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Research article

Remarks on radial symmetry and monotonicity for solutions of semilinear higher order elliptic equations†

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Abstract: Half a century after the appearance of the celebrated paper by Serrin about overdetermined boundary value problems in potential theory and related symmetry properties, we reconsider semilinear polyharmonic equations under Dirichlet boundary conditions in the unit ball of $\mathbb{R}^n$. We discuss radial properties (symmetry and monotonicity) of positive solutions of such equations and we show that, in *conformal dimensions*, the associated Green function satisfies elegant reflection and symmetry properties related to a suitable Kelvin transform (inversion about a sphere). This yields an alternative formula for computing the partial derivatives of solutions of polyharmonic problems. Moreover, it gives some hints on how to modify a counterexample by Sweers where radial monotonicity fails: we numerically recover strict radial monotonicity for the biharmonic equation in the unit ball of $\mathbb{R}^4$.

Keywords: polyharmonic operators; Green function; radial symmetry; conformal dimensions

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$, possibly $n = 1971$) be a bounded smooth domain and consider the problem

$$\begin{cases} -\Delta u = 1 \quad \text{in} \quad \Omega\,, \\ u = 0\,, \qquad \dfrac{\partial u}{\partial \nu} = c \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.1}$$

where $c \in \mathbb{R}$ is a constant and ν denotes the outward unit normal to $\partial\Omega$. Imposing both Dirichlet and Neumann conditions on $\partial\Omega$ makes the problem overdetermined so that, in general, (1.1) has no solution. In a celebrated paper published in 1971, Serrin [15] proved that if (1.1) admits a smooth solution, then Ω must necessarily be a ball. This paper has raised a great interest and, nowadays (half a century later), it has reached almost 600 citations in the Mathscinet. Serrin's original proof combines analytic arguments, such as the Maximum Principle and a refinement of Hopf's boundary Lemma, with geometric techniques such as the moving plane method inspired to the Alexandrov characterization of spheres [1, 2]. Starting from [18], several different approaches have been devised as an alternative to Serrin's original proof, see [14] and the references therein for a fairly complete survey.

The moving plane method has been fruitfully used in symmetry results for semilinear elliptic equations, see [4, 10] for second order equations and [3] for higher order problems. Let $\mathbf{B} \subset \mathbb{R}^n$ be the unit ball and consider the semilinear polyharmonic problem under Dirichlet boundary conditions:

$$\begin{cases} (-\Delta)^m u = f(u) & \text{in } \mathbf{B}\,, \\ u = \dfrac{\partial u}{\partial r} = \cdots = \dfrac{\partial^{m-1} u}{\partial r^{m-1}} = 0 & \text{on } \partial\mathbf{B}\,. \end{cases} \tag{1.2}$$

Here $r = |x|$ denotes the radial variable and, hence, the outward normal direction to $\partial\mathbf{B}$. The following result, valid for second order equations with $m = 1$, is a restatement of [10, Theorem 1], combined with deep remarks by Spruck [10, Remark 1].

Theorem 1.1. ([4, 10]) *Let $n \geq 2$ and $m = 1$. Assume that*

$$\textit{either} \quad f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}) \quad \textit{or} \quad f \in C(\mathbb{R}_+; \mathbb{R}) \textit{ is nondecreasing.} \tag{1.3}$$

Then, every positive strong solution $u \in W^{2,n}_{\mathrm{loc}}(\mathbf{B}) \cap C(\overline{\mathbf{B}})$ ($u > 0$ in $\mathbf{B}$) of (1.2) *is radially symmetric ($u = u(r)$) and $u'(r) < 0$ for all $r \in (0, 1)$. Moreover, if*

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}) \quad \textit{and} \quad f(0) \geq 0\,, \tag{1.4}$$

then every nonnegative and non-trivial strong solution $u \in W^{2,n}_{\mathrm{loc}}(\mathbf{B}) \cap C(\overline{\mathbf{B}})$ ($u \geq 0$, $u \not\equiv 0$ in $\mathbf{B}$) of (1.2) *is radially symmetric ($u = u(r)$) and $u'(r) < 0$ for all $r \in (0, 1)$.*

This result was extended in [3, Theorem 1] to the case $m \geq 2$, in the following form:

Theorem 1.2. ([3]) *Let $n \geq 2$ and $m \geq 1$. Assume that*

$$f \in C(\mathbb{R}_+; \mathbb{R}) \textit{ is nondecreasing with } f(0) \geq 0\,. \tag{1.5}$$

Then, every nonnegative and non-trivial strong solution $u \in H^m_0(\mathbf{B}) \cap L^\infty(\mathbf{B})$ ($u \geq 0$, $u \not\equiv 0$ in $\mathbf{B}$) of (1.2) *is radially symmetric ($u = u(r)$) and $u'(r) < 0$ for all $r \in (0, 1)$.*

In its original version, Theorem 1.2 was stated by requiring that the solution u was strictly positive, $u > 0$ in $\mathbf{B}$. But this is not necessary since [9, Theorem 5.1] ensures that if $u \geq 0$ is nontrivial, then $u > 0$ (recall that $f(u) \geq 0$ in view of (1.5)). Assumption (1.5) is stronger than (1.3) and one may wonder whether Theorem 1.2 also holds under weaker assumptions. A nice example by Sweers [16] shows that, for a smooth and decreasing f, positive radial solutions may not be radially decreasing. Therefore, (1.3) *is not enough* to obtain the full statement of Theorem 1.1 (radial symmetry and monotonicity) in the higher order case $m \geq 2$. This discrepancy of assumptions between the cases $m = 1$ and $m \geq 2$ is due to the lack of a maximum principle for polyharmonic operators; see [9, Section 1.2].

To find conditions, different from (1.5), ensuring both radial symmetry and radial monotonicity for (1.2) in the case $m \geq 2$ is a challenging problem. The first purpose of this paper is precisely to discuss some conditions on the source f which ensure the radial symmetry of the solutions of (1.2) and their radial monotonicity. In Section 2, particular attention is devoted to the regularity and monotonicity of f: for the biharmonic equation in $\mathbf{B} \subset \mathbb{R}^n$, in Theorem 2.3 we exhibit a Hölder-continuous function f for which the radial symmetry of solutions fails, while Proposition 2.4 displays a strictly decreasing and sign-changing f that still ensures the existence of a radially symmetric and strictly decreasing solution.

Contrary to the case of Navier boundary conditions, considered in [17], (1.2) cannot be reduced to a second order system when $m \geq 2$. This is why the moving plane procedure used in the proof of Theorem 1.2 was carried on by using fine estimates of the Green function G associated to the polyharmonic operator $(-\Delta)^m$, see [3, Section 2]. The second purpose of the present paper is to obtain new properties of the Green function in the so-called *conformal dimensions* $n = 2m$. In Section 3 we prove that some estimates become explicit identities in conformal dimensions, see formula (3.6). Combined with the inversion in the sphere and the Kelvin transform, this enables us to obtain an elegant symmetry property of the Green function, see Theorem 3.2 and the sketchy representation in Figures 3 and 4. In turn, this result is used in Corollary 3.5 where we present an alternative formula for the computation of the partial derivatives of solutions of (1.2): since the sign of these partial derivatives is the fundamental feature to implement the moving plane procedure, this formula can become useful under suitable assumption on the source f. Indeed, based on this formula, Proposition 3.7 suggests that the radial symmetry and monotonicity of the solutions of (1.2) is ensured if $\|f'\|_{L^\infty(\mathbb{R}_+)}$ is small.

Finally, as an application of Proposition 3.7, in Section 4 we revisit the counterexample by Sweers [16] where radial monotonicity of the solution fails: by appropriately modifying the source f (enforcing the conditions given in Proposition 3.7) we numerically obtain radially symmetric and strictly decreasing solutions of the biharmonic equation in the unit ball of $\mathbb{R}^4$ for a decreasing function f with sufficiently small derivative. Throughout this work, some open problems and questions are posed.

