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INTRODUCTION

The explanatory and heuristic power of mathematics

Introduction to a topical collection in synthese

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Mathematics is, and has been for a very long time, one of the most successful autonomous fields of research. However, the last five centuries have seen it become so deeply interwoven in virtually every area of scientific inquiry to convince Kant that "in any special doctrine of nature there can be only as much *proper* science as there is *mathematics* therein" (Kant 1786/2004, 6; emphases in original).

While the distinction between pure and applied mathematics remains somewhat elusive, philosophers have been interested in understanding the nature of each. Moreover, the idea that there are actually two uses of mathematics, an *explanatory* and a *heuristic* one, has begun to feature more and more prominently in recent philosophical debates. This topical collection showcases papers discussing both these uses, with some of the contributions also tackling the way they are related to each other.

When it comes to explanation, these debates have touched upon a variety of questions. For example, philosophers have wondered how mathematical theorems, and mathematical formalisms more generally, can be used in explanations in other scientific fields such as physics, life sciences, or the social sciences. Another question is how certain parts of mathematics can be used to provide explanations of results belonging to other parts of mathematics. Relatedly, some interesting debates concern the way in which a given philosophical-metaphysical attitude towards mathematical entities (for instance, a nominalist conception) affects the explanatory power of

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a scientific theory. One can also ask whether it is the mathematical elements, rather than non-mathematical ones, that carry most of the explanatory burden in certain scientific explanations. And finally, one might wonder how we ought to assess such comparative claims.

The main questions regarding heuristics focus on how mathematics can act as an *inter-field* heuristic engine; that is, how certain mathematical results are employed to generate hypotheses and solve problems in other scientific domains. Additionally, since mathematics can act also as an *intra-field* heuristic engine, one may wonder how it is possible that certain pieces of mathematics are employed to make discoveries in other parts of mathematics. Another important line of inquiry taken up in the papers below is the ramifications that a heuristic view of mathematics has, both for different accounts of the nature of mathematical objects, and for the method of mathematical research (among other things).

The idea for this collection was inspired by a conference organized by Emiliano Ippoliti in June 2019 at the Sapienza University in Rome. Since not all the speakers present at the event could contribute a paper, the collection includes other philosophers as well. Meanwhile, one of the most prominent authors in these two areas, Mark Steiner, sadly passed away from Covid in April 2020. We dedicate this collection to his memory.

1 The explanatory role of mathematics

As noted above, modern science, especially physics, is permeated with mathematics. This relationship between the two disciplines has constituted a subject of sustained reflection since at least the days of Kepler and Galileo. A substantial amount of philosophical work has been devoted to discussing it, and controversial philosophical theses have been advanced by scientists and philosophers alike. For example, the effectiveness of mathematics in science has been called ‘unreasonable’, even a ‘mystery’, by major physicists like Eugene Wigner (1967). This sentiment was further echoed by philosophers, most notably by Mark Steiner (1998). Although much has been said on this topic, it is generally acknowledged that it is still far from clear how such claims (if true) should be understood.[1]

We recalled Kant’s remark at the outset and, in the same spirit, it is worth mentioning what Paolo Mancosu (2018) characterizes as “a change of criteria of explanation and intelligibility” that occurred roughly during the time when the mathematized Newtonian physics became the dominant paradigm in natural science. Historians of science such as Yves Gingras have drawn attention to this development and pointed out that mathematization “transform[ed] the very meaning of the term ‘explanation’ as it was used by philosophers in the seventeenth century” (2001, 385; cited in Mancosu 2018). As Gingras also remarks, the idea to provide an explanation of a physical phenomenon in mathematical terms “was something new.” (ibid.) Without presenting any “physical mechanism involved in [the] production” of a phenomenon, “Newton’s

[1] Works by Morrison (2000; 2014), Batterman (2002; 2022), and Wilson (2008; 2017) seek to further illuminate these issues.

Principia marks the beginning of this shift where mathematical explanations came to be preferred to mechanical explanations when the latter did not conform to calculations." (Gingras, 2001, 398. See also Dorato 2017).

This historical context is worth sketching, since it is in many respects a good starting point for a reflection on the contemporary debates in the philosophy of science and mathematics.

