A Carleman estimate and an energy method for a first-order symmetric hyperbolic system

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Abstract

For a symmetric hyperbolic system of the first order, we prove a Carleman estimate under some positivity condition concerning the coefficient matrices. Next, applying the Carleman estimate, we prove an observability L^2-estimate for initial values by boundary data.

Key words. Carleman estimate, symmetric hyperbolic system, energy estimate

AMS subject classifications. 35R30, 35L40

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$, and let $N \in \mathbb{N}$ and

$$Q := \Omega \times (0, T).$$

For any $k \in \{0, 1, 2, ..., d\}$, let $H^k = H^k(x,t)$ be $N \times N$ matrices whose elements belong to $C^1(\overline{Q})$, and P be an $N \times N$ matrix with elements in $L^\infty(Q)$. We write $x = (x_1, ..., x_d) \in \mathbb{R}^d$, and

$$\partial_0 := \frac{\partial}{\partial t}, \quad \partial_k := \frac{\partial}{\partial x_k}, \quad k = 1, ..., d.$$

Here and henceforth $\cdot^T$ means the transpose of vectors and matrices under consideration.

Throughout this article, we assume that $H^k(x,t)$, $k = 0, 1, ..., d$, are symmetric matrices, that is,

$$(H^k)^T = H^k, \quad k = 0, 1, ..., d.$$

Let an $\mathbb{R}^N$-valued function $F(x,t)$ be in $L^2(Q)$.

We consider a symmetric hyperbolic system of the first order:

$$Lu(x,t) := H^0(x,t)\partial_t u + \sum_{k=1}^{d} H^k(x,t)\partial_k u + P(x,t)u = F(x,t), \quad x \in \Omega,\ 0 < t < T, \tag{1.1}$$

where $u = (u_1, ..., u_N)^T$.

Whenever there is no fear of confusion, we understand that $(U \cdot V)$ denotes a scalar product in $\mathbb{R}^N$ or $L^2(\Omega)$ for $U = (U_1, ..., U_N)^T$ and $V = (V_1, ..., V_N)^T$:

$$(U \cdot V) := \sum_{j=1}^{N} U_j V_j \quad \text{or} \quad (U \cdot V) := \int_\Omega \sum_{j=1}^{N} U_j(x) V_j(x) dx.$$

By $\|\cdot\|_X$ we denote the norm in a normed space X for specifying X, and let $\|\cdot\|$ mean $\|\cdot\|_{L^2(\Omega)}$ if we do not specify extra, and $|U|_{\mathbb{R}^N} := \left(\sum_{j=1}^N U_j^2\right)^{\frac{1}{2}}$.

The system (1.1) can describe several important equations in mathematical physics such as Maxwell's equations and the elasticity equations.

The main purpose of this articile is to establish a Carleman estimate for the symmetric hyperbolic system (1.1), and apply it to an energy estimate called an observability inequality.

A Carleman estimate is a weighted L^2-estimate for solutions to (1.1), and originates from Carleman [6], and provides an effective method to the unique continuation problem and inverse problems for partial differential equation. As for the applications to the unique continuation, we can refer to Hörmander [11], Isakov [16] for example. On the other hand, as pioneerng work for an application of a Carleman estimate to inverse problems, see Bukhgeim and Klibanov [3]. See also Beilina and Klibanov [1], Bellassoued and Yamamoto [2], Imanuvilov and Yamamoto [14], [15], Klibanov [18], Klibanov and Timonov [21], Yamamoto [27]. Here we do not intend any comprehensive lists of works, and the readers can consult also the references therein.

We emphasize that Carleman estimates are essential ingredients for solving the unique continuation and mathematically analyzing inverse problems.

For $N = 1$, as related works by Carleman estimates, we refer to Gaitan and Ouzzane [9], Cannarsa, Floridia, Gölgeleyen, and Yamamoto [4], Cannarsa, Floridia and Yamamoto [5], Floridia and Takase [7], [8], Gölgeleyen and Yamamoto [10], and see also Klibanov and Pamyatnykh [19], [20], Klibanov and Yamamoto [22], Machida and Yamamoto [24] as for related problems for the radiative transport equation called the Boltzmann equation. On the other hand, for a symmteric hyperbolic system of the first order, the research for the direct problem is completed (e.g., Mizohata [25], Petrovsky [26]). However, to the best knowledge of the authors, Carleman estimates are not available for (1.1) and so works for inverse problems by Carleman esimates have not been published.

