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Some Results on a New Refinable Class Suitable for Fractional Differential Problems

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Abstract: In recent years, we found that some multiscale methods applied to fractional differential problems, are easy and efficient to implement, when we use some fractional refinable functions introduced in the literature. In fact, these functions not only generate a multiresolution on $\mathbb{R}$, but also have fractional (non-integer) derivative satisfying a very convenient recursive relation. For this reason, in this paper, we describe this class of refinable functions and focus our attention on their approximating properties.

Keywords: fractional refinable functions; fractional differential problems; collocation method

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1. Introduction

In the last decades, fractional calculus has increased in popularity, owing to the awareness that many physical problems, such as viscoelasticity, Brownian motion, medical issues and so forth, require fractional derivatives to be modeled appropriately. For a better understanding, please see [1,2].

Analytical solutions for certain problems have been found. They are expressed through the Mittag–Leffler function [3], which is a series expansion, and thus require numerical tools to be computed. For this issue and for the other unsolved problems, the literature provides many ways to numerically solve fractional differential problems. Most of the methods employ the quadrature rule to compute the fractional derivatives [4]; others use spectral or Galerkin methods [5].

In recent papers [5–7], the authors proved that the multiscale collocation methods are easy and efficient to implement, when using certain fractional refinable functions introduced in [6,8]. In fact, these functions not only generate a multiresolution on $\mathbb{R}$, but also satisfy a fractional derivative convenient formula. Moreover, the collocation technique allows one to obtain an algebraic system from a differential problem.

The coefficient matrix is given by the collocation of basis functions into the collocation nodes. In this way, the result is given by the solution (often in a least-squares sense) of a linear algebraic system. The goal of this paper is to prove further approximating properties of this class of fractional refinable functions with respect to [6,8], suitable to the solution of fractional differential problems.

More precisely, in [6,8], we proved the basis properties of the class $\varphi_{\alpha,h}$, for $\alpha > 2$. The novelty of this paper, is that here we prove that these properties are also valid for $\alpha > 1$ and that other important approximating and smoothing properties can be proved, e.g., the order of polynomial reproducibility. In this way, we enlarge the class of fractional refinable functions from $\alpha > 2$ to $\alpha > 1$ and thus, also its applicability to *a wider class of fractional differential problems* . Furthermore, we prove that all the properties derive from a suitable convolution formula. Note that when we apply these functions to a differential problem with fractional derivative γ, we have to choose refinable functions of approximation order α such that $\alpha - \gamma > 1$.

The paper is organized as folllows. Section 3 introduces some fractional derivative definitions, that can be computed by numerical quadrature rules. We choose the Caputo derivative for several reasons: computational efficiency, minor regularity required, stability [2]. Section 4 explains Multiresolution Analysis (MRA) properties on $\mathbb{R}$ and on the interval. Section 5 describes the collocation and the Galerkin methods constructed with MRA. Section 6 lists the main properties of the fractional B-splines, introduced in [9,10], emphasizing the fractional derivative properties. Section 7 describes the new class of fractional refinable functions constructed introduced by [6,8], through a convolution formula involving the functions in [11] and with a continuos dependence from a parameter h. We prove that these functions satisfy new properties that are similar to those of the fractional B-splines, such as, for example, the polynomial reproducibility. Furthermore, we prove a differentiation formula that makes them particularly interesting in the fractional derivative context. In the conclusions, we explain all the advantages of this new class of fractional refinable functions, including an example on polynomial reproducibility.

2. Fractional Derivatives

The fractional derivative can be defined in many ways: for example, in the Caputo sense or in the Riemann Liouville way.

