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Spacetime Fermions in Light-cone Gauge Superstring Field Theory and Dimensional Regularization

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Abstract

We consider the dimensional regularization of the light-cone gauge type II superstring field theories in the NSR formalism. In the previous work, we have calculated the tree-level amplitudes with external lines in the (NS,NS) sector using the regularization and shown that the desired results are obtained without introducing contact term interactions. In this work, we study the tree-level amplitudes with external lines in the Ramond sector. In order to deal with them, we propose a worldsheet theory to be used instead of that for the naive dimensional regularization. With the worldsheet theory, we regularize and define the tree-level amplitudes by analytic continuation. We show that the results coincide with those of the first quantized formulation.

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1 Introduction

It is desirable to find a good way to regularize ultraviolet and infrared divergences in string field theory. Although regularization did not play essential roles in calculating scattering amplitudes, it will be needed to deal with amplitudes which involve off-shell quantities such as boundary states. In the superstring field theory, regularization is needed to deal with the contact term problem [1, 2, 3, 4, 5].

In the previous works [6, 7, 8, 9], we have proposed to dimensionally regularize the light-cone gauge string field theory [10, 11, 12, 13, 14, 15] to deal with the divergences. We have shown that the light-cone gauge string theory in noncritical dimensions corresponds to a conformal gauge string theory in a Lorentz noninvariant background which preserves the BRST symmetry on the worldsheet. This implies that the dimensional regularization in the light-cone gauge string field theory preserves the gauge symmetry of the string field. We have calculated the tree-level amplitudes for the type II superstring field theories in the NSR formalism using this regularization, when the external lines are in the (NS,NS) sector. The results of the first quantized formulation are reproduced without any need for introducing the contact term interactions in the analytic continuation $d \to 10$ performed in the end of the calculation.

In this work, we would like to generalize our analysis to the amplitudes with external lines in the Ramond sector. In Ref. [16], we have discussed how we can deal with the Ramond sector for superstrings in noncritical dimensions. It is possible to define BRST invariant vertex operators for the Ramond sector and obtain the BRST invariant form of the amplitudes using them. However, it has also been pointed out that the naive dimensional regularization has a problem in dealing with the type II theories. Namely, it turns out that the regularized theory cannot have spacetime fermions in the spectrum.

In this paper, we propose an alternative to the naive dimensional regularization. We modify the worldsheet theory so that the theory includes spacetime fermions and at the same time the divergences are regularized. Using the worldsheet theory, we calculate the amplitudes and analytically continue the results to $d = 10$. We show that this procedure yields tree-level amplitudes which coincide with those of the first quantized formalism, without any need for adding the contact term interactions.

This paper is organized as follows. In section 2, we present the worldsheet theory to be used for the regularization. We show that it can be used to regularize the superstring theory with spacetime fermions. In section 3, we explain how to perform the analytic continuation of the amplitudes to define them for $d = 10$. We show that the results obtained coincide with those of the first quantized formalism. Section 4 is devoted to conclusions and discussions. In particular, we discuss how we can apply our dimensional regularization scheme to deal with the divergences of the multi-loop amplitudes. In appendix A, we present the action of the light-cone gauge superstring field theory and the calculation of the amplitudes.

2 Dimensional regularization

In the previous works, we have considered dimensional regularization of the light-cone gauge string field theory exactly as in the field theory for particles. It is possible to define the action in d dimensional spacetime if $d \geq 2$ is an integer. We have calculated the amplitudes perturbatively and analytically continued them as functions of d. This procedure works for bosonic strings and for superstrings as long as we deal with only fields in the (NS,NS) sector. However, as was pointed out in Ref. [16], such a naive procedure does not work if one wants to incorporate spacetime fermions. Naive dimensional continuation implies that the level-matching condition for the (R,NS) sector becomes

$$\mathcal{N} + \frac{d-2}{16} = \tilde{\mathcal{N}} \ , \tag{2.1}$$

where $\mathcal{N}$ and $\tilde{\mathcal{N}}$ denote the left and the right mode numbers of the light-cone gauge string state, and there are no states satisfying it for general d. The same argument applies to the (NS,R) sector. Therefore the dimensionally regularized theory cannot have spacetime fermions in the spectrum and we cannot use it to regularize the type II superstring theories. It may be used to deal with the type 0 theories.

In order to deal with spacetime fermions, we need to find an alternative to the naive dimensional regularization. In the calculation presented in Ref. [9], one can see that the contact term divergences are regularized by the factor $e^{-\frac{d-2}{16}\Gamma}$ in the light-cone gauge amplitudes which comes from the conformal anomaly.[1] Therefore what matters in regularization is the Virasoro central charge of the worldsheet theory. Instead of considering the theory in d dimensional spacetime, we can consider the light-cone gauge worldsheet theory with the Virasoro central charge $\frac{3}{2}(d-2)$ to make the amplitudes finite. As will be discussed in section 4, in contrast to particle theory, the ultraviolet behavior of the multi-loop amplitudes in string theory is also affected by the central charge rather than the number of the momentum variables.

[1] The explicit form of Γ is given in eq.(3.4).

2.1 Worldsheet theory

For dimensional regularization, we need to make the Virasoro central charge less than the critical one. This can be achieved by adding a superconformal field theory with negative central charge to the usual worldsheet theory of the transverse variables $X^i, \psi^i, \tilde{\psi}^i$ ($i = 1, \ldots, 8$). Actually we have considered such a worldsheet theory in Ref. [6]. As long as we are dealing with the (NS,NS) sector, the amplitudes essentially depends only on the value of the Virasoro central charge of the added superconformal field theory. However, the Ramond sector depends on the details of the theory. In the following, we would like to show that it is possible to choose the superconformal field theory so that the difficulty mentioned above can be avoided and we can deal with the spacetime fermions.

The value of $\frac{d-2}{16}$ on the left hand side of eq.(2.1) comes from the conformal dimension of the spin field, which depends on the number of the fermionic coordinates. If we add a free superconformal field theory to the worldsheet theory for dimensional regularization, the Virasoro central charge and the number of the fermionic coordinates are both fixed by the spacetime dimensions. By choosing an interacting superconformal field theory, one can change the Virasoro central charge and the dimension of the spin field independently. Then it would be possible to deal with the divergences and the level-matching condition simultaneously. An example of such a theory is the super WZW model. By changing the level, we can change the value of the Virasoro central charge keeping the dimension of the group manifold and therefore the dimension of the spin field. However, adding a super WZW model alone to the transverse variables is not enough, because the dimension of the spin field is still positive and the spectrum of the spacetime fermions becomes quite different from that for $d = 10$. One can deal with this problem by adding variables corresponding to coordinates with opposite statistics, namely a ghost-like system. Such coordinates can be considered as those in the directions with "negative dimensions" and have the effect of reducing the dimension of the spin field. Therefore adding a superconformal field theory which consists of a super WZW model and a ghost-like system will do the job.

In this paper, as an example of such a superconformal field theory, we consider the one with the action

$$S = S_G + S_{\mathrm{gh}} \ , \tag{2.2}$$

where

$$\begin{aligned} S_G &= kS_{\mathrm{WZW}}[g] + \frac{1}{\pi}\int d^2z \left(\lambda^a\bar{\partial}\lambda^a + \tilde{\lambda}^a\partial\tilde{\lambda}^a\right) \ , \\ S_{\mathrm{gh}} &= \frac{1}{\pi}\int d^2z \left(b^A\bar{\partial}c^A + \tilde{b}^A\partial\tilde{c}^A + \beta^A\bar{\partial}\gamma^A + \tilde{\beta}^A\partial\tilde{\gamma}^A\right) \ . \end{aligned} \tag{2.3}$$

S_G denotes the action of the super WZW model of level k for the group $G = (SU(2))^{2M}$, where $M \geq 0$ and $k > 0$ are integers. $S_{\text{WZW}}[g]$ is the action for the $(SU(2))^{2M}$ WZW model, λ^a and $\tilde{\lambda}^a$ are the fermions with weight $\left(\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right)$ respectively, and $a = 1, \cdots, 6M$ are the indices in the adjoint representation of $(SU(2))^{2M}$. The ghost-like variables $b^A, c^A, \tilde{b}^A, \tilde{c}^A$, $\beta^A, \gamma^A, \tilde{\beta}^A, \tilde{\gamma}^A$ are with $A = 1, \cdots, 3M$. $b^A, c^A, \tilde{b}^A, \tilde{c}^A$ are Grassmann odd variables with conformal weight $(1,0), (0,0), (0,1), (0,0)$ respectively and $\beta^A, \gamma^A, \tilde{\beta}^A, \tilde{\gamma}^A$ are Grassmann even and with conformal weight $\left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right), \left(0, \frac{1}{2}\right)$ respectively. We take the boundary conditions of γ^A, β^A (resp. $\tilde{\gamma}^A, \tilde{\beta}^A$) to be the same as those of ψ^i, λ^a (resp. $\tilde{\psi}^i, \tilde{\lambda}^a$). Thus the theory is a superconformal field theory consisting of the super WZW model and a ghost-like system.

We take the worldsheet theory to be that of the transverse coordinates $X^i, \psi^i, \tilde{\psi}^i$ combined with the CFT defined above. Therefore the total central charge is

$$c = \underset{X^i,\psi^i}{12} + \underset{\text{WZW}}{\frac{6Mk}{k+2}} + \underset{\lambda^a}{3M} + \underset{b^A,c^A,\beta^A,\gamma^A}{(-9M)} = 12 - \frac{12M}{k+2}, \tag{2.4}$$

and we may regard the resultant worldsheet theory as the "transverse part" of a theory in the effective spacetime dimension

$$d \equiv \frac{2}{3}c + 2 = 10 - \frac{8M}{k+2}. \tag{2.5}$$

By varying M with k fixed, one can realize superconformal field theories with largely negative central charge. For such theories, the amplitudes are finite because of the factor $e^{-\frac{d-2}{16}\Gamma}$. Using these theories, one can get an expression of the amplitudes which can be analytically continued as a function of M, as will be explained in section 3. The critical dimension is recovered in the limit $M \to 0$.