2. Assumptions ensuring radial symmetry and monotonicity

In this section we discuss radial properties (symmetry and monotonicity) of solutions of (1.2). Let us recall the weak formulation of (1.2) within the Sobolev space $H^m_0(\mathbf{B})$, which is a Hilbert space if endowed with the scalar product

$$(u, v)_m = \begin{cases} \displaystyle\int_{\mathbf{B}} (\Delta^{m/2} u)(\Delta^{m/2} v) & \text{if } m \text{ is even} \\ \displaystyle\int_{\mathbf{B}} (\nabla \Delta^{(m-1)/2} u) \cdot (\nabla \Delta^{(m-1)/2} v) & \text{if } m \text{ is odd} \end{cases} \qquad \forall u, v \in H^m_0(\mathbf{B}) .$$

We denote the induced norm by $\| \cdot \|_m$; in particular, $\| \cdot \|_0$ is the $L^2(\mathbf{B})$-norm. A function $u \in H^m_0(\mathbf{B})$ is called a *weak solution* of (1.2) if $f(u) \in H^{-m}(\mathbf{B})$ (the dual space of $H^m_0(\mathbf{B})$) and (1.2) is satisfied in a weak sense, that is

$$(u,v)_m = \langle f(u), v\rangle \qquad \text{for all } v \in H^m_0(\mathbf{B}),$$

where $\langle\cdot,\cdot\rangle$ stands for the duality product between $H^{-m}(\mathbf{B})$ and $H^m_0(\mathbf{B})$. In the present article, however, we will mostly deal with slightly more regular solutions: when f is continuous and $u \in H^m_0(\mathbf{B}) \cap L^\infty(\mathbf{B})$, we say that u is a *strong solution* of (1.2) if

$$(u,v)_m = \int_{\mathbf{B}} f(u)v \qquad \text{for all } v \in H^m_0(\mathbf{B})\,; \tag{2.1}$$

the integral exists since $u \in L^\infty(\mathbf{B})$ and f is continuous. By elliptic regularity, any such strong solution u belongs to $C^{2m-1,\alpha}(\overline{\mathbf{B}})$, for some $\alpha \in (0,1)$, and all partial derivatives of order less than m vanish on $\partial\mathbf{B}$. Moreover, if f is Hölder continuous, then $u \in C^{2m,\alpha}(\overline{\mathbf{B}})$ is a classical solution, see [9]. In the sequel, we always take (at least) $f \in C(\mathbb{R};\mathbb{R})$.

Our first (elementary) result is a restatement of [3, Remark (iv)] and gives a different condition for radial symmetry (but not for radial monotonicity).

Theorem 2.1. *Let $n \geq 2$ and $m \geq 1$. Let λ_1 be the first Dirichlet eigenvalue for $(-\Delta)^m$ in $\mathbf{B}$. If*

$$f \in C(\mathbb{R};\mathbb{R}) \quad \textit{satisfies} \quad \frac{f(t)-f(s)}{t-s} < \lambda_1 \qquad \forall t > s\,, \tag{2.2}$$

then there exists at most one strong solution $u \in H^m_0(\mathbf{B}) \cap L^\infty(\mathbf{B})$ of problem (1.2) which is, moreover, radially symmetric.

Proof. By contradiction, assume that (1.2) admits two strong solutions $u_1, u_2 \in H^m_0(\mathbf{B}) \cap L^\infty(\mathbf{B})$. Then by (2.1) we have both that

$$(u_1,v)_m = \int_{\mathbf{B}} f(u_1)v \qquad \text{and} \qquad (u_2,v)_m = \int_{\mathbf{B}} f(u_2)v \qquad \text{for all } v \in H^m_0(\mathbf{B})\,.$$

Set $w \doteq u_1 - u_2$, subtract these two equations and choose $v = w$ as a test function to obtain

$$(w,w)_m = \int_{\mathbf{B}} \big(f(u_1) - f(u_2)\big)w\,. \tag{2.3}$$

By (2.2) we know that

$$\big(f(u_1(x)) - f(u_2(x))\big)w(x) \leq \lambda_1 w(x)^2 \qquad \forall x \in \mathbf{B}\,,$$

with strict inequality in a set of positive measure since $w \not\equiv 0$ (recall $u_2 \not\equiv u_1$). Therefore, (2.3) yields

$$\|w\|_m^2 < \lambda_1 \|w\|_0^2\,,$$

which contradicts the Poincaré inequality and proves uniqueness.

Once uniqueness is established, it suffices to remark that if u is a nonradial solution of (1.2), then for any given nontrivial rotation $A \in SO(n)$, also the function $u_A \doteq u \circ A$ is a (different) strong solution of (1.2), which contradicts the just proved uniqueness statement.