From the 1950s onwards, several philosophers, among whom Quine and Putnam stand out, have argued that fundamental questions about the nature of mathematical truth and the ontological status of mathematical objects can be answered by drawing on the indispensable role of mathematics in science. They are generally credited with proposing what is now called the *Indispensability Argument* for mathematical realism. Heavily (and, many believe, convincingly) criticized initially by Field (1980) and Maddy (1992; 1997), among others, the renewed interest in an *enhanced* version of the argument is due in large part to the work of Colyvan (1998, 2001, 2010) and Baker (2005; 2009). These two authors defended it by recasting it in terms of the indispensability of mathematics in formulating *explanations of physical phenomena*. A large body of literature ensued, mostly critical of the argument (Leng, 2010, 2012; Saatsi, 2011; Bueno, 2012; Yablo, 2012), but also supportive of it (Bangu, 2012, 2013; Baron, 2014, 2016). In addition to the fascinating case-studies introduced by Colyvan and Baker, Pincock (2007; 2012) proposed an analysis of the famous *Konigsberg bridges* case-study. (An updated review of these debates is Paseau and Baker (forthcoming)). In this context, Marc Lange's 2017 book *Because without Cause* deserves a special mention. Although he abandons the pursuit of drawing metaphysical-ontological consequences from such explanations, he articulates a detailed and insightful counterfactual approach to such 'distinctively mathematical explanations', characterized as explanations 'by constraint'. For critical assessments of Lange's position, see Craver and Povich 2017, Bangu 2021, and Leng (this volume).

Mathematical explanations of mathematical results, and explanatory mathematical proofs in particular, have also been the object of philosophical investigation. Although intuitively clear, the distinction between explanatory and non-explanatory proofs is surprisingly difficult to pin down. As both mathematicians and philosophers have remarked, there seem to be proofs which show *that* a result holds, and others which do more: they also show *why* it holds. As Mancosu (1996; 2000) documented in detail, this distinction has been acknowledged since antiquity. Yet, it wasn't until the late 1970s when the first systematic analyses of it were proposed. The first one of these was Steiner's (1978), and central to this account is the notion of a "characterizing property", i.e., "a property unique to a given entity or structure within a family or domain of such entities or structures" (1978, 143). Thus, in Steiner's view, explanatory proofs differ from non-explanatory ones because they are given in terms of this kind of property "such that from the proof it is evident that the result depends on the property" (*ibidem*).

This theory has been criticized, e.g., by Resnik and Kushner (1987), and more recently by Hafner and Mancosu (2005), as well as Baker (2009). An alternative systematic analysis of the notion of explanatory proof has been articulated by Kitcher (1989), drawing on his unificationism about explanation in natural science. (For more discussion, see Tappenden (2005)). Further recent developments in this area are due to Pincock (2015), Lange (2017, Ch. 7 and 8), Colyvan et al. (2018) and d'Alessandro (2020).

2 The double connection of heuristics and mathematics

The heuristic role of mathematics has drawn increasing attention in the past few decades, with a starting point in the works of Lakatos (1976, 1978), Laudan (1977, 1981), followed by the contributions of Nickles (1980a, 1980b, 1981, 2014) and Cellucci (2013, 2022). The "heuristic side" of mathematics – where "heuristics" is the study of the methods and rules of discovery and invention (see Pólya 1954), and not the process by which humans use mental shortcuts for decision-making[2] – should be understood in this context in the following two senses, although these clarifications should not be taken as an exhaustive definition: (i) how mathematical knowledge advances by solving mathematical problems, and (ii) how mathematics acts as a tool for other disciplines.

Moving from a problem-based perspective (Russell, 1912; Popper, 1999), whose roots can be found notably in Aristotle (Quarantotto, 2017), this debate has shifted towards the search for an account of rational hypotheses-generation. In this context, mathematical discovery consists in generating hypotheses that are sufficient conditions to solve problems; it is thus a process that can be investigated and used to guide us. This tenet is the cornerstone of the "heuristic view" (e.g., Cellucci 2022, Ippoliti 2018a, Jaccard-Jacoby 2010, Polya 1954, Shelley 2003), which has significantly evolved in the last two decades, providing us with improved theoretical frameworks to account for the process of generation of mathematical hypotheses.