The Caputo definition of the *fractional derivative* is:

$$ {}_cD_t^\gamma\, y(t) := \big(\mathcal{J}^{(k-\gamma)} y^{(k)}\big)(t)\,, \qquad k-1<\gamma<k\,, \quad k\in\mathbb{N}\,, \qquad t>0\,, \tag{1}$$

where $\mathcal{J}^{(\beta)}$ is the *Riemann–Liouville integral operator*

$$ \big(\mathcal{J}^{(\beta)} y\big)(t) := \frac{1}{\Gamma(\beta)} \int_0^t y(\tau)\,(t-\tau)^{\beta-1}\,d\tau \quad \beta\in\mathbb{R}, \tag{2}$$

and Γ denotes Euler's gamma function

$$ \Gamma(\beta) := \int_0^\infty \tau^{\beta-1}\,\mathrm{e}^{-\tau}\,d\tau\,. \tag{3}$$

Hence,

$$ {}_cD_t^\gamma\, y(t) := \frac{1}{\Gamma(k-\gamma)} \int_0^t y^{(k)}(\tau)\,(t-\tau)^{k-\gamma-1}\,d\tau, \quad k=\lceil\gamma\rceil. \tag{4}$$

For example, if $\gamma=0.5$ then $k=1$ and:

$$ {}_cD_t^{0.5} y(t) := \frac{1}{\Gamma(0.5)} \int_0^t \frac{y'(\tau)}{\sqrt{(t-\tau)}}\,d\tau\,. \tag{5}$$

If, for example, $y(t)=t^n$ then

$$ {}_cD_t^\gamma y(t) := \frac{\Gamma(n+1-\gamma)}{\Gamma(n+1)} t^{n-\gamma}. \tag{6}$$

Riemann–Liouville definition is instead

$$ {}_{RL}D\gamma\, y(t) := \frac{d^k}{dt^k}\,\big(\mathcal{J}^{(\gamma)} y\big)(t)\,, \qquad t>0\,. \tag{7}$$

They both reduce to the usual differential operator when $\gamma\in\mathbb{N}$. In the general case, we have the following relation between the Caputo and the Riemann derivatives

$$ {}_cD^\gamma\, y(t) ={}_{RL} D^\gamma \left(y(t) - \sum_{l=0}^{k} \frac{t^l}{l!} y^{(l)}(0^+) \right). \tag{8}$$

The definitions coincide for homogenous boundary initial conditions.

In the Fourier domain one has

$$\mathcal{F}\big(D^\gamma, y(t)\big) = (i\omega)^\gamma \mathcal{F}\big(y(t)\big), \qquad \gamma \in \mathbb{R}^+, \omega \in \mathbb{C} \tag{9}$$

where $\mathcal{F}(y)$ is the Fourier transform of the function y.

3. MRA and Refinable Spaces

A sequence of functional spaces $\{V_j, j \in \mathbb{Z}\} \subset L^2(\mathbb{R})$, forms a multiresolution analysis (MRA) of $L^2(\mathbb{R})$ if

1. $V_j \subset V_{j+1}, j \in \mathbb{Z}$,
2. $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;
3. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
4. $f(t) \in V_j \leftrightarrow f(2t) \in V_{j+1}, j \in \mathbb{Z}$;
5. there exists a $L^2(\mathbb{R})$-stable basis in V_0.

MRA Based on Refinable Functions

An MRA can be generated by a *refinable function* ϕ, i.e., a function that satisfies a *refinement functional equation*

$$\phi(t) = \sum_{k\in\mathbb{Z}} a_k\, \phi(2\,t - k), \qquad t \in \mathbb{R}. \tag{10}$$

It is known that if the mask coefficients $\{a_k,\ k \in \mathbb{Z}\}$ form a finite sequence and have some particular properties, then the existence of a unique solution to (10) in $L^2(\mathbb{R})$, can be proved [12]. Moreover, the shifted refinable functions $\{\phi(t-k), k \in \mathbb{Z}\}$ give rise to a stable basis in V_0, i.e., the space they span.

It is important to associate (10) with its symbol

$$b^n(z) = \sum_k a_k\, z^k$$

When the mask is an infinity sequence, under suitable conditions the solution exists as proved in [8].

Now, we can define the spaces V_j of the multiresolution:

$$V_j := \overline{\text{span}\,\{\phi_{jk}(t) := \phi(2^j\,t - k)\,, k \in \mathbb{Z}\}}\,, \qquad j \in \mathbb{Z}\,, \qquad t \in \mathbb{R}\,. \tag{11}$$

Since we are taking into account differential problems of order n with initial conditions, it is also important to define an MRA on an interval $[0, T]$, belonging to Ł$^2([0, T])$.