Now for the statements of our main results, we introduce some notations and conditions. Let $\nu = (\nu_1, ..., \nu_d)^T$ be the unit outward normal vector to $\partial\Omega$ at x. For $0 \le t \le T$, we set

$$\partial\Omega_+(t) := \left\{ x \in \partial\Omega;\ \left(\left(\sum_{k=1}^{d} H^k(x,t)\nu_k \right) v \cdot v \right) > 0 \quad \text{for all } v \in \mathbb{R}^N \right\} \tag{1.2}$$

and

$$\partial\Omega_-(t) := \left\{ x \in \partial\Omega;\ \left(\left(\sum_{k=1}^{d} H^k(x,t)\nu_k \right) v \cdot v \right) \le 0 \quad \text{for all } v \in \mathbb{R}^N \right\}. \tag{1.3}$$

For t-independent H^k, we note that $\partial\Omega_+(t)$ and $\partial\Omega_-(t)$ are independent of t. We remark that $\partial\Omega \supsetneqq \partial\Omega_+(t) \cup \partial\Omega_-(t)$, in general.

Let $\varphi = \varphi(x,t) \in C^1(\overline{Q})$. Now we can state our first main result.

Theorem 1 (Carleman estimate for a symmetric hyperbolic system of the first order).
We assume that there exists a constant $\delta > 0$ such that

$$\left(\left(\sum_{k=0}^{d} (\partial_k \varphi) H^k(x,t) \right) v \cdot v \right) \ge \delta |v|^2_{\mathbb{R}^N} \quad \textit{for all } (x,t) \in Q \textit{ and } v \in \mathbb{R}^N. \tag{1.4}$$

Then there exist constants $C > 0$ and $s_0 > 0$ such that

$$\begin{aligned} & s\int_\Omega (H^0(x,0)u(x,0) \cdot u(x,0)) e^{2s\varphi(x,0)} dx + s^2 \int_Q |u|^2 e^{2s\varphi} dxdt \\ & + s\int_0^T \int_{\partial\Omega_-(t)} \left| \left(\left(\sum_{k=1}^{d} H^k(x,t)\nu_k \right) u \cdot u \right) \right| e^{2s\varphi} dSdt \\ & \le C\int_Q |F|^2 e^{2s\varphi} dxdt + Cs\int_0^T \int_{\partial\Omega \setminus \partial\Omega_-(t)} |u|^2 e^{2s\varphi} dSdt \\ & + s\int_\Omega (H^0(x,T)u(x,T) \cdot u(x,T)) e^{2s\varphi(x,T)} dx \end{aligned} \tag{1.5}$$

for all $s \ge s_0$ and $u \in H^1(Q)$ satisfying (1.1) in Q.

Theorem 1 is a direct generalization to the case of a system, where the case $N = 1$ is considered by [4] and [10], and the weight function φ is different from [9] and is of the same type as [4], [10], [24]. In general, Carleman estimates do not estimate $u(\cdot, 0)$, but the left-hand side of (1.5) can include the values $u(x, 0)$.

The general theory for the Carleman estimate was designed mostly for functions with compact supports (e.g., [11], [16]), while our Carleman estimate does not assume that u has compact supports, which allows us to apply a simplified argument by Huang, Imanuvilov and Yamamoto [12]. If we apply Carleman estimates with compact supports, then we need a cut-off function, which is a quite conventional way. However, the cut-off procedure makes the total arguments more complicated. The application of a general theory for constructing Carleman estimates for functions without compact supports, is also very complicated and the direct way is more relevant for the derivation of a Carleman estimate for (1.1).

In the case where H^k, $k = 0, 1, ..., d$ are diagonal matrices, system (1.1) is not coupled with the first-order terms but only with the zeroth order terms. Then under adequate conditions, thanks to the large parameter $s > 0$, the Carleman estimate can be directly derived from the case for $N = 1$, so that our main interest in Theorem 1 is for the case of non-diagonal matrices H^k for $0 \le k \le d$.

Next we apply Theorem 1 to the observability inequality of estimating initial value by boundary data. First in Theorem 1, we choose the weight function $\varphi(x, t)$ by

$$\varphi(x, t) = \eta(x) - \beta t,$$

where $\beta > 0$ and $\eta \in C^1(\overline{\Omega})$ are chosen later. We assume that there exist constants $\delta_0 > 0$, $\delta_1 > 0$ and $M > 0$ such that

$$\left(\sum_{k=1}^{d} (\partial_k \eta)(x) H^k(x, t)v \cdot v \right) \ge \delta_0 |v|^2_{\mathbb{R}^N} \quad \text{for all } (x, t) \in Q \text{ and } v \in \mathbb{R}^N. \tag{1.6}$$

and

$$\delta_1 |v|^2_{\mathbb{R}^N} \le (H^0(x, t)v \cdot v) \le M |v|^2_{\mathbb{R}^N} \quad \text{for all } (x, t) \in Q \text{ and } v \in \mathbb{R}^N. \tag{1.7}$$

Theorem 2 (Observability inequality).
We further assume (1.6), (1.7) and

$$T > \frac{M}{\delta_0}\left(\max_{x\in\overline{\Omega}}\eta(x) - \min_{x\in\overline{\Omega}}\eta(x)\right). \tag{1.8}$$

Then there exists a constant $C > 0$ such that

$$\|u(\cdot,0)\| \le C\|u\|_{L^2(\partial\Omega\times(0,T))}$$

for all $u \in H^1(Q)$ satisfying $H^0(x,t)\partial_t u + \sum_{k=1}^d H^k(x,t)\partial_k u + Pu = 0$ in Q.

