ASYMPTOTIC ANALYSIS FOR NON-LOCAL PROBLEMS IN COMPOSITES WITH DIFFERENT IMPERFECT CONTACT CONDITIONS

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ABSTRACT. We consider a composite material made up of a hosting medium containing an ε-periodic array of perfect thermal conductors. Comparing with the previous contributions in the literature, in the present paper, the inclusions are completely disconnected and form two families with dissimilar physical behaviour. More specifically, the imperfect contact between the hosting medium and the inclusions obeys two different laws, according to the two different types of inclusions. The contact conditions involve the small parameter ε and two positive constants D_1, D_2. We investigate the homogenization limit $\varepsilon \to 0$ and the limits for D_1, D_2 going to 0 or $+\infty$, taken in any order, with the aim to find out the cases in which the two limits commute.

KEYWORDS: Homogenization, time-dependent unfolding, total flux boundary conditions, imperfect contact conditions, parabolic problems, exchange of asymptotic limits.
AMS-MSC: 35B27, 35Q79, 35K20

1. INTRODUCTION

In this paper, we consider a composite made by a plastic material containing a periodic array of heat conductors, having infinite thermal conductivity. Such materials are widely used in practical applications, as remarked in [27, 30, 34, 35, 37, 39]. For the mathematical aspects behind the behaviour of these kinds of materials, we refer

to [2, 14, 15, 16, 17, 22, 24, 28, 29, 31, 32, 33], where such problems are studied in various contexts like heat diffusion, electric conduction, petroleum engineering industry, wave equations or elastic properties of perforated materials.

A crucial feature of our and similar models is represented by the fact that, while the temperature $u_\varepsilon^{\text{out}}$ in the hosting medium satisfies a standard heat equation, inside each inclusion the temperature $u_\varepsilon^{\text{int}}$ depends only on time and is governed by an ordinary differential equation, involving a non-standard condition of non-local type, in which the time-variation of the temperature of the inner phase is determined by the global thermal flux coming from the outer phase (see, for instance, [6, 7, 11]). However, differently from the previous contributions, where only one type of inclusions was considered, in the present paper we address the case in which we have two types of fillers, having dissimilar thermal features.

The thermal potentials $u_\varepsilon^{\text{out}}$ of the hosting medium and $u_\varepsilon^{\text{int},1}$, $u_\varepsilon^{\text{int},2}$ of the two fillers, respectively, are coupled through imperfect contact transmission conditions across the interface between the different conductive phases of the medium. Actually, the models treated in the references quoted at the beginning of the Introduction deal with perfect contact conditions which can be seen as an approximation of the more realistic imperfect contact conditions. The imperfect contact conditions we propose here involve the microscopic geometry of the problem through the characteristic length ε of the inclusions and two amplitude factors, similarly as described in [11], where a hierarchy of possible scalings in the period of the microscopic geometry is considered. As a consequence of the investigation in [11], only two scalings result to be critical, in the sense that they preserve memory of the amplitude factor in the macroscopic model, obtained after passing to the limit $\varepsilon \to 0$. It is worthwhile to notice that hierarchies of scalings are often considered in homogenization problems (see, for instance, [5, 13, 16, 25, 29, 36] and the references therein).

Therefore, in the present paper, we focus our analysis on the two critical scalings mentioned above, prescribing the first one in the contact condition of the first family of inclusions and the second one in the contact condition of the other family of inclusions. Moreover, in each one of these two contact conditions it appears a different amplitude factor, D_1 and D_2, respectively. More precisely, the jump $[u_\varepsilon]$ between the hosting material and the inclusions is proportional to the external heat flux through the coefficient $D_1 \varepsilon^{-1}$ and $D_2 \varepsilon$, respectively (see (2.6), (2.7)).

The presence of these two types of parameters (the microscopic scaling and the amplitude factors) justifies the need to perform different limits.

One of the goals of the paper is to perform the homogenization limit $\varepsilon \to 0$, in order to obtain the macroscopic description of the material. To this end, we apply a procedure based on the periodic unfolding technique (see, among others, [21]). The other goal of the paper is to perform the limit in D_1 and D_2. We consider not only the case where D_1 and/or D_2 go to 0, for which the perfect contact condition is recovered (see [6, 11]), but also the case where D_1 and/or D_2 go to $+\infty$, which accounts for insulated materials.

The limits in ε and in D_1, D_2 are considered in different orders, with the aim of comparing the final results and identifying the cases in which the two limits commute. This happens only in the case $D_1 \to +\infty$ and $D_2 \to 0$, in which we obtain a sort

of bidomain model, where the temperature of the leading phase is governed by a parabolic equation and in the other phase the temperature satisfies an algebraic condition. This can be explained by taking into account that, in this case, the behaviour of the two amplitude parameters is in agreement with the one of the scaling factor depending on ε, i.e. we have $D_1\varepsilon^{-1}$ and $D_2\varepsilon$ in (2.6) and (2.7), respectively. The commutativity of the limits makes the starting model more robust, because one can choose the three parameters (ε, D_1 and D_2) essentially independently (close to their limiting value). For all the rest of the cases, there is no commutation and the limit problems involve a monodomain, a bidomain or even a tridomain structure (see Section 8, in which we provide a summary of all our results).

From the technical point of view, the main difficulty of the paper, also due to the non-locality of the problem, resides in combining different homogenization techniques, in order to treat the different contributions of the two inclusions. The homogenization procedure leads to a non-variational limit problem, which requires a careful analysis. Besides, even the limits in D_1, D_2 involve some non-standard arguments, used in the investigation of the cell functions behaviour.

The paper is organized as follows: in Section 2, we define our geometrical setting and we state the microscopic problem. Section 3 contains some preliminaries about the unfolding techniques and the main compactness results. Section 4 is devoted to the construction of the cell functions needed for the homogenization procedure. In Sections 5 and 6, we provide the homogenization result and the limits of the obtained problem with respect to the two amplitude parameters, respectively. In Section 7, we perform the limit in D_1, D_2 and then in ε, namely in the opposite order with respect to what we have done in the previous two sections. Finally, in Section 8, we summarize and compare the results we obtained in Sections 6 and 7.

2. Statement of the problem

Let $Y = (0,1)^N$ denote the unit cell of $\mathbb{R}^N$, $N \geq 2$, and consider a smooth bounded subset $E \subset \mathbb{R}^N$, which we assume to be periodic in the sense that $E + z = E$ for all $z \in \mathbb{Z}^N$, setting also $E_{\text{int}} = E \cap Y$, $E_{\text{out}} = Y \setminus \overline{E}$, $\Gamma = \partial E \cap Y$. We suppose that $\overline{E_{\text{int}}} \subset Y$, implying that $\partial E_{\text{int}} = \Gamma$, and we assume that $E_{\text{int}} = E_{\text{int},1} \cup E_{\text{int},2}$, where $\overline{E_{\text{int},1}}$ and $\overline{E_{\text{int},2}}$ are two disjoint connected sets, whose boundaries are Γ_1 and Γ_2, respectively, so that $\Gamma = \Gamma_1 \cup \Gamma_2$. Notice that, in particular, also Γ_1 and Γ_2 are disjoint sets.

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and, for a given $T > 0$, define $\Omega_T = \Omega \times (0, T)$. For any $\varepsilon \in (0,1)$, we define the set

$$\Xi^\varepsilon = \left\{ \xi \in \mathbb{Z}^N, \quad \varepsilon(\xi + Y) \subset \Omega \right\},$$

and, for $\xi \in \Xi^\varepsilon$ and $i = 1, 2$, we introduce

$$E_{\text{int},i}^{\varepsilon,\xi} := \varepsilon(E_{\text{int},i} + \xi), \qquad \Gamma_{\xi,i}^\varepsilon := \partial E_{\text{int},i}^{\varepsilon,\xi}.$$

Moreover, for $i = 1, 2$, we let

$$\Omega_{\text{int},i}^\varepsilon = \bigcup_{\xi \in \Xi^\varepsilon} E_{\text{int},i}^{\varepsilon,\xi}, \quad \Gamma_i^\varepsilon = \partial \Omega_{\text{int},i}^\varepsilon = \bigcup_{\xi \in \Xi^\varepsilon} \Gamma_{\xi,i}^\varepsilon, \quad \Omega_{\text{int}}^\varepsilon = \Omega_{\text{int},1}^\varepsilon \cup \Omega_{\text{int},2}^\varepsilon,$$

$$\Omega_{\text{out}}^\varepsilon = \Omega \setminus \overline{\Omega_{\text{int}}^\varepsilon}, \quad \Gamma^\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon.$$

We remark that $\Omega_{\text{out}}^\varepsilon$ is connected, while $\Omega_{\text{int}}^\varepsilon$ is disconnected. According to the applications outlined in the Introduction, $\Omega_{\text{out}}^\varepsilon$ and $\Omega_{\text{int}}^\varepsilon$ represent the hosting medium and the perfect conductive inclusions, respectively (see Figure 1). Finally, let ν denote the normal unit vector to Γ pointing into E_{out}, extended by periodicity to the whole of $\mathbb{R}^N$, so that $\nu_\varepsilon(x) = \nu(x/\varepsilon)$ denotes the normal unit vector to Γ^ε pointing into $\Omega_{\text{out}}^\varepsilon$.
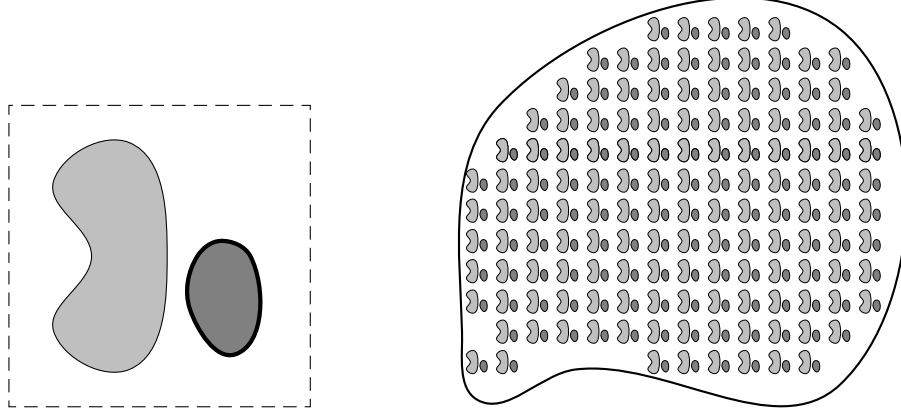

FIGURE 1. *On the left:* the unit cell Y. *On the right:* the domain Ω.

In the following, by γ we shall denote a strictly positive constant, independent of ε, which may vary from line to line, and we denote by χ_O the characteristic function of the set $O \subseteq \mathbb{R}^N$.

For the unknown u_ε, we introduce the following piecewise notation:

$$u_\varepsilon = \begin{cases} u_\varepsilon^{\text{int},1}, & \text{in } \Omega_{\text{int},1}^\varepsilon \times (0,T), \\ u_\varepsilon^{\text{int},2}, & \text{in } \Omega_{\text{int},2}^\varepsilon \times (0,T), \\ u_\varepsilon^{\text{out}}, & \text{in } \Omega_{\text{out}}^\varepsilon \times (0,T), \end{cases} \tag{2.1}$$

where $u_\varepsilon^{\text{int},i}$, $i = 1, 2$, and $u_\varepsilon^{\text{out}}$ represent the thermal potentials (or the temperatures) of the three phases. Accordingly, the jump of such a function across $\Gamma_i^\varepsilon \times (0,T)$, $i = 1, 2$, is denoted by $[u_\varepsilon] = u_\varepsilon^{\text{out}} - u_\varepsilon^{\text{int},i}$. The same notation will be used for other functions, namely for test functions φ_ε which may exhibit jumps across $\Gamma^\varepsilon \times (0,T)$. The thermal diffusion properties of the medium under study are described by the Y-periodic symmetric matrix

$$\kappa_\varepsilon(x) = \kappa\left(\frac{x}{\varepsilon}\right),$$

where $\kappa = (\kappa_{ij})$, with $\kappa_{ij} \in L^\infty(Y)$, is such that there exists a constant $\gamma_0 \geq 1$ satisfying

$$\gamma_0^{-1}|\zeta|^2 \leq \kappa(y)\zeta \cdot \zeta \leq \gamma_0|\zeta|^2, \qquad \text{for a.e. } y \in Y, \ \forall \zeta \in \mathbb{R}^N.$$

Further, for two given constants $\lambda_1, \lambda_2 > 0$, we set

$$a_\varepsilon(x) = \begin{cases} \dfrac{\lambda_1}{|E_{\text{int},1}|}, & \text{in } \Omega_{\text{int},1}^\varepsilon, \\[2mm] \dfrac{\lambda_2}{|E_{\text{int},2}|}, & \text{in } \Omega_{\text{int},2}^\varepsilon, \\[2mm] 1, & \text{in } \Omega_{\text{out}}^\varepsilon, \end{cases} \tag{2.2}$$

which accounts for the capacities of the three phases.

We state, now, the problem for u_ε:

$$\frac{\partial u_\varepsilon^{\text{out}}}{\partial t} - \text{div}(\kappa_\varepsilon \nabla u_\varepsilon^{\text{out}}) = f, \qquad\qquad \text{in } \Omega_{\text{out}}^\varepsilon \times (0,T); \tag{2.3}$$

$$\lambda_1 \frac{\partial u_\varepsilon^{\text{int},1}}{\partial t} = \frac{1}{\varepsilon^N} \int_{\Gamma_{\xi,1}^\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon^{\text{out}} \cdot \nu_\varepsilon \, d\sigma, \qquad \text{in } E_{\text{int},1}^{\varepsilon,\xi} \times (0,T), \ \xi \in \Xi^\varepsilon; \tag{2.4}$$

$$\lambda_2 \frac{\partial u_\varepsilon^{\text{int},2}}{\partial t} = \frac{1}{\varepsilon^N} \int_{\Gamma_{\xi,2}^\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon^{\text{out}} \cdot \nu_\varepsilon \, d\sigma, \qquad \text{in } E_{\text{int},2}^{\varepsilon,\xi} \times (0,T), \ \xi \in \Xi^\varepsilon; \tag{2.5}$$

$$[u_\varepsilon] = \frac{D_1}{\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon^{\text{out}} \cdot \nu_\varepsilon, \qquad\qquad \text{on } \Gamma_1^\varepsilon \times (0,T); \tag{2.6}$$

$$[u_\varepsilon] = D_2 \varepsilon \kappa_\varepsilon \nabla u_\varepsilon^{\text{out}} \cdot \nu_\varepsilon, \qquad\qquad \text{on } \Gamma_2^\varepsilon \times (0,T); \tag{2.7}$$

$$u_\varepsilon = 0, \qquad\qquad \text{on } \partial\Omega \times (0,T); \tag{2.8}$$

$$u_\varepsilon(x,0) = \overline{u_\varepsilon}(x), \qquad\qquad \text{in } \Omega. \tag{2.9}$$

We assume $D_1, D_2 \in (0, +\infty)$, $f \in L^2(\Omega_T)$ and $\overline{u_\varepsilon} \in L^2(\Omega)$. The initial datum $\overline{u_\varepsilon}$ is assumed to be constant in each $E_{\text{int},i}^{\varepsilon,\xi}$, $i = 1,2$, with possibly different values, and such that $\overline{u_\varepsilon} \to \overline{u}$ strongly in $L^2(\Omega)$, as $\varepsilon \to 0$.

This choice of the initial datum $\overline{u_\varepsilon}$, together with (2.4) and (2.5), implies that the function u_ε depends only on (ξ,t) in each component $E_{\text{int},i}^{\varepsilon,\xi}$ of $\Omega_{\text{int}}^\varepsilon$, so that it is piecewise constant in $\Omega_{\text{int}}^\varepsilon$.

Remark 2.1. We point out that one can consider also other scalings with respect to ε in (2.6) and (2.7) (see [11]), leading to different macroscopic problems; however, we decided to treat here only the present case, which seems to be the most interesting one, as pointed out in the Introduction. We also remark that our homogenization results can be obtained, without additional difficulties, even for a diffusion matrix $\kappa = \kappa(x,y)$, Y-periodic in the second variable and sufficiently smooth in the first one. However, the extension of the results proved in Section 6 to this case seems to require a more technical analysis. $\qquad\square$

By integrating, formally, by parts, we arrive at the following rigorous formulation of the problem (2.3)–(2.9).