2.2 External lines

We would like to write down the dimensionally regularized version of the amplitudes using the modified worldsheet theory. In order to do so, we consider the amplitudes with the external lines specified in the following way. Suppose that for $d = 10$ each external line corresponds to a state $|\ \rangle_{X^i,\psi^i,\tilde{\psi}^i}$ in the Fock space of the transverse variables. We bosonize $\lambda^a, \beta^A, \gamma^A$ as

$$\frac{1}{\sqrt{2}}\left(\lambda^{2A-1} \pm i\lambda^{2A}\right) = e^{\pm i\varphi^A},$$

$$
\begin{aligned}
\gamma^A &= \eta^A e^{\phi^A} , \\
\beta^A &= e^{-\phi^A} \partial \xi^A ,
\end{aligned}
\tag{2.6}
$$

in the usual way so that we can express the vertex operators in the Ramond sector. Here φ^A, ϕ^A are chiral bosons and η^A, ξ^A are Grassmann odd variables with weight $(1,0), (0,0)$. As the regularized version for the amplitudes, we consider those with the external lines

$$
\begin{aligned}
| \ \rangle_{X^i,\psi^i,\tilde{\psi}^i} \otimes |0\rangle_L \otimes |0\rangle_R & \qquad (\mathrm{NS},\mathrm{NS})\,\mathrm{sector} , \\
| \ \rangle_{X^i,\psi^i,\tilde{\psi}^i} \otimes |\pm\rangle_L \otimes |0\rangle_R & \qquad (\mathrm{R},\mathrm{NS})\,\mathrm{sector} , \\
| \ \rangle_{X^i,\psi^i,\tilde{\psi}^i} \otimes |0\rangle_L \otimes |\pm\rangle_R & \qquad (\mathrm{NS},\mathrm{R})\,\mathrm{sector} , \\
| \ \rangle_{X^i,\psi^i,\tilde{\psi}^i} \otimes |\pm\rangle_L \otimes |\pm\rangle_R & \qquad (\mathrm{R},\mathrm{R})\,\mathrm{sector} ,
\end{aligned}
\tag{2.7}
$$

where $|0\rangle_{L,R}$ denote the left and right moving $SL(2,\mathbb{C})$ invariant vacua of the superconformal field theory defined above and $|\pm\rangle_L$ denote the states corresponding to the primary fields

$$
\prod_A e^{\pm\frac{1}{2}\phi^A} \prod_A e^{\pm\frac{i}{2}\varphi^A} ,
\tag{2.8}
$$

whose conformal weight is 0. $|\pm\rangle_R$ are defined in the same way. Therefore the level-matching conditions for the (R,NS) and (NS,R) sectors are the same as those for the transverse part. Thus all the spacetime fermions in the critical theory have their cousins in the regularized theory. Given any amplitudes for $d = 10$, one can write down a regularized version by choosing the external lines from eq.(2.7) so that the amplitudes do not vanish identically. With this choice of the external lines, we do not have any trouble in dealing with spacetime fermions.

3 Dimensionally regularized amplitudes

Let us consider the tree-level amplitudes of the light-cone gauge string field theory corresponding to the worldsheet theory defined in the previous section. We obtain them as functions of M, with M non-negative integer. For M big enough, the divergences are regularized and the amplitudes are well-defined. We would like to define the amplitudes as analytic functions of M and obtain the amplitudes for $d = 10$ by taking the limit $M \to 0$. Since the amplitudes are given only for non-negative integer M, we should specify the way to preform the analytic continuation.[2]

[2] This problem was not discussed in the previous works [6, 9], in which we considered amplitudes with (NS,NS) external lines. If all the external lines are in the (NS,NS) sector, it is possible to actually realize the

3.1 Light-cone gauge amplitudes

The string field theory action and the calculation of the amplitudes are explained in appendix A. The regularized tree-level amplitudes for N strings can be expressed as an integral over the moduli space of the string diagram:

$$\mathcal{A}_N = (4ig)^{N-2} \int \left(\prod_{\mathcal{I}=1}^{N-3} \frac{d^2\mathcal{T}_\mathcal{I}}{4\pi} \right) F_N \left(\mathcal{T}_\mathcal{I}, \bar{\mathcal{T}}_\mathcal{I} \right) , \tag{3.1}$$

where $\mathcal{T}_\mathcal{I}, \bar{\mathcal{T}}_\mathcal{I}$ $(\mathcal{I} = 1, \ldots, N-3)$ are the moduli parameters, which correspond to the Schwinger parameters for the propagators in each channel. The integrand $F_N \left(\mathcal{T}_\mathcal{I}, \bar{\mathcal{T}}_\mathcal{I} \right)$ is described by using the correlation function $\langle \cdots \rangle_{\mathrm{LC}}$ for the light-cone gauge worldsheet theory given in the last section as

$$\begin{aligned} F_N \left(\mathcal{T}_\mathcal{I}, \bar{\mathcal{T}}_\mathcal{I} \right) \;&=\; (2\pi)^2 \delta \left(\sum_{r=1}^{N} p_r^+ \right) \delta \left(\sum_{r=1}^{N} p_r^- \right) \mathrm{sgn} \left(\prod_{r=1}^{N} \alpha_r \right) e^{-\frac{d-2}{16}\Gamma} f\left(\alpha_r; Z_r\right) \\ &\quad \times \left\langle \prod_A \left(c^A(z_0)\, \tilde{c}^A(\bar{z}_0) \right) \prod_{I=1}^{N-2} \left| \left(\partial^2\rho\right)^{-\frac{3}{4}} T_F^{\mathrm{LC}}(z_I) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle_{\mathrm{LC}} , \end{aligned} \tag{3.2}$$

where the function f is given in eq.(A.17), T_F^{LC} denotes the supercurrent of the worldsheet theory, and V_r^{LC} are the vertex operators corresponding to the states in eq.(2.7). V_r^{LC} can be obtained by combining the left and right contributions which take the form given in Ref. [16] but this time they involve the operators in eq.(2.8) from the superconformal field theory defined in section 2. $\rho(z)$ is the Mandelstam mapping defined as

$$\rho(z) = \sum_{r=1}^{N} \alpha_r \ln(z - Z_r) , \tag{3.3}$$

which maps the complex z-plane to the N-string light-cone diagram parametrized by the complex coordinate ρ as usual. z_I $(I = 1, \ldots, N-2)$ are the interaction points determined by $\partial\rho(z_I) = 0$. Γ is defined by

$$e^{-\Gamma} = \left| \sum_{r=1}^{N} \alpha_r Z_r \right|^4 \prod_{r=1}^{N} \left(|\alpha_r|^{-2} e^{-2\,\mathrm{Re}\,\bar{N}_{00}^{rr}} \right) \prod_{I=1}^{N-2} \left| \partial^2\rho(z_I) \right|^{-1} , \tag{3.4}$$

where $\bar{N}_{00}^{rr}$ is a Neumann coefficient given as

$$\bar{N}_{00}^{rr} = \frac{\tau_0^{(r)} + i\beta_r}{\alpha_r} - \sum_{s \neq r} \frac{\alpha_s}{\alpha_r} \ln(Z_r - Z_s) , \qquad \tau_0^{(r)} + i\beta_r \equiv \rho(z_{I^{(r)}}) . \tag{3.5}$$

amplitudes for noninteger d by choosing the worldsheet theory with the Virasoro central charge $c = \frac{3}{2}(d-2)$. Then the analytic continuation is obvious.

Here $z_{I^{(r)}}$ denotes the interaction point where the r-th string interacts. The amplitudes (3.1) are defined for non-negative integer M.

Let us see that by taking M large enough, one can regularize the divergences in the amplitudes caused by the colliding supercurrents $T_F^{\mathrm{LC}}(z_I)$ inserted at the interaction points, $T_F^{\mathrm{LC}}(z_I)T_F^{\mathrm{LC}}(z_J) \sim (z_I - z_J)^{-3}$. Because of the identity

$$\partial^2 \rho(z_I) = \left(\sum_{s=1}^{N} \alpha_s Z_s\right) \frac{\prod_{J \neq I}(z_I - z_J)}{\prod_{r=1}^{N}(z_I - Z_r)} , \tag{3.6}$$

one can find that $e^{-\frac{d-2}{16}\Gamma}$ obtained from eq.(3.4) behaves as $e^{-\frac{d-2}{16}\Gamma} \sim |z_I - z_J|^{-\frac{d-2}{8}}$ for $z_I \sim z_J$. It follows that the most divergent term in the integrand F_N in eq.(3.2) behaves as

$$e^{-\frac{d-2}{16}\Gamma} \prod_I \left|(\partial^2\rho)^{-\frac{3}{4}} T_F^{\mathrm{LC}}(z_I)\right|^2 \sim |z_I - z_J|^{-9-\frac{d-2}{8}} \tag{3.7}$$

for $z_I \sim z_J$. Therefore if we take $d < -70$, the integrand on the right hand side of eq.(3.1) does not diverge but vanishes for $z_I = z_J$. Due to eq.(2.5), this implies that if $M > 10(k+2)$, the divergences are indeed regularized.

Therefore we can define $\mathcal{A}_N$ for M large enough and may analytically continue it to $M = 0$. In order to do so, we should specify the way to define $\mathcal{A}_N$ as an analytic function of M. It is easy to see that the M dependence of F_N arises in the following way:

- Through the total central charge c in eq.(2.4) which appears in various places. One is in the anomaly contribution to the amplitudes

$$e^{-\frac{d-2}{16}\Gamma} = e^{-\frac{1}{2}\Gamma + M\frac{\Gamma}{2(k+2)}} . \tag{3.8}$$

  The on-shell condition also depends on c. One can see that each vertex operator V_r^{LC} involves a factor

$$\exp\left(-\frac{M}{2\,(k+2)}\frac{\tau_0^{(r)}}{p_r^+}\right) , \tag{3.9}$$

  which comes from the on-shell condition.