This article is composed of three sections. In Section 2 we prove Theorem 1 and Section 3 is devoted to the proof of Theorem 2.

2 Proof of Theorem 1

The proof is based on integration by parts, which is applicable also for direct proofs of Carleman estimates for parabolic and hyperbolic equations. The main steps are described as follows.

Step I. We transform system (1.1) in u to a system in terms of $w := ue^{s\varphi}$.

Step II. Calculating the squared L^2-norms of the transformed system, we make a lower estimate of the cross terms composed of products of w and $\partial_k w$ for $k = 0, 1, ..., d$.

Step III. The lower estimate in Step II cannot dierctly produce the Carleman estimate, which is the same for parabolic and hypebolic equations. Therefore, we take scalar products of the transformed system by $B(x,t)w$ with suitably chosen matrix function $B(x,t)$ and we add the resulting equality to the lower estimate obtained in Step II, which leads us to the conclusion.

Before starting the proof, we remark that it is sufficient to prove Theorem 1 for $P \equiv 0$. Indeed let Theorem 1 be proved in the case of $P \equiv 0$. Then

$$s\int_\Omega (H^0(x,0)u(x,0)\,\cdot\,u(x,0))e^{2s\varphi(x,0)}dx + s^2\int_Q |u|^2 e^{2s\varphi}dxdt$$

$$
\begin{aligned}
&+s\int_0^T\int_{\partial\Omega_-(t)}\left|\left(\left(\sum_{k=1}^d H^k(x,t)\nu_k\right)u\,\cdot\,u\right)\right|e^{2s\varphi}dSdt\\
\le&C\int_Q|F-Pu|^2e^{2s\varphi}dxdt+\int_0^T\int_{\partial\Omega\setminus\partial\Omega_-(t)}|u|^2e^{2s\varphi}dSdt\\
&+s\int_\Omega(H^0(x,T)u(x,T)\,\cdot\,u(x,T))e^{2s\varphi(x,T)}dx.
\end{aligned}
$$

Since

$$
|F-Pu|^2\le 2|F|^2+2|Pu|^2\le 2|F|^2+2\|P\|^2_{L^\infty(Q)}|u|^2
$$

in Q, we can estimate $\int_Q|F-Pu|^2e^{2s\varphi}dxdt$ by

$$
C_0\int_Q|F|^2e^{2s\varphi}dxdt+C_0\int_Q|u|^2e^{2s\varphi}dxdt
$$

with some constant $C_0>0$. Choosing $s>0$ large, we can absorb the term $C_0\int_Q|u|^2e^{2s\varphi}dxdt$ into $s^2\int_Q|u|^2e^{2s\varphi}dxdt$ on the left-hand side. Therefore, (1.5) with $P\in L^\infty(Q)$ is derived from (1.5) in the case of $P\equiv 0$.

Moreover we show

$$
(R(x,t)\partial_k w\,\cdot\,w)_{\mathbb{R}^N}=\frac{1}{2}\partial_k(Rw\,\cdot\,w)_{\mathbb{R}^N}-\frac{1}{2}(\partial_kRw\cdot w)_{\mathbb{R}^N},\quad k=0,1,...,d\quad\text{in } Q,\quad (2.1)
$$

where $R\in C^1(\overline{Q})$ is an $N\times N$ symmetric matrix and $w=(w_1,...,w_N)^T\in H^1(Q)$.

Indeed, by the symmetry of R, we have $(R\partial_kw\cdot w)_{\mathbb{R}^N}=(\partial_kw\cdot Rw)_{\mathbb{R}^N}$, and so

$$
\begin{aligned}
&\partial_k(Rw\cdot w)_{\mathbb{R}^N}=(R\partial_kw\cdot w)_{\mathbb{R}^N}+(Rw\cdot\partial_kw)_{\mathbb{R}^N}+((\partial_kR)w\cdot w)_{\mathbb{R}^N}\\
=&2(R\partial_kw\cdot w)_{\mathbb{R}^N}+((\partial_kR)w\cdot w)_{\mathbb{R}^N}\quad\text{in } Q,
\end{aligned}
$$

which verifies (2.1).

Step I.