Definition 2.2. Let the function u_ε be as in (2.1) and, with a slight abuse of notation, $u_\varepsilon^{\mathrm{int},i}(x,t) = u_\varepsilon^{\mathrm{int},i}(\xi,t)$ in $E_{\mathrm{int},i}^{\varepsilon,\xi}$ and $u_\varepsilon^{\mathrm{int},i} \in L^2(\Omega_{\mathrm{int},i}^\varepsilon \times (0,T))$, $i = 1,2$, $u_\varepsilon^{\mathrm{out}} \in L^2(0,T;H^1(\Omega_{\mathrm{out}}^\varepsilon))$, $[u_\varepsilon] \in L^2(\Gamma^\varepsilon \times (0,T))$, $u_\varepsilon^{\mathrm{out}} = 0$ on $\partial\Omega$. Then, u_ε is said to be a weak solution of problem (2.3)–(2.9) if

$$\int_0^T \int_\Omega \{-a_\varepsilon u_\varepsilon \varphi_{\varepsilon,t} + \chi_{\Omega_{\mathrm{out}}^\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon^{\mathrm{out}} \cdot \nabla \varphi_\varepsilon^{\mathrm{out}}\} \,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon}{D_1} \int_0^T \int_{\Gamma_1^\varepsilon} [u_\varepsilon][\varphi_\varepsilon] \,\mathrm{d}\sigma\,\mathrm{d}t$$

$$+ \frac{1}{D_2\varepsilon} \int_0^T \int_{\Gamma_2^\varepsilon} [u_\varepsilon][\varphi_\varepsilon] \,\mathrm{d}\sigma\,\mathrm{d}t = \int_0^T \int_{\Omega_{\mathrm{out}}^\varepsilon} f\varphi_\varepsilon^{\mathrm{out}} \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega a_\varepsilon \overline{u_\varepsilon} \varphi_\varepsilon(0) \,\mathrm{d}x, \quad (2.10)$$

for all test functions φ_ε such that $\varphi_\varepsilon^{\mathrm{int},i}$ is constant with respect to x in each $E_{\mathrm{int},i}^{\varepsilon,\xi}$, $\varphi_\varepsilon^{\mathrm{int},i} \in L^2(\Omega_{\mathrm{int},i}^\varepsilon; H^1(0,T))$, $i = 1,2$, $\varphi_\varepsilon^{\mathrm{out}} \in H^1(\Omega_{\mathrm{out}}^\varepsilon \times (0,T))$, $[\varphi_\varepsilon] \in L^2(\Gamma^\varepsilon \times (0,T))$, $\varphi_\varepsilon(\cdot, T) = 0$ and $\varphi_\varepsilon = 0$ on $\partial\Omega \times (0,T)$. $\qquad\square$

Here, by $\varphi_{\varepsilon,t}$ we denote the time derivative of the function φ_ε and similar notation will be used in what follows.

Existence of solutions to (2.3)–(2.9), for each fixed $\varepsilon > 0$, can be proven by means of an approximation argument with strictly parabolic equations in the whole domain, by replacing (2.4) and (2.5) with equations similar to (2.3) with $\kappa_\varepsilon = 1/\delta$ and $f = 0$, and then letting δ go to 0. The well-posedness result for the δ-approximating problems follows from the abstract parabolic theory as in [38, Theorem 23.A]. Finally, the uniqueness follows from the energy inequality.

By means of routine approximation procedures (as in [9, Proposition 2.2]), we arrive at the following energy inequality:

$$\sup_{0<t<T} \int_\Omega u_\varepsilon^2(t) \,\mathrm{d}x + \int_0^T \int_{\Omega_{\mathrm{out}}^\varepsilon} |\nabla u_\varepsilon^{\mathrm{out}}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon}{D_1} \int_0^T \int_{\Gamma_1^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma\,\mathrm{d}t$$

$$+ \frac{1}{D_2\varepsilon} \int_0^T \int_{\Gamma_2^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma\,\mathrm{d}t \leq \gamma(\|f\|_{L^2(\Omega_T)}^2 + \|\overline{u_\varepsilon}\|_{L^2(\Omega)}^2), \quad (2.11)$$

which implies, since $\overline{u_\varepsilon}$ converges strongly,

$$\sup_{0<t<T} \int_\Omega u_\varepsilon^2(t) \,\mathrm{d}x + \int_0^T \int_{\Omega_{\mathrm{out}}^\varepsilon} |\nabla u_\varepsilon^{\mathrm{out}}|^2 \,\mathrm{d}x\,\mathrm{d}t$$

$$+ \frac{\varepsilon}{D_1} \int_0^T \int_{\Gamma_1^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma\,\mathrm{d}t + \frac{\varepsilon}{D_2} \int_0^T \int_{\Gamma_2^\varepsilon} \left(\frac{[u_\varepsilon]}{\varepsilon}\right)^2 \,\mathrm{d}\sigma\,\mathrm{d}t \leq \gamma, \quad (2.12)$$

where γ does not depend on ε or D_1, D_2.

3. Time-dependent unfolding operator

We start this section by briefly recalling the definitions and the main properties of a space-time version of the unfolding operators studied in [18, 21, 25, 26] (see, also, [3, 4, 6, 7, 8, 12]). For a complete survey on the unfolding technique, we refer directly to [21].

If $[r]$ and $\{r\}$ denote the integer and the fractional part of $r \in \mathbb{R}$, respectively, for any $x \in \mathbb{R}^N$, we define

$$\left[\frac{x}{\varepsilon}\right]_Y = \left(\left[\frac{x_1}{\varepsilon}\right], \ldots, \left[\frac{x_N}{\varepsilon}\right] \right) \quad \text{and} \quad \left\{\frac{x}{\varepsilon}\right\}_Y = \left(\left\{\frac{x_1}{\varepsilon}\right\}, \ldots, \left\{\frac{x_N}{\varepsilon}\right\} \right),$$

so that

$$x = \varepsilon \left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y \right).$$

Moreover, let $Y^\varepsilon(x) = \varepsilon \left(\left[\frac{x}{\varepsilon}\right]_Y + Y \right)$ be the space cell containing x.

For $\xi \in \Xi^\varepsilon$, we set

$$\widehat{\Omega}^\varepsilon = \operatorname{interior} \left\{ \bigcup_{\xi \in \Xi^\varepsilon} \varepsilon(\xi + \overline{Y}) \right\}, \qquad \Lambda_T^\varepsilon = \widehat{\Omega}^\varepsilon \times (0, T).$$

Definition 3.1. Let w be a Lebesgue-measurable function on Ω_T. Then, the (time-dependent) unfolding operator $\mathcal{T}_\varepsilon$ is defined as

$$\mathcal{T}_\varepsilon(w)(x, t, y) = \begin{cases} w\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, t \right), & (x, t, y) \in \Lambda_T^\varepsilon \times Y, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, if w is a Lebesgue-measurable function on Γ_T^ε, the (time-dependent) boundary unfolding operator $\mathcal{T}_\varepsilon^b$ is defined as

$$\mathcal{T}_\varepsilon^b(w)(x, t, y) = \begin{cases} w\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, t \right), & (x, t, y) \in \Lambda_T^\varepsilon \times \Gamma, \\ 0, & \text{otherwise.} \end{cases}$$

$\square$

Definition 3.2. If w is an integrable function on Ω_T, the local (time-dependent) space average operator is defined by

$$\mathcal{M}_\varepsilon(w)(x, t) = \begin{cases} \dfrac{1}{\varepsilon^N} \displaystyle\int_{Y^\varepsilon(x)} w(\zeta, t) \, \mathrm{d}\zeta, & \text{if } (x, t) \in \Lambda_T^\varepsilon, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

$\square$

One can see that $\mathcal{M}_\varepsilon(w) = \mathcal{M}_Y(\mathcal{T}_\varepsilon(w))$, where, for a general set O, $\mathcal{M}_O(\cdot)$ denotes the integral average on O.

In the following, we will use the subscript # to denote spaces of periodic functions.

Proposition 3.3. *Let $w \in L^2(0, T; H^1(\Omega))$. Then,*

$$\frac{1}{\varepsilon} \left[\mathcal{T}_\varepsilon(w) - \mathcal{M}_\varepsilon(w) \right] \to y^c \cdot \nabla w, \quad strongly\ in\ L^2(\Omega_T \times Y), \quad (3.2)$$

$$\frac{1}{\varepsilon} \left[\mathcal{T}_\varepsilon^b(w) - \mathcal{M}_\varepsilon(w) \right] \to y^c \cdot \nabla w, \quad strongly\ in\ L^2(\Omega_T \times \Gamma), \quad (3.3)$$

where $y^c = (y^{c1}, \dots, y^{cN}) = y - \mathcal{M}_Y(y)$.

We recall the following general compactness result, which can be found, for instance, in [21, Chapters 1 and 4], [25, Theorems 2.17, 2.18 and 2.19] and [26, Theorem 4.3].

Proposition 3.4. *Let $w_\varepsilon = (w_\varepsilon^{\mathrm{int},1}, w_\varepsilon^{\mathrm{int},2}, w_\varepsilon^{\mathrm{out}})$, with $w_\varepsilon^{\mathrm{int},i} \in L^2(0, T; H^1(\Omega_{\mathrm{int},i}^\varepsilon))$, $i = 1, 2$, and $w_\varepsilon^{\mathrm{out}} \in L^2(0, T; H^1(\Omega_{\mathrm{out}}^\varepsilon))$. Assume that there exists $\gamma > 0$ (independent of ε) such that*

$$\int_{\Omega_T} |w_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} |\nabla w_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \gamma, \quad \forall \varepsilon > 0. \quad (3.4)$$

Then, there exist $w^{\mathrm{int},i} \in L^2(\Omega_T)$, for $i = 1, 2$, $w^{\mathrm{out}} \in L^2(0, T; H^1(\Omega))$, $\widehat{w} \in L^2(\Omega_T; H^1_\#(E_{\mathrm{out}}))$ and $\overline{w}^{\mathrm{int},i} \in L^2(\Omega_T; H^1(E_{\mathrm{int},i}))$, $i = 1, 2$ such that, up to a subsequence,

$$\mathcal{T}_\varepsilon(\chi_{\Omega_{\mathrm{int},i}^\varepsilon} w_\varepsilon) \rightharpoonup w^{\mathrm{int},i}, \quad weakly\ in\ L^2(\Omega_T \times E_{\mathrm{int},i}),\ i = 1, 2; \quad (3.5)$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega_{\mathrm{out}}^\varepsilon} w_\varepsilon) \rightharpoonup w^{\mathrm{out}}, \quad weakly\ in\ L^2(\Omega_T \times E_{\mathrm{out}}); \quad (3.6)$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega_{\mathrm{int},i}^\varepsilon} \nabla w_\varepsilon^{\mathrm{int},i}) \rightharpoonup \nabla_y \overline{w}^{\mathrm{int},i}, \quad weakly\ in\ L^2(\Omega_T \times E_{\mathrm{int},i}),\ i = 1, 2; \quad (3.7)$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega_{\mathrm{out}}^\varepsilon} \nabla w_\varepsilon) \rightharpoonup \nabla w^{\mathrm{out}} + \nabla_y \widehat{w}, \quad weakly\ in\ L^2(\Omega_T \times E_{\mathrm{out}}), \quad (3.8)$$

for $\varepsilon \to 0$. Moreover, due to (3.4), we have

$$\varepsilon \int_{\Gamma_T^\varepsilon} [w_\varepsilon]^2 \,\mathrm{d}\sigma\,\mathrm{d}t \leq \gamma, \quad \forall \varepsilon > 0, \quad (3.9)$$

with γ independent of ε, and then (see [10, Proposition 2])

$$\mathcal{T}_\varepsilon^b([w_\varepsilon]) \rightharpoonup w^{\mathrm{out}} - w^{\mathrm{int},i}, \quad weakly\ in\ L^2(\Omega_T \times \Gamma_i),\ i = 1, 2. \quad (3.10)$$

Remark 3.5. We recall that, when $w_\varepsilon \to w$, strongly in $L^2(\Omega_T)$, then $\mathcal{T}_\varepsilon(w_\varepsilon) \to w$ strongly in $L^2(\Omega_T \times Y)$. However, the only classes for which the strong convergence of the unfolding $\mathcal{T}_\varepsilon(w_\varepsilon)$ is known to hold in $L^2(\Omega_T \times Y)$, without assuming the strong convergence of w_ε, are sums of the following cases: $w_\varepsilon(x, t) = f_1(x, t)f_2(\varepsilon^{-1}x)$ with f_1, f_2 suitable Lebesgue-measurable functions, $w_\varepsilon(x, t) = w(x, \varepsilon^{-1}x, t)$ with $w \in L^2(Y; \mathcal{C}(\overline{\Omega}_T))$ or $w \in L^2(\Omega_T; \mathcal{C}(\overline{Y}))$ (see [1, 19, 20] and [4, Remark 2.9]). $\square$

Next, we collect some compactness results for the unique solution u_ε of problem (2.3)–(2.9). As usual, the convergences below hold up to extracting subsequences. As a consequence of the energy estimate (2.12) and Proposition 3.4, applied to the solution u_ε, and using the fact that $u_\varepsilon^{\mathrm{int},i}$ is piecewise constant in $\Omega_{\mathrm{int},i}^\varepsilon$, $i = 1, 2$, it follows that there exist suitable functions $u^{\mathrm{int},1}, u^{\mathrm{int},2} \in L^2(\Omega_T)$, $u^{\mathrm{out}} \in L^2(0, T; H^1_0(\Omega))$

and $\widehat{u} \in L^2(\Omega_T; H^1_\#(E_{\text{out}}))$ such that

$$\mathcal{T}_\varepsilon(\chi_{\Omega^\varepsilon_{\text{int},1}} u_\varepsilon) \rightharpoonup u^{\text{int},1}, \qquad \text{weakly in } L^2(\Omega_T \times E_{\text{int},1}); \qquad (3.11)$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^\varepsilon_{\text{int},2}} u_\varepsilon) \rightharpoonup u^{\text{int},2}, \qquad \text{weakly in } L^2(\Omega_T \times E_{\text{int},2}); \qquad (3.12)$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^\varepsilon_{\text{out}}} u_\varepsilon) \rightharpoonup u^{\text{out}}, \qquad \text{weakly in } L^2(\Omega_T \times E_{\text{out}}); \qquad (3.13)$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^\varepsilon_{\text{out}}} \nabla u_\varepsilon) \rightharpoonup \nabla u^{\text{out}} + \nabla_y \widehat{u}, \qquad \text{weakly in } L^2(\Omega_T \times E_{\text{out}}); \qquad (3.14)$$

$$\mathcal{T}^b_\varepsilon([u_\varepsilon]) \rightharpoonup u^{\text{out}} - u^{\text{int},i}, \qquad \text{weakly in } L^2(\Omega_T \times \Gamma_i), \; i = 1,2. \qquad (3.15)$$

We point out that the limit functions $u^{\text{int},1}, u^{\text{int},2} \in L^2(\Omega_T)$ are independent of y. More precisely, still using (2.12), we get

$$\int_{\Omega_T \times \Gamma_1} \mathcal{T}^b_\varepsilon([u_\varepsilon])^2 \, \mathrm{d}\sigma_y \, \mathrm{d}x \, \mathrm{d}t \leq \gamma, \qquad (3.16)$$

$$\int_{\Omega_T \times \Gamma_2} \mathcal{T}^b_\varepsilon([u_\varepsilon])^2 \, \mathrm{d}\sigma_y \, \mathrm{d}x \, \mathrm{d}t \leq \gamma\varepsilon^2. \qquad (3.17)$$

As a consequence of (3.15) and (3.16), we obtain

$$\mathcal{T}^b_\varepsilon([u_\varepsilon]) \rightharpoonup u^{\text{out}} - u^{\text{int},1}, \qquad \text{weakly in } L^2(\Omega_T \times \Gamma_1). \qquad (3.18)$$

On the other hand, (3.17) immediately implies

$$\mathcal{T}^b_\varepsilon([u_\varepsilon]) \to 0, \qquad \text{strongly in } L^2(\Omega_T \times \Gamma_2), \qquad (3.19)$$

and therefore, $u^{\text{int},2} = u^{\text{out}} =: u \in L^2(0,T; H^1_0(\Omega))$, again in virtue of (3.15). More precisely, we have

$$u_\varepsilon \rightharpoonup (|E_{\text{out}}| + |E_{\text{int},2}|)u + |E_{\text{int},1}|u^{\text{int},1}, \qquad \text{weakly in } L^2(\Omega_T). \qquad (3.20)$$

Moreover, we get

$$\mathcal{T}^b_\varepsilon\left(\frac{[u_\varepsilon]}{\varepsilon}\right) \rightharpoonup \widehat{u} + y^c \cdot \nabla u + \zeta, \qquad \text{weakly in } L^2(\Omega_T \times \Gamma_2), \qquad (3.21)$$

for a suitable function $\zeta \in L^2(\Omega_T)$. For the convergence in (3.21), we refer to [25, Theorem 2.20] and [26, Theorem 4.3].