- Since the superconformal field theory defined in section 2 consists of M copies of $SU(2)^2$ super WZW model and ghost-like system, the correlation functions of the theory depend on M. With the external lines given in eq.(2.7), the correlation functions on the complex plane depend on M through the combinatorial factors, which can be expressed as a polynomial of M.

Altogether we can see that F_N is given in the form

$$F_N = P_N(M)\, e^{MQ_N} \ , \tag{3.10}$$

where

$$Q_N \equiv \frac{1}{2(k+2)} \left(\Gamma - \sum_{r=1}^{N} \frac{\mathcal{T}_0^{(r)}}{p_r^+} \right) \ , \tag{3.11}$$

and $P_N(M)$ is a polynomial of M. Thus we define F_N for noninteger M by analytically continuing the right hand side of eq.(3.10) and obtain $\mathcal{A}_N$ as an analytic function of M.

3.2 BRST invariant expression of the amplitudes

It is obvious that in the limit $M \to 0$, $\mathcal{A}_N$ in eq.(3.1) coincides with the expression of the amplitudes for $d = 10$. We define the amplitudes $\mathcal{A}_N$ when M is large enough, analytically continue them and in the end of the calculation take the limit $M \to 0$. We would like to prove that the results are finite and coincide with those of the first quantized formalism. In order to do so, we first rewrite the light-cone gauge amplitude (3.1) into the conformal gauge one. Using the BRST invariance of the conformal gauge expression, we can prove the above mentioned facts. Since the arguments are rather lengthy, we first illustrate the proof for a simple case and treat the general case later.

Four point amplitudes

Let us consider the four point tree-level amplitudes with two external lines in the (R,NS) sector and two in the (NS,NS) sector. The regularized amplitude is given as

$$\mathcal{A}_4 = (4ig)^2 \int \frac{d^2\mathcal{T}}{4\pi} F_4\left(\mathcal{T}, \bar{\mathcal{T}}\right) \ , \tag{3.12}$$

where the integrand $F_4\left(\mathcal{T}, \bar{\mathcal{T}}\right)$ is described by using the correlation function as

$$\begin{aligned} F_4\left(\mathcal{T}, \bar{\mathcal{T}}\right) &= (2\pi)^2 \delta\left(\sum_{r=1}^4 p_r^+\right) \delta\left(\sum_{r=1}^4 p_r^-\right) \mathrm{sgn}\left(\prod_{r=1}^4 \alpha_r\right) e^{-\frac{d-2}{16}\Gamma} f(\alpha_r; Z_r) \\ &\quad \times \left\langle \prod_A \left(c^A(z_0)\, \tilde{c}^A(\bar{z}_0)\right) \prod_{I=1}^2 \left| \left(\partial^2\rho\right)^{-\frac{3}{4}} T_F^{\mathrm{LC}}(z_I) \right|^2 \prod_{r=1}^4 V_r^{\mathrm{LC}} \right\rangle_{\mathrm{LC}} . \end{aligned} \tag{3.13}$$

Here we have used the Mandelstam mapping $\rho(z)$ depicted in Fig. 1 to express the correlation function. The complex moduli parameter $\mathcal{T}$ is given as

$$\mathcal{T} = \rho(z_2) - \rho(z_1) \ . \tag{3.14}$$

Let us consider the case where $V_1^{\mathrm{LC}}, V_2^{\mathrm{LC}}$ are in the (R,NS) sector.

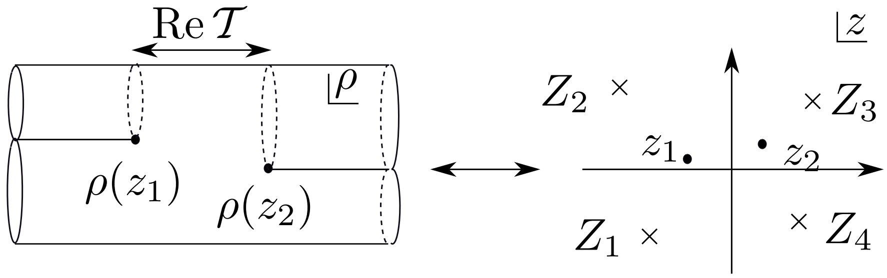

Figure 1: Four point tree-level amplitudes

Conformal gauge expression

We would like to express the correlation function in eq.(3.13) using the conformal gauge variables. In the conformal gauge, the worldsheet theory consists of the light-cone gauge one given in section 2 combined with the $X^{\pm}$ CFT [8] and the super-reparametrization ghosts. With all the field content, one can define the nilpotent BRST charge Q_{B}. In order to rewrite eq.(3.13), we need the following identity:

$$
\begin{aligned}
&(2\pi)^2\delta\left(\sum_{r=1}^{4}p_r^+\right)\delta\left(\sum_{r=1}^{4}p_r^-\right)e^{-\frac{d-2}{16}\Gamma}\prod_{I=1}^{2}\left|\partial^2\rho\left(z_I\right)\right|^{-\frac{3}{2}}f\left(\alpha_r;Z_r\right)\prod_{r=1}^{4}V_r^{\mathrm{LC}} \\
&\sim\ \Bigg\langle\left|\partial\rho c(\infty)\right|^2\prod_{I=1}^{2}\left|\oint_{z_I}\frac{dz}{2\pi i}\frac{b}{\partial\rho}(z)e^{\phi}(z_I)\right|^2\prod_{r=1}^{4}\mathcal{S}_r^{-1} \\
&\qquad\times V_1^{(-\frac{3}{2},-1)}(Z_1,\bar{Z}_1)V_2^{(-\frac{1}{2},-1)}(Z_2,\bar{Z}_2)V_3^{(-1,-1)}(Z_3,\bar{Z}_3)V_4^{(-1,-1)}(Z_4,\bar{Z}_4)\Bigg\rangle_{X^{\pm},\mathrm{ghosts}} \\
&\qquad\times\alpha_1^{\frac{1}{2}}\alpha_2^{\frac{1}{2}}\ ,
\end{aligned}
\tag{3.15}
$$

where the $\sim$ means that the left and right hand sides coincide up to an overall numerical constant. Since such overall constants are irrelevant to show the BRST invariance, we will henceforth ignore them. On the right hand side, the expectation value is taken with respect to the $X^{\pm}$ CFT and the super-reparametrization ghosts. $V_r^{(p_{L,r},p_{R,r})}$ denotes the vertex operator in the picture $(p_{L,r},p_{R,r})$, which can be obtained by combining the left and right ones defined in Ref. [16]. $\mathcal{S}_r^{-1}$ is given in terms of superfields [16] as

$$
\mathcal{S}_r^{-1}\equiv\oint_{z_{I(r)}}\frac{d\mathbf{z}}{2\pi i}D\Phi(\mathbf{z})\oint_{\bar{z}_{I(r)}}\frac{d\bar{\mathbf{z}}}{2\pi i}\bar{D}\Phi(\bar{\mathbf{z}})\,e^{\frac{d-10}{16}\frac{i}{p_r^+}\mathcal{X}^+}(\mathbf{z},\bar{\mathbf{z}})\ ,
\tag{3.16}
$$

where

$$
\Phi\equiv\ln\left(-4\left(D\Theta^+\right)^2\left(\bar{D}\tilde{\Theta}^+\right)^2\right)\ ,\qquad\Theta^+\left(\mathbf{z}\right)\equiv\frac{D\mathcal{X}^+}{\left(\partial\mathcal{X}^+\right)^{\frac{1}{2}}}\left(\mathbf{z}\right)\ ,
$$

$$\mathcal{X}^+(\mathbf{z},\bar{\mathbf{z}}) = X^+(z,\bar{z}) + i\theta\psi^+(z) + i\bar{\theta}\tilde{\psi}^+(\bar{z}) + i\theta\bar{\theta}F^+(z,\bar{z})\ . \tag{3.17}$$

The identity (3.15) was proved in Ref. [16] when all the external lines are spacetime bosons. Actually it can be proved in almost the same way even if spacetime fermions are involved. The only subtlety is that with spacetime fermions the superghost correlation function becomes left-right asymmetric and in the course of the calculation we encounter phase factors which are constant but depend on how we take the cuts to define the correlation function. We can proceed as in Ref. [16] but obtain eq.(3.15) only up to such a phase factor. However, in the final form (3.15) we can see that the phases add up to become just a numerical constant, because neither the left nor the right hand side has cuts as a function of $Z_r, \bar{Z}_r, z_I, \bar{z}_I$. Thus eq.(3.15) holds also in our case.