Not only for functions in $L^2(\Omega)$, but also for $u=(u_1,...,u_N)^T$, $v=(v_1,...,v_N)^T\in L^2(Q)$, we use the notation

$$
(u\cdot v):=\int_Q\sum_{j=1}^N u_j(x,t)v_j(x,t)dxdt.
$$

We set

$$Lu := H^0(x,t)\partial_t u + \sum_{k=1}^{d} H^k(x,t)\partial_k u = F \quad \text{in } Q. \tag{2.2}$$

We set

$$w := e^{s\varphi}u, \quad Pw := e^{s\varphi}L(e^{-s\varphi}w).$$

Then

$$Pw = H^0\partial_t w + \sum_{k=1}^{d} H^k\partial_k w - s\left(\sum_{k=1}^{d}(\partial_k\varphi)H^k + (\partial_t\varphi)H^0\right)w$$

in Q. For short description, we set

$$A := \sum_{k=0}^{d}(\partial_k\varphi)H^k \quad \text{in } Q,$$

where $\partial_0\varphi = \partial_t\varphi$. Then we can write (1.1) in terms of the transformed system in terms of w:

$$Pw = H^0\partial_t w + \sum_{j=1}^{n} H^k\partial_k w - sAw = Fe^{s\varphi} \quad \text{in } Q. \tag{2.3}$$

Step II.

By the definition, we have

$$\|Pw\|^2 = \int_Q |Lw|^2 e^{2s\varphi}dxdt = \int_Q |F|^2 e^{2s\varphi}dxdt.$$

Therefore, we have to make a lower estimate of $\|Pw\|^2$ and extract the terms on the left-hand side of (1.5). Direct calculations yield

$$\|Pw\|^2 = 2\left(\left(H^0\partial_t w + \sum_{k=1}^{d} H^k\partial_k w\right) \cdot -sAw\right) + s^2\|Aw\|^2 + \left\|H^0\partial_t w + \sum_{k=1}^{d} H^k\partial_k w\right\|^2$$

$$\geq -2s\int_Q \left(\sum_{k=0}^{d} H^k\partial_k w \cdot Aw\right)dxdt + s^2\|Aw\|^2.$$

Hence,

$$-2s\int_Q \left(\sum_{k=0}^{d} H^k\partial_k w \cdot Aw\right)dxdt + s^2\|Aw\|^2 \leq \|Fe^{s\varphi}\|^2. \tag{2.4}$$

The last term $s^2\|Aw\|^2$ can estimate $s^2\|w\|^2$ suitably, but we cannot directly estimate the cross term

$$-2s\int_Q \left(\sum_{k=0}^{d} H^k\partial_k w \cdot Aw\right)dxdt.$$

The difficulty comes from the non-symmetry in the form $\left(\sum_{k=0}^{d} H^k\partial_k w \cdot Aw\right)$, so that we cannot treat the derivatives $\partial_k w$. We remark that H^k and A are symmetric but AH^k and H^kA are not necessarily symmetric. Thus we need some argument in Step III.

Step III.

For compensating for the above cross term, we take the scalar product of (2.3) with $B(x,t)w$, where we choose B later:

$$\left(\sum_{k=0}^{d} H^k\partial_k w \,\cdot\, Bw\right) - s(Aw\cdot Bw) = (Fe^{s\varphi}\cdot Bw).$$

For adjusting the power of s, we multiply by $2s$:

$$2s\left(\sum_{k=0}^{d} H^k\partial_k w \,\cdot\, Bw\right) - 2s^2(Aw\cdot Bw) = 2s(Fe^{s\varphi}\cdot Bw). \tag{2.5}$$

Adding (2.4) and (2.5), we obtain

$$\begin{aligned}&2s\left(\sum_{k=0}^{d} H^k\partial_k w \,\cdot\, (B-A)w\right) + s^2(\|Aw\|^2 - 2(Aw\cdot Bw))\\ \le&\|Fe^{s\varphi}\|^2 + 2s(Fe^{s\varphi}\cdot Bw).\end{aligned}$$

Applying the Cauchy-Schwarz inequality to the right-hand side and using

$$2s(Fe^{s\varphi}\cdot Bw) = 2\left(\frac{Fe^{s\varphi}}{\varepsilon}\,\cdot\, s\varepsilon Bw\right) \le \frac{1}{\varepsilon^2}\|Fe^{s\varphi}\|^2 + s^2\varepsilon^2\|Bw\|^2,$$

we obtain

$$\begin{aligned}2s&\left(\sum_{k=0}^{d} H^k\partial_k w \,\cdot\, (B-A)w\right) + s^2(\|Aw\|^2 - 2(Aw\cdot Bw))\\ &\le \left(1+\frac{1}{\varepsilon^2}\right)\|Fe^{s\varphi}\|^2 + s^2\varepsilon^2\|Bw\|^2.\end{aligned} \tag{2.6}$$