In particular, three different problem-based views have been formulated and refined in the last few decades (see Ippoliti 2018b):

(1) The Popper-Laudan position.
(2) The Poincaré-Simon position.
(3) The Lakatos-Cellucci position.

Crucially, these three approaches tried, but some have not fully succeeded, to dispense with the romantic idea of the 'mathematical genius', idea according to which the production of new hypotheses is entirely subjective and inscrutable – and thus only an individual with mathematically exceptional abilities can generate new such hypotheses to be rationally tested. This romantic idea was appealed to by many logicians, philosophers, and scientists who pursued an account of hypothesis-generation based on notions like 'intuition' or 'insight' (see e.g., Frege 1960; Einstein 2010). However, this is an appeal to a conceptual black-box and does not offer actionable explanations. As such, this approach constituted one of the main obstacles to the development of a rational account of hypothesis-generation. While historically and conceptually interwoven, it has been argued (Ippoliti, 2018b) that the above-mentioned three approaches differ substantially.

Popper's (1999) and Laudan's (1981) positions are still very close to the romantic idea; they hold that hypothesis-generation is a subjective and idiosyncratic process,

[2] For this, see Gigerenzer and Todd 1999, Kahneman, Slovic and Tversky 1982, Chow 2015.

and cannot be rationally or logically investigated, reconstructed, and transmitted. For them, problem-solving proceeds by random trial and error. In this respect, this approach falls within the romantic tradition: we can't have a rational account, much less a method, for hypothesis-generation and scientific discovery. This thesis is challenged by other approaches, such as that advocated by Poincaré and Simon, which maintain that, on the contrary, hypothesis-generation is within the purview of rationality.

Poincaré (1908) argues that it is possible to investigate and systematize the subconscious processes that underlie the search for a solution to a problem. When conscious attempts to solve a problem fail, unconscious processes come into play by generating all possible combinations of given ideas and select a small subset (or a single one) of them as candidate solution to the problem at hand, if any. Nevertheless, Poincaré's account lacks a cogent methodology for analysing how these combinations are formed and selected in a rational way.

To bridge this gap, Simon's account (Simon, 1977; Simon et al., 1987) sharpened Poincaré's approach by conceptualizing it in terms of the 'problem space', and by providing a formalization of the process of hypothesis-generation. Simon developed his influential theory of problems in five main steps:

1) Human brains and computers are alike: both are information-processing systems (see Simon et al. 1987, 8).
2) Thinking and problem-solving are a form of computation: "copying and reorganizing symbols in memory, receiving and outputting symbols, and comparing symbol structures for identity or difference" (ibid.).
3) Problem-solving is a rational process performed by creating a symbolic representation of the problem (the 'problem space') as well as of its operators and constraints.
4) Several rational *heuristic* procedures guide conscious thinking throughout the problem-space to generate a hypothesis to solve the problem. Furthermore, Simon identifies two classes of heuristics, weak and strong.
5) Such heuristics can be implemented in algorithms, so the process can be 'mechanized' into software (called BACON, versions 1 through 6). A famous case study of these heuristics involves the discovery of Kepler's law.

Unfortunately, this view has important shortcomings (see e.g., Nickles 1980a; Kantorovich 1993; Gillies 1995; Weisberg 2006). Firstly, it has been argued that algorithms do not perform the crucial part of hypothesis-generation, i.e., the choice of the relevant variables and data. This choice must be made prior to the launching of the BACON software: the relevant data and variables are selected and supplied by the programmers, and the BACON heuristic procedures only iterate through them. As Donald Gillies (1996, 24) has stressed, the software package BACON.1 already knows both (i) which two variables to relate, and (ii) the general form of the law that it should look for. Moreover, it has also been objected that BACON only does what a computer does better than humans, namely computation. In fact, BACON finds patterns in datasets when given the *right* data. Thus, this algorithm simply does better the last, and in a sense most trivial, stage of hypothesis-generation: it identifies statistical regularities, correlations, and does parameter-estimation. Of course, carrying out these tasks is much harder for humans alone, yet this advantage does not amount to an *explanation* of hypothesis-generation.