Substituting eq.(3.15) into eq.(3.13) we obtain the conformal gauge expression of the amplitude:

$$\begin{aligned}\mathcal{A}_4 \sim \int d^2\mathcal{T} \Biggl\langle \left|\partial\rho c(\infty)\right|^2 \prod_A \left(c^A(z_0)\,\tilde{c}^A(\bar{z}_0)\right) \\ \times \prod_{I=1}^{2} \left| \oint_{z_I} \frac{dz}{2\pi i}\frac{b}{\partial\rho}(z)\, e^{\phi}T_F^{\mathrm{LC}}(z_I) \right|^2 \prod_{r=1}^{4} \left[\mathcal{S}_r^{-1} V_r^{(p_{L,r},p_{R,r})}\right] \Biggr\rangle\, \alpha_1^{\frac{1}{2}}\alpha_2^{\frac{1}{2}}\ .\end{aligned} \tag{3.18}$$

Here $\langle\cdots\rangle$ denotes the correlation function for the conformal gauge worldsheet theory. In eq.(3.18), we can replace $e^{\phi}T_F^{\mathrm{LC}}$ by the picture changing operator

$$X(z) \equiv \{Q_{\mathrm{B}}, \xi(z)\}\ , \tag{3.19}$$

as was proved in Ref. [16]. The factor $\alpha_1^{\frac{1}{2}}\alpha_2^{\frac{1}{2}}$ is attributed to the normalization of the wave function for the fermionic fields and will be ignored in the following. Thus we eventually obtain

$$\begin{aligned}\mathcal{A}_4 \sim \int d^2\mathcal{T} \Biggl\langle \left|\partial\rho c(\infty)\right|^2 \prod_A \left(c^A(z_0)\,\tilde{c}^A(\bar{z}_0)\right) \\ \times \prod_{I=1}^{2} \left| \oint_{z_I} \frac{dz}{2\pi i}\frac{b}{\partial\rho}(z)\, X(z_I) \right|^2 \prod_{r=1}^{4} \left[\mathcal{S}_r^{-1} V_r^{(p_{L,r},p_{R,r})}\right] \Biggr\rangle .\end{aligned} \tag{3.20}$$

One can show that the expression on the right hand side of eq.(3.20) is BRST invariant, if M is large enough. Deforming the contour of $\oint_{z_I}\frac{dz}{2\pi i}\frac{b}{\partial\rho}(z)$, one easily gets the expression

$$\mathcal{A}_4 \sim \int d^2\mathcal{T} \Biggl\langle \prod_A \left(c^A(z_0)\,\tilde{c}^A(\bar{z}_0)\right) \left| \oint_{C} \frac{dz}{2\pi i}\frac{b}{\partial\rho}(z) \right|^2 \prod_{I=1}^{2} \left|X(z_I)\right|^2 \prod_{r=1}^{4} \left[\mathcal{S}_r^{-1} V_r^{(p_{L,r},p_{R,r})}\right] \Biggr\rangle , \tag{3.21}$$

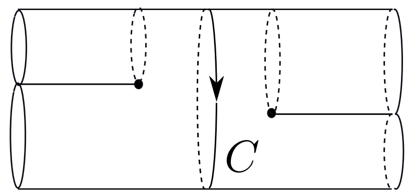

Figure 2: Contour C

where C is the contour depicted in Fig. 2. By using the "small Hilbert space" mentioned at the end of appendix A, the integrand on the right hand side of eq.(3.21) is written as

$$\left\langle \left| \oint_C \frac{dz}{2\pi i} \frac{b}{\partial\rho}(z) \right|^2 \prod_{I=1}^{2} |X(z_I)|^2 \prod_{r=1}^{4} \left[\mathcal{S}_r^{-1} V_r^{(p_{L,r}, p_{R,r})} \right] \right\rangle_{\text{small Hilbert space}} . \tag{3.22}$$

In this form, it is easy to show the BRST invariance.[3] We note that

$$\left\{ Q_{\mathrm{B}} \, , \oint_C \frac{dz}{2\pi i} \frac{b}{\partial\rho}(z) \right\} = \oint_C \frac{dz}{2\pi i} \frac{T}{\partial\rho}(z) \; . \tag{3.23}$$

The right hand side coincides with L_0 on the light-cone diagram and therefore

$$\left\{ Q_{\mathrm{B}} \, , \oint_C \frac{dz}{2\pi i} \frac{b}{\partial\rho}(z) \right\} \sim -\partial_{\mathcal{T}} \; . \tag{3.24}$$

If M is large enough, we can ignore the surface terms and the total derivative terms vanish[4]. Since the other insertions are BRST invariant, the right hand side of eq.(3.20) is BRST invariant with large enough M.

The BRST invariant expression (3.20) can also be defined for noninteger M. The integrand on the right hand side of eq.(3.21) coincides with F_4 in eq.(3.13) and can be written as eq.(3.10). Therefore we analytically continue it in terms of M as in the previous subsection. We would like to show that the limit $M \to 0$ of the analytically continued amplitudes are finite and coincide with those in the first quantized formalism. To do so, we first consider eq.(3.21) for non-negative integer M and move the picture changing operators $X(z_I), \tilde{X}(\bar{z}_I)$

[3] In the large Hilbert space expression, the BRST variation of $\prod_A \left(c^A(z_0) \tilde{c}^A(\bar{z}_0) \right)$ does not vanish. However, the correlation functions including it vanish if the other insertions are in the "small Hilbert space".

[4] For $z_I \to z_J$, the factor $e^{-\frac{d-10}{16}\Gamma}$ which comes from the $X^{\pm}$ CFT dominates the divergent powers of $|z_I - z_J|$, if we take $-d$ to be large enough. Other surface terms are dealt with by analytically continuing the external momenta, as is usually done.

to the positions of external lines as was done in Ref. [17]. Using eq.(3.24), we obtain[5]

$$\begin{aligned}\mathcal{A}_4 \sim \int d^2\mathcal{T}\Bigg[&\left\langle \prod_A \left(c^A(z_0)\,\tilde{c}^A(\bar{z}_0)\right)\left|\oint_C \frac{dz}{2\pi i}\frac{b}{\partial\rho}(z)\right|^2 \prod_{r=1}^4\left[\mathcal{S}_r^{-1}V_r^{\left(p'_{L,r},p'_{R,r}\right)}\right]\right\rangle \\ &+\partial_{\mathcal{T}}\left\langle \prod_A \left(c^A(z_0)\,\tilde{c}^A(\bar{z}_0)\right)\oint_C \frac{d\bar{z}}{2\pi i}\frac{\tilde{b}}{\bar{\partial}\rho}(\bar{z})\tilde{X}(\bar{z}_1)\left|X(z_2)\right|^2 \right. \\ &\qquad\left. \times\prod_{r=1}^4\left[\mathcal{S}_r^{-1}V_r^{\left(p_{L,r},p_{R,r}\right)}\right]\right\rangle \\ &+\cdots\Bigg]\,. \end{aligned} \tag{3.25}$$

As a result of this manipulation, the pictures of the vertex operators applied by picture changing operators are shifted to be $\left(p'_{L,r}, p'_{R,r}\right)$, up to total derivative terms. The ellipses on the right hand side denote the other total derivative terms which appear in the course of the deformation of the contours of the BRST currents. For M large enough, the total derivative terms do not contribute to the integral and the right hand side of eq.(3.21) is equal to

$$\int d^2\mathcal{T}\left\langle \prod_A \left(c^A(z_0)\,\tilde{c}^A(\bar{z}_0)\right)\left|\oint_C \frac{dz}{2\pi i}\frac{b}{\partial\rho}(z)\right|^2 \prod_{r=1}^4\left[\mathcal{S}_r^{-1}V_r^{\left(p'_{L,r},p'_{R,r}\right)}\right]\right\rangle\,. \tag{3.26}$$

In this form, the amplitudes are finite and coincide with those of the first quantized formalism when $M = 0$. The problem is whether we can show eq.(3.25) and express $\mathcal{A}_4$ as eq.(3.26) even for noninteger M.

Amplitudes for noninteger M

In order to define eqs.(3.25) and (3.26) for noninteger M, we would like to show that all the correlation functions which appear in these equations are of the form

$$(\text{polynomial of } M) \times e^{MQ_4}\,, \tag{3.27}$$

as functions of M, where Q_4 is given in eq.(3.11) with $N = 4$. These correlation functions consist of the contributions from the $X^\pm$ CFT, the transverse part, the super-reparametrization ghost part and the superconformal field theory in section 2. M dependence comes either from the $X^\pm$ CFT or the superconformal field theory. With the external states (2.7), the

[5] Of course, such a manipulation is not valid if the external lines are off-shell. It is possible to define dimensionally regularized off-shell amplitudes using eq.(3.20), but they cannot be written in the form (3.26).

contributions from the latter are the combinatorial factors again and can be expressed as a polynomial of M. Therefore what we should show is that the contributions from the $X^{\pm}$ CFT are of the form (3.27).

The M dependence of the contributions from the $X^{\pm}$ CFT can be seen as follows. In these correlation functions, the variables $X^{\pm}, \psi^{\pm}, \tilde{\psi}^{\pm}$ appear in

- the picture changing operators $X, \tilde{X}$,
- $\mathcal{S}_r^{-1}$
- the vertex operators $V_r^{\left(p_{L,r}, p_{R,r}\right)}$.

$V_r^{\left(p_{L,r}, p_{R,r}\right)}$ involves a factor

$$e^{-ip_r^+X^- - i\left(p_r^- - \frac{\mathcal{N}}{p_r^+} + \frac{d-10}{16}\frac{1}{p_r^+}\right)X^+}\left(Z_r, \bar{Z}_r\right) = e^{-ip_r^+X^- - \frac{i}{2p_r^+}\left(\vec{p}_r^{\,2} - 1\right)X^+}\left(Z_r, \bar{Z}_r\right), \tag{3.28}$$

and the correlation functions are given by the path integral of the form

$$\begin{aligned}
&\int \left[dX^{\pm}d\psi^{\pm}d\tilde{\psi}^{\pm}\right] e^{-S_{X^{\pm}}} \prod_r e^{-ip_r^+X^-}\left(Z_r, \bar{Z}_r\right)\cdots \\
&= \int \left[dX^{\pm}d\psi^{\pm}d\tilde{\psi}^{\pm}\right] e^{\frac{1}{\pi}\int d^2z\left(\partial X^+\bar{\partial}X^- + \psi^+\bar{\partial}\psi^- + \tilde{\psi}^+\partial\tilde{\psi}^-\right)} \\
&\qquad\qquad \times e^{-\frac{d-10}{8}\Gamma_{\text{super}}\left[X^+, \psi^+, \tilde{\psi}^+\right] - \sum_r ip_r^+X^-\left(Z_r, \bar{Z}_r\right)}\cdots,
\end{aligned} \tag{3.29}$$

where the ellipses denote the other insertions, $S_{X^{\pm}}$ is the action of the $X^{\pm}$ CFT given in Ref. [8] and

$$\Gamma_{\text{super}}\left[X^+, \psi^+, \tilde{\psi}^+\right] \equiv -\frac{1}{2\pi}\int d^2\mathbf{z}\bar{D}\Phi D\Phi\,. \tag{3.30}$$