Remark 3.6. We point out that all the above convergences take place as $\varepsilon \to 0$ and D_1 and D_2 are fixed. In Sections 6 and 7, we will discuss also the limits on D_1 and D_2, after and before the homogenization procedure, respectively. $\qquad \square$

4. Cell functions

In this section, we assume all the hypotheses introduced in Section 2 and we gather the problems for the cell functions used in what follows.
We start by recalling the following well-known result (see, e.g., [23, Section 2]).

Lemma 4.1. *For each $j = 1, \ldots, N$, there exists a unique $\bar{\chi}^j \in H^1_\#(E_{\text{out}})$ which satisfies*

$$-\operatorname{div}_y(\kappa(y)\,\nabla_y(\bar{\chi}^j(y) - y^j)) = 0\,, \quad in\ E_{\text{out}}; \tag{4.1}$$

$$\kappa(y)\,\nabla_y(\bar{\chi}^j(y) - y^j) \cdot \nu = 0\,, \quad on\ \Gamma = \Gamma_1 \cup \Gamma_2; \tag{4.2}$$

$$\int_{E_{\text{out}}} \bar{\chi}^j(y)\,\mathrm{d}y = 0\,. \tag{4.3}$$

Then, we recall the following lemma stated in [7] and [11], in the case of a single hole and, for instance, in [9, Proposition 2.6] and [22, Section 5], for the case of multiple holes.

Lemma 4.2. *For each $j = 1, \ldots, N$, there exists a unique $\chi^j \in H^1_\#(E_{\text{out}})$ which satisfies*

$$-\operatorname{div}_y(\kappa(y)\,\nabla_y(\chi^j(y) - y^j)) = 0\,, \qquad\qquad in\ E_{\text{out}}; \tag{4.4}$$

$$\int_{\Gamma_i} \kappa(y)\,\nabla_y(\chi^j(y) - y^j) \cdot \nu\,\mathrm{d}\sigma_y = 0\,, \qquad\qquad i = 1, 2; \tag{4.5}$$

$$\chi^j(y) - y^j \qquad\qquad\qquad\qquad is\ constant\ on\ \Gamma_1; \tag{4.6}$$

$$\chi^j(y) - y^j \qquad\qquad\qquad\qquad is\ constant\ on\ \Gamma_2; \tag{4.7}$$

$$\int_{E_{\text{out}}} \chi^j(y)\,\mathrm{d}y = 0\,. \tag{4.8}$$

Arguing as in [7, Lemma 4.7], one can prove also the following result.

Lemma 4.3. *There exists a unique $\hat{\chi}^j_o \in H^1_\#(E_{\text{out}} \cup E_{\text{int},2})$ which, for each $j = 1, \ldots, N$, satisfies*

$$-\operatorname{div}_y\left(\kappa(y)\nabla_y(\hat{\chi}^j_o(y) - y^{cj})\right) = 0 \qquad\qquad in\ E_{\text{out}}; \tag{4.9}$$

$$\kappa(y)\nabla_y(\hat{\chi}^j_o(y) - y^{cj}) \cdot \nu = 0 \qquad\qquad on\ \Gamma_1; \tag{4.10}$$

$$\int_{\Gamma_2} \kappa(y)\,\nabla_y(\hat{\chi}^j_o(y) - y^{cj}) \cdot \nu\,\mathrm{d}\sigma_y = 0\,; \tag{4.11}$$

$$\hat{\chi}^j_o(y) - y^{cj} \qquad\qquad\qquad\qquad is\ constant\ in\ E_{\text{int},2}; \tag{4.12}$$

$$\int_{E_{\text{out}} \cup E_{\text{int},2}} \hat{\chi}^j_o(y)\,\mathrm{d}y = 0\,. \tag{4.13}$$

We conclude this section by constructing a new type of cell functions which will be used in the main homogenization theorem (see Theorem 5.1).

Lemma 4.4. *For $j = 1, \ldots, N$, let us consider the problem*

$$- \operatorname{div}_y(\kappa(y)\nabla_y(\hat{\chi}^j(y) - y^{cj})) = 0 \,, \qquad\qquad\qquad in \; E_{\text{out}}; \quad (4.14)$$

$$\kappa(y)\nabla_y(\hat{\chi}^j(y) - y^{cj})\cdot\nu = 0, \qquad\qquad\qquad on \; \Gamma_1; \quad (4.15)$$

$$\kappa(y)\nabla_y(\hat{\chi}^j(y) - y^{cj})\cdot\nu = \frac{1}{D_2}\Big(\hat{\chi}^j(y) - y^{cj} - \frac{1}{|\Gamma_2|}\int_{\Gamma_2}(\hat{\chi}^j(z) - z^{cj})\,\mathrm{d}\sigma_z\Big), \; on \; \Gamma_2; \quad (4.16)$$

$$\int_{E_{\text{out}}} \hat{\chi}^j(y)\,\mathrm{d}y = 0\,. \qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.17)$$

Then, problem (4.14)–(4.17) admits a unique solution $\hat{\chi}^j \in H^1_\#(E_{\text{out}})$.

Proof. For $j = 1, \ldots, N$, consider the Neumann-Robin problem

$$- \operatorname{div}_y(\kappa(y)\nabla_y(\widetilde{\chi}^j(y) - y^{cj})) = 0\,, \qquad\qquad in \; E_{\text{out}}; \quad (4.18)$$

$$\kappa(y)\nabla_y(\widetilde{\chi}^j(y) - y^{cj})\cdot\nu = 0, \qquad\qquad on \; \Gamma_1; \quad (4.19)$$

$$\kappa(y)\nabla_y(\widetilde{\chi}^j(y) - y^{cj})\cdot\nu - \frac{1}{D_2}(\widetilde{\chi}^j - y^{cj}) = 0\,, \qquad on \; \Gamma_2, \quad (4.20)$$

whose unique solution $\widetilde{\chi}^j \in H^1_\#(E_{\text{out}})$ can be obtained by a standard application of Lax-Milgram Lemma. Set

$$\hat{\zeta}^j = \frac{1}{|E_{\text{out}}|}\int_{E_{\text{out}}} \widetilde{\chi}^j(y)\,\mathrm{d}y \qquad\qquad\qquad\qquad (4.21)$$

and define

$$\hat{\chi}^j(y) = \widetilde{\chi}^j(y) - \hat{\zeta}^j\,. \qquad\qquad\qquad\qquad (4.22)$$

Integrating by parts the differential equation (4.18) and taking into account (4.20) and (4.22), it follows that

$$\hat{\zeta}^j = -\frac{1}{|\Gamma_2|}\int_{\Gamma_2}(\hat{\chi}^j(y) - y^{cj})\,\mathrm{d}\sigma_y\,. \qquad\qquad (4.23)$$

Replacing $\widetilde{\chi}^j$ with $\hat{\chi}^j + \hat{\zeta}^j$ in (4.18)–(4.20), we get

$$- \operatorname{div}_y(\kappa(y)\nabla_y(\hat{\chi}^j(y) - y^{cj})) = 0\,, \qquad\qquad in \; E_{\text{out}}; \quad (4.24)$$

$$\kappa(y)\nabla_y(\hat{\chi}^j(y) - y^{cj})\cdot\nu = 0\,, \qquad\qquad on \; \Gamma_1; \quad (4.25)$$

$$\kappa(y)\nabla_y(\hat{\chi}^j(y) - y^{cj})\cdot\nu = \frac{1}{D_2}(\hat{\chi}^j(y) - y^{cj} + \hat{\zeta}^j)\,, \qquad on \; \Gamma_2. \quad (4.26)$$

By (4.23)–(4.26), it is not difficult to see that $\hat{\chi}^j$ solves the problem (4.14)–(4.17). Its uniqueness is a consequence of routine energy estimates. $\qquad\square$

Remark 4.5. We remark that (4.23) can be also written as

$$\int_{\Gamma_2}(\hat{\chi}^j(y) - y^{cj} + \hat{\zeta}^j)\,\mathrm{d}\sigma_y = 0\,. \qquad\qquad\qquad (4.27)$$

5. HOMOGENIZATION OF THE ε-PROBLEM FOR D_1 AND D_2 FIXED

In this section, we study the homogenization of problem (2.3)–(2.9), for D_1 and D_2 fixed. In the next section, we will perform the limit of the homogenized problem for the parameters D_1 and D_2 going to 0 or $+\infty$. The converse procedure will be addressed in Section 7.

Theorem 5.1. *The limiting pair* $(u^{\mathrm{int},1}, u) \in L^2(\Omega_T) \times L^2(0,T; H_0^1(\Omega))$, *appearing in* (3.20), *is the unique solution of*

$$- (|E_{\mathrm{out}}| + \lambda_2) \int_{\Omega_T} u\varphi_{2,t} \, \mathrm{d}x \, \mathrm{d}t - \lambda_1 \int_{\Omega_T} u^{\mathrm{int},1} \varphi_{1,t} \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_{\Omega_T} A_{\mathrm{hom}}^{D_2} \nabla u \cdot \nabla \varphi_2 \, \mathrm{d}x \, \mathrm{d}t + \frac{|\Gamma_1|}{D_1} \int_{\Omega_T} (u - u^{\mathrm{int},1})(\varphi_2 - \varphi_1) \, \mathrm{d}x \, \mathrm{d}t$$

$$= |E_{\mathrm{out}}| \int_{\Omega_T} f \varphi_2 \, \mathrm{d}x \, \mathrm{d}t + (|E_{\mathrm{out}}| + \lambda_2) \int_{\Omega} \overline{u} \varphi_2(0) \, \mathrm{d}x + \lambda_1 \int_{\Omega} \overline{u} \varphi_1(0) \, \mathrm{d}x \,, \quad (5.1)$$

for all $\varphi = (\varphi_1, \varphi_2)$, *with* $\varphi_2 \in H^1(\Omega_T)$, $\varphi_2 = 0$ *on* $\partial\Omega \times (0,T)$ *and for* $t = T$, *and all* $\varphi_1 \in H^1(0,T; L^2(\Omega))$, *with* $\varphi_1 = 0$ *for* $t = T$. *Here, the constant homogenized matrix* $A_{\mathrm{hom}}^{D_2}$ *is defined by*

$$A_{\mathrm{hom}}^{D_2} = -\int_{E_{\mathrm{out}}} \kappa \, \nabla_y (\hat{\chi} - y^c) \cdot \nabla_y y^c \, \mathrm{d}y - \frac{1}{D_2} \int_{\Gamma_2} (\hat{\chi} - y^c + \hat{\zeta}) \otimes y^c \, \mathrm{d}\sigma_y \,, \qquad (5.2)$$

where $\hat{\chi} = (\hat{\chi}^1, \dots, \hat{\chi}^N)$ *and* $\hat{\zeta} = (\hat{\zeta}^1, \dots, \hat{\zeta}^N)$ *have been introduced in Lemma 4.4.*

Proof. Without loss of generality, we can first choose the admissible test function $\varphi_\varepsilon(x,t) = \varepsilon\phi(x, x/\varepsilon, t)$, where

$$\phi(x,y,t) = \begin{cases} \Psi(x,y)z(t)\,, & \text{in } \Omega_T \times E_{\mathrm{out}}, \\ 0\,, & \text{in } \Omega_T \times (E_{\mathrm{int},1} \cup E_{\mathrm{int},2}). \end{cases}$$

Here, $z \in \mathcal{C}^1([0,T])$, $z(T) = 0$, the function $\Psi \in \mathcal{C}^\infty(\overline{\Omega \times E_{\mathrm{out}}})$ vanishes near $\partial\Omega$ and is Y-periodic. By unfolding the integrals appearing in the weak formulation (2.10),

we arrive at

$$- \varepsilon \int\limits_{\Omega_T \times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(u_\varepsilon) \mathcal{T}_\varepsilon(z') \mathcal{T}_\varepsilon(\Psi) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \varepsilon \int\limits_{\Omega_T \times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(z) \mathcal{T}_\varepsilon(\kappa_\varepsilon) \mathcal{T}_\varepsilon(\nabla u_\varepsilon) \cdot \mathcal{T}_\varepsilon(\nabla_x \Psi) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int\limits_{\Omega_T \times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(z) \mathcal{T}_\varepsilon(\kappa_\varepsilon) \mathcal{T}_\varepsilon(\nabla u_\varepsilon) \cdot \mathcal{T}_\varepsilon(\nabla_y \Psi) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \frac{\varepsilon}{D_1} \int\limits_{\Omega_T \times \Gamma_1} \mathcal{T}_\varepsilon^b([u_\varepsilon]) \mathcal{T}_\varepsilon^b(\Psi) \mathcal{T}_\varepsilon^b(z) \, \mathrm{d}\sigma_y \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \frac{1}{D_2} \int\limits_{\Omega_T \times \Gamma_2} \mathcal{T}_\varepsilon^b \left(\frac{[u_\varepsilon]}{\varepsilon} \right) \mathcal{T}_\varepsilon^b(\Psi) \mathcal{T}_\varepsilon^b(z) \, \mathrm{d}\sigma_y \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int\limits_{\Omega_T \times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(f) \mathcal{T}_\varepsilon(\varphi_\varepsilon) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t - \int\limits_{\Omega \times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(\overline{u_\varepsilon}) \mathcal{T}_\varepsilon(\varphi_\varepsilon(0)) \, \mathrm{d}y \, \mathrm{d}x \to 0 \,, \qquad \varepsilon \to 0 \,.$$

However, it is easily seen that all the terms above, except the third and the fifth integral, vanish in the limit independently of the previous relation, which therefore yields, together with (3.14) and (3.21),

$$\int\limits_{\Omega_T \times E_{\mathrm{out}}} \kappa(y)(\nabla u + \nabla_y \widehat{u}) \cdot \nabla_y \Psi \, z(t) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \frac{1}{D_2} \int\limits_{\Omega_T \times \Gamma_2} (\widehat{u} + y^c \cdot \nabla u + \zeta) \Psi \, z(t) \, \mathrm{d}\sigma_y \, \mathrm{d}x \, \mathrm{d}t = 0 \,. \quad (5.3)$$

The distributional formulation of (5.3) is given by

$$- \operatorname{div}_y(\kappa(\nabla u + \nabla_y \widehat{u})) = 0 \,, \qquad\qquad \text{in } \Omega_T \times E_{\mathrm{out}}; \qquad (5.4)$$

$$\kappa(\nabla u + \nabla_y \widehat{u}) \cdot \nu = 0 \,, \qquad\qquad \text{on } \Omega_T \times \Gamma_1; \qquad (5.5)$$

$$\kappa(\nabla u + \nabla_y \widehat{u}) \cdot \nu = \frac{1}{D_2}(\widehat{u} + y^c \cdot \nabla u + \zeta) \,, \qquad \text{on } \Omega_T \times \Gamma_2. \qquad (5.6)$$

Notice that (5.4)–(5.6) imply

$$\int\limits_{\Omega_T \times \Gamma_2} (\widehat{u} + y^c \cdot \nabla u + \zeta) \, \mathrm{d}\sigma_y \, \mathrm{d}x \, \mathrm{d}t = 0 \,. \qquad (5.7)$$

