$V_c(M) \leq 4\pi \left[\frac{\gamma + 3}{2} \right] \quad (41)$$

and

$$\lambda''_{1,1}(M) = \mu(M)$$

where $\lfloor \cdot \rfloor$ stands for the floor function and $\mu(M)$ is the first positive eigenvalue of the Laplacian of M acting on functions, with Neumann boundary conditions if $\partial M \neq \emptyset$. Thus, the inequality (22) leads to the following:

Corollary 13. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a domain Ω of a compact orientable Riemannian surface M of genus γ . One has*

$$\lambda_2(H_{A,q})|\Omega| \leq 8\pi \left\lfloor \frac{\gamma + 3}{2} \right\rfloor + \frac{\|B\|^2}{\mu(\Omega)} + d(h, \mathcal{L}_{\mathbf{Z}})^2 + \int_{\Omega} qv_g. \quad (42)$$

where $\mu(\Omega)$ is the first positive eigenvalue of the Laplacian on functions, with Neumann b. c. if $\Omega \subsetneq M$.

The following corollary extends Hersch's inequality

Corollary 14. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a compact orientable Riemannian surface M of genus zero. One has*

$$\lambda_2(H_{A,q})|M| \leq 8\pi + \frac{\|B\|^2}{\mu(M)} + \int_M qv_g. \quad (43)$$

In [10], we have proved that if a Riemannian manifold M admits an isometric immersion in a Euclidean space whose components are first eigenfunctions of the Laplacian, then

$$\left(\frac{V_c(M)}{|M|} \right)^{\frac{2}{n}} = \frac{\mu(M)}{n}. \quad (44)$$

In particular, the equality (44) holds for any compact rank one symmetric space. Such a space is Einstein and satisfies $H^1(M, \mathbf{R}) = \{0\}$. Thus, combining with Lemma 8, we get the following

Corollary 15. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a domain Ω (with convex boundary if $\Omega \subsetneq M$) of a compact rank one symmetric space M of (real) dimension n . One has*

$$\lambda_2(H_{A,q}) \leq \mu(M) \left(\frac{|M|}{|\Omega|} \right)^{\frac{2}{n}} + \frac{1}{|\Omega|} \left(\frac{\|B\|^2}{2c_M} + \int_{\Omega} qv_g \right) \quad (45)$$

where c_M is the Ricci curvature constant of M and $B = dA$ is the magnetic field.

When M is a closed immersed submanifold in a Riemannian space form of curvature $\kappa = -1, 0, +1$ it was established ([12, 14]) the following relationship between the second eigenvalue of a scalar Schrödinger operator $\Delta + W$ and the L^2 -norm mean curvature h_M of M :

$$\lambda_2(\Delta + W) \leq \frac{1}{|M|} \int_M (n|h_M|^2 + n\kappa + W) v_g$$

This inequality is known as Reilly inequality when $\kappa = 0$ and $W = 0$.

The same arguments as before enable us to obtain the following

Corollary 16. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a closed immersed sub-manifold M of a space-form of curvature $\kappa = -1, 0, +1$. One has*

$$\lambda_2(H_{A,q})|M| \leq \int_M (n|h_M|^2 + n\kappa) v_g + \frac{1}{\lambda''_{1,1}(M)} \|B\|^2 + d(h, \mathcal{L}_Z)^2 + \int_M qv_g. \quad (46)$$

4 Upper bounds for higher order eigenvalues of $H_{A,q}$

In order to prove Theorem 6, we use again the relation (16)

$$\lambda_k(H_{A,q}) \leq \lambda_k(H_{0,|A|^2+q}).$$

In order to prove the inequality (23), we use the recent [20] Theorem 1.1: for a scalar Schrödinger operator $\Delta + W$ on a compact Riemannian manifold without boundary. From this result, we deduce that

$$\lambda_k(\Delta + W) \leq \frac{1}{|M|} \int_M Wv_g + c([g]) \left(\frac{k}{|M|} \right)^{\frac{2}{n}}$$

where $c([g])$ is a constant depending only on the conformal class $[g]$ of g , and the conclusion follows as before because $W = |A|^2 + q$.