Another criticism maintains that BACON is, at best, an ex-post *reconstruction* of a historical finding. The charge is that it exploits the knowledge of the result, i.e., it uses the benefit of knowing what the problem is, knowing that the data can be dealt with by certain heuristics, and also knowing that the problem has been solved. As Nickels rightly points out, BACON does "not need to ask itself conceptually profound questions or consider deep reformulations of the problem, because the primary constraints (and therefore the problem itself) are programmed in from the beginning. Only certain empirical constraints remain to be fed in" (Nickles 1980b, 38).

The third position, advocated by Lakatos and Cellucci, construes "heuristics" as the methods and rules of discovery and invention *à la* Polya, and aims at overcoming the weaknesses of the other approaches. On this view, there is a method for generating hypotheses, but it is not fully algorithmic. This view also provides an account of hypothesis-generation that reunites problem-finding and problem-solving, as well as discovery and justification, within a single coherent framework. Their conception accomplishes this by showing that they are not separated as the standard view maintains.

The most recent version of this standard view argues that the method for hypothesis-generation is an appropriately revised version of the 'analytic method' proposed by Plato (see e.g. Cellucci 2013), a method which must be supplemented with a set of heuristic rules, namely types of inferences like induction, analogy, metaphor, and combinations thereof (e.g. Cellucci 2013, Jaccard-Jacoby 2010, Ippoliti 2018b). These rules build and shape the problem space which, contrary to Simon's approach, does not exist until heuristic procedures are applied. On this view, a hypothesis, once introduced, must satisfy the requirement of 'plausibility', and not 'truth'. Or, as Celluci (2013, 56) put it, "the arguments for the hypothesis must be stronger than the arguments against it, on the basis of experience".[3] This, on the other hand, implies that the solutions to problems are revisable.

This position is characterized by a specific set of ontological, epistemological, and methodological tenets. For example, mathematical objects are conceived of as hypotheses introduced to solve given mathematical problems (to illustrate how this works, recall that imaginary numbers were introduced by Bombelli to solve cubic equations; see Cellucci 2022). Moreover, these objects have the potential to acquire new properties over time, and can be considered from different points of view (e.g., a circle can be regarded as a geometrical, algebraic, topological, etc. object). Then, when certain mathematical objects turn out to be useful to solve further problems, "they consolidate and acquire a stability that makes them independent of the problem for which they were originally introduced, and become subjects of study themselves" (Cellucci, 2022, 15).

[3] One hypothesis is subjected to the following plausibility test (see Cellucci 2013, 56):(1) deduce conclusions from the hypothesis → (2) compare these conclusions with each other, in order to check that they do not produce a contradiction → (3) compare the conclusions with known results of observations or experiments, and with other hypotheses already known to be plausible.

This approach has clearly boosted the in-depth, systematic study of the rational means used to generate mathematical hypotheses, and paved the way for a fine-grained theory of hypothesis-generation through means such as multiple representations, ampliative inferences, or models.

3 The papers in this collection

Recent work on the topic of heuristic has greatly improved our understanding of how mathematical knowledge advances by solving problems (see e.g., Paseau 2015; Ulazia 2016; Ippoliti 2018a, 2020; Clement 2020; Cellucci 2022; also Grosholz 2007). This work show how mathematics, and mathematical objects in particular, play a major role in generating hypotheses, and serve as a heuristic tool at the intra-mathematical and extra-mathematical levels (e.g., the use of algebra to solve problems in geometry, and the use of topology to solve problems in biology). However, due to the richness of mathematical and scientific practice, many open questions concerning mathematical heuristics remain. Moreover, the rapid development of the literature on mathematical explanation has vastly enriched our understanding of the difficulties involved in formulating an account of both intra- and extra-mathematical explanation, as well as the role of the explanatory virtues such as simplicity, generality, depth, etc. We believe that the papers in this topical issue contribute to these investigations in substantial and original ways.