Then, we will apply below the factorization

$$\widehat{u}(x, y, t) = -\hat{\chi}(y) \cdot \nabla u(x, t) - \hat{\zeta} \cdot \nabla u(x, t) - \zeta(x, t) \,, \qquad (5.8)$$

where $\hat{\chi}$ is given in Lemma 4.4 and $\zeta, \hat{\zeta}$ are given by (3.21) and (4.23). Next, we select the test function $\tilde{\varphi}_\varepsilon(x, x/\varepsilon, t)$, where

$$\tilde{\varphi}_\varepsilon(x, y, t) = \begin{cases} z(t)w(x), & \text{in } \Omega_T \times E_{\text{out}}, \\ z_1(t)w_1(x), & \text{in } \Omega_T \times E_{\text{int},1}, \\ z(t)\mathcal{M}_\varepsilon(w)(x), & \text{in } \Omega_T \times E_{\text{int},2}, \end{cases}$$

with $z, z_1 \in \mathcal{C}^1([0, T])$, $z(T) = z_1(T) = 0$, $w \in \mathcal{C}_0^\infty(\Omega)$ and $w_1 \in \mathcal{C}^\infty(\overline{\Omega})$. We get

$$-\int_{\Omega_T \times E_{\text{out}}} \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(z'w)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t - \frac{\lambda_1}{|E_{\text{int},1}|}\int_{\Omega_T \times E_{\text{int},1}} \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(z_1')\mathcal{T}_\varepsilon(w_1)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$

$$-\frac{\lambda_2}{|E_{\text{int},2}|}\int_{\Omega_T \times E_{\text{int},2}} \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(z')\mathcal{M}_\varepsilon(w)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T \times E_{\text{out}}} \mathcal{T}_\varepsilon(z\kappa_\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \cdot \mathcal{T}_\varepsilon(\nabla w)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$

$$+\frac{1}{D_1}\int_{\Omega_T \times \Gamma_1} \mathcal{T}_\varepsilon^b([u_\varepsilon])\mathcal{T}_\varepsilon^b(zw - z_1 w_1)\,\mathrm{d}\sigma_y\,\mathrm{d}x\,\mathrm{d}t$$

$$+\frac{1}{D_2}\int_{\Omega_T \times \Gamma_2} \mathcal{T}_\varepsilon^b(z)\mathcal{T}_\varepsilon^b\Big(\frac{[u_\varepsilon]}{\varepsilon}\Big)\mathcal{T}_\varepsilon^b\Big(\frac{w - \mathcal{M}_\varepsilon(w)}{\varepsilon}\Big)\,\mathrm{d}\sigma_y\,\mathrm{d}x\,\mathrm{d}t$$

$$-\int_{\Omega_T \times E_{\text{out}}} \mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(zw)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$

$$-\int_\Omega \Big(\chi_{E_{\text{out}}} + \frac{\lambda_1}{|E_{\text{int},1}|}\chi_{E_{\text{int},1}} + \frac{\lambda_2}{|E_{\text{int},2}|}\chi_{E_{\text{int},2}}\Big)\mathcal{T}_\varepsilon(\overline{u}_\varepsilon)\mathcal{T}_\varepsilon(\tilde{\varphi}_\varepsilon(0))\,\mathrm{d}y\,\mathrm{d}x \to 0,$$

as $\varepsilon \to 0$. Thus, in the limit, we obtain

$$-|E_{\text{out}}|\int_{\Omega_T} uz'w\,\mathrm{d}x\,\mathrm{d}t - \lambda_1\int_{\Omega_T} u^{\text{int},1}z_1'w_1\,\mathrm{d}x\,\mathrm{d}t - \lambda_2\int_{\Omega_T} uz'w\,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_{\Omega_T \times E_{\text{out}}} z\kappa(\nabla u + \nabla_y \widehat{u}) \cdot \nabla w\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$

$$+\frac{1}{D_1}\int_{\Omega_T \times \Gamma_1} (u - u^{\text{int},1})(zw - z_1 w_1)\,\mathrm{d}\sigma_y\,\mathrm{d}x\,\mathrm{d}t \qquad (5.9)$$

$$+\frac{1}{D_2}\int_{\Omega_T \times \Gamma_2} z(\widehat{u} + y^c \cdot \nabla u + \zeta)y^c \cdot \nabla w\,\mathrm{d}\sigma_y\,\mathrm{d}x\,\mathrm{d}t$$

$$= |E_{\text{out}}|\int_{\Omega_T} fzw\,\mathrm{d}x\,\mathrm{d}t + (|E_{\text{out}}| + \lambda_2)\int_\Omega \overline{u}z(0)w\,\mathrm{d}x + \lambda_1\int_\Omega \overline{u}z_1(0)w_1\,\mathrm{d}x.$$

Equation (5.9) is, up to the usual density argument, the weak formulation of the limiting problem. Next, we insert into it the factorization for $\widehat{u}$ given in (5.8), obtaining

(5.1). By Proposition 5.2 below, it follows that the homogenized matrix $A_{\text{hom}}^{D_2}$ is symmetric and positive definite. Moreover, following the same ideas as in [11, Section 7], the pair $(u^{\text{int},1}, u)$ solution of (5.1) is unique, and then all the above convergences hold true for the whole sequence. $\square$

Proposition 5.2. *The constant matrix $A_{\text{hom}}^{D_2}$ can be rewritten as*

$$(A_{\text{hom}}^{D_2})^{ji} = \int_{E_{\text{out}}} \kappa \, \nabla_y(\hat{\chi}^j - y^{cj}) \cdot \nabla_y(\hat{\chi}^i - y^{ci}) \, \mathrm{d}y$$

$$+ \frac{1}{D_2} \int_{\Gamma_2} (\hat{\chi}^j - y^{cj} + \hat{\zeta}^j)(\hat{\chi}^i - y^{ci} + \hat{\zeta}^i) \, \mathrm{d}\sigma_y \,, \quad (5.10)$$

for $i, j = 1, \ldots, N$. It follows that $A_{\text{hom}}^{D_2}$ is symmetric and positive definite.

Proof. On using $\hat{\chi}^i$ in (4.14) as test function and taking into account (4.23), we find

$$\int_{E_{\text{out}}} \kappa \, \nabla_y(\hat{\chi}^j - y^{cj}) \cdot \nabla_y \, \hat{\chi}^i \, \mathrm{d}y + \frac{1}{D_2} \int_{\Gamma_2} (\hat{\chi}^j - y^{cj} + \hat{\zeta}^j)\hat{\chi}^i \, \mathrm{d}\sigma_y = 0 \,. \quad (5.11)$$

On the other hand,

$$\int_{\Gamma_2} (\hat{\chi}^j - y^{cj} + \hat{\zeta}^j)\hat{\zeta}^i \, \mathrm{d}\sigma_y = 0 \,, \quad (5.12)$$

owing to (4.27), since $\hat{\zeta}^i$ is a constant. By adding (5.11) and (5.12) to (5.2), we prove (5.10). The positive definiteness of $A_{\text{hom}}^{D_2}$ is then standard. $\square$

Remark 5.3. It can be easily proved that the following energy estimate

$$\sup_{0<t<T} \left(\int_{\Omega} u^2 \, \mathrm{d}x + \int_{\Omega} (u^{\text{int},1})^2 \, \mathrm{d}x \right)$$

$$+ \int_{\Omega_T} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{D_1} \int_{\Omega_T} (u - u^{\text{int},1})^2 \, \mathrm{d}x \, \mathrm{d}t \leq \gamma \quad (5.13)$$

holds, for a suitable $\gamma > 0$, independent of D_1. Moreover, if the ellipticity constant of the matrix $A_{\text{hom}}^{D_2}$ is independent of D_2 (as we will prove in Lemma 6.3, for D_2 small or large enough), we get that γ is also independent of D_2. In addition, the following estimate

$$\sup_{0<t<T} \left(\int_{\Omega} u^2 \, \mathrm{d}x + \int_{\Omega} (u^{\text{int},1})^2 \, \mathrm{d}x \right) + \frac{1}{D_1} \int_{\Omega_T} (u - u^{\text{int},1})^2 \, \mathrm{d}x \, \mathrm{d}t \leq \gamma \quad (5.14)$$

holds with $\gamma > 0$ always independent of D_1 and D_2. $\square$

Remark 5.4. Notice that the distributional formulation of the problem (5.1) is

$$
\begin{aligned}
& (|E_{\text{out}}| + \lambda_2)\, u_t - \text{div}(A_{\text{hom}}^{D_2} \nabla u) + \frac{|\Gamma_1|}{D_1}(u - u^{\text{int},1}) = |E_{\text{out}}|f\,, && \text{in } \Omega_T; \\
& \lambda_1 u_t^{\text{int},1} - \frac{|\Gamma_1|}{D_1}(u - u^{\text{int},1}) = 0\,, && \text{in } \Omega_T; \\
& u = 0\,, && \text{on } \partial\Omega \times (0,T); \\
& u(x,0) = \overline{u}\,, && \text{in } \Omega; \\
& u^{\text{int},1}(x,0) = \overline{u}\,, && \text{in } \Omega.
\end{aligned}
\tag{5.15}
$$

$\square$

We point out that the homogenized diffusion matrix $A_{\text{hom}}^{D_2}$ depends only on D_2, while both equations depend on D_1. This calls for a second limit for D_1 and D_2 going to 0 or $+\infty$, which will be performed in the next section.

6. Limit with respect to D_1 and D_2 of the homogenized problem

This section is devoted to perform the limit for D_1 and D_2 going to 0 or $+\infty$ of the homogenized problem (5.15). For the sake of simplicity, the dependence on D_1, D_2 of some quantities is not denoted explicitly, but it is let understood from the context.

6.1. **Limit $D_1, D_2 \to 0$.** Let us define the functional space

$$
\mathcal{X}_{\#}^{\Gamma_2}(E_{\text{out}}) := \{\psi \in H_{\#}^1(E_{\text{out}}) \,:\, \psi \text{ is independent of } y \text{ on } \Gamma_2\}\,.
\tag{6.1}
$$

Theorem 6.1. *For $D_1, D_2 \to 0$, there exists $u_o \in L^2(0,T;H_0^1(\Omega))$ such that*

$$
\begin{aligned}
& u^{\text{int},1} \rightharpoonup u_o\,, && \text{weakly in } L^2(\Omega_T); \\
& u \rightharpoonup u_o\,, && \text{weakly in } L^2(0,T;H_0^1(\Omega)); \\
& u - u^{\text{int},1} \to 0\,, && \text{strongly in } L^2(\Omega_T),
\end{aligned}
\tag{6.2}
$$

where u_o is the unique solution of the problem

$$
\begin{aligned}
-\,(|E_{\text{out}}| + \lambda_1 + \lambda_2)\int_{\Omega_T} u_o \varphi_t \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} A_{\text{hom}}^0 \nabla u_o \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\
= |E_{\text{out}}| \int_{\Omega_T} f\varphi \,\mathrm{d}x\,\mathrm{d}t + (|E_{\text{out}}| + \lambda_1 + \lambda_2)\int_{\Omega} \overline{u}\varphi(0)\,\mathrm{d}x\,,
\end{aligned}
\tag{6.3}
$$

for any $\varphi \in H^1(\Omega_T)$ with $\varphi = 0$ on $\partial\Omega \times (0,T)$ and for $t = T$. Here, A_{hom}^0 is defined by

$$
(A_{\text{hom}}^0)^{ij} = -\int_{E_{\text{out}}} \kappa_{ik}(y)\partial_{y_k}(\hat{\chi}_o^j - y^{cj})\,\mathrm{d}y - \int_{\Gamma_2} \kappa \nabla_y(\hat{\chi}_o^j - y^{cj}) \cdot \nu y^{ci} \,\mathrm{d}\sigma_y\,,
\tag{6.4}
$$

for $i,j = 1, \ldots, N$, where $\hat{\chi}_o = (\hat{\chi}_o^1, \ldots, \hat{\chi}_o^N)$ is given in Lemma 4.3.

Proof. Similarly as in the proof of Theorem 5.4 in [11], we consider the weak formulation of problem (4.14)–(4.17) given by

$$\int\limits_{E_{\text{out}}} \kappa(y)\nabla_y(\hat{\chi}^j - y^{cj}) \cdot \nabla_y\phi \,\mathrm{d}y + \frac{1}{D_2}\int\limits_{\Gamma_2} (\hat{\chi}^j - y^{cj} + \hat{\zeta}^j)\phi \,\mathrm{d}\sigma_y = 0\,, \qquad (6.5)$$

for every $\phi \in H^1_\#(E_{\text{out}})$. However, problem (4.14)–(4.17) can be written in a variational form. For $j = 1, \ldots, N$, we can introduce the function $U^j(y) = \hat{\chi}^j(y) - y^{cj}\Psi(y)$, where $\Psi \in \mathcal{C}^\infty(\overline{E_{\text{out}}})$ is a function such that $\Psi = 0$ on ∂Y, $\Psi \equiv 1$ on Γ_2 and $\int_{E_{\text{out}}} y^{cj}\Psi \,\mathrm{d}y = 0$. Clearly, $U^j = \hat{\chi}^j - y^{cj} + y^{cj}(1 - \Psi)$. Moreover, set

$$g_1^j = -\operatorname{div}_y\left(\kappa\nabla_y(y^{cj}(1 - \Psi))\right)\,, \qquad \text{in } E_{\text{out}}$$

and

$$g_2^j = \kappa\nabla_y\left(y^{cj}(1 - \Psi)\right) \cdot \nu\,, \qquad \text{on } \Gamma_1 \cup \Gamma_2.$$

By (4.14)–(4.17), we obtain that $U^j \in H^1_\#(E_{\text{out}})$ and it satisfies the following variational problem

$$-\operatorname{div}_y(\kappa\nabla_y U^j) = g_1^j\,, \qquad \text{in } E_{\text{out}}; \qquad (6.6)$$

$$\kappa\nabla_y U^j \cdot \nu = g_2^j\,, \qquad \text{on } \Gamma_1; \qquad (6.7)$$

$$\kappa\nabla_y U^j \cdot \nu = \frac{1}{D_2}(U^j + \hat{\zeta}^j) + g_2^j\,, \quad \text{on } \Gamma_2; \qquad (6.8)$$

$$\int\limits_{E_{\text{out}}} U^j(y) \,\mathrm{d}y = 0\,. \qquad (6.9)$$

Multiplying equation (6.6) by U^j (which is an admissible test function), integrating by parts and using (6.8), we arrive at

$$\int\limits_{E_{\text{out}}} \kappa\nabla_y U^j \cdot \nabla_y U^j \,\mathrm{d}y + \frac{1}{D_2}\int\limits_{\Gamma_2} (U^j + \hat{\zeta}^j)U^j \,\mathrm{d}\sigma_y$$

$$= \int\limits_{E_{\text{out}}} g_1^j U^j \,\mathrm{d}y - \int\limits_{\Gamma_1\cup\Gamma_2} g_2^j U^j \,\mathrm{d}\sigma_y = \int\limits_{E_{\text{out}}} \kappa\nabla_y(y^{cj}(1 - \Psi)) \cdot \nabla_y U^j \,\mathrm{d}y\,. \quad (6.10)$$

Taking into account that, by construction, $U^j = \hat{\chi}^j - y^{cj}$ on Γ_2 and that $\hat{\zeta}^j$ is a constant, by (4.27), it follows

$$\int\limits_{\Gamma_2} (U^j + \hat{\zeta}^j)\hat{\zeta}^j \,\mathrm{d}\sigma_y = \int\limits_{\Gamma_2} (\hat{\chi}^j - y^{cj} + \hat{\zeta}^j)\hat{\zeta}^j \,\mathrm{d}\sigma_y = 0\,,$$

so that the equality (6.10) can be rewritten in the form

$$\int\limits_{E_{\text{out}}} \kappa\nabla_y U^j \cdot \nabla_y U^j \,\mathrm{d}y + \frac{1}{D_2}\int\limits_{\Gamma_2} (U^j + \hat{\zeta}^j)(U^j + \hat{\zeta}^j) \,\mathrm{d}\sigma_y = \int\limits_{E_{\text{out}}} \kappa\nabla_y(y^{cj}(1 - \Psi)) \cdot \nabla_y U^j \,\mathrm{d}y\,.$$

This leads to the energy estimate

$$\int\limits_{E_{\mathrm{out}}} |\nabla_y U^j|^2 \, \mathrm{d}y + \frac{1}{D_2} \int\limits_{\Gamma_2} (U^j + \hat{\zeta}^j)^2 \, \mathrm{d}\sigma_y \leq \gamma \,, \qquad (6.11)$$

where γ depends on γ_0 and $\|\Psi\|_{\mathcal{C}^1(E_{\mathrm{out}})}$, but it is independent of D_2. Therefore, we obtain that, for $j = 1, \ldots, N$,

$$\hat{\chi}^j - y^{cj} + \hat{\zeta}^j \to 0 \,, \qquad \text{strongly in } L^2(\Gamma_2), \qquad (6.12)$$

when $D_2 \to 0$; moreover, there exists $\hat{\chi}_o^j \in H^1_{\#}(E_{\mathrm{out}})$ such that, up to a subsequence,

$$\hat{\chi}^j \rightharpoonup \hat{\chi}_o^j \,, \qquad \text{weakly in } H^1_{\#}(E_{\mathrm{out}}), \qquad (6.13)$$

when $D_2 \to 0$. In particular, from (6.12) and (6.13), it follows that $\hat{\chi}_o^j - y^{cj}$ is independent of y on Γ_2.