In order to apply (2.1) to the first term on the left-hand side of (2.6) for estimating the cross term, we choose B suitably, for example, such that

$$B - A = -E_N,$$

where E_N is the $N\times N$ identity matrix. Then

$$-2s\left(\sum_{k=0}^{d} H^k\partial_k w \,\cdot\, w\right) + s^2(2(Aw\,\cdot\, w) - \|Aw\|^2)$$

$$\leq \left(1+\frac{1}{\varepsilon^2}\right)\|Fe^{s\varphi}\|^2 + s^2\varepsilon^2\|(A-E_N)w\|^2. \tag{2.7}$$

With a parameter $\mu > 0$, we set

$$\widetilde{\varphi} := \mu\varphi, \quad \widetilde{A}(x,t) := \sum_{k=0}^{d}(\partial_k\widetilde{\varphi})H^k = \mu A \quad \text{in } Q.$$

Henceforth $C > 0$ denotes generic constants which are independent of $s > 0$ and $\mu > 0$. Setting $\widetilde{w} := ue^{s\widetilde{\varphi}}$, we repeat the calculations to obtain

$$-2s\left(\sum_{k=0}^{d} H^k\partial_k\widetilde{w}\,\cdot\,\widetilde{w}\right) + s^2(2(\widetilde{A}\widetilde{w}\,\cdot\,\widetilde{w}) - \|\widetilde{A}\widetilde{w}\|^2)$$

$$\leq \left(1+\frac{1}{\varepsilon^2}\right)\|Fe^{s\widetilde{\varphi}}\|^2 + s^2\varepsilon^2\|(\widetilde{A}-E_N)\widetilde{w}\|^2. \tag{2.8}$$

We can directly verify

$$\|\widetilde{A}\widetilde{w}\|^2 = \mu^2\|A\widetilde{w}\|^2 \leq C\mu^2\|\widetilde{w}\|^2$$

and

$$\|(\widetilde{A}-E_N)\widetilde{w}\|^2 \leq 2\|\widetilde{A}\widetilde{w}\|^2 + 2\|E_N\widetilde{w}\|^2 \leq 2(C\mu^2+1)\|\widetilde{w}\|^2.$$

Moreover, by (1.4), we see $(\widetilde{A}\widetilde{w}\cdot\widetilde{w}) \geq \mu\delta\|\widetilde{w}\|^2$. Hence, (2.8) yields

$$-2s\sum_{k=0}^{d}(H^k\partial_k\widetilde{w}\,\cdot\,\widetilde{w}) + s^2(2\delta\mu - C\mu^2)\|\widetilde{w}\|^2 \leq \left(1+\frac{1}{\varepsilon^2}\right)\|Fe^{s\widetilde{\varphi}}\|^2 + 2s^2\varepsilon^2(C\mu^2+1)\|\widetilde{w}\|^2. \tag{2.9}$$

Since H^k, $k = 0, 1, ..., d$ are symmetric, by (2.1) we have

$$-\sum_{k=1}^{d}(H^k\partial_k\widetilde{w}\,\cdot\,\widetilde{w}) = -\frac{1}{2}\sum_{k=1}^{d}\int_Q \partial_k(H^k\widetilde{w}\,\cdot\,\widetilde{w})dxdt + \frac{1}{2}\sum_{k=1}^{d}\int_Q((\partial_k H^k)\widetilde{w}\,\cdot\,\widetilde{w})dxdt \tag{2.10}$$

$$= -\frac{1}{2}\int_{\partial\Omega\times(0,T)}\sum_{k=1}^{d}((H^k\nu_k)\widetilde{w}\,\cdot\,\widetilde{w})dSdt + \frac{1}{2}\sum_{k=1}^{d}\int_Q((\partial_k H^k)\widetilde{w}\,\cdot\,\widetilde{w})dxdt$$

$$\geq -\frac{1}{2}\int_0^T\int_{\partial\Omega\setminus\partial\Omega_-(t)}\sum_{k=1}^{d}|H^k\nu_k||\widetilde{w}|^2 dSdt$$

$$+\frac{1}{2}\int_0^T\int_{\partial\Omega_-(t)}\left|\sum_{k=1}^{d}((H^k\partial_k)\widetilde{w}\cdot\widetilde{w})\right| dSdt - C\|\widetilde{w}\|^2.$$