In her paper *Diagrams and "Free Rides" in Mathematics*, Jessica Carter investigates the heuristic role of visual representations. Focusing on diagrams, Carter applies Atsushi Shimojima's (1996) notion of "free rides" to mathematics in order to better understand the fruitfulness of mathematical representations that exploit spatial properties as opposed to symbolic or sentential representations. The notion of a "free ride" characterises those phenomena in which novel information about the properties of a mathematical object — information which was not contained in its construction — can be "read off" an external representation of that object in a way that would not be possible from a merely sentential representation. Through examples of fruitful representations employed to solve problems in mathematical practice, Carter illustrates a variety of uses of "free rides" in mathematics, and demonstrates how mathematical "free rides" have additional properties that differentiate them from those characterised by Shimojima. On the basis of this analysis, Carter maintains that free rides in mathematics come in different forms, and sometimes require reasoning from the syntactic and semantic properties of diagrams to convey the additional information.

Emiliano Ippoliti's paper, *On the Heuristic Power of Mathematical Representations*, argues that mathematical representations can have heuristic power since their construction can be ampliative. He examines how the representation of a constructed object introduces elements and properties into an object which are not contained at the beginning of the object's construction. Moreover, he discusses how representation guides the manipulation of the represented object in ways that restructure its components by gradually adding new pieces of information. These pieces, in turn, then produce a hypothesis to solve a problem. Finally, he shows that these representations draw on ampliative inferences that form the basis for gradually building hypotheses to solve a problem both *inside* mathematics (the construction of an algebraic representation of 3-manifolds), and *outside* mathematics (the construction of a topological representation of DNA supercoiling).

In *Arithmetic Enumerative Induction and Order Bias*, Alexander Paseau aims at vindicating the use of non-deductive methods in mathematics. The paper proposes an analysis of the general mistrust mathematicians have towards enumerative inductive evidence. This phenomenon is particularly evident in the case of arithmetical generalizations, where enumerative inductive evidence is deemed heuristically useful but is thought to lack justificatory force. According to Paseau's analysis (which is supported by several novel case studies) the principled reason for scepticism about the value of enumerative inductive evidence in arithmetic is that known instances of an arithmetical conjecture are usually 'small': they appear in the initial segment of the natural number sequence (e.g., Goldbach's Conjecture was checked up to value 4×10^{18}). Such evidence consequently suffers from size bias. Paseau distinguishes between different varieties of size-scepticism as advocated by both mathematicians and philosophers of mathematics, analyses their motivation, and poses a series of challenges to each kind of scepticism. He concludes that in many cases of arithmetical generalizations size-scepticism is in fact *not* warranted. The paper closes by providing some remarks about enumerative inductive evidence for the consistency of set theory in the light of the consideration drawn from the previous discussion of size scepticism.

In his *The Derivation of Poiseuille's Law: Heuristic and Explanatory Considerations*, Chris Pincock examines the historical development and the debate surrounding Poiseuille's law. Pincock focuses on this case study because it helps us better understand the character and value of mathematical explanation in science, especially in those cases where the search for a theoretical explanation of an experimental result has heuristic value in facilitating further experimental discoveries. Poiseuille's law, which relates the rate of flow for some fluid through a cylinder to the change in pressure, was conclusively supported by Poiseuille's own measurements in 1846. However, a lack of clarity as to why the law would hold in controlled laboratory conditions (while failing in a broader range of circumstances), together with the highly mathematical character of the law, led the scientific community to search for a theoretical explanation. Further mathematical investigations eventually pointed to a widely accepted explanation: Poiseuille's law is derivable from an idealized model of steady, laminar flow. This derivation contributed (together with strong experimental evidence) to the establishment of the so-called "no slip" boundary condition: as a fluid flows through such a cylinder, its velocity becomes nil at the walls. Pincock's paper addresses two important questions concerning this explanation: (1) How can a derivation supported by an idealized model explain – when the idealizing assumptions fail to apply to the real-world systems used by Poiseuille to derive his law? (2) In such cases, what is the heuristic value of an assumption like the no slip condition, which plays a key role in the measurement of a fluid's viscosity? Pincock argues that it was explanatory considerations which both led to the discovery of the derivation, and drove further discoveries concerning turbulent flows where Poiseuille's law ceases to apply.