In order to pass to the limit in the weak formulation (6.5), let us take a test function $\psi \in \mathcal{X}^{\Gamma_2}_{\#}(E_{\mathrm{out}})$. Then, using (4.27), we get

$$\int\limits_{E_{\mathrm{out}}} \kappa(y) \nabla_y (\hat{\chi}_o^j - y^{cj}) \cdot \nabla_y \psi \, \mathrm{d}y = 0 \,, \qquad \forall \psi \in \mathcal{X}^{\Gamma_2}_{\#}(E_{\mathrm{out}}) \,. \qquad (6.14)$$

Taking into account that $\hat{\chi}_o^j - y^{cj}$ is independent of y on Γ_2, we can extend $\hat{\chi}_o^j$ to $E_{\mathrm{int},2}$ in such a way that $\hat{\chi}_o^j - y^{cj}$ remains independent of y (still denoting this extension by $\hat{\chi}_o^j$). Hence, it is easy to see that such a $\hat{\chi}_o^j$ belongs to the space $H^1_{\#}(E_{\mathrm{out}} \cup E_{\mathrm{int},2})$ and, if we add to it a suitable constant, it satisfies problem (4.9)–(4.13), for $j = 1, \ldots, N$. Now, proceeding as in the proof of Theorem 5.4 in [11], we pass to the limit, for $D_2 \to 0$, in the homogenized matrix obtaining that $A^{D_2}_{\mathrm{hom}} \to A^0_{\mathrm{hom}}$.

By Remark 6.2 below, it follows that the matrix A^0_{hom} is symmetric and positive definite. Thus, as shown in Lemma 6.3, we have that $A^{D_2}_{\mathrm{hom}}$ is positive definite, independently of D_2, for D_2 small enough. Hence, by taking into account Remark 5.3, we have that, as $D_1, D_2 \to 0$, $u - u^{\mathrm{int},1} \to 0$, strongly in $L^2(\Omega_T)$; moreover, up to a subsequence, we get that there exists $u_o \in L^2(0, T; H^1_0(\Omega))$ such that

$$u \rightharpoonup u_o \,, \qquad \text{weakly in } L^2(0, T; H^1_0(\Omega)).$$

Therefore, also $u^{\mathrm{int},1} \rightharpoonup u_o$ in $L^2(\Omega_T)$. Thus, passing to the limit for $D_1, D_2 \to 0$ in the weak formulation (5.1), where we have taken $\varphi_1 = \varphi_2 = \varphi$, we arrive at (6.3), whose uniqueness is standard. Hence, the above convergences hold true for the whole sequences. $\qquad \square$

Remark 6.2. The distributional formulation of the limit problem (6.3) reads like

$$\begin{aligned}
(|E_{\mathrm{out}}| + \lambda_1 + \lambda_2) \, u_{o,t} - \mathrm{div}(A^0_{\mathrm{hom}} \nabla u_o) &= |E_{\mathrm{out}}| f \,, & &\text{in } \Omega_T; \\
u_o &= 0 \,, & &\text{on } \partial\Omega \times (0, T); \qquad (6.15) \\
u_o(x, 0) &= \overline{u} \,, & &\text{in } \Omega.
\end{aligned}$$

Following the same ideas as in [11, Remark 5.5], we can rewrite the homogenized matrix in the symmetric form given by

$$A_{\text{hom}}^0 = \int\limits_{E_{\text{out}} \cup E_{\text{int},2}} \kappa(y) \nabla_y(\hat{\chi}_o - y^c) \cdot \nabla_y(\hat{\chi}_o - y^c) \, \mathrm{d}y$$

$$= \int\limits_{E_{\text{out}}} \kappa(y) \nabla_y(\hat{\chi}_o - y^c) \cdot \nabla_y(\hat{\chi}_o - y^c) \, \mathrm{d}y, \quad (6.16)$$

whose positive definiteness is a standard matter. $\qquad \square$

Lemma 6.3. *Let $A_{\text{hom}}^{D_2}$ be the matrix defined in (5.2). Assume that there exists a positive definite matrix A such that $A_{\text{hom}}^{D_2} \to A$, for $D_2 \to 0$ (or $D_2 \to +\infty$). Then, for D_2 sufficiently small (or sufficiently large), also $A_{\text{hom}}^{D_2}$ is positive definite with ellipticity constant independent of D_2.*

Proof. Since, by assumption, the matrix A is positive definite, there exists $\gamma_A > 0$ such that

$$A\zeta \cdot \zeta \geq \gamma_A |\zeta|^2 \qquad \forall \zeta \in \mathbb{R}^N.$$

Moreover, for any $\varepsilon > 0$, if D_2 is sufficiently small (or sufficiently large), we have that

$$|A_{\text{hom}}^{D_2} - A| < \varepsilon \qquad \Longrightarrow \qquad |(A_{\text{hom}}^{D_2} - A)\zeta \cdot \zeta| \leq \varepsilon |\zeta|^2.$$

Choosing $\varepsilon = \gamma_A/2$, it follows

$$A_{\text{hom}}^{D_2}\zeta \cdot \zeta \geq A\zeta \cdot \zeta - \frac{\gamma_A}{2}|\zeta|^2 \geq \frac{\gamma_A}{2}|\zeta|^2.$$

Then, the thesis is achieved. $\qquad \square$

We point out that, passing to the limit for $D_1, D_2 \to 0$ in the right-hand side of (3.20), we obtain

$$\lim_{D_1,D_2 \to 0}\left(\lim_{\varepsilon \to 0} u_\varepsilon\right) = \lim_{D_1,D_2 \to 0}\left((|E_{\text{out}}| + |E_{\text{int},2}|)u + |E_{\text{int},1}|u^{\text{int},1}\right)$$

$$= (|E_{\text{out}}| + |E_{\text{int},2}| + |E_{\text{int},1}|)u_o = u_o. \quad (6.17)$$

6.2. **Limit $D_1 \to +\infty, D_2 \to 0$.**

Theorem 6.4. *For $D_1 \to +\infty$ and $D_2 \to 0$, there exist $u^{\text{int}} \in L^2(\Omega_T)$ and $u_{\infty,o} \in L^2(0,T;H_0^1(\Omega))$ such that*

$$u^{\text{int},1} \rightharpoonup u^{\text{int}}, \qquad\qquad weakly\ in\ L^2(\Omega_T);$$
$$u \rightharpoonup u_{\infty,o}, \qquad\qquad weakly\ in\ L^2(0,T;H_0^1(\Omega)), \qquad (6.18)$$

where the pair $(u^{\text{int}}, u_{\infty,o})$ is the unique solution of the problem

$$-(|E_{\text{out}}|+\lambda_2)\int\limits_{\Omega_T} u_{\infty,o}\varphi_{2,t}\,\mathrm{d}x\,\mathrm{d}t - \lambda_1\int\limits_{\Omega_T} u^{\text{int}}\varphi_{1,t}\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega_T} A_{\text{hom}}^0 \nabla u_{\infty,o}\cdot\nabla\varphi_2\,\mathrm{d}x\,\mathrm{d}t$$

$$= |E_{\text{out}}|\int\limits_{\Omega_T} f\varphi_2\,\mathrm{d}x\,\mathrm{d}t + (|E_{\text{out}}|+\lambda_2)\int\limits_\Omega \overline{u}\varphi_2(0)\,\mathrm{d}x + \lambda_1\int\limits_\Omega \overline{u}\varphi_1(0)\,\mathrm{d}x, \quad (6.19)$$

for any $\varphi = (\varphi_1, \varphi_2)$, with $\varphi_2 \in H^1(\Omega_T)$, $\varphi_2 = 0$ on $\partial\Omega \times (0,T)$ and for $t = T$ and $\varphi_1 \in H^1(0,T; L^2(\Omega))$, with $\varphi_1 = 0$ for $t = T$. Here, A^0_{hom} is the same matrix defined in (6.16).

Proof. As in the proof of Theorem 6.1, for $D_2 \to 0$, we obtain that, for every $j = 1, \ldots, N$, up to a subsequence, $\hat{\chi}^j \rightharpoonup \hat{\chi}^j_o$, weakly in $H^1_\#(E_{\mathrm{out}})$, where $\hat{\chi}^j_o$ satisfies (4.9)–(4.13). Thus, the homogenized matrix $A^{D_2}_{\mathrm{hom}} \to A^0_{\mathrm{hom}}$. Moreover, from Lemma 6.3 and Remark 5.3, there exist $u_{\infty,o} \in L^2(0,T; H^1_0(\Omega))$ and $u^{\mathrm{int}} \in L^2(\Omega_T)$, such that (6.18) holds. Hence, passing to the limit in (5.1), we get (6.19).
The uniqueness is a consequence of the symmetry and the positive definiteness of the matrix A^0_{hom} (see Remark 6.2 above). Hence, the above convergences hold true for the whole sequences. $\qquad\square$

Remark 6.5. The distributional formulation of the limit problem (6.19) reads like

$$
\begin{aligned}
(|E_{\mathrm{out}}| + \lambda_2)\, \partial_t u_{\infty,o} - \mathrm{div}(A^0_{\mathrm{hom}} \nabla u_{\infty,o}) &= |E_{\mathrm{out}}| f\,, &&\text{in } \Omega_T; \\
u^{\mathrm{int}}_t &= 0\,, &&\text{in } \Omega_T; \\
u_{\infty,o} &= 0\,, &&\text{on } \partial\Omega \times (0,T); \qquad (6.20)\\
u_{\infty,o}(x,0) &= \overline{u}\,, &&\text{in } \Omega; \\
u^{\mathrm{int}}(x,0) &= \overline{u}\,, &&\text{in } \Omega.
\end{aligned}
$$

Notice that we obtained a decoupled bidomain problem made by a parabolic equation for $u_{\infty,o}$, similar to the one found in Subsection 6.1, but with a different capacity, and an ordinary differential equation for u^{int}, which leads to $u^{\mathrm{int}}(x,t) = u^{\mathrm{int}}(x) = \overline{u}(x)$ a.e. in Ω_T. No memory of λ_1 is preserved. $\qquad\square$

We point out that, passing to the limit for $D_1 \to +\infty$ and $D_2 \to 0$ in the right-hand side of (3.20), we obtain

$$
\lim_{\substack{D_1 \to +\infty \\ D_2 \to 0}} \left(\lim_{\varepsilon \to 0} u_\varepsilon \right) = \lim_{\substack{D_1 \to +\infty \\ D_2 \to 0}} \left((|E_{\mathrm{out}}| + |E_{\mathrm{int},2}|) u + |E_{\mathrm{int},1}| u^{\mathrm{int},1} \right)
$$

$$
= (|E_{\mathrm{out}}| + |E_{\mathrm{int},2}|) u_{\infty,o} + |E_{\mathrm{int},1}| \overline{u}\,. \quad (6.21)
$$

6.3. Limit $D_1 \to 0, D_2 \to +\infty$.

Theorem 6.6. *For $D_1 \to 0$ and $D_2 \to +\infty$, there exists $u_{o,\infty} \in L^2(0,T; H^1_0(\Omega))$ such that*

$$
\begin{aligned}
u^{\mathrm{int},1} &\rightharpoonup u_{o,\infty}\,, &&\text{weakly in } L^2(\Omega_T); \\
u &\rightharpoonup u_{o,\infty}\,, &&\text{weakly in } L^2(0,T; H^1_0(\Omega)); \qquad (6.22)\\
u - u^{\mathrm{int},1} &\to 0\,, &&\text{strongly in } L^2(\Omega_T),
\end{aligned}
$$

and $u_{o,\infty}$ is the unique solution of the problem

$$
-(|E_{\mathrm{out}}| + \lambda_1 + \lambda_2) \int_{\Omega_T} u_{o,\infty} \varphi_t \, dx \, dt + \int_{\Omega_T} A^\infty_{\mathrm{hom}} \nabla u_{o,\infty} \cdot \nabla \varphi \, dx \, dt
$$

$$
= |E_{\mathrm{out}}| \int_{\Omega_T} f\varphi \, dx \, dt + (|E_{\mathrm{out}}| + \lambda_1 + \lambda_2) \int_{\Omega} \overline{u}\varphi(0) \, dx\,, \quad (6.23)
$$

for any $\varphi \in H^1(\Omega_T)$ with $\varphi = 0$ on $\partial\Omega \times (0, T)$ and for $t = T$. Here, A_{hom}^{∞} is the matrix defined by

$$(A_{\text{hom}}^{\infty})^{ij} = -\int\limits_{E_{\text{out}}} \kappa_{ik}(y)\partial_{y_k}(\bar{\chi}^j - y^{cj})\,\mathrm{d}y, \qquad (6.24)$$

for i, $j = 1, \ldots, N$, where $\bar{\chi} = (\bar{\chi}^1, \ldots, \bar{\chi}^N)$ is given in Lemma 4.1.

Proof. As in the proof of Theorem 6.1, we arrive at the energy estimate

$$\int\limits_{E_{\text{out}}} |\nabla_y U^j|^2\,\mathrm{d}y + \frac{1}{D_2}\int\limits_{\Gamma_2} (U^j + \hat{\zeta}^j)^2\,\mathrm{d}\sigma_y \le \gamma, \qquad (6.25)$$

where γ does not depend on D_2. Therefore, for any $j = 1, \ldots, N$, there exists a function $\hat{\chi}_{\infty}^j \in H_{\#}^1(E_{\text{out}})$ such that, up to a subsequence, $\hat{\chi}^j \rightharpoonup \hat{\chi}_{\infty}^j$, weakly in $H_{\#}^1(E_{\text{out}})$. Then, passing to the limit in the weak formulation (6.5), we arrive at

$$\int\limits_{E_{\text{out}}} \kappa(y)\nabla_y(\hat{\chi}_{\infty}^j - y^{cj}) \cdot \nabla_y\phi\,\mathrm{d}y = 0, \qquad (6.26)$$

for every $\phi \in H_{\#}^1(E_{\text{out}})$, whose distributional form is given by

$$\begin{aligned} -\operatorname{div}(\kappa(\nabla_y(\hat{\chi}_{\infty}^j - y^{cj}))) &= 0, && \text{in } E_{\text{out}}; \\ \kappa(\nabla_y(\hat{\chi}_{\infty}^j - y^{cj})) \cdot \nu &= 0, && \text{on } \Gamma = \Gamma_1 \cup \Gamma_2; \\ \int\limits_{E_{\text{out}}} \hat{\chi}_{\infty}^j &= 0. && \end{aligned} \qquad (6.27)$$

This implies that $\hat{\chi}_{\infty}$ coincides with the cell function $\bar{\chi}$ given in Lemma 4.1. Now, we can pass to the limit in (5.2), obtaining $A_{\text{hom}}^{D_2} \to A_{\text{hom}}^{\infty}$, with

$$A_{\text{hom}}^{\infty} = -\int\limits_{E_{\text{out}}} \kappa(y)\,\nabla_y(\bar{\chi} - y^c) \cdot \nabla_y\,y^c\,\mathrm{d}y, \qquad (6.28)$$

where we have taken into account that $\hat{\chi}_{\infty} = \bar{\chi}$. By Remark 6.7 below, it follows that the matrix A_{hom}^{∞} is symmetric and positive definite; hence, by Lemma 6.3, we can assure that also $A_{\text{hom}}^{D_2}$ is positive definite independently of D_2, for D_2 large enough. Now, invoking Remark 5.3, we have that (6.22) is in force and, hence, we can pass to the limit in (5.1) with $\varphi_1 = \varphi_2 = \varphi$, arriving at (6.23), whose uniqueness is standard and, so, the above convergences hold true for the whole sequences. $\qquad\square$

Remark 6.7. The distributional formulation of the limit problem (6.23) reads like

$$\begin{aligned} (|E_{\text{out}}| + \lambda_1 + \lambda_2)\,\partial_t u_{o,\infty} - \operatorname{div}(A_{\text{hom}}^{\infty}\nabla u_{o,\infty}) &= |E_{\text{out}}|f, && \text{in } \Omega_T; \\ u_{o,\infty} &= 0, && \text{on } \partial\Omega \times (0, T); \\ u_{o,\infty}(x, 0) &= \overline{u}, && \text{in } \Omega. \end{aligned} \qquad (6.29)$$

Moreover, the matrix A_{hom}^{∞} is the standard matrix appearing in the homogenization of perforated domains. Therefore, it is well-known that it can be written in the

symmetric form

$$A_{\text{hom}}^\infty = \int\limits_{E_{\text{out}}} \kappa(y) \nabla_y (\bar\chi - y^c) \cdot \nabla_y (\bar\chi - y^c) \, \mathrm{d}y \tag{6.30}$$

and it is positive definite. Notice that problem (6.29) represents formally the same monodomain obtained in Subsection 6.1, but with a different diffusion matrix, corresponding to the one given in a perforated domain with two insulated holes. □

We point out that, passing to the limit for $D_1 \to 0$ and $D_2 \to +\infty$ in the right-hand side of (3.20), we obtain

$$\lim_{\substack{D_1 \to 0 \\ D_2 \to +\infty}} \left(\lim_{\varepsilon \to 0} u_\varepsilon \right) = \lim_{\substack{D_1 \to 0 \\ D_2 \to +\infty}} \left((|E_{\text{out}}| + |E_{\text{int},2}|) u + |E_{\text{int},1}| u^{\text{int},1} \right)$$

$$= (|E_{\text{out}}| + |E_{\text{int},2}| + |E_{\text{int},1}|) u_{o,\infty} = u_{o,\infty} . \tag{6.31}$$

6.4. **Limit** $D_1, D_2 \to +\infty$. Combining the results obtained in Theorem 6.4, for $D_1 \to +\infty$, and in Theorem 6.6, for $D_2 \to +\infty$, we arrive at the following statement.