Let us suppose that the support of ϕ is compact, i.e., $supp\,\phi = [0, \sigma]$. Then, we can define an MRA on the interval.

$$V_j^0[0,T] = \overline{\text{span}\,\{\phi^0_{jk}(t)\,, k \in \mathcal{N}_j\}}\,, \quad j \geq j_0\,, \quad t \in [0,T]\,, \tag{12}$$

where

$\phi^0_{jk}(t) := \{\phi_{jk}|_{[0,T]} : \phi_{jk}(0) = \phi'_{jk}(0) = \cdots = \phi^{(n-1)}_{jk}(0) = 0\}$;

$\mathcal{N}_j \subset \mathbb{Z}$, with $\#\mathcal{N}_j = N_j = 2^j + \sigma - 1$, is the set of admissible index k and j_0 is the initial multiresolution scale, i.e., the minimal index such that $\text{supp}\,\phi^0_{j_0 0} \subset [0, \sigma]$.

4. The Collocation Method and the Galerkin Method

We use the MRA to approximate each fractional differential problem by the collocation method and Galerkin method. For both methods, we pose the solution in the following form:

$$y_j(t) = \sum_{k \in \mathcal{N}_j} c_{jk}\, \zeta_{jk}(t) \tag{13}$$

where ζ_{jk} is a refinable function generating an MRA.

- In the collocation method, we substitute (13) in the differential problem and we collocate it in the dyadic nodes $\{t_p = p/2^s,\, p = 0, \ldots, N_s, s \geq j\}$. So, we obtain a linear algebraic system in N_s equations and N_j unknowns $\{c_{jk}\}_{k=1}^{N_j}$. Usually, we solve the system by least-squares method.
- In the Galerkin method, we rewrite the differential problem in a weak form, and we substitute (13), using ζ_{jk} as trial and test functions. In this way, the resulting linear algebraic system, will contain as the coefficient matrix, the integrals between u and the test (trial) functions ζ_{jk} (Stiffness matrix).

In this way, the differential problem is converted into a system of algebraic equations that is suitable for computer programming.

Note.

If u also depends on x, i.e., $u(t,x)$, then the coefficients are $c_{jk} = c_{jk}(x)$ [5,6].

5. Fractional B-Splines

A particular class of refinable functions is provided by the cardinal B-Splines of degree n, i.e., functions that are positive and compactly supported in $[0, n+1]$, in each interval of the partition are polynomials of degree at most n and in $\mathbb{R}$ have regularity $C^{n-1}(\mathbb{R})$. The Fourier transform of the classical B-Splines is:

$$\hat{B}_n(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega}\right)^{n+1}, \qquad n = 0, 1, \cdots \tag{14}$$

We can define a *fractional B-Spline* starting with its Fourier transform obtained introducing a fractional (non-integer) exponent in (14):

$$\hat{B}_\alpha(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega}\right)^{\alpha+1}, \quad \alpha > -1 \tag{15}$$

It is proven that for $\alpha > -1$, the antitransform B_α is in $L^1(\mathbb{R})$, while B_α is in $L^2(\mathbb{R})$ for $\alpha > -1/2$ [9].

In the time domain, the cardinal B-Splines B_n, are defined in the following way. Let $t_+ := \max(0, t)$ be the usual truncated power function and the finite difference operator

$$\Delta^n\, v(t) := \sum_{k \in \mathbb{N}_0} (-1)^k \binom{n}{k} v(t-k)\,, \qquad n \in \mathbb{N} \tag{16}$$

Then, $B_n(t)$ can be defined as:

$$B_n(t) := \frac{\Delta^{n+1}\, t_+^n}{(n+1)!}, \tag{17}$$

whose *symbol* is

$$b^n(z) = \frac{1}{2^n}(1+z)^{n+1}$$

In the *non-integer case*, we define the *generalized finite difference operator*

$$\Delta^\gamma v(t) := \sum_{k\in\mathbb{N}_0} (-1)^k \binom{\gamma}{k} v(t-k), \qquad \gamma \in \mathbb{R}^+. \tag{18}$$

When $\gamma \in \mathbb{N}$, $\{\binom{\gamma}{k}\}_k$ is a compactly supported sequence and we get the usual finite difference operator.