In order to prove Inequality (24), one can make use of the estimates obtained by A. Hassannezhad [23] for a scalar Schrödinger operator $\Delta + W$ on a compact Riemannian manifold: If $\lambda_1(\Delta + W) \geq 0$ (which is in particular the case if $W \geq 0$ as in our situation), then

$$\lambda_k(\Delta + W) \leq \frac{c_1}{|M|} \int_M Wv_g + c_2 \left(\frac{V([g])}{|M|} \right)^{\frac{2}{n}} + c_3 \left(\frac{k-1}{|M|} \right)^{\frac{2}{n}}$$

where c_1, c_2 and c_3 are constants which depend only on the dimension n and $V([g])$ is the infimum of the volume of M with respect to all Riemannian metrics g_0 conformal to g and such that $\text{Ric}_{g_0} \geq -(n-1)g_0$.

In particular, if $\text{Ric}_g \geq -(n-1)a^2g$ for some $a \neq 0$, then the metric $g_0 = a^2g$ satisfies $\text{Ric}_{g_0} \geq -(n-1)g_0$ and $|M|_{g_0} = a^n|M|_g$. Thus, $V([g]) \leq a^n|M|_g$.

So, we can conclude by observing that

$$\int_M Wv_g = \int_M (|A|^2 + q)v_g \leq \Gamma(M, A, q).$$

As a corollary, on a compact orientable surface M of genus $\gamma \geq 2$, every Riemannian metric g is conformal to a hyperbolic metric g_0 which implies $V([g]) \leq |M|_{g_0} = 4\pi(\gamma - 1)$. The same observations as before lead to the following

Corollary 17. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a compact orientable surface of genus γ , then*

$$\lambda_k(H_{A,q})|M| \leq ak + b\gamma + c\Gamma(M, A, q).$$

where a , b and c are universal constants.

Let us now consider a magnetic Schrödinger operator $H_{A,q}$ on a bounded domain of an Euclidean space (here, as precised before, we consider Neumann condition on the boundary). The following estimates for the sum of eigenvalues (generalizing that of Kröger for $H_{0,0}$), for the Riesz means and for the trace of the magnetic heat kernel (generalizing that of Kac for $H_{0,0}$) are consequences of the considerations above and the estimates obtained in [13]. For convenience and as before, we use the notation $\Gamma(\Omega, A, q) := \frac{1}{|\Omega|} \left(d(h, \mathcal{L}\mathbf{z})^2 + \frac{\|B\|^2}{\lambda_{1,1}'(\Omega)} + \int_{\Omega} qv_g \right)$

Theorem 18. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a bounded domain Ω of \mathbb{R}^n . One has*

(1) *For all $z \in \mathbb{R}$,*

$$\sum_{j \geq 1} (z - \lambda_j(H_{A,q}))_+ \geq \frac{2|\Omega|}{n+2} \mathcal{W}_n^{-\frac{n}{2}} (z - \Gamma(\Omega, A, q))_+^{1+\frac{n}{2}}, \quad (47)$$

where $\mathcal{W}_n = 4\pi^2/\omega_n^{\frac{2}{n}}$ is the Weyl constant.

(2) *For all $k \geq 1$,*

$$\frac{1}{k} \sum_{j=1}^k \lambda_j(H_{A,q}) \leq \frac{n}{n+2} \mathcal{W}_n \left(\frac{k-1}{|\Omega|} \right)^{\frac{2}{n}} + \Gamma(\Omega, A, q). \quad (48)$$

and, if $\sum_{j=1}^k \lambda_j(\Delta + q) \geq 0$,

$$\lambda_k(H_{A,q}) \leq \max \left(2(n+2)^{\frac{2}{n}} \mathcal{W}_n \left(\frac{k-1}{|\Omega|} \right)^{\frac{2}{n}}, 2\Gamma(\Omega, A, q) \right). \quad (49)$$