Theorem 6.8. *For $D_1, D_2 \to +\infty$, there exist $u_\infty \in L^2(0, T; H_0^1(\Omega))$ and $u^{\text{int}} \in L^2(\Omega_T)$ such that*

$$\begin{aligned} u &\rightharpoonup u_\infty , && \text{weakly in } L^2(0, T; H_0^1(\Omega)); \\ u^{\text{int},1} &\rightharpoonup u^{\text{int}} , && \text{weakly in } L^2(\Omega_T), \end{aligned} \tag{6.32}$$

where the pair $(u_\infty, u^{\text{int}})$ is the unique solution of the problem

$$- (|E_{\text{out}}| + \lambda_2) \int\limits_{\Omega_T} u_\infty \varphi_{2,t} \, \mathrm{d}x \, \mathrm{d}t - \lambda_1 \int\limits_{\Omega_T} u^{\text{int}} \varphi_{1,t} \, \mathrm{d}x \, \mathrm{d}t + \int\limits_{\Omega_T} A_{\text{hom}}^\infty \nabla u_\infty \cdot \nabla \varphi_2 \, \mathrm{d}x \, \mathrm{d}t$$

$$= |E_{\text{out}}| \int\limits_{\Omega_T} f \varphi_2 \, \mathrm{d}x \, \mathrm{d}t + (|E_{\text{out}}| + \lambda_2) \int\limits_\Omega \overline{u} \varphi_2(0) \, \mathrm{d}x + \lambda_1 \int\limits_\Omega \overline{u} \varphi_1(0) \, \mathrm{d}x , \tag{6.33}$$

for any $\varphi = (\varphi_1, \varphi_2)$, with $\varphi_2 \in H^1(\Omega_T)$, $\varphi_2 = 0$ on $\partial\Omega \times (0, T)$ and for $t = T$ and any $\varphi_1 \in H^1(0, T; L^2(\Omega))$ with $\varphi_1 = 0$ for $t = T$. Here, A_{hom}^∞ is the same matrix defined in (6.24).

Remark 6.9. The distributional formulation of the limit problem (6.33) reads like

$$\begin{aligned} (|E_{\text{out}}| + \lambda_2) \, u_{\infty,t} - \text{div}(A_{\text{hom}}^\infty \nabla u_\infty) &= |E_{\text{out}}| f , && \text{in } \Omega_T; \\ u_t^{\text{int}} &= 0 , && \text{in } \Omega_T; \\ u_\infty &= 0 , && \text{on } \partial\Omega \times (0, T); \\ u_\infty(x, 0) &= \overline{u} , && \text{in } \Omega; \\ u^{\text{int}}(x, 0) &= \overline{u} , && \text{in } \Omega. \end{aligned} \tag{6.34}$$

Notice that we obtained a decoupled bidomain problem similar to the one found in Subsection 6.2, but with a different diffusion matrix. Again, no memory of λ_1 is preserved. □

We point out that, passing to the limit for $D_1, D_2 \to +\infty$ in the right-hand side of (3.20), we obtain

$$\lim_{D_1, D_2 \to +\infty} \left(\lim_{\varepsilon \to 0} u_\varepsilon \right) = \lim_{D_1, D_2 \to +\infty} \left((|E_{\text{out}}| + |E_{\text{int},2}|)u + |E_{\text{int},1}|u^{\text{int},1} \right)$$
$$= (|E_{\text{out}}| + |E_{\text{int},2}|)u_\infty + |E_{\text{int},1}|\overline{u} . \quad (6.35)$$

7. Limit with respect to D_1, D_2 followed by homogenization

In this section, we shall perform the limit with respect to the parameters D_1, D_2 and ε in the opposite order with respect to what we have done in Sections 5 and 6, i.e. we first pass to the limit with respect to D_1, D_2 and, then, we homogenize the resulting systems.

7.1. Limit $D_1, D_2 \to 0$.

Theorem 7.1. *Let $\varepsilon > 0$ be fixed and u_ε be the unique solution of (2.10). Then, there exists $u_\varepsilon^o \in L^2(0, T; H_0^1(\Omega))$ such that, for $D_1, D_2 \to 0$, we have*

$$u_\varepsilon^{\text{out}} \rightharpoonup u_\varepsilon^o , \qquad \text{weakly in } L^2(0, T; H^1(\Omega_{\text{out}}^\varepsilon)); \quad (7.1)$$

$$u_\varepsilon^{\text{int}} \rightharpoonup u_\varepsilon^o , \qquad \text{weakly in } L^2(\Omega_{\text{int}}^\varepsilon \times (0, T)); \quad (7.2)$$

$$[u_\varepsilon] \to 0 , \qquad \text{strongly in } L^2(\Gamma_T^\varepsilon), \quad (7.3)$$

where u_ε^o is the unique solution of problem

$$\frac{\partial u_\varepsilon^o}{\partial t} - \text{div}(\kappa_\varepsilon \nabla u_\varepsilon^o) = f , \qquad \text{in } \Omega_{\text{out}}^\varepsilon \times (0, T); \quad (7.4)$$

$$\lambda_i \frac{\partial u_\varepsilon^o}{\partial t} = \frac{1}{\varepsilon^N} \int_{\Gamma_{\xi,i}^\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon^o \cdot \nu_\varepsilon \, d\sigma , \qquad \text{in } E_{\text{int},i}^{\varepsilon,\xi} \times (0, T), \, \xi \in \Xi^\varepsilon, \, i = 1, 2; \quad (7.5)$$

$$u_\varepsilon^o = 0 , \qquad \text{on } \partial\Omega \times (0, T); \quad (7.6)$$

$$u_\varepsilon^o(x, 0) = \overline{u_\varepsilon}(x) , \qquad \text{in } \Omega. \quad (7.7)$$

Proof. From the energy estimate (2.12), we get (7.1)–(7.3), which, in particular, also imply that $u_\varepsilon^{\text{int}} \rightharpoonup u_\varepsilon^o$ weakly in $L^2(\Gamma_T^\varepsilon)$. Thus, taking into account that $u_\varepsilon^{\text{int}}$ is independent of x on each $E_{\text{int},i}^{\varepsilon,\xi}$, from (7.2) it follows that the same property is satisfied by u_ε^o. Now, in order to pass to the limit in the weak formulation (2.10), we choose a test function $\varphi_\varepsilon \in H^1(\Omega_T)$ such that $\varphi_\varepsilon = 0$ on $\partial\Omega \times (0, T)$ and for $t = T$, and φ_ε is independent of x on each $E_{\text{int},i}^{\varepsilon,\xi}$. Then, we obtain

$$\int_0^T \int_\Omega \{-a_\varepsilon u_\varepsilon^o \varphi_{\varepsilon,t} + \chi_{\Omega_{\text{out}}^\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon^0 \cdot \nabla \varphi_\varepsilon\} \, dx \, dt = \int_0^T \int_{\Omega_{\text{out}}^\varepsilon} f \varphi_\varepsilon \, dx \, dt + \int_\Omega a_\varepsilon \overline{u_\varepsilon} \varphi_\varepsilon(0) \, dx . \quad (7.8)$$

Recalling (2.2), we notice that (7.8) is the weak formulation of problem (7.4)–(7.7).

$\square$

Remark 7.2. We remark that in (7.5) we can replace the set $E_{\text{int},i}^{\varepsilon,\xi} \times (0,T)$ with $\Gamma_{\xi,i}^{\varepsilon} \times (0,T)$, so that problem (7.4)–(7.7) is the same problem studied in [6] (see problem (2)-(5)), the only difference being the fact that here we have two isolated inclusions in each periodic cell. Therefore, the procedure carried out in [6] leads to a homogenized limit function u^o satisfying the problem

$$
\begin{cases}
(|E_{\text{out}}| + \lambda_1 + \lambda_2)\, u_t^o - \operatorname{div}(A_{\text{hom}}\nabla u^o) = |E_{\text{out}}| f\,, & \text{in } \Omega_T; \\
u^o = 0\,, & \text{on } \partial\Omega \times (0,T); \\
u^o(x,0) = \overline{u}\,, & \text{in } \Omega.
\end{cases} \quad (7.9)
$$

Problem (7.9) is analogous to [6, Problem (50)], with a new capacity $\lambda = \lambda_1 + \lambda_2$ and a new diffusion matrix A_{hom} given by

$$
A_{\text{hom}} = \int_{E_{\text{out}}} \kappa(y)\nabla_y(\chi - y^c) \cdot \nabla_y(\chi - y^c)\,\mathrm{d}y
$$

$$
= \int_Y \kappa(y)\nabla_y(\chi - y^c) \cdot \nabla_y(\chi - y^c)\,\mathrm{d}y\,, \quad (7.10)
$$

where χ is the cell function obtained in Lemma 4.2. In the last equality, we have extended the function $\chi - y^c$ to the whole of Y, by taking it constant in $E_{\text{int},1}$ and $E_{\text{int},2}$. □

We point out that, taking the limit of u_ε, first for $D_1, D_2 \to 0$ and, then, for $\varepsilon \to 0$, we obtain

$$
\lim_{\varepsilon \to 0}\left(\lim_{D_1,D_2 \to 0} u_\varepsilon\right) = \lim_{\varepsilon \to 0}\left(\lim_{D_1,D_2 \to 0}(\chi_{\Omega_{\text{out}}^{\varepsilon}} u_\varepsilon + \chi_{\Omega_{\text{int},1}^{\varepsilon}} u_\varepsilon + \chi_{\Omega_{\text{int},2}^{\varepsilon}} u_\varepsilon)\right)
$$

$$
= \lim_{\varepsilon \to 0}\left((\chi_{\Omega_{\text{out}}^{\varepsilon}} + \chi_{\Omega_{\text{int},1}^{\varepsilon}} + \chi_{\Omega_{\text{int},2}^{\varepsilon}})u_\varepsilon^o\right) = \lim_{\varepsilon \to 0} u_\varepsilon^o = u^o\,. \quad (7.11)
$$

7.2. **Limit $D_1 \to +\infty$ and $D_2 \to 0$.** For later use, we introduce the space

$$
\mathcal{X}_0^\varepsilon(\Omega) := \{v = (v^{\text{int},1}, v^{\text{out},2}) : v^{\text{out},2} \in H^1(\Omega_{\text{out}}^\varepsilon \cup \overline{\Omega_{\text{int},2}^\varepsilon}),\ v^{\text{out},2} \text{ is constant}
$$

$$
\text{in } x \text{ in each } E_{\text{int},2}^{\varepsilon,\xi},\ \xi \in \Xi^\varepsilon,\ v^{\text{int},1} \in L^2(\Omega_{\text{int},1}^\varepsilon),\ v^{\text{int},1} \text{ is constant in } x
$$

$$
\text{in each } E_{\text{int},1}^{\varepsilon,\xi},\ \xi \in \Xi^\varepsilon,\ v = v^{\text{out},2} = 0 \text{ on } \partial\Omega\}. \quad (7.12)
$$

Theorem 7.3. *Let $\varepsilon > 0$ be fixed and u_ε be the unique solution of (2.10). Then, there exists $u_\varepsilon^{\infty,o} \in L^2(0,T;\mathcal{X}_0^\varepsilon(\Omega))$ such that, for $D_1 \to +\infty$ and $D_2 \to 0$, we have*

$$
u_\varepsilon^{\text{out}} \rightharpoonup u_\varepsilon^{\infty,o}, \qquad \text{weakly in } L^2(0,T;H^1(\Omega_{\text{out}}^\varepsilon)); \qquad (7.13)
$$

$$
u_\varepsilon^{\text{int},1} \rightharpoonup u_\varepsilon^{\infty,o}, \qquad \text{weakly in } L^2(\Omega_{\text{int},1}^\varepsilon \times (0,T)); \qquad (7.14)
$$

$$
u_\varepsilon^{\text{int},2} \rightharpoonup u_\varepsilon^{\infty,o}, \qquad \text{weakly in } L^2(\Omega_{\text{int},2}^\varepsilon \times (0,T)); \qquad (7.15)
$$

$$
[u_\varepsilon] \to 0\,, \qquad \text{strongly in } L^2(\Gamma_2^\varepsilon \times (0,T)), \qquad (7.16)
$$

where $u_\varepsilon^{\infty,o}$ is the unique solution of problem

$$\frac{\partial u_\varepsilon^{\infty,o}}{\partial t} - \operatorname{div}(\kappa_\varepsilon \nabla u_\varepsilon^{\infty,o}) = f\,, \qquad in\ \Omega_{\text{out}}^\varepsilon \times (0,T); \tag{7.17}$$

$$\lambda_2 \frac{\partial u_\varepsilon^{\infty,o}}{\partial t} = \frac{1}{\varepsilon^N} \int_{\Gamma_{\xi,2}^\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon^{\infty,o} \cdot \nu_\varepsilon \, \mathrm{d}\sigma\,, \qquad in\ E_{\text{int},2}^{\varepsilon,\xi} \times (0,T),\ \xi \in \Xi^\varepsilon; \tag{7.18}$$

$$\kappa_\varepsilon \nabla u_\varepsilon^{\infty,o} \cdot \nu_\varepsilon = 0\,, \qquad in\ \Gamma_{\xi,1}^\varepsilon \times (0,T),\ \xi \in \Xi^\varepsilon; \tag{7.19}$$

$$\lambda_1 \frac{\partial u_\varepsilon^{\infty,o}}{\partial t} = 0\,, \qquad in\ E_{\text{int},1}^{\varepsilon,\xi} \times (0,T),\ \xi \in \Xi^\varepsilon; \tag{7.20}$$

$$u_\varepsilon^{\infty,o} = 0\,, \qquad on\ \partial\Omega \times (0,T); \tag{7.21}$$

$$u_\varepsilon^{\infty,o}(x,0) = \overline{u_\varepsilon}(x)\,, \qquad in\ \Omega. \tag{7.22}$$