On the other hand, when $\gamma \in \mathbb{R}^+ \backslash \mathbb{N}$, then

$$\binom{\gamma}{k} := \frac{\Gamma(\gamma+1)}{k!\,\Gamma(\gamma-k+1)} = O(k^{-\gamma-1}), \qquad k \in \mathbb{N}_0, \quad \gamma \in \mathbb{R}^+$$

and thus the sequence $\{\binom{\gamma}{k}\}_k$ is absolutely summable and the limit of the series (18) exists under suitable hypothesis on v. [9]

The *fractional B-spline*, i.e., the B-spline of non-integer order, in the time domain is defined as:

$$B_\alpha(t) := \frac{\Delta^{\alpha+1}\, t_+^\alpha}{\Gamma(\alpha+1)}, \qquad \alpha > -\frac{1}{2}, \tag{19}$$

The following theorem writes, with a different proof with respect to [9].

Theorem 1. *The fractional derivative of a B-Spline is a fractional B-Spline. More precisely,*

$$D^\gamma B_n(x) = \frac{\Delta^{n+1} t_+^{n-\gamma}(x)}{\Gamma(n+1-\gamma)} = \Delta^\gamma B_{n-\gamma}(x) \tag{20}$$

In fact, one has

$$D^\gamma B_n(x) = D^\gamma \frac{\Delta^{n+1} t_+^n(x)}{(n+1)!} = \Delta^{n+1} \frac{D^\gamma t_+^n(x)}{(n+1)!} =$$

$$\frac{\Gamma(n+1)}{\Gamma(n+1-\gamma)} \frac{\Delta^{n+1} t_+^{n-\gamma}(x)}{(n+1)!} = \frac{\Delta^{n+1} t_+^{n-\gamma}(x)}{\Gamma(n+1-\gamma)}$$

Proof. Now, for the rule of the difference finite operator composition

$$\Delta^\gamma \Delta^{n-\gamma+1} = \Delta^{n+1},$$

it is easy to verify that

$$\frac{\Delta^{n+1} t_+^{n-\gamma}(x)}{\Gamma(n+1-\gamma)} = \frac{\Delta^\gamma \Delta^{n-\gamma+1} t_+^{n-\gamma}(x)}{\Gamma(n+1-\gamma)} = \Delta^\gamma B_{n-\gamma}(x).$$

The theorem is proved. □

It is also worthwhile to define the symbol b^α of B_α, i.e.,

$$b^\alpha(z) = \frac{1}{2^\alpha}(1+z)^{\alpha+1}$$

5.1. Main Properties of Fractional B-Splines

In the study by [9], fractional B-splines are introduced for the first time and their main properties are proved. We summarize these properties in the following propositions avoiding the proof.

Proposition 1. *The fractional B-Splines B_α belong to $L^1(\mathbb{R})$, for $\alpha > -1$ and to the Sobolev space $W_2^r(\mathbb{R})$, $0 \leq r < \alpha + \frac{1}{2}$, for $\alpha > -\frac{1}{2}$; where $W_2^r(\mathbb{R})$, represents the Banach subspace of $L^2(\mathbb{R})$, equipped with the norm*

$$||f||_r = ||f||_2 + ||D^r f||_2$$

Proposition 2. *When $\alpha > -1/2$, the fractional B-Splines are α-order continuous, i.e., they can be derived up to the order α but ∂^α is in general only bounded.*

Moreover, they generate an MRA of $L^2(\mathbb{R})$

Proposition 3. *The fractional B-splines reproduce polynomials up to degree $\lceil \alpha \rceil$, but they do not satisfy Strang and Fix theory. In fact, they have fractional approximation order $\alpha + 1$, instead of $\lceil \alpha \rceil + 1$.*

For the CAGD and isogeometric context, it is important to know that they form a partition of unity for $\alpha > -1$.