Moreover,

$$-(H^0\partial_t\widetilde{w}\ \cdot\ \widetilde{w}) = -\frac{1}{2}\int_Q \partial_t(H^0\widetilde{w}\cdot\widetilde{w})dxdt + \frac{1}{2}\int_Q ((\partial_t H^0)\widetilde{w}\cdot\widetilde{w})dxdt \tag{2.11}$$

$$\begin{aligned}
=&\frac{1}{2}\int_\Omega (H^0(x,0)\widetilde{w}(x,0)\ \cdot\ \widetilde{w}(x,0))dx - \frac{1}{2}\int_\Omega (H^0(x,T)\widetilde{w}(x,T)\ \cdot\ \widetilde{w}(x,T))dx \\
+&\frac{1}{2}\int_Q ((\partial_t H^0)\widetilde{w}\cdot\widetilde{w})dxdt \\
\ge&\frac{1}{2}\int_\Omega (H^0(x,0)\widetilde{w}(x,0)\ \cdot\ \widetilde{w}(x,0))dx - \frac{1}{2}\int_\Omega (H^0(x,T)\widetilde{w}(x,T)\ \cdot\ \widetilde{w}(x,T))dx - C\|\widetilde{w}\|^2.
\end{aligned}$$

Applying (2.10) and (2.11) in (2.9), we obtain

$$\begin{aligned}
&s^2[(2\delta\mu - C\mu^2) - 2\varepsilon^2(C\mu^2+1)]\|\widetilde{w}\|^2 - Cs\|\widetilde{w}\|^2 \\
&+s\int_0^T\int_{\partial\Omega_-(t)}\left|\sum_{k=1}^d((H^k\nu_k)\widetilde{w}\ \cdot\ \widetilde{w})\right|dSdt + s\int_\Omega (H^0(x,0)\widetilde{w}(x,0)\ \cdot\ \widetilde{w}(x,0))dx \\
&\le\left(1+\frac{1}{\varepsilon^2}\right)\|Fe^{s\widetilde{\varphi}}\|^2 + Cs\int_0^T\int_{\partial\Omega\setminus\partial\Omega_-(t)}|\widetilde{w}|^2dSdt \\
&+s\int_\Omega (H^0(x,T)\widetilde{w}(x,T)\ \cdot\ \widetilde{w}(x,T))dx.
\end{aligned}$$

We choose $\mu > 0$ small so that $2\delta\mu - C\mu^2 > 0$. For this $\mu > 0$, we choose $\varepsilon > 0$ sufficiently small such that

$$(2\delta\mu - C\mu^2) - 2\varepsilon^2(C\mu^2+1) > 0.$$

Then, for $\mu > 0$ and $\varepsilon > 0$, we can find a constant $C_0 = C_0(\mu,\varepsilon) > 0$ such that

$$\begin{aligned}
&C_0s^2\|\widetilde{w}\|^2 - Cs\|\widetilde{w}\|^2 + Cs\int_0^T\int_{\partial\Omega_-(t)}\left|\sum_{k=1}^d((H^k\nu_k)\widetilde{w}\ \cdot\ \widetilde{w})\right|dSdt \\
&+s\int_\Omega (H^0(x,0)\widetilde{w}(x,0)\ \cdot\ \widetilde{w}(x,0))dx \\
&\le C_0\|Fe^{s\widetilde{\varphi}}\|^2 + Cs\int_0^T\int_{\partial\Omega\setminus\partial\Omega_-(t)}|\widetilde{w}|^2dSdt + s\int_\Omega (H^0(x,T)\widetilde{w}(x,T)\ \cdot\ \widetilde{w}(x,T))dx.
\end{aligned}$$

Setting $\widetilde{s}_0 := \frac{2C}{C_0(\mu,\varepsilon)}$, we obtain

$$C_0s^2\|\widetilde{w}\|^2 - Cs\|\widetilde{w}\|^2 \ge \frac{1}{2}C_0s^2\|\widetilde{w}\|^2$$

for all $s > \widetilde{s}_0$. Rewriting in terms of $\widetilde{w} = e^{s\widetilde{\varphi}}u = e^{s\mu\varphi}u$, we have

$$\frac{1}{2}C_0 s^2 \int_Q |u|^2 e^{2s\mu\varphi} dxdt + s\int_0^T \int_{\partial\Omega_-(t)} \left| \sum_{k=1}^d ((H^k \nu_k)u \cdot u) \right| e^{2s\mu\varphi} dSdt$$
$$+ s\int_\Omega (H^0(x,0)u(x,0) \cdot u(x,0)) e^{2s\mu\varphi(x,0)} dx$$
$$\le C_0 \|F\|^2 e^{2s\mu\varphi} dxdt + Cs\int_0^T \int_{\partial\Omega\setminus\partial\Omega_-(t)} |u|^2 e^{2s\mu\varphi} dSdt$$
$$+ s\int_\Omega (H^0(x,T)u(x,T) \cdot u(x,T)) e^{2s\mu\varphi(x,T)} dx$$

for all $s > \widetilde{s}_0$. Therefore, rewriting $\mu\varphi$ by φ in the above inequality, we complete the proof of Theorem 1. ■

Remark. In Step III, the choice of B is not unique. For example, we can similarly argue with

$$B := A - \gamma E_N$$

with large constant $\gamma > 0$. The auxiliary estimate in Step III for comleting the proof is traditional and conventional for establishing L^2-estmates for partial differetial equations. We can refer for example to the descriptions on p.8 in Bellassoued and Yamamoto [2], p.37 in Komornik [23], p.10 in Yamamoto [27].