Proof. From the energy estimate (2.12), we get (7.13)–(7.16) which, in particular, imply that $u_\varepsilon^{\infty,o}$ has no jump across $\Gamma_{\xi,2}^\varepsilon \times (0,T)$, for $\xi \in \Xi^\varepsilon$. Moreover, taking into account that $u_\varepsilon^{\text{int}}$ is independent of x on each $E_{\text{int},i}^{\varepsilon,\xi}$, from (7.14)–(7.15) it follows that the same property is satisfied by $u_\varepsilon^{\infty,o}$. Then, $u_\varepsilon^{\infty,o} \in L^2(0,T;\mathcal{X}_0^\varepsilon(\Omega))$. Now, in order to pass to the limit in the weak formulation (2.10), we choose a test function $\varphi_\varepsilon \in L^2(\Omega; H^1(0,T)) \cap L^2(0,T;\mathcal{X}_0^\varepsilon(\Omega))$, such that $\varphi_\varepsilon = 0$ for $t = T$. Then, in the limit, we obtain

$$\int_0^T\!\!\!\int_\Omega \{-a_\varepsilon u_\varepsilon^{\infty,o}\varphi_{\varepsilon,t} + \chi_{\Omega_{\text{out}}^\varepsilon}\kappa_\varepsilon \nabla u_\varepsilon^{\infty,o}\cdot\nabla\varphi_\varepsilon\}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\!\int_{\Omega_{\text{out}}^\varepsilon} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega a_\varepsilon\overline{u_\varepsilon}\varphi_\varepsilon(0)\,\mathrm{d}x\,. \tag{7.23}$$

Recalling (2.2), we notice that (7.23) is the weak formulation of problem (7.17)–(7.22). By standard computations and exploiting the linearity of the problem, as a consequence of the associated energy estimate, we get that system (7.17)–(7.22) is well-posed. Hence, all the above convergences hold true for the whole sequence. $\square$

Remark 7.4. We remark that problem (7.17)–(7.22) is a decoupled system, where the solution $u_\varepsilon^{\infty,o} = \overline{u_\varepsilon}$ in $\Omega_{\text{int},1}^\varepsilon \times (0,T)$. Then, for $\varepsilon \to 0$, there exists $u_{\text{int},1}^{\infty,o} \in L^2(\Omega_T)$ such that $u_\varepsilon^{\infty,o}\chi_{\Omega_{\text{int},1}^\varepsilon} \rightharpoonup u_{\text{int},1}^{\infty,o}$, weakly in $L^2(\Omega_T)$, where, from our assumptions on the initial datum, it is easy to see that $u_{\text{int},1}^{\infty,o} = |E_{\text{int},1}|\overline{u}$. $\square$

Theorem 7.5. *Let $u_\varepsilon^{\infty,o}$ be the function appearing in Theorem 7.3. Then, there exists $u^{\infty,o} \in L^2(0,T;H_0^1(\Omega))$ such that, for $\varepsilon \to 0$, we have*

$$u_\varepsilon^{\infty,o} \rightharpoonup (|E_{\text{out}}| + |E_{\text{int},2}|)u^{\infty,o} + |E_{\text{int},1}|\overline{u}\,, \quad weakly\ in\ L^2(\Omega_T), \tag{7.24}$$

and $u^{\infty,o}$ is the unique solution of

$$-(|E_{\text{out}}| + \lambda_2)\int_{\Omega_T} u^{\infty,o}\varphi_t\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} A_{\text{hom}}^0 \nabla u^{\infty,o}\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t$$

$$= |E_{\text{out}}|\int_{\Omega_T} f\varphi\,\mathrm{d}x\,\mathrm{d}t + (|E_{\text{out}}| + \lambda_2)\int_\Omega \overline{u}\varphi(0)\,\mathrm{d}x\,, \tag{7.25}$$

for all $\varphi \in H^1(\Omega_T)$, with $\varphi = 0$ on $\partial\Omega \times (0, T)$ and for $t = T$. Here, A^0_{hom} is the same symmetric and positive definite homogenized matrix defined in (6.16).

Proof. From the weak formulation (7.23), we get the energy estimate

$$\sup_{0 < t < T} \int_{\Omega} |u^{\infty,o}_\varepsilon(t)|^2 \, dx + \int_0^T \int_{\Omega^\varepsilon_{\text{out}}} |\nabla u^{\infty,o}_\varepsilon|^2 \, dx \, dt \leq \gamma, \qquad (7.26)$$

with γ independent of ε. Then, by Proposition 3.4 and the fact that $u^{\infty,o}_\varepsilon$ is piecewise constant in $\Omega^\varepsilon_{\text{int},2}$, there exist $u^{\infty,o} \in L^2(0, T; H^1_0(\Omega))$ and $\hat{u} \in L^2(\Omega_T; H^1_\#(E_{\text{out}}))$ such that, up to a subsequence, we have

$$\mathcal{T}_\varepsilon(\chi_{\Omega^\varepsilon_{\text{out}}} u^{\infty,o}_\varepsilon) \rightharpoonup u^{\infty,o}, \qquad \text{weakly in } L^2(\Omega_T \times E_{\text{out}}); \qquad (7.27)$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^\varepsilon_{\text{int},2}} u^{\infty,o}_\varepsilon) \rightharpoonup u^{\infty,o}, \qquad \text{weakly in } L^2(\Omega_T \times E_{\text{int},2}); \qquad (7.28)$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^\varepsilon_{\text{out}}} \nabla u^{\infty,o}_\varepsilon) \rightharpoonup \nabla u^{\infty,o} + \nabla_y \hat{u}, \qquad \text{weakly in } L^2(\Omega_T \times E_{\text{out}}). \qquad (7.29)$$

The two limit functions in (7.27) and (7.28) are equal, invoking (3.10) and the fact that $[u^{\infty,o}_\varepsilon] = 0$ on $\Gamma^\varepsilon_{\xi,2} \times (0, T)$, since $u^{\infty,o}_\varepsilon \in L^2(0, T; \mathcal{X}^\varepsilon_0(\Omega))$. Moreover, taking into account Remark 7.4 and the properties of the unfolding operator (see [21]), we obtain that also convergence (7.24) holds true.

In order to pass to the limit in the weak formulation (7.23), we choose first the test function $\varphi_\varepsilon(x, t) = \varepsilon\varphi(x, x/\varepsilon, t)$, where

$$\varphi(x, y, t) = \begin{cases} \psi(y)\mathcal{M}_\varepsilon(w(x, t)) + w(x, t)\phi(y), & \text{in } \Omega_T \times (E_{\text{out}} \cup E_{\text{int},2}), \\ 0, & \text{in } \Omega_T \times E_{\text{int},1}, \end{cases}$$

with $w \in \mathcal{C}^1(\overline{\Omega}_T)$, $w(x, t) = 0$ on $\partial\Omega \times (0, T)$ and for $t = T$; the function $\psi \in \mathcal{C}^\infty(\overline{Y} \setminus E_{\text{int},1})$ vanishes near ∂Y and it is constant in $E_{\text{int},2}$, and the function $\phi \in \mathcal{C}^\infty_\#(\overline{Y} \setminus E_{\text{int},1})$ vanishes in $E_{\text{int},2}$. Then, unfolding, passing to the limit and using a density argument, we get

$$\int_{\Omega_T \times E_{\text{out}}} \kappa(y)(\nabla u^{\infty,o} + \nabla_y \hat{u}) \cdot \nabla_y \Psi \, dy \, dx \, dt = 0, \qquad (7.30)$$

for every $\Psi \in \mathcal{C}^\infty(\overline{\Omega_T \times (E_{\text{out}} \cup E_{\text{int},2})})$, with $\Psi(x, t, y) = 0$ on $\partial\Omega \times (0, T) \times (E_{\text{out}} \cup E_{\text{int},2})$ and for $t = T$ and Ψ independent of y in $E_{\text{int},2}$. By standard computations, we obtain that $\hat{u}$ can be factorized as $\hat{u}(x, t, y) = -\hat{\chi}_o(y) \cdot \nabla u^{\infty,o}(x, t)$, where $\hat{\chi}_o$ is given in Lemma 4.3. Now, we take as test function $\tilde{\varphi}_\varepsilon(x, x/\varepsilon, t)$, with

$$\tilde{\varphi}_\varepsilon(x, y, t) = \begin{cases} \psi(y)\mathcal{M}_\varepsilon(w(x, t)) + w(x, t)(1 - \psi(y)), & \text{in } \Omega_T \times (E_{\text{out}} \cup E_{\text{int},2}), \\ 0, & \text{in } \Omega_T \times E_{\text{int},1}, \end{cases}$$

where w and ψ are as before, but $\psi(y) = 1$ in $E_{\text{int},2}$. Then, following the same idea as in [7] and [22], unfolding the weak formulation (7.23), passing to the limit for $\varepsilon \to 0$

and adding (7.30) to the resulting formulation, we arrive at

$$-\iint\limits_{\Omega_T E_{\text{out}}}\int u^{\infty,o}w_t\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t+\iint\limits_{\Omega_T E_{\text{out}}}\int \kappa(\nabla u^{\infty,o}+\nabla_y\hat{u})\cdot[\nabla w+\nabla_y(\Psi-(y^c\cdot\nabla w)\psi)]\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$

$$-\frac{\lambda_2}{|E_{\text{int},2}|}\iint\limits_{\Omega_T E_{\text{int},2}}\int u^{\infty,o}w_t\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t=\iint\limits_{\Omega_T E_{\text{out}}}\int fw\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$

$$+\iint\limits_{\Omega E_{\text{out}}}\int \overline{u}_0w(x,0)\,\mathrm{d}y\,\mathrm{d}x+\frac{\lambda_2}{|E_{\text{int},2}|}\iint\limits_{\Omega E_{\text{int},2}}\int \overline{u}_0w(x,0)\,\mathrm{d}y\,\mathrm{d}x\,. \quad (7.31)$$

Now, inserting the factorization of $\hat{u}$ in (7.31) and following the same ideas as in [7, Theorem 4.10], by a density argument, we obtain (7.25) where A^0_{hom} is the symmetric and positive definite matrix given in (6.16). Then, we have that problem (7.25) is well-posed and, hence, all the above convergences hold true for the whole sequence. $\quad\square$

Remark 7.6. Taking into account Remark 7.4 and Theorem 7.5, we arrive at the distributional formulation of the homogenized problem, which reads like

$$\begin{aligned}
(|E_{\text{out}}|+\lambda_2)\,u_t^{\infty,o}-\text{div}(A^0_{\text{hom}}\nabla u^{\infty,o})&=|E_{\text{out}}|f\,, &&\text{in }\Omega_T;\\
u_{\text{int},1}^{\infty,o}&=|E_{\text{int},1}|\overline{u}\,, &&\text{in }\Omega_T;\\
u^{\infty,o}&=0\,, &&\text{on }\partial\Omega\times(0,T);\\
u^{\infty,o}(x,0)&=\overline{u}\,, &&\text{in }\Omega.
\end{aligned} \quad (7.32)$$

Notice that, starting from a decoupled system (see (7.17)–(7.22)), we arrived at a decoupled bidomain system, made by a parabolic equation and an algebraic one. $\quad\square$

We point out that, taking the limit of u_ε, first for $D_1\to+\infty, D_2\to 0$ and, then, for $\varepsilon\to 0$, using (7.24), we obtain

$$\lim_{\varepsilon\to 0}\left(\lim_{\substack{D_1\to+\infty\\D_2\to 0}}u_\varepsilon\right)=\lim_{\varepsilon\to 0}\left(\lim_{\substack{D_1\to+\infty\\D_2\to 0}}(u_\varepsilon\chi_{\Omega_{\text{out}}^\varepsilon}+u_\varepsilon\chi_{\Omega_{\text{int},1}^\varepsilon}+u_\varepsilon\chi_{\Omega_{\text{int},2}^\varepsilon})\right)$$

$$=\lim_{\varepsilon\to 0}\left((\chi_{\Omega_{\text{out}}^\varepsilon}+\chi_{\Omega_{\text{int},2}^\varepsilon})u_\varepsilon^{\infty,o}+\chi_{\Omega_{\text{int},1}^\varepsilon}u_\varepsilon^{\infty,o}\right)$$

$$=(|E_{\text{out}}|+|E_{\text{int},2}|)u^{\infty,o}+|E_{\text{int},1}|\overline{u}\,. \quad (7.33)$$

7.3. **Limit $D_1\to 0$ and $D_2\to+\infty$.** Proceeding in the same way as in the previous subsection, with Γ_1 and Γ_2 interchanged, we obtain that, letting first $D_1\to 0$ and $D_2\to+\infty$ and, then, $\varepsilon\to 0$, there exists a pair of limit functions $u^{o,\infty}\in L^2(0,T;H_0^1(\Omega))$ and $u_{\text{int},2}^{o,\infty}\in L^2(\Omega_T)$ such that

$$\begin{aligned}
(|E_{\text{out}}|+\lambda_1)\,u_t^{o,\infty}-\text{div}(B^0_{\text{hom}}\nabla u^{o,\infty})&=|E_{\text{out}}|f\,, &&\text{in }\Omega_T;\\
u_{\text{int},2}^{o,\infty}&=|E_{\text{int},2}|\overline{u}\,, &&\text{in }\Omega_T;\\
u^{o,\infty}&=0\,, &&\text{on }\partial\Omega\times(0,T);\\
u^{o,\infty}(x,0)&=\overline{u}\,, &&\text{in }\Omega,
\end{aligned} \quad (7.34)$$

where B^0_{hom} is defined as in (6.16), but with respect to a cell function $\hat{\chi}^o$ constructed as in Lemma 4.3, with $E_{\text{int},2}$ and Γ_1 replaced by $E_{\text{int},1}$ and Γ_2, respectively.

Moreover, as in the previous subsection, taking the limit of u_ε, first for $D_1 \to 0, D_2 \to +\infty$ and, then, for $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon \to 0} \left(\lim_{\substack{D_1 \to 0 \\ D_2 \to +\infty}} u_\varepsilon \right) = (|E_{\text{out}}| + |E_{\text{int},1}|)u^{o,\infty} + |E_{\text{int},2}|\overline{u}. \qquad (7.35)$$

7.4. Limit $D_1, D_2 \to +\infty$.

Theorem 7.7. *Let $\varepsilon > 0$ be fixed and u_ε be the unique solution of (2.10). Then, there exist $u^{\infty,\text{int}}_{\varepsilon,i} \in L^2(\Omega^\varepsilon_{\text{int},i} \times (0,T))$, $i = 1, 2$, and $u^{\infty,\text{out}}_\varepsilon \in L^2(0,T; H^1(\Omega^\varepsilon_{\text{out}}))$, with $u^{\infty,\text{out}}_\varepsilon = 0$ on $\partial\Omega \times (0,T)$, such that, for $D_1, D_2 \to +\infty$, we have*

$$u^{\text{int}}_\varepsilon \rightharpoonup u^{\infty,\text{int}}_{\varepsilon,i}, \qquad \text{weakly in } L^2(\Omega^\varepsilon_{\text{int},i} \times (0,T)), \ i = 1, 2; \qquad (7.36)$$

$$u^{\text{out}}_\varepsilon \rightharpoonup u^{\infty,\text{out}}_\varepsilon, \qquad \text{weakly in } L^2(0,T; H^1(\Omega^\varepsilon_{\text{out}})), \qquad (7.37)$$

where $u^\infty_\varepsilon := (u^{\infty,\text{int}}_{\varepsilon,1}, u^{\infty,\text{int}}_{\varepsilon,2}, u^{\infty,\text{out}}_\varepsilon)$ is the unique solution of problem

$$\frac{\partial u^{\infty,\text{out}}_\varepsilon}{\partial t} - \text{div}(\kappa_\varepsilon \nabla u^{\infty,\text{out}}_\varepsilon) = f, \qquad \text{in } \Omega^\varepsilon_{\text{out}} \times (0,T); \qquad (7.38)$$

$$\frac{\partial u^{\infty,\text{int}}_{\varepsilon,i}}{\partial t} = 0, \qquad \text{in } \Omega^\varepsilon_{\text{int},i} \times (0,T), \ i = 1, 2; \qquad (7.39)$$

$$\kappa_\varepsilon \nabla u^{\infty,\text{out}}_\varepsilon \cdot \nu_\varepsilon = 0, \qquad \text{in } \Gamma^\varepsilon \times (0,T); \qquad (7.40)$$

$$u^{\infty,\text{out}}_\varepsilon = 0, \qquad \text{on } \partial\Omega \times (0,T); \qquad (7.41)$$

$$u^{\infty,\text{out}}_\varepsilon(x,0) = \overline{u_\varepsilon}(x), \qquad \text{in } \Omega; \qquad (7.42)$$

$$u^{\infty,\text{int}}_{\varepsilon,i}(x,0) = \overline{u_\varepsilon}(x), \qquad \text{in } \Omega. \qquad (7.43)$$