Proposition 4. *It is also important to consider the following fractional derivation rule that is a generalization of the Formula (20)*

$$D^\gamma (B_\alpha) = \Delta^\gamma B_{\alpha-\gamma} \tag{21}$$

where D^γ is the usual derivative of order γ.

There is also a formula that allows us to assume that a fractional B-spline preserves the order of approximation of any refinable function of order α.

Proposition 5. *Let ϕ_α be a refinable function generating an MRA in $L^2(\mathbb{R})$, of order of approximation α. Then, ϕ_α can be factorized as*

$$\phi_\alpha = B_\alpha * \phi_0, \tag{22}$$

$\alpha \geq 0$ and ϕ_0 is a distribution such that $\int \phi_0 = 1$ [10].

Let us observe that all the previous propositions can be proved by starting from Proposition 5.

5.2. *Fractional Derivative of Refinable Functions*

If we consider a generic function ϕ_α of order α, it is possible to generalize the differentiation rule (21).

In fact, let it be that $\phi_0 \in \mathcal{C}^0(\mathbb{R})$, then $\phi_\alpha \in \mathcal{C}^{\lceil \alpha \rceil}(\mathbb{R})$ and

$$D^\gamma \phi_\alpha = D^\gamma (B_\alpha * \phi_0) = \Delta^\gamma (B_{\alpha-\gamma} * \phi_0) = \Delta^\gamma \phi_{\alpha-\gamma}, \qquad 0 < \gamma \leq \alpha. \tag{23}$$

The claim follows from some results in [10].

For shifted functions $\phi_{\alpha,k}(t)$, we obtain a similar result.

Proposition 6. *Let $\phi_{\alpha,k}(t) := \phi_\alpha(t-k)$. Then,*

$$D_t^\gamma \phi_{\alpha,k} = \Delta^\gamma \phi_{\alpha-\gamma,k}, \qquad 0 < \gamma \leq \alpha. \tag{24}$$

Let us note that since $\phi \in L^2(\mathbb{R})$ and the generalized binomial coefficients decay similar to $k^{-\gamma-1}$ as $k \to +\infty$, thus the series in (24) converges. Thus, in practical computation, $\Delta^\gamma \phi_{\alpha-\gamma,k}$ is a finite sum.

6. Fractional GP Refinable Functions

We present here the main results regarding a new class of refinable functions of fractional order α, obtained starting by a suitable refinable function (of support $[0,2]$) introduced in [11]. We consider

$$\phi_{\alpha,h} = \frac{1}{2} B_{\alpha-2} * \phi_{1,\hat{h}}\,, \qquad 0 \le \alpha \le h\,, \tag{25}$$

where $\phi_{\hat{h}} \in L^2(\mathbb{R})$ is the *elementary refinable function*, solution of the refinement equation

$$\phi_{1,\hat{h}}(t) = \sum_{k=0}^{2} a_{\hat{h},k}\, \phi_h(2t-k)\,, \qquad t \in \mathbb{R}\,, \tag{26}$$

with mask coefficients in [11] and $\hat{h} = h - \alpha + 1$. h is a real *shape parameter* that controls the shape of $\phi_{\alpha,h}$. The symbol of $\phi_{n,h}$ in general is

$$b_h^n(z) = \frac{1}{2^h}[(1+z)^{n+1} + 4(2^{h-n} - 1)z(1+z)^{n-1}].$$

that, for $n = 1$ reduces to

$$b_h^1(z) = \frac{1}{2^h}[(1+z)^2 + 4(2^{h-1} - 1)z].$$

In the Fourier domain, the definition of $\phi_{\alpha,h}$ becomes:

$$\mathcal{F}(\phi_{\alpha,h})(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega}\right)^{\alpha-1} \hat{\phi}_{\hat{h}}(\omega)$$

We observe that when $\alpha \in \mathbb{N}$, $\alpha \ge 0$, then $\phi_{\alpha,h}$ is compactly supported, belongs to $\in C^{\alpha-1}(\mathbb{R})$ and is a GP function as in [11]; in particular for $h = \alpha$ it reduces to a cardinal B-Spline. Instead, when α is not an integer but $h = \alpha$, then $\phi_{\alpha,\alpha}$ is a fractional B-spline in [9].