3 Proof of Theorem 2

It is known that a relevant Carleman estimate produces an L^2-estimate for initial value (e.g., Kazemi and Klibanov [17], Klibanov and Timonov [21]). Here we apply a method in Huang, Imanuvilov and Yamamoto [12] which simplifies the argument based on Carleman estimate in [17], [21].

First Step.

Let $H^k \in C^1(\overline{Q})$, $k = 0, 1, ..., d$ be symmetric matrices. We prove

Lemma 1 (energy estimate for an initial value problem).

We assume (1.7). Then there exists a constant $C > 0$ *such that*

$$\|u(\cdot,t)\|^2 + \int_0^T \int_{\partial\Omega_+(\xi)} \left(\sum_{k=1}^d (H^k \nu_k) u \cdot u \right) dS d\xi$$

$$\le C \left(\|u(\cdot,0)\|^2 + \int_0^T \int_{\partial\Omega \setminus \partial\Omega_+(\xi)} |u|^2 dS d\xi \right) \tag{3.1}$$

for all $u \in H^1(Q)$ *satisfying*

$$H^0 \partial_t u + \sum_{k=1}^d H^k \partial_k u + Pu = 0 \quad \textit{in } Q. \tag{3.2}$$

Proof.

The proof is by the following standard way. We take the scalar product (3.2) with u and integrate in $x \in \Omega$ to have

$$(H^0(\cdot,\xi)\partial_\xi u(\cdot,\xi) \cdot u(\cdot,\xi)) + \sum_{k=1}^d (H^k(\cdot,\xi)\partial_k u(\cdot,\xi) \cdot u(\cdot,\xi))$$

$$+(P(\cdot,\xi)u(\cdot,\xi) \cdot u(\cdot,\xi)) = 0, \quad 0 \le \xi \le t.$$

In the same way as (2.10) and (2.11), we see

$$\frac{1}{2}\partial_\xi \int_\Omega (H^0(x,\xi)u(x,\xi) \cdot u(x,\xi))dx + \frac{1}{2}\int_{\partial\Omega} \left(\sum_{k=1}^d \nu_k H^k(x,\xi) \cdot u(x,\xi) \right) dS$$

$$-\frac{1}{2}\int_\Omega ((\partial_t H^0(x,\xi))u(x,\xi) \cdot u(x,\xi))dx - \frac{1}{2}\int_\Omega \left(\left(\sum_{k=1}^d \partial_k H^0(x,\xi) \right) u(x,\xi) \cdot u(x,\xi) \right) dx$$

$$+\int_\Omega (P(x,\xi)u(x,\xi) \cdot u(x,\xi))dx = 0, \quad 0 \le \xi \le t.$$

We integrate in $\xi \in (0,t)$ to have

$$\int_\Omega (H^0(x,t)u(x,t) \cdot u(x,\xi))dx + \int_0^t \int_{\partial\Omega_+(\xi)} \left(\left(\sum_{k=1}^d \nu_k H^k \right) u(x,\xi) \cdot u(x,\xi) \right) dS d\xi$$

$$= \int_\Omega (H^0(x,0)u(x,0) \cdot u(x,0))dx - \int_0^t \int_{\partial\Omega \setminus \partial\Omega_+(\xi)} \left(\left(\sum_{k=1}^d \nu_k H^k \right) u(x,\xi) \cdot u(x,\xi) \right) dS d\xi$$

$$+\int_0^t \int_\Omega (\partial_t H^0(x,\xi)u(x,\xi) \cdot u(x,\xi))dxd\xi$$

$$+\int_0^t\int_\Omega\sum_{k=1}^d((\partial_k H^k)u(x,\xi)\,\cdot u(x,\xi))dxd\xi-2\int_0^t\int_\Omega(Pu(x,\xi)\,\cdot u(x,\xi))dxd\xi,\quad 0\le t\le T.$$