(3) *For all $t > 0$,*

$$\sum_{j \geq 1} e^{-t\lambda_j(H_{A,q})} \geq \frac{|\Omega|}{(4\pi t)^{\frac{n}{2}}} e^{-t\Gamma(\Omega, A, q)}. \quad (50)$$

Proof. Taking $A = \delta\psi + h$. As we have seen before,

$$\lambda_k(H_{A,q}) \leq \lambda_k(\Delta + |\delta\psi + h|^2 + q). \quad (51)$$

In [13], the authors obtained estimates for the eigenvalues, their Riesz means, their sum and the heat trace of a general elliptic operator. For a scalar Schrödinger operator $\Delta + W$ on a bounded Euclidean domain $\Omega \subset \mathbb{R}^n$, these estimates take the following form :

(1) For all $z \in \mathbb{R}$,

$$\sum_{j \geq 1} (z - \lambda_j(\Delta + W))_+ \geq \frac{2|\Omega|}{n+2} \mathcal{W}_n^{-\frac{n}{2}} \left(z - \frac{1}{|\Omega|} \int_{\Omega} W dx \right)_+^{1+\frac{n}{2}}, \quad (52)$$

(2) For all $k \geq 1$,

$$\frac{1}{k} \sum_{j=1}^k \lambda_j(\Delta + W) \leq \frac{n}{n+2} \mathcal{W}_n \left(\frac{k-1}{|\Omega|} \right)^{\frac{2}{n}} + \frac{1}{|\Omega|} \int_{\Omega} W dx. \quad (53)$$

and if $\sum_{j=1}^k \lambda_j(\Delta + W) \geq 0$, then

$$\lambda_k(\Delta + W) \leq \max \left(2(n+2)^{\frac{2}{n}} \mathcal{W}_n \left(\frac{k-1}{|\Omega|} \right)^{\frac{2}{n}}, \frac{2}{|\Omega|} \int_{\Omega} W dx \right). \quad (54)$$

(3) For all $t > 0$,

$$\sum_{j \geq 1} e^{-t\lambda_j(\Delta+W)} \geq \frac{|\Omega|}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{t}{|\Omega|} \int_{\Omega} W dx}. \quad (55)$$

To conclude the proof, we simply apply these inequalities to the Schrödinger operator $\Delta + W$ with $W = |\delta\psi + h|^2 + q$ and observe that, using the same arguments as before,

$$\frac{1}{|\Omega|} \int_{\Omega} W dx \leq \Gamma(\Omega, A, q).$$

□

Estimates such as (52) ... (55) are also available in [13] for a bounded domain Ω of a Riemannian manifold M . However, in this case the constants which involve the geometry of Ω are less explicit than in the Euclidean case. Therefore, we can deduce that there exist constants $c_1(\Omega), \dots, c_4(\Omega)$, depending only on Ω such that, for all $z \in \mathbb{R}$, $k \geq 1$ and $t > 0$

$$\sum_{j \geq 1} (z - \lambda_j(H_{A,q}))_+ \geq c_1(\Omega) (z - \Lambda(A, q))_+^{1+\frac{n}{2}}, \quad (56)$$

$$\frac{1}{k} \sum_{j=1}^k \lambda_j(H_{A,q}) \leq c_2(\Omega) \left(\frac{k-1}{|\Omega|} \right)^{\frac{2}{n}} + \Gamma(\Omega, A, q), \quad (57)$$

anf if $\sum_{j=1}^k \lambda_j(\Delta + q) \geq 0$, then

$$\lambda_k(H_{A,q}) \leq \max \left(c_3(\Omega) \left(\frac{k-1}{|\Omega|} \right)^{\frac{2}{n}}, 2\Gamma(\Omega, A, q) \right), \quad (58)$$

$$\sum_{j \geq 1} e^{-t\lambda_j(H_{A,q})} \geq \frac{c_4(\Omega)}{t^{\frac{n}{2}}} e^{-t\Gamma(\Omega, A, q)}. \quad (59)$$

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