In *The Mathematical Stance*, Alan Baker defends the Enhanced Indispensability Argument (EIA), considered by many the strongest argument currently available for mathematical platonism. He tackles two important objections against this argument. First, according to the EIA, we should commit to the existence of any entity that is explanatorily indispensable to science. Since mathematical objects are indispensable to our best scientific explanations, we should therefore commit to their existence. However, a critic may complain here that the argument assumes that there is a *unique* structure that underpins a given physical phenomenon. Another criticism is that the argument over-generates, by committing us to idealized entities—such as perfectly continuous fluids and frictionless surfaces—which play an indispensable explanatory role in our scientific theories. This would imply that explanatory indispensability is not after all a sufficient condition for ontological commitment.[4] According to Baker, these two objections can be overcome by appealing to the notion of "the mathematical stance", modelled after Dennett's notion of "the intentional stance". Namely, we should regard a given physical phenomenon as an abstract mathematical structure in a way that facilitates prediction and explanation of the puzzling aspects of its behavior. Baker argues that such a stance-based account of the application of mathematics can lay the grounds for achieving a satisfactory general philosophical account of mathematical explanation in science.

In *On the Plains and Prairies of Minnesota: The Role of Mathematical Statistics in Biological Explanation*, Emily Grosholz considers the explanatory and heuristic aspects of the use of mathematical statistics in the research on plant population. She considers as a case study the work of population geneticist Ruth Geyer Shaw and her group on prairie restoration in Minnesota. In order to adequately capture the complexity of this biological phenomenon, this type of field work involves a combination of intensive collection of empirical data and development of appropriate mathematical modelling. This, in turn, requires measuring both macroscopic and genetic features of the population involved in their interactions over time. Thus, it involves selecting the significant variables to record, the study of the interaction of different causal mechanisms in the different stages of development, as well as the analysis of the genetic variance, with special attention to avoiding sampling bias. According to Grosholz, the complex issues at the interplay between statistics and biology have been neglected by many philosophers: often, physics and deductive explanations are presented as a regulative ideal for biology (here, a typical example is Newtonian mechanics). On the other hand, explanations in biology are usually partial and provisional: living beings have heterogeneous and unpredictable behaviors, and there is an unbridgeable gap between the phenomenon of interest and the limitations of the compiled dataset (it is simply impossible to study every aspect of every member of a population). Grosholz focuses on the use of Aster Models (a class of statistical models that jointly analyse the measurements of separate, sequential, non-normally distributed components of fitness), and examines three case studies. These show why the use of mathematical statistics in biology, if expanded to employ a variety of non-normal distributions, offers insightful explanations and reveals itself to be highly

[4] This critique is due to Maddy (1997) and has been also addressed in Bangu (2012).

valuable in generating predictions. These, in turn, can be the grounds for the development of local and national environmental policies.

Finally, Mary Leng's paper *Models, Structures, and the Explanatory Role of Mathematics in Empirical Science* argues that we should think of explanations by constraint as structural explanations. "Explanations by constraints" are characterised by Lange (2017) as being, roughly speaking, explanations which show how a physical phenomenon follows from mathematics together with physical background conditions. "Structural explanations", on the other hand, show how a physical phenomenon necessarily follows from the features of the mathematical structure instantiated in the physical system under consideration. As such, they are presented in the form of necessitated conditionals. According to Leng, these types of explanations genuinely enhance our understanding of the physical world by providing modal information about their explananda, and by showing how apparently disparate phenomena are instances of a common structure. She then considers the notion of a structural explanation in the context of the debate on the enhanced indispensability argument, and contends that since structural explanations can be understood in modal structural terms, they do not engender a commitment to mathematical platonism. Subsequently, Leng discusses case studies where a mathematical structure is only instantiated in an idealised model of a physical description, arguing that even in these cases, structural explanations do not lend support to mathematical platonism. Toward the end, Leng also maintains that viewing mathematical explanations as structural explanations provides a novel perspective on the explanatory role played by mathematics in our understanding of the world. This role reveals how mathematical-structural dependencies in mathematical models of physical phenomena reflect mathematical-structural dependencies in the physical world.

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