We set $E(t)=\int_\Omega|u(x,t)|^2dx$. By (1.7), $H^k\in C^1(\overline{Q})$, and $P\in L^\infty(Q)$, we obtain

$$E(t)+\int_0^t\int_{\partial\Omega_+(\xi)}\left(\left(\sum_{k=1}^d\nu_kH^k\right)u(x,\xi)\,\cdot u(x,\xi)\right)dSd\xi \tag{3.3}$$

$$\le CE(0)+C\int_0^T\int_{\partial\Omega\setminus\partial\Omega_+(\xi)}|u(x,\xi)|^2dSd\xi+C\int_0^tE(\xi)d\xi,\quad 0\le t\le T.$$

In particular,

$$E(t)\le CE(0)+C\int_0^T\int_{\partial\Omega\setminus\partial\Omega_+(\xi)}|u(x,\xi)|^2dSd\xi+C\int_0^tE(\xi)d\xi,\quad 0\le t\le T.$$

The Gronwall inequality yields

$$E(t)\le CE(0)+C\int_0^T\int_{\partial\Omega\setminus\partial\Omega_+(\xi)}|u(x,\xi)|^2dSd\xi,\quad 0\le t\le T.$$

Substituting this into (3.3), we reach

$$\int_0^t\int_{\partial\Omega_+(\xi)}\left(\left(\sum_{k=1}^d\nu_kH^k\right)u(x,\xi)\,\cdot u(x,\xi)\right)dSd\xi$$
$$\le CE(0)+C\int_0^T\int_{\partial\Omega\setminus\partial\Omega_+(\xi)}|u|^2dSd\xi,\quad 0\le t\le T.$$

The proof of Lemma 1 is complete.

Second Step.

We complete the proof of Theorem 2. By (1.8) we can choose $\beta\in\left(0,\frac{\delta_0}{M}\right)$ which is close to $\frac{\delta_0}{M}$ such that

$$T>\frac{1}{\beta}(\max_{x\in\overline{\Omega}}\eta(x)-\min_{x\in\overline{\Omega}}\eta(x)),$$

that is,

$$\min_{x\in\overline{\Omega}}\varphi(x,0)-\max_{x\in\overline{\Omega}}\varphi(x,T)>\delta_2>0, \tag{3.4}$$

where $\delta_2>0$ is some constant.

Moreover, by $\beta<\frac{\delta_0}{M}$, (1.6) and (1.7), we see that

$$\sum_{k=0}^d((\partial_k\varphi)H^kv\cdot v)=-\beta(H^0v\cdot v)+\sum_{k=1}^d((\partial_k\eta)H^kv\cdot v)\ge(-\beta M+\delta_0)|v|^2_{\mathbb{R}^N},$$

which verifies (1.4) with $\delta := \delta_0 - \beta M$. Therefore, Theorem 1 is applicable with the weight function $\varphi(x,t) := \eta(x) - \beta t$.

With this δ_2, we apply Theorem 1. In terms of (1.7), the first term on the left-hand side of (1.5) is replaced by

$$s\delta_1 \int_\Omega |u(x,0)|^2 e^{2s\varphi(x,0)} dx \ge s\delta_1 e^{2s\min_{x\in\overline{\Omega}}\varphi(x,0)} \|u(\cdot,0)\|^2.$$

We neglect the second and the third terms on the left-hand side of (1.5) and further replace the second term on the right-hand side by $Cse^{C_1 s}\|u\|^2_{L^2(\partial\Omega\times(0,T))}$ where $C_1 := \max_{(x,t)\in\overline{Q}} \varphi(x,t)$.

Thus

$$se^{2s\min_{x\in\overline{\Omega}}\varphi(x,0)}\|u(\cdot,0)\|^2 \le Ce^{C_1 s}\|u\|^2_{L^2(\partial\Omega\times(0,T))} + Cse^{2s\max_{x\in\overline{\Omega}}\varphi(x,T)}\|u(\cdot,T)\|^2$$

for all $s \ge s_0$. Applying (3.4), we obtain

$$\|u(\cdot,0)\|^2 \le Ce^{C_1 s}\|u\|^2_{L^2(\partial\Omega\times(0,T))} + Ce^{-2s\delta_2}\|u(\cdot,T)\|^2 \tag{3.5}$$

for all $s \ge s_0$.

Lemma 1 implies

$$\|u(\cdot,T)\|^2 \le C(\|u(\cdot,0)\|^2 + \|u\|^2_{L^2(\partial\Omega\times(0,T))}).$$

Substituting this into (3.5), we reach

$$\|u(\cdot,0)\|^2 \le Ce^{Cs}\|u\|^2_{L^2(\partial\Omega\times(0,T))} + Ce^{-2s\delta_2}\|u(\cdot,0)\|^2 + Ce^{-2s\delta_2}\|u\|^2_{L^2(\partial\Omega\times(0,T))}$$

for all $s \ge s_0$. Choosing $s > 0$ sufficicently large, we can absorb the second term on the right-hand side into the left-side hand, and so we complete the proof of Theorem 2. ■

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