Proof. From the energy estimate (2.12), we get (7.36) and (7.37). Now, in order to pass to the limit in the weak formulation (2.10), we choose a test function $\varphi_\varepsilon = (\varphi^{\text{int},i}_\varepsilon, \varphi^{\text{out}}_\varepsilon)$ such that $\varphi^{\text{int},i}_\varepsilon \in L^2(\Omega^\varepsilon_{\text{int},i}; H^1(0,T))$ with $\varphi^{\text{int},i}_\varepsilon$ constant with respect to x in each $E^{\varepsilon,\xi}_{\text{int},i}$, $i = 1, 2$, $\varphi^{\text{out}}_\varepsilon \in H^1(\Omega^\varepsilon_{\text{out}} \times (0,T))$, $[\varphi_\varepsilon] \in L^2(\Gamma^\varepsilon \times (0,T))$, and $\varphi_\varepsilon(\cdot,T) = 0$, $\varphi_\varepsilon = \varphi^{\text{out}}_\varepsilon = 0$ on $\partial\Omega \times (0,T)$. Then, in the limit, we obtain

$$\int_0^T \!\!\! \int_\Omega \{-a_\varepsilon u^\infty_\varepsilon \varphi_{\varepsilon,t} + \chi_{\Omega^\varepsilon_{\text{out}}} \kappa_\varepsilon \nabla u^{\infty,\text{out}}_\varepsilon \cdot \nabla \varphi_\varepsilon\} \, dx \, dt = \int_0^T \!\!\! \int_{\Omega^\varepsilon_{\text{out}}} f\varphi_\varepsilon \, dx \, dt + \int_\Omega a_\varepsilon \overline{u_\varepsilon} \varphi_\varepsilon(0) \, dx,$$

$$(7.44)$$

where we have taken into account that $u^{\text{int},i}_\varepsilon$ is constant on $\Omega^\varepsilon_{\text{int},i} \times (0,T)$, $i = 1, 2$. Recalling (2.2), we notice that, by a density argument, (7.44) is the weak formulation of problem (7.38)–(7.42). Since uniqueness for system (7.38)–(7.43) is a standard matter, all the above convergences hold true for the whole sequence. $\qquad \square$

Remark 7.8. We remark that problem (7.38)–(7.42) is a decoupled system, where the solution $u^{\infty,\text{int}}_{\varepsilon,i} = \overline{u_\varepsilon}$ in $\Omega^\varepsilon_{\text{int}} \times (0,T)$. Then, for $\varepsilon \to 0$, there exists $u^\infty_{\text{int}} \in L^2(\Omega_T)$ such

that $u_{\varepsilon,1}^{\infty,\text{int}}\chi_{\Omega_{\text{int},1}^{\varepsilon}}+u_{\varepsilon,2}^{\infty,\text{int}}\chi_{\Omega_{\text{int},2}^{\varepsilon}} \rightharpoonup u_{\text{int}}^{\infty}$, weakly in $L^2(\Omega_T)$, where, from our assumptions on the initial datum, it is easy to see that $u_{\text{int}}^{\infty} = (|E_{\text{int},1}| + |E_{\text{int},2}|)\overline{u}$. $\qquad\square$

Theorem 7.9. *Let u_{ε}^{∞} be the function appearing in Theorem 7.7. Then, there exists $u^{\infty} \in L^2(0,T;H_0^1(\Omega))$ such that, for $\varepsilon \to 0$, we have*

$$u_{\varepsilon}^{\infty} \rightharpoonup |E_{\text{out}}|u^{\infty} + (|E_{\text{int},1}| + |E_{\text{int},2}|)\overline{u}, \quad \text{weakly in } L^2(\Omega_T), \qquad (7.45)$$

and u^{∞} is the unique solution of

$$-|E_{\text{out}}| \int\limits_{\Omega_T} u^{\infty}\varphi_t \,\mathrm{d}x \,\mathrm{d}t + \int\limits_{\Omega_T} A_{\text{hom}}^{\infty} \nabla u^{\infty} \cdot \nabla \varphi \,\mathrm{d}x \,\mathrm{d}t$$

$$= |E_{\text{out}}| \int\limits_{\Omega_T} f\varphi \,\mathrm{d}x \,\mathrm{d}t + |E_{\text{out}}| \int\limits_{\Omega} \overline{u}\varphi(0) \,\mathrm{d}x, \quad (7.46)$$

for all $\varphi \in H^1(\Omega_T)$, with $\varphi = 0$ on $\partial\Omega \times (0,T)$ and for $t = T$. Here, A_{hom}^{∞} is the same symmetric and positive definite homogenized matrix defined in (6.30).

Proof. From the weak formulation (7.44), it follows that an energy estimate analogous to the one in (7.26) is still in force with $u_{\varepsilon}^{\infty,o}$ replaced by $(u_{\varepsilon,1}^{\infty,\text{int}}, u_{\varepsilon,2}^{\infty,\text{int}}, u_{\varepsilon}^{\infty,\text{out}})$. Then, there exist $u^{\infty} \in L^2(0,T;H_0^1(\Omega))$ and $\hat{u} \in L^2(\Omega_T;H_{\#}^1(E_{\text{out}}))$ such that, up to a subsequence, we have

$$\mathcal{T}_{\varepsilon}(\chi_{\Omega_{\text{out}}^{\varepsilon}} u_{\varepsilon}^{\infty,\text{out}}) \rightharpoonup u^{\infty}, \qquad\qquad \text{weakly in } L^2(\Omega_T \times E_{\text{out}}); \qquad (7.47)$$

$$\mathcal{T}_{\varepsilon}(\chi_{\Omega_{\text{out}}^{\varepsilon}} \nabla u_{\varepsilon}^{\infty,\text{out}}) \rightharpoonup \nabla u^{\infty} + \nabla_y\hat{u}, \quad \text{weakly in } L^2(\Omega_T \times E_{\text{out}}), \qquad (7.48)$$

and, taking into account Remark 7.8, we obtain that also convergence (7.45) holds true. We notice that (7.38) and (7.40)–(7.42) give a standard Neumann problem in a perforated domain with two insulated holes in any reference unit cell; then, the homogenization limit for such a Neumann problem is a well-known result and leads to (7.46) (see [21] and the references therein). $\qquad\square$

Remark 7.10. Taking into account Remark 7.8 and Theorem 7.9, we arrive at the distributional formulation of the homogenized problem, which reads like

$$\begin{aligned}
|E_{\text{out}}|u_t^{\infty} - \text{div}(A_{\text{hom}}^{\infty}\nabla u^{\infty}) &= |E_{\text{out}}|f\,, & &\text{in } \Omega_T; \\
u_{\text{int},1}^{\infty} &= |E_{\text{int},1}|\overline{u}\,, & &\text{in } \Omega_T; \\
u_{\text{int},2}^{\infty} &= |E_{\text{int},2}|\overline{u}\,, & &\text{in } \Omega_T; \quad (7.49) \\
u^{\infty} &= 0\,, & &\text{on } \partial\Omega \times (0,T); \\
u^{\infty}(x,0) &= \overline{u}\,, & &\text{in } \Omega.
\end{aligned}$$

Notice that, starting from a decoupled system (see (7.38)–(7.42)), we arrived at a decoupled tridomain system, made by a parabolic equation and two algebraic ones.

$\qquad\square$

We point out that, taking the limit of u_ε, first for $D_1, D_2 \to +\infty$ and, then, for $\varepsilon \to 0$, using (7.45), we obtain

$$
\lim_{\varepsilon \to 0} \left(\lim_{D_1, D_2 \to +\infty} u_\varepsilon \right) = \lim_{\varepsilon \to 0} \left(\lim_{D_1, D_2 \to +\infty} (\chi_{\Omega_{\text{out}}^\varepsilon} u_\varepsilon + \chi_{\Omega_{\text{int},1}^\varepsilon} u_\varepsilon + \chi_{\Omega_{\text{int},2}^\varepsilon} u_\varepsilon) \right)
$$
$$
= \lim_{\varepsilon \to 0} \left(\chi_{\Omega_{\text{out}}^\varepsilon} u_\varepsilon^{\infty,\text{out}} + \chi_{\Omega_{\text{int},1}^\varepsilon} u_{\varepsilon,1}^{\infty,\text{int}} + \chi_{\Omega_{\text{int},2}^\varepsilon} u_{\varepsilon,2}^{\infty,\text{int}} \right)
$$
$$
= |E_{\text{out}}| u^\infty + (|E_{\text{int},1}| + |E_{\text{int},2}|) \overline{u} . \quad (7.50)
$$

8. Summary and comparison of the results

We end the paper by providing a scheme of our obtained models, with the aim to better emphasize the different features of the limit problems.

- **Limit $D_1, D_2 \to 0$, two monodomains, no commutation.** If we first perform homogenization and then we take the limit for $D_1, D_2 \to 0$, the final function u_o satisfies the parabolic equation (see (6.15))

$$
(|E_{\text{out}}| + \lambda_1 + \lambda_2) \partial_t u_o - \text{div}(A_{\text{hom}}^0 \nabla u_o) = |E_{\text{out}}| f ,
$$

  with

$$
A_{\text{hom}}^0 = \int_{E_{\text{out}}} \kappa(y) \nabla_y(\hat{\chi}_0 - y^c) \cdot \nabla_y(\hat{\chi}_0 - y^c) \, dy .
$$

  Instead, if we first perform the limit for $D_1, D_2 \to 0$ and then we homogenize, the final limit u^o satisfies (see (7.9))

$$
(|E_{\text{out}}| + \lambda_1 + \lambda_2) \partial_t u^o - \text{div}(A_{\text{hom}} \nabla u^o) = |E_{\text{out}}| f ,
$$

  with

$$
A_{\text{hom}} = \int_{E_{\text{out}}} \kappa(y) \nabla_y(\chi - y^c) \cdot \nabla_y(\chi - y^c) \, dy .
$$

  We have formally the same monodomain, described by two parabolic equations with the same capacity, but in which the two diffusion matrices, though formally analogous, are obtained by means of different cell functions.

- **Limit $D_1 \to +\infty$ and $D_2 \to 0$, two bidomains, commutation.** If we first perform homogenization and then we take the limit for $D_1 \to +\infty$ and $D_2 \to 0$, from (6.21), we have that

$$
u_\varepsilon \to (|E_{\text{out}}| + E_{\text{int},2}|) u_{\infty,o} + |E_{\text{int},1}| \overline{u} , \quad (8.1)
$$

  and we get a bidomain where the two limit functions $u_{\infty,o}$ and $u^{\text{int}} = \overline{u}$ satisfy, respectively, the parabolic equation and the ordinary differential equation given by (see (6.20))

$$
(|E_{\text{out}}| + \lambda_2) \partial_t u_{\infty,o} - \text{div}(A_{\text{hom}}^0 \nabla u_{\infty,o}) = |E_{\text{out}}| f ;
$$
$$
\partial_t u^{\text{int}} = 0 ,
$$

with the same diffusion matrix A_{hom}^0 as in the previous case. Instead, if we first perform the limit for $D_1 \to +\infty$ and $D_2 \to 0$ and, then, we homogenize, from (7.33), we have that

$$u_\varepsilon \to (|E_{\text{out}}| + E_{\text{int},2}|)u^{\infty,o} + |E_{\text{int},1}|\overline{u}\,, \tag{8.2}$$

and we get a bidomain where the two limit functions $u^{\infty,o}$ and $u_{\text{int},1}^{\infty,o}$ satisfy the parabolic equation and the algebraic one given, respectively, by (see (7.32))

$$(|E_{\text{out}}| + \lambda_2)\partial_t u^{\infty,o} - \text{div}(A_{\text{hom}}^0 \nabla u^{\infty,o}) = |E_{\text{out}}|f\,;$$
$$u_{\text{int},1}^{\infty,o} = |E_{\text{int},1}|\overline{u}\,.$$

We have the same equation for the leading phase (i.e., $u_{\infty,o} = u^{\infty,o}$) and then, comparing (8.1) and (8.2), we get that the two limits commute.

- **Limit $D_1 \to 0$ and $D_2 \to +\infty$, monodomain+bidomain, no commutation.** If we first perform homogenization and then we take the limit for $D_1 \to 0$ and $D_2 \to +\infty$, the final limit $u_{o,\infty}$ satisfies the parabolic equation (see (6.29))

$$(|E_{\text{out}}| + \lambda_1 + \lambda_2)\partial_t u_{o,\infty} - \text{div}(A_{\text{hom}}^\infty \nabla u_{o,\infty}) = |E_{\text{out}}|f\,,$$

with

$$A_{\text{hom}}^\infty = \int_{E_{\text{out}}} \kappa(y)\nabla_y(\bar{\chi}_0 - y^c) \cdot \nabla_y(\bar{\chi}_0 - y^c)\,dy\,.$$

Instead, if we first perform the limit for $D_1 \to 0$ and $D_2 \to +\infty$ and, then, we homogenize, from (7.35), we have

$$u_\varepsilon \to (|E_{\text{out}}| + |E_{\text{int},1}|)u^{o,\infty} + |E_{\text{int},2}|\overline{u}\,,$$

and we get a bidomain where the two limit functions $u^{o,\infty}$ and $u_{\text{int},2}^{o,\infty}$ satisfy the parabolic equation and the algebraic one given, respectively, by (see (7.34))

$$(|E_{\text{out}}| + \lambda_1)\partial_t u^{o,\infty} - \text{div}(B_{\text{hom}}^0 \nabla u^{o,\infty}) = |E_{\text{out}}|f\,;$$
$$u_{\text{int},2}^{o,\infty} = |E_{\text{int},2}|\overline{u}\,,$$

where

$$B_{\text{hom}}^0 = \int_{E_{\text{out}}} \kappa(y)\nabla(\hat{\chi}^o - y^c) \cdot \nabla(\hat{\chi}^o - y^c)\,dy\,,$$

with $\hat{\chi}^o$ defined as in Lemma 4.3, with $E_{\text{int},2}$ and Γ_1 replaced by $E_{\text{int},1}$ and Γ_2, respectively.

Notice that the parabolic equation for the leading phase differs from the one governing the monodomain, both for the capacity and for the diffusion matrix.

- **Limit $D_1, D_2 \to +\infty$, bidomain + tridomain, no commutation.** If we first perform homogenization and then we take the limit for $D_1, D_2 \to +\infty$, from (6.35), we have

$$u_\varepsilon \to (|E_{\text{out}}| + |E_{\text{int},2}|)u_\infty + |E_{\text{int},1}|\overline{u}\,,$$

and we get a bidomain where the two limit functions u_∞ and $u^{\text{int}} = \overline{u}$ satisfy, respectively, the parabolic equation and the ordinary differential equation given

by (see (6.34))

$$(|E_{\text{out}}| + \lambda_2)\partial_t u_\infty - \text{div}(A_{\text{hom}}^\infty \nabla u_\infty) = |E_{\text{out}}|f \,;$$
$$\partial_t u^{\text{int}} = 0 \,,$$

with the same diffusion matrix A_{hom}^∞ as in the case $D_1 \to 0$ and $D_2 \to +\infty$. Instead, if we first perform the limit for $D_1, D_2 \to +\infty$ and, then, we homogenize, from (7.50), we have

$$u_\varepsilon \to |E_{\text{out}}|u^\infty + (|E_{\text{int},1}| + |E_{\text{int},2}|)\overline{u} \,,$$

and we get a tridomain where the three limit functions $u^\infty, u_{\text{int},1}^\infty, u_{\text{int},2}^\infty$ satisfy the parabolic equation and the two algebraic ones given, respectively, by (see (7.49))

$$|E_{\text{out}}|\partial_t u^\infty - \text{div}(A_{\text{hom}}^\infty \nabla u^\infty) = |E_{\text{out}}|f \,;$$
$$u_{\text{int},1}^\infty = |E_{\text{int},1}|\overline{u} \,, \qquad u_{\text{int},2}^\infty = |E_{\text{int},2}|\overline{u} \,.$$

Notice that the two non degenerate parabolic equations differ, since though they present the same diffusion matrix, they have different capacities. Therefore, the two limits do not commute.

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