It is easy to show that $\phi_{\alpha,h}$ can be also obtained by placing a fractional index in the mask of $\phi_{n,h}$, i.e., $a_{\alpha,h,k} = \frac{1}{h}\left[\binom{\alpha+1}{k} + 4(2^{h-\alpha} - 1)\binom{\alpha-1}{k-1}\right]$ and, in this case $\phi_{\alpha,h}$ becomes:

$$b_h^\alpha(z) = \frac{1}{2^h}[(1+z)^{\alpha+1} + 4(2^{h-\alpha} - 1)z(1+z)^{\alpha-1}]$$

Therefore, it is not difficult to prove that:

$$b_h^\alpha(z) = \frac{1}{2} b^{\alpha-2}(z)\, b_{\hat{h}}^1(z) \tag{27}$$

where

$$b^{\alpha-2}(z) = \frac{1}{2^{\alpha-2}}(1+z)^{\alpha-1} \quad \text{and } \hat{h} = h - \alpha + 1$$

In fact,

$$b_h^\alpha(z) = \frac{1}{2^h}(1+z)^{\alpha-1}[(1+z)^2 + (2^{h-\alpha+2}) - 2^2)z] =$$
$$= \frac{1}{2^{\alpha-2}} \frac{1}{2^{h-\alpha+2}}(1+z)^{\alpha-1}[(z^2 + (2^{h-\alpha+2}) - 2)z + 1].$$

Observe that from (29), we deduce that $B_{\alpha-2}$ carries all the approximation properties of $\varphi_{\alpha,h}$. In fact, since $\phi_{\hat{h}}$ is summable, the convolution preserves all the properties of $B_{\alpha-2}$. So, we have the following theorem,

Theorem 2. *For any admissible α and h, $\phi_{\alpha,h}$ belongs to $C^{\lfloor\alpha\rfloor-2}(\mathbb{R})$ (and decays to the infinity rather rapidly so that in practice it can be assumed compactly supported).*

Moreover, it has derivative $\partial^{\alpha-1}$, but it is only bounded, not necessary continuous; one says that it is α- continous. As for the order of approximation, $\phi_{\alpha,h}$ has order of approximation $\alpha-1$ *and order of polynomial reproducibility $\lceil\alpha\rceil-1$; so it does not verify the Strang and Fix theory.*

Finally, the differentiation rule is specified in

$$D_t^\gamma\,\phi_{\alpha,h}(t)=\Delta^\gamma\,\phi_{\alpha-\gamma,h-\alpha+2}(t)=$$

$$\textstyle\sum_{k\in\mathbb{N}_0}(-1)^k\binom{\alpha}{k}\phi_{\alpha-\gamma,h-\alpha+2}(t-k),\quad 0<\gamma\le\alpha\,.$$

Proof. The properties of $\phi_{\alpha,h}$ are the same properties of $B_{\alpha-2}$ [9] that are preserved through the convolution Formula (27) since $\phi_{1,\hat{h}}$ is summable. □

7. Conclusions

Since, as in the classical B-spline case, the fractional derivative of a GP refinable function is a GP fractional refinable function, we deal in this paper with fractional GP functions stemming from the fractional derivative of GP refinable functions. In this way, we obtain a class of refinable functions, closed with respect to the fractional derivative.

Another advantage of these fractional GP refinable functions ϕ_h^α with respect to the GP refinable function, is that, in practice, due to the rapid decay of ϕ_h^α, their supports appear strictly contained in the supports $[0,n+1]$ of $\phi_{n,h}$, but the order of exactness is the same, i.e., $n-1$. This property, in addition to derivative Formulas (23) and (24), renders them highly suitable for solving fractional differential problems, as shown in [5,6].

More precisely, if, for example, we consider ϕ_h^α, with $\alpha=1.5$, then the order of polynomial reproducibility is $\lceil\alpha\rceil-1=1$, that is the straight line can be reproduced, in the same manner as classical GP refinable, when $n=1$, and support $[0,2]$.

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