With the insertion $\prod_r e^{-\sum_r ip_r^+X^-}\left(Z_r, \bar{Z}_r\right)$, the variable X^+ possesses an expectation value. Dividing the variable X^+ and the functional Γ_{super} into their expectation values and fluctuations as

$$\begin{aligned}
X^+ &= -\frac{i}{2}\left(\rho(z) + \bar{\rho}(\bar{z})\right) + \delta X^+, \\
\Gamma_{\text{super}}\left[X^+, \psi^+, \tilde{\psi}^+\right] &= \langle\Gamma_{\text{super}}\rangle + \delta\Gamma_{\text{super}}, \\
\langle\Gamma_{\text{super}}\rangle &\equiv \Gamma_{\text{super}}\left[-\frac{i}{2}\left(\rho(z) + \bar{\rho}(\bar{z})\right), 0, 0\right],
\end{aligned} \tag{3.31}$$

eq.(3.29) becomes

$$\begin{aligned}
&\int \left[d\delta X^+dX^-d\psi^{\pm}d\tilde{\psi}^{\pm}\right] e^{\frac{1}{\pi}\int d^2z\left(\partial\delta X^+\bar{\partial}X^- + \psi^+\bar{\partial}\psi^- + \tilde{\psi}^+\partial\tilde{\psi}^-\right)} \\
&\qquad\qquad \times e^{-\frac{d-10}{8}\langle\Gamma_{\text{super}}\rangle}\sum_{n=0}^{\infty}\frac{\left(-\frac{d-10}{8}\delta\Gamma_{\text{super}}\right)^n}{n!}\cdots.
\end{aligned} \tag{3.32}$$

$\langle \Gamma_{\rm super} \rangle$ coincides with $\frac{1}{2}\Gamma$. We can calculate the correlation functions perturbatively with $d-10$ as the coupling constant, by contracting $(\delta\Gamma_{\rm super})^n$ and $\cdots$ by using the free propagators for $\delta X^+, X^-, \psi^\pm, \tilde{\psi}^\pm$.[6]

It is convenient to introduce an additive quantum number which we refer to as the $\pm$ number assigned to the fields as [9]

fields	δX^+	X^-	$\psi^\pm$	$\tilde{\psi}^\pm$	others
$\pm$number	$+1$	-1	± 1	± 1	0

In eq.(3.32) only the terms in $\sum_{n=0}^\infty \frac{\left(-\frac{d-10}{8}\delta\Gamma\right)^n}{n!}\cdots$ with vanishing $\pm$ number have nonvanishing contributions.

In the perturbative calculation, the contribution of each ingredient given above is summarized as

- $X, \tilde{X}$ contribute factors of $\partial\delta X^+, \bar{\partial}\delta X^+, \partial X^-, \bar{\partial}X^-, \psi^\pm, \tilde{\psi}^\pm$.

- $\mathcal{S}_r^{-1}$ has the expectation value $e^{\frac{d-10}{16}\frac{\tau_0^{(r)}}{p_r^+}}$ and the contribution is of the form

$$e^{\frac{d-10}{16}\frac{\tau_0^{(r)}}{p_r^+}}\left(1+\sum_{l=1}^{\infty}(d-10)^l\,\mathcal{O}_l\left(z_{I^{(r)}},\bar{z}_{I^{(r)}}\right)\right)\,, \tag{3.33}$$

  where $\mathcal{O}_l$ denotes a local operator made from derivatives of $\delta X^+, X^-, \psi^\pm, \tilde{\psi}^\pm$ with the $\pm$ number larger than or equal to l.

- $V_r^{\left(p_{L,r},p_{R,r}\right)}$ includes spin fields. They are defined by using the free field description given in Ref. [16]. They can be rewritten in terms of $\delta X^+, X^-, \psi^\pm, \tilde{\psi}^\pm$, but the expression is quite complicated. However, here we do not need the exact form of the operator. The $-\frac{3}{2}$ picture vertex operator involves a spin field of the form

$$e^{\sigma-\frac{3}{2}\phi}e^{\pm\frac{i}{2}H}\left(1+\sum_{l=1}^{\infty}(d-10)^l\,\mathcal{O}'_l\left(Z_r,\bar{Z}_r\right)\right)\,, \tag{3.34}$$

  where $i\partial H = \psi^-\psi^+$, which can be used for bosonization in the perturbative treatment. $\mathcal{O}'_l$ is a local operator made from derivatives of $\delta X^+, X^-, \psi^\pm, \tilde{\psi}^\pm$ with the $\pm$ number larger than or equal to l. The vertex operators in other pictures in the Ramond sector

[6] The correlation functions in the $X^\pm$ CFT was defined and calculated in Ref. [8]. Since the perturbative expansion here terminates at a finite order and one can obtain exact results, they coincide with those given there. It is also possible to interpret the definition in Ref. [8] directly in terms of the perturbative calculations.

are expressed by acting the picture changing operators on the $-\frac{3}{2}$ picture ones. The picture changing operators contribute factors of $\partial\delta X^+, \bar{\partial}\delta X^+, \partial X^-, \bar{\partial}X^-, \psi^\pm, \tilde{\psi}^\pm$, as stated above.

Altogether, the correlation functions are given as sums of the terms

$$
\begin{aligned}
& e^{-\frac{d-10}{8}\langle\Gamma_{\text{super}}\rangle+\sum_r \frac{d-10}{16}\frac{\tau_0^{(r)}}{p_r^+}} (d-10)^{n+l} \\
& \times \int \left[d\delta X^+ dX^- d\psi^\pm d\tilde{\psi}^\pm\right] e^{\frac{1}{\pi}\int d^2z\left(\partial\delta X^+\bar{\partial}X^- + \psi^+\bar{\partial}\psi^- + \tilde{\psi}^+\partial\tilde{\psi}^-\right)} \\
& \qquad \times \left(\delta\Gamma_{\text{super}}\right)^n \mathcal{O}_l \\
& \qquad \times \partial X^- \cdots \bar{\partial}X^- \cdots \psi^- \cdots \tilde{\psi}^- \cdots .
\end{aligned}
\tag{3.35}
$$

The factors $\partial X^- \cdots \bar{\partial}X^- \cdots \psi^- \cdots \tilde{\psi}^- \cdots$ come from the picture changing operators, and the number of these factors is less than that of the picture changing operators in the correlation functions. In our case, there are five picture changing operators including the one to express $V_2^{\left(-\frac{1}{2},-1\right)}(Z_2,\bar{Z}_2)$ as

$$
V_2^{\left(-\frac{1}{2},-1\right)}(Z_2,\bar{Z}_2) = XV_2^{\left(-\frac{3}{2},-1\right)}(Z_2,\bar{Z}_2) . \tag{3.36}
$$

Since only the terms with vanishing $\pm$ number can contribute, the perturbative expansion terminates at a finite order. Using

$$
d-10 = -\frac{8M}{k+2}, \qquad -\frac{d-10}{8}\langle\Gamma_{\text{super}}\rangle + \sum_{r=1}^{4}\frac{d-10}{16}\frac{\tau_0^{(r)}}{p_r^+} = MQ_4 , \tag{3.37}
$$

we can see that the correlation functions are of the form (3.27). It is easy to generalize the arguments here to more general correlation functions, such as those which appear in N point amplitudes.

Thus we have proved that all the correlation functions which appear on the right hand side of eq.(3.25) are expressed as eq.(3.27). Therefore we define them as analytic functions of M using this expression. We define eq.(3.26) for noninteger M in the same way.

The integrand on the right hand side of eq.(3.25) can be written in the form

$$
R_4(M)\, e^{MQ_4} + \partial_{\mathcal{T}}\left(S_4(M)\, e^{MQ_4}\right) + \partial_{\bar{\mathcal{T}}}\left(\bar{S}_4(M)\, e^{MQ_4}\right) , \tag{3.38}
$$

where $R_4(M), S_4(M), \bar{S}_4(M)$ are polynomials of M. Here we make only the M dependence explicit but the coefficients of these polynomials of course depend on $\mathcal{T}, \bar{\mathcal{T}}$ and other kinematic parameters. Since the integrand on the right hand side of eq.(3.20) is written as $P_4(M)\, e^{MQ_4}$, the relation

$$
P_4(M)\, e^{MQ_4} = R_4(M)\, e^{MQ_4} + \partial_{\mathcal{T}}\left(S_4(M)\, e^{MQ_4}\right) + \partial_{\bar{\mathcal{T}}}\left(\bar{S}_4(M)\, e^{MQ_4}\right) \tag{3.39}
$$

holds if M is a non-negative integer. It implies that

$$P_4(M) = R_4(M) + \partial_{\mathcal{T}} S_4(M) + S_4(M) M \partial_{\mathcal{T}} Q_4 + \partial_{\bar{\mathcal{T}}} \bar{S}_4(M) + \bar{S}_4(M) M \partial_{\bar{\mathcal{T}}} Q_4 \,. \quad (3.40)$$

Since $P_4(M), R_4(M), S_4(M), \bar{S}_4(M)$ are all polynomials, eq.(3.40) can be written as

$$\sum_{k=0}^{n} c_k M^k = 0\,. \quad (3.41)$$

If such an identity holds for any non-negative integer M, then $c_k = 0$ and thus it holds for any complex number M. Therefore eq.(3.25) holds for the analytically continued correlation functions.

Now that the right hand side of eq.(3.25) is defined as an analytic function of M, discarding the total derivative terms, we obtain

$$\mathcal{A}_4 \sim \int d^2\mathcal{T} \left\langle \prod_A \left(c^A(z_0)\, \tilde{c}^A(\bar{z}_0)\right) \left| \oint_C \frac{dz}{2\pi i} \frac{b}{\partial\rho}(z) \right|^2 \prod_{r=1}^{4} \left[\mathcal{S}_r^{-1} V_r^{(p'_{L,r}, p'_{R,r})} \right] \right\rangle , \quad (3.42)$$

as an analytic function of M. In the limit $M \to 0$, $\mathcal{A}_4$ coincides with the result of the first quantized formalism.

General case

It is straightforward to generalize the above arguments to the case of N point amplitudes. The conformal gauge expression for the N point amplitudes (3.1) with eq.(3.2) becomes

$$\begin{aligned}\mathcal{A}_N \;\sim\; & \int \prod_{\mathcal{I}=1}^{N-3} d^2\mathcal{T}_{\mathcal{I}} \left\langle \prod_A \left(c^A(z_0)\, \tilde{c}^A(\bar{z}_0)\right) \right. \\ & \left. \times \prod_{\mathcal{I}=1}^{N-3} \left| \oint_{C_{\mathcal{I}}} \frac{dz}{2\pi i} \frac{b}{\partial\rho} \right|^2 \prod_{I=1}^{N-2} |X(z_I)|^2 \prod_{r=1}^{N} \left[\mathcal{S}_r^{-1} V_r^{(p_{L,r}, p_{R,r})} \right] \right\rangle, \end{aligned} \quad (3.43)$$

where the integration contour $C_{\mathcal{I}}$ is taken to go around the $\mathcal{I}$-th internal propagator. The BRST invariance of this expression can be shown by using

$$\left\{ Q_{\mathrm{B}}, \oint_{C_{\mathcal{I}}} \frac{dz}{2\pi i} \frac{b}{\partial\rho} \right\} \sim -\partial_{\mathcal{T}_{\mathcal{I}}}\,. \quad (3.44)$$

One can show that the right hand side of eq.(3.43) is equal to

$$\int \prod_{\mathcal{I}=1}^{N-3} d^2\mathcal{T}_{\mathcal{I}} \left\langle \prod_A \left(c^A(z_0)\, \tilde{c}^A(\bar{z}_0)\right) \prod_{\mathcal{I}=1}^{N-3} \left| \oint_{C_{\mathcal{I}}} \frac{dz}{2\pi i} \frac{b}{\partial\rho} \right|^2 \prod_{r=1}^{N} \left[\mathcal{S}_r^{-1} V_r^{(p'_{L,r}, p'_{R,r})} \right] \right\rangle , \quad (3.45)$$

up to total derivatives with respect to $\mathcal{T}_{\mathcal{I}}, \bar{\mathcal{T}}_{\mathcal{I}}$, by moving the picture changing operators $X(z_I), \tilde{X}(\bar{z}_I)$ to the positions of external lines. Since eq.(3.45) coincides with the results of the first quantized formalism for $M=0$, what we should show is that $\mathcal{A}_N$ is equal to (3.45) even for noninteger M.

The correlation functions which appear in eqs.(3.43) and (3.45) are respectively expressed as $P_N(M)\, e^{MQ_N}$ and $R_N(M)\, e^{MQ_N}$ using polynomials $P_N(M)$ and $R_N(M)$, as we have mentioned above. We define them as analytic functions of M using these expressions. For M non-negative integer, by moving the picture changing operators, we can see that

$$(P_N(M) - R_N(M))\, e^{MQ_N} \tag{3.46}$$

is expressed as derivatives of correlation functions with respect to the moduli parameters. Those correlation functions are also written in the form

$$(\text{polynomial of } M) \times e^{MQ_N} \,. \tag{3.47}$$

Thus, when M is a non-negative integer, there exist the polynomials $S_N^{\mathcal{I}}(M), \bar{S}_N^{\mathcal{I}}(M)$ of M such that

$$(P_N(M) - R_N(M))\, e^{MQ_N} = \sum_{\mathcal{I}=1}^{N-3} \left[\partial_{\mathcal{T}_{\mathcal{I}}} \left(S_N^{\mathcal{I}}(M)\, e^{MQ_N} \right) + \partial_{\bar{\mathcal{T}}_{\mathcal{I}}} \left(\bar{S}_N^{\mathcal{I}}(M)\, e^{MQ_N} \right) \right] \,. \tag{3.48}$$

Eq.(3.48) implies that a polynomial equation

$$P_N(M) - R_N(M) = \sum_{\mathcal{I}=1}^{N-3} \left[\partial_{\mathcal{T}_{\mathcal{I}}} S_N^{\mathcal{I}} + S_N^{\mathcal{I}} M \partial_{\mathcal{T}_{\mathcal{I}}} Q_N + \partial_{\bar{\mathcal{T}}_{\mathcal{I}}} \bar{S}_N^{\mathcal{I}} + \bar{S}_N^{\mathcal{I}} M \partial_{\bar{\mathcal{T}}_{\mathcal{I}}} Q_N \right] \tag{3.49}$$

is satisfied for all non-negative integers M, and we can conclude that it holds for any complex number M. Therefore the analytically continued amplitudes are given as eq.(3.45). Thus the dimensionally regularized amplitudes are finite and become equal to the first quantized results as $M \to 0$.

4 Conclusions and discussions

In this paper, we have shown that the previously proposed dimensional regularization scheme [6, 7, 8, 9] also works for the tree-level amplitudes involving the external lines in the Ramond sector in the light-cone gauge type II superstring field theories. In order to overcome the difficulty pointed out in Ref. [16], we have modified the worldsheet theory. We have shown that the tree-level amplitudes calculated using the modified worldsheet theory can be analytically continued and used to regularize the superstring field theories. The resultant amplitudes turn out to coincide with those of the first quantized formulation without introducing any contact terms as counterterms.

Therefore, with the regularization, the light-cone gauge closed superstring field theory with only cubic interaction terms correctly describes the tree-level amplitudes. One obvious thing to be done is to examine if our dimensional regularization scheme can be employed to regularize the ultraviolet divergences of the multi-loop amplitudes. One can argue that the dimensional regularization can be used to regularize the divergences of the multi-loop amplitudes as in the field theory for particles, although the way how the regularization works in string field theory is a little bit different. In string theory, ultraviolet divergences in some channel correspond to infrared divergences in another channel. Therefore what we should do is to regularize the infrared divergences. Infrared divergences come from the long tube-like worldsheet depicted in Fig. 3. If the worldsheet theory is taken to be a CFT with central charge c, the contribution of a worldsheet which includes such a cylinder behaves as

$$\sim \exp\left(T\left(\frac{c}{12}-2\triangle_{min}\right)\right) , \tag{4.1}$$

for $T \gg 1$. Here $\triangle_{min}$ denotes the conformal dimension of the primary field with the lowest dimension in the worldsheet CFT. Therefore, if one can use a CFT with $\frac{c}{12}-2\triangle_{min}$ largely negative in the dimensional regularization, we expect that the divergences are regularized. In the case of the superconformal field theory proposed in section 2,

$$\begin{aligned} \frac{c}{12}-2\triangle_{min} &= \frac{c}{12} \\ &= 1-\frac{M}{k+2} , \end{aligned} \tag{4.2}$$

and one can make the amplitudes finite by taking M large enough. Besides such a stringy aspect, our dimensional regularization scheme has a lot in common with the dimensional reduction scheme employed in the field theory for particles. Therefore the regularization scheme we propose in this paper will be useful to regularize the ultraviolet divergences of the multi-loop amplitudes. It will be intriguing to compare the amplitudes defined by using our regularization with those given in Ref. [18].

One important problem to be addressed about the multi-loop calculations is the definition of $(-1)^F$. In the field theory for particles, the dimensional regularization has problems in treating γ^5. In order to keep $\mathrm{tr}\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ nonvanishing for general d, the definition of γ^5 should be taken so that it does not anticommute with γ^μ [19, 20, 21]. In superstring theory,

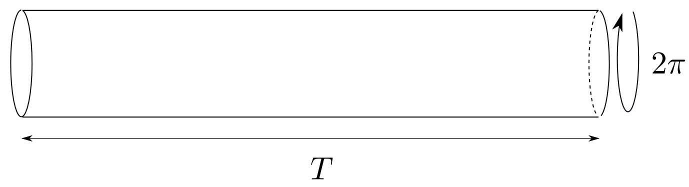

Figure 3: Tube

this problem is related to the definition of $(-1)^F$ which appears in the definition of the GSO projection operator for the Ramond sector. $(-1)^F$ should anticommute with ψ^μ for the BRST invariance and it seems that we will have trouble in defining it for general d. Indeed, the superconformal field theory defined in section 2 is not well-defined for odd spin structure. The partition function of the $\beta\gamma$ system is the inverse of that of the ψ system and therefore it is divergent. The odd spin structure arises when we insert $(-1)^F$ and the problem is related to the definition of this operator. One way to deal with the divergence is to modify the definition of $(-1)^F$. Doing so breaks the gauge invariance but it may be possible to prove that the procedure does not cause any problems if the theory is not anomalous.

Whatever the outcome of the multi-loop calculations will be, the cubic light-cone gauge action is valid classically. Such a formulation may be useful in exploring classical solutions of closed superstring field theories. For many reasons, it would be better to have a gauge invariant version of such a theory. Since we have written down the conformal gauge expression of the amplitudes, we can infer what the gauge invariant version would be. For the bosonic string theory, as we have pointed out in Ref. [7], it should be the $\alpha = p^+$ HIKKO theory [22]. Construction of such a theory for superstrings will be a problem to be pursued.

We would like to come back to these problems elsewhere.

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A Action and amplitudes for $d \neq 10$

In this appendix, we present the action of the light-cone gauge superstring field theory for $d \neq 10$, and calculate the tree-level amplitudes perturbatively by using this action.

The string fields are taken to be GSO even and satisfy the level-matching condition. Those in the bosonic sector, namely the (NS,NS) or the (R,R) sector, are Grassmann even, whereas those in the fermionic sector, namely the (R,NS) or the (NS,R) sector, are Grassmann odd.

A.1 The kinetic term and the propagator

The kinetic terms and the propagators for the string fields in the bosonic sector are given in Refs. [9, 16].[7] For the string fields in the fermionic sector, the kinetic term is given as

$$\frac{1}{2}\int dt \int d1d2 \left\langle R\left(1,2\right)\right| \Phi(t)\rangle_1 \left(i\frac{\partial}{\partial t} - \frac{L_0^{\mathrm{LC}(2)} + \tilde{L}_0^{\mathrm{LC}(2)} - \frac{d-2}{8}}{\alpha_2} \right) |\Phi(t)\rangle_2 \ . \tag{A.1}$$

Here $t = x^+$, dr and $\alpha_r = 2p_r^+$ are the zero-mode measure and the string-length parameter for the r-th string respectively, and $\langle R(1,2)|$ is the reflector. $L_0^{\mathrm{LC}(r)}$ denotes the Virasoro zero mode of the light-cone gauge worldsheet theory. While the kinetic term (A.1) is in the same form as that for the strings in the bosonic sector, this time we should differently define the zero-mode measure and the reflector. The zero-mode measure is defined as

$$dr = \frac{d\alpha_r}{4\pi}\frac{d^{d-2}p_r}{(2\pi)^{d-2}} = \frac{dp_r^+}{2\pi}\frac{d^{d-2}p_r}{(2\pi)^{d-2}} \ . \tag{A.2}$$

Compared with dr for the bosonic sector, a factor of α_r is absent in this case. The reflector is defined as

$$\langle R\left(1,2\right)| = \delta\left(1,2\right)\sum_n {}_2\langle n|_1\langle n| \ , \tag{A.3}$$

where

$$\delta\left(1,2\right) = 4\pi\delta\left(\alpha_1+\alpha_2\right)\left(2\pi\right)^{d-2}\delta^{d-2}\left(p_1+p_2\right) \ , \tag{A.4}$$

and we have introduced a basis $\{|n\rangle\}$ of the projected Fock space for the non-zero modes and the fermionic zero modes. $|n\rangle$ is Grassmann even and normalized as

$$\langle n|n'\rangle = \delta_{n,n'} \ . \tag{A.5}$$

[7] Although the definitions in those references are given for the naive dimensional regularization, it is straightforward to generalize them to any worldsheet theory as will be mentioned at the end of this subsection.

As in the bosonic case, by using the basis $\{|n\rangle\}$, the string field $|\Phi\rangle$ can be expanded as

$$|\Phi(t)\rangle = \sum_n \psi_n\left(t,\alpha,\vec{p}\right)|n\rangle\,, \tag{A.6}$$

and

$$\left(L_0^{\mathrm{LC}} + \tilde{L}_0^{\mathrm{LC}} - \frac{d-2}{8}\right)|\Phi(t)\rangle = \sum_n \left(\vec{p}^{\,2} + m_n^2\right)\psi_n\left(t,\alpha,\vec{p}\right)|n\rangle\,. \tag{A.7}$$

This time, ψ_n is Grassmann odd. In terms of ψ_n, the kinetic term (A.1) is written as

$$\frac{1}{2}\sum_n \int \frac{d^dp}{(2\pi)^d}\tilde{\psi}_n\left(-p\right)\left(p^- - \frac{\vec{p}^2 + m_n^2}{2p^+}\right)\tilde{\psi}_n\left(p\right)\ , \tag{A.8}$$

where

$$\tilde{\psi}_n(p) \equiv \int dt\, e^{ip^-t}\psi_n(t,\alpha,\vec{p})\ . \tag{A.9}$$

In order for the kinetic term (A.8) to be consistent with the fact that ψ_n is Grassmann odd, we have chosen dr different from that for the bosonic case. With the kinetic term thus defined the propagator becomes

$$\begin{aligned}
\tilde{\psi}_n\left(p\right)\tilde{\psi}_{n'}\left(p'\right) &= \alpha\delta_{n,n'}\left(2\pi\right)^d\delta^d\left(p+p'\right)\frac{-i}{p^2+m_n^2}\ ,\\
\left|\tilde{\Phi}\left(p_1^-\right)\right\rangle_1\left|\tilde{\Phi}\left(p_2^-\right)\right\rangle_2 &= \frac{1}{\alpha_1}\left(2\pi\right)^d\delta^d\left(p_1+p_2\right)\\
&\quad\times\int\frac{d^2\mathcal{T}}{4\pi}\,e^{-\frac{\mathcal{T}}{|\alpha_1|}\left(L_0^{\mathrm{LC}(1)}-\frac{d-2}{16}\right)-\frac{\bar{\mathcal{T}}}{|\alpha_1|}\left(\tilde{L}_0^{\mathrm{LC}(1)}-\frac{d-2}{16}\right)}\\
&\quad\times e^{\frac{\alpha_1}{|\alpha_1|}p_1^-T}\sum_n|n\rangle_1|n\rangle_2\ ,\\
\tilde{\psi}_n\left(p\right)\left|\tilde{\Phi}\left(p'\right)\right\rangle &= \alpha\left(2\pi\right)^d\delta^d\left(p+p'\right)\frac{-i}{p^2+m_n^2}|n\rangle\ .
\end{aligned} \tag{A.10}$$

Because of the different choice of dr, the propagators include an extra factor of α compared with those for the bosons.

The kinetic terms and the propagators can be defined for any worldsheet CFT. Here we would like to define the string field theory using the worldsheet theory defined in section 2. Since the ghost-like variables $b^A, c^A, \tilde{b}^A, \tilde{c}^A$ have zero modes, we should mention how to treat them in defining the string field theory. We take the string field $|\Phi\rangle$ to satisfy

$$b_0^A\,|\Phi\rangle = \tilde{b}_0^A\,|\Phi\rangle = 0\,. \tag{A.11}$$

Then the propagators come with the projection operator corresponding to the condition (A.11).

A.2 The three-string vertex

The three-string vertex for the bosonic fields is given as

$$\begin{aligned}
&\int dt \int d1d2d3 \left\langle V_3\left(1,2,3\right)|\Phi(t)\rangle_1 |\Phi(t)\rangle_2 |\Phi(t)\rangle_3 \right. \\
&= \int \prod_{r=1}^{3} \left(\frac{d^d p_r}{(2\pi)^d} \alpha_r \right) (2\pi)^d \delta^d \left(\sum_{r=1}^{3} p_r \right) e^{-\Gamma^{[3]}(1,2,3)} \\
&\qquad \times \left\langle V_3^{\mathrm{LPP}}\left(1,2,3\right)\right| P_{123} \left| \tilde{\Phi}\left(p_1^-\right) \right\rangle_1 \left| \tilde{\Phi}\left(p_2^-\right) \right\rangle_2 \left| \tilde{\Phi}\left(p_3^-\right) \right\rangle_3 .
\end{aligned} \tag{A.12}$$

Here $\left\langle V_3^{\mathrm{LPP}}\left(1,2,3\right)\right|$ denotes the LPP vertex [23, 24]. $\Gamma^{(3)}(1,2,3)$ and P_{123} are defined in eqs.(A.4) and (A.6) of Ref. [9]. With the vertex (A.12) and the propagator, one can calculate the amplitudes [9]. They are expressed in terms of the correlation functions of vertex operators in the worldsheet theory.

The propagators for the fermions come with extra factors of α. In order to make the theory Lorentz invariant in the critical dimension, we should define the three-string vertex so as to compensate for these extra factors of α. Thus the three-string vertex should be defined as

$$\begin{aligned}
&\int dt \int d1d2d3 \left\langle V_3\left(1,2,3\right)|\Phi(t)\rangle_1 |\Phi(t)\rangle_2 |\Phi(t)\rangle_3 \right. \\
&\propto \int \prod_{r=1}^{3} \left(\frac{d^d p_r}{(2\pi)^d} \alpha_r \right) (2\pi)^d \delta^d \left(\sum_{r=1}^{3} p_r \right) e^{-\Gamma^{[3]}(1,2,3)} \\
&\qquad \times \left\langle V_3^{\mathrm{LPP}}\left(1,2,3\right)\right| P_{123} \alpha_1^{-\frac{1}{2}} \left| \tilde{\Phi}\left(p_1^-\right) \right\rangle_1 \alpha_2^{-\frac{1}{2}} \left| \tilde{\Phi}\left(p_2^-\right) \right\rangle_2 \left| \tilde{\Phi}\left(p_3^-\right) \right\rangle_3 ,
\end{aligned} \tag{A.13}$$

for $|\Phi\rangle_1, |\Phi\rangle_2$ fermionic and $|\Phi\rangle_3$ bosonic. Here the problem is how to define the phase of $\alpha^{-\frac{1}{2}}$. Unless we define the phase properly, the contributions to the amplitudes from various channels do not connect smoothly and the theory will not become Lorentz invariant when $d = 10$. As far as we know, this problem has never been discussed in the literature and we would like to discuss it in the following.

Phase of $\alpha^{-\frac{1}{2}}$

Suppose we perturbatively calculate the amplitude obtained by amputating the external legs of the following correlation function in the string field theory:

$$\left\langle\!\left\langle \tilde{\psi}_{n_1}(p_1)\, \tilde{\psi}_{n_2}(p_2) \cdots \tilde{\psi}_{n_{2F}}(p_{2F})\, \tilde{\phi}_{n_{2F+1}}(p_{2F+1}) \cdots \tilde{\phi}_{n_N}(p_N) \right\rangle\!\right\rangle . \tag{A.14}$$

Here $\langle\langle \cdots \rangle\rangle$ denotes the expectation value in the string field theory. $\tilde{\phi}_n$ are the bosonic modes of the string field given in Ref. [16], whereas $\tilde{\psi}_n$ are the fermionic ones in eq.(A.7). The correlation function (A.14) can be evaluated by using the propagators and the three-string vertices (A.12) and (A.13). The amplitude this yields can be described as a sum of integrals of the correlation function

$$\left\langle \prod_I \left(T_F(z_I) \tilde{T}_F(\bar{z}_I) \right) \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle_{\mathrm{LC}} \tag{A.15}$$

for the worldsheet theory on the z-plane, over the Schwinger parameters. The vertex operator V_r^{LC} corresponds to the r-th external line and is a local operator at $z = Z_r$. Due to the spin fields involved in V_r^{LC} and the supercurrent insertions at the interaction points z_I, this correlation function can be defined only up to sign. Those spin fields are expressed in the form $e^{iaH}(z)$, where $H(z)$ is a chiral free boson and a is a half integer. Suppose we always define $e^{iaH}(z)$ by

$$e^{iaH}(z) \equiv \lim_{\Lambda \to \infty} \sum_{n=0}^{\infty} \frac{1}{n!} : \left(ia \int_{-\Lambda}^{0} dx \partial H(z+x) \right)^n : , \tag{A.16}$$

where the integration contour for x is taken along the real axis. Then the correlation function (A.15) is defined on the Riemann surface with cuts depicted in Fig. 4.

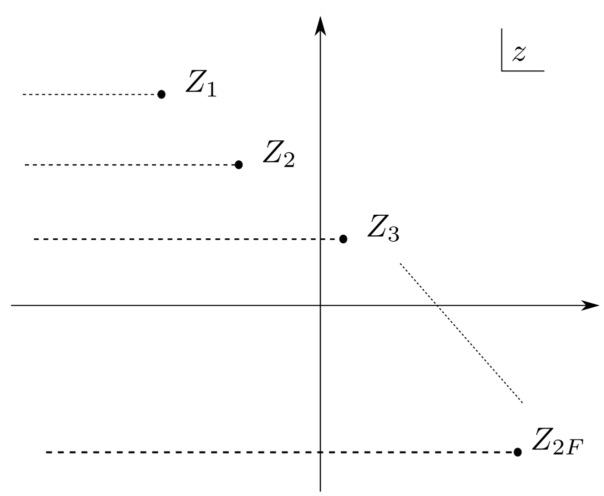

Figure 4: Cuts are parallel to the real axis and indicated by the dotted lines.

If one varies Z_r continuously, the correlation function changes its sign when some of z_I cross the cuts. In order to remedy this, we introduce

$$f(\alpha_r; Z_r) \equiv \prod_{n=1}^{F} \alpha_{2n} \prod_I \prod_{n=1}^{F} (z_I - Z_{2n-1})^{\frac{1}{2}} \prod_I \prod_{n=1}^{F} (z_I - Z_{2n})^{-\frac{1}{2}}$$

$$\times \prod_{r=2F+1}^{N} \prod_{n=1}^{F} (Z_r - Z_{2n-1})^{-\frac{1}{2}} \prod_{r=2F+1}^{N} \prod_{n=1}^{F} (Z_r - Z_{2n})^{\frac{1}{2}}$$
$$\times \prod_{F \geq n > n' \geq 1} (Z_{2n-1} - Z_{2n'-1})^{-1} \prod_{F \geq n > n' \geq 1} (Z_{2n} - Z_{2n'}) \ , \qquad \text{(A.17)}$$

which coincides with

$$\prod_{r=1}^{2F} |\alpha_r|^{\frac{1}{2}} \ , \qquad \text{(A.18)}$$

up to phase. Here we take the square-roots to be also defined with the cuts depicted in Fig. 4. Then the value of the combination

$$f(\alpha_r; Z_r) \left\langle \prod_I \left(T_F(z_I) \tilde{T}_F(\bar{z}_I) \right) \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle_{\mathrm{LC}} \qquad \text{(A.19)}$$

does not jump if we vary Z_r continuously. This combination can be proved to be invariant under $SL(2,\mathbb{C})$ up to sign. Since it is not discontinuous, it is $SL(2,\mathbb{C})$ invariant. It is easy to see that f becomes $-f$ under the exchange $(\alpha_{2n-1}, Z_{2n-1}) \leftrightarrow (\alpha_{2n'-1} Z_{2n'-1})$ or $(\alpha_{2n}, Z_{2n}) \leftrightarrow (\alpha_{2n'}, Z_{2n'})$ $(n \neq n')$. Since

$$-\frac{\alpha_{2n-1}}{\alpha_{2n'}} = \frac{\prod_I (z_I - Z_{2n-1}) \prod_{r \neq 2n-1, 2n'} (Z_r - Z_{2n'})}{\prod_I (z_I - Z_{2n'}) \prod_{r \neq 2n-1, 2n'} (Z_r - Z_{2n-1})} \ , \qquad \text{(A.20)}$$

one can prove that f becomes $-f$ under the exchange $(\alpha_{2n-1}, Z_{2n-1}) \leftrightarrow (\alpha_{2n'}, Z_{2n'})$. Therefore the combination (A.19) transforms in the same way as the correlation function (A.14) under the permutation $r \leftrightarrow s$. Namely, the combination (A.19) has the right properties to be used to express the amplitude obtained from the correlation function (A.14).

We would like to arrange the three-string vertex so that the worldsheet correlation functions always appear in the form (A.19). This can be achieved by defining the three-string vertex (A.13) which involves fermions as

$$\int dt \int d1d2d3 \left\langle V_3(1,2,3) \right| \Phi(t) \rangle_1 \left| \Phi(t) \right\rangle_2 \left| \Phi(t) \right\rangle_3$$
$$= \int \prod_{r=1}^{3} \left(\frac{d^d p_r}{(2\pi)^d} \right) (2\pi)^d \delta^d \left(\sum_{r=1}^{3} p_r \right) e^{-\Gamma^{[3]}(1,2,3)} \alpha_3 f(\alpha_1, \alpha_2, \alpha_3; Z_1, Z_2, Z_3)$$
$$\times \left\langle V_3^{\mathrm{LPP}}(1,2,3) \right| P_{123} \left| \tilde{\Phi}(p_1^-) \right\rangle_1 \left| \tilde{\Phi}(p_2^-) \right\rangle_2 \left| \tilde{\Phi}(p_3^-) \right\rangle_3 \ , \qquad \text{(A.21)}$$

where $\left\langle V_3^{\mathrm{LPP}}(1,2,3) \right| P_{123}$ is defined using the correlation functions of the vertex operators at $z = Z_r$ $(r = 1,2,3)$ and with cuts in Fig. 4. Then it is straightforward to show that amplitudes can be given by the integral of eq.(A.19) by checking the factorization properties. Since the combination (A.19) is specified by $Z_r, \bar{Z}_r$ and information on the external lines, we can see that contributions from various channels to the amplitudes are smoothly connected. Thus a tree-level amplitude can be given as an integral over the moduli space.

LPP vertex V_3^{LPP}

In order to define V_3^{LPP} for the worldsheet theory in section 2, we need to specify the way to treat the zero modes of the ghost-like variables. We define the LPP vertex so that

$$\left\langle V_3^{\mathrm{LPP}}(1,2,3)\right| \mathcal{O}\left|0\right\rangle_1 \left|0\right\rangle_2 \left|0\right\rangle_3 = \left\langle \prod_A \left(c^A(z_0)\,\tilde{c}^A(\bar{z}_0)\right)\mathcal{O}\right\rangle_{\mathrm{LC}}\,, \tag{A.22}$$

for $\mathcal{O}$ which depends on $c^A, \tilde{c}^A$ only through their derivatives. Since the string field $|\Phi\rangle$ satisfies the condition (A.11), eq.(A.22) for such $\mathcal{O}$ is enough to define the string field action. With the condition (A.11) and the vertex (A.22), the correlation functions on the worldsheet are with insertions of $\prod_A \left(c^A(z_0)\,\tilde{c}^A(\bar{z}_0)\right)$ and $\prod_{A,C}\left(\oint_C \frac{dz}{2\pi i} b^A(z) \oint_C \frac{d\bar{z}}{2\pi i}\tilde{b}^A(\bar{z})\right)$, which soak up the zero modes. The contours C are taken to be the ones depicted in Fig. 5 on light-cone diagrams for multi-loop amplitudes. The insertion of $\prod_A \left(c^A(z_0)\,\tilde{c}^A(\bar{z}_0)\right)$ soaks up the zero modes of $c^A, \tilde{c}^A$ with weight $(0,0)$ and it works as the insertion of ξ in the bosonized superghost system. The condition that the operator $\mathcal{O}$ depends on $c^A, \tilde{c}^A$ only through their derivatives means that it is in the "small Hilbert space", in which the modes c_0^A, $\tilde{c}_0^A$ are absent. Therefore the right hand side of eq.(A.22) can be expressed as

$$\langle \mathcal{O}\rangle_{\text{small Hilbert space}}\,. \tag{A.23}$$

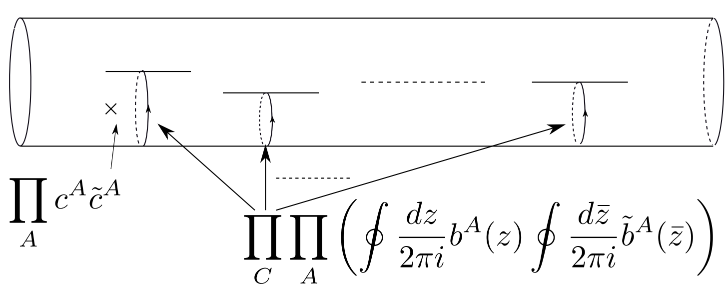

Figure 5: Insertions of $b^A, c^A, \tilde{b}^A, \tilde{c}^A$

The Fock vacua of $\beta^A, \gamma^A, \tilde{\beta}^A, \tilde{\gamma}^A$ are defined so that the positive frequency modes annihilate them. There exist zero modes on the worldsheet when the spin structure is odd and the correlation functions become infinite. This will be a problem in calculating multi-loop amplitudes as is mentioned in section 4.

Putting all the above results together, we obtain the tree-level N-string amplitudes $\mathcal{A}_N$ in eq.(3.1).

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