We have so shown that, if (2.2) holds, then there exists at most a unique solution of (1.2) which is necessarily radially symmetric. This proves the theorem. □

Theorem 2.1 states that one cannot expect radial monotonicity under the sole assumption (2.2). Note also that Theorem 2.1 does not require the positivity of the solution and its elementary proof is based on the invariance properties of polyharmonic operators under rotations, a fact that ensures uniqueness. Let us recall that symmetry is also ensured if $f'(s) < \lambda_2$, for all $s \in \mathbb{R}$ (with λ_2 being the second Dirichlet eigenvalue for $(-\Delta)^m$ in $\mathbf{B}$, see [6]), and uniqueness is guaranteed if $f \in C(\mathbb{R}_+;\mathbb{R})$ is sublinear at 0 and $+\infty$ (see [7]). All this raises a natural question:

Problem 2.2. *Is radial symmetry related to uniqueness? Are there examples of multiple radial positive solutions? Except, of course, when $f(u) = \lambda_1 u$!*

Some attention also deserves the *regularity* of f, as this feature is related to the *existence of non-radial solutions*. The counterexample in [10, Remark 1] shows that Theorem 1.1 ($m = 1$) does not hold if the Lipschitz continuity assumption on f is relaxed to Hölder continuity. We extend this example to the biharmonic equation ($m = 2$), showing that the monotonicity requirement in (1.5) cannot be dropped even in an arbitrarily small interval. We only deal with the biharmonic operator since the amount of computations grows very quickly as m increases.

Theorem 2.3. *Let $n \geq 2$. For every $\varepsilon > 0$ there exists a Hölder-continuous function $g_\varepsilon : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that*

$$g_\varepsilon(0) = 0\,, \qquad g_\varepsilon(w) > -\varepsilon \quad \forall w \in \mathbb{R}_+\,, \qquad g_\varepsilon(\varepsilon) > 0\,, \qquad g_\varepsilon \text{ is strictly increasing over } [\varepsilon, \infty),$$

and such that the problem

$$\begin{cases} \Delta^2 u = g_\varepsilon(u) & \text{in} \quad \mathbf{B} \\ u = \dfrac{\partial u}{\partial r} = 0 & \text{on} \quad \partial\mathbf{B}, \end{cases}$$

admits both

- *a radial solution ($u = u(r)$) in $C^4(\mathbf{B})$, nonnegative (but not strictly positive) and radially decreasing (but not strictly radially decreasing): namely, $u(r_0) = 0$ and $u'(r_0) = 0$ for some $r_0 \in (0, 1)$;*
- *infinitely many nonnegative nontrivial solutions in $C^4(\mathbf{B})$ which are not radially symmetric.*

Proof. For any $p > 4$ define the function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$ by

$$\begin{aligned} g(w) = 4p(p-1)w^{1-\frac{4}{p}}\Big[&4(p-3)(p-2) - 4\left(n(p-2) + 2(p-2)^2\right)w^{1/p} \\ &+ \left(n^2 + 4(p-2)(p-1) + 2n(2p-3)\right)w^{2/p}\Big] \qquad \forall w \geq 0\,. \end{aligned} \tag{2.4}$$

Then $g \in C^{0,1-\frac{4}{p}}([0,\infty);\mathbb{R}) \cap C^\infty((0,\infty);\mathbb{R})$ and

$$\begin{aligned} g'(w) = 4(p-2)(p-1)w^{-4/p}\Big[&4(p-4)(p-3) + 4\left(3(n-4) - 2p^2 - p(n-3) + 7p\right)w^{1/p} \\ &+ \left(n^2 - 6n + 8 + 4p^2 + 4p(n-3)\right)w^{2/p}\Big] \qquad \forall w > 0\,. \end{aligned}$$

The quantities between brackets in both the expressions of $g(w)$ and $g'(w)$ can be seen as second order polynomials in the variable $w^{1/p}$. In particular, there must exist finite numbers $M = M(n, p) > 0$, $\Gamma_1 = \Gamma_1(n, p) > 0$, $\Gamma_2 = \Gamma_2(n, p) \in (0, \Gamma_1)$ such that

$$-M(n,p) \doteq \min_{w\geq 0} g(w)\,, \qquad g(w) > 0 \Leftrightarrow w^{1/p} > \Gamma_1(n,p)\,, \qquad g'(w) > 0 \Leftrightarrow w^{1/p} > \Gamma_2(n,p)\,. \tag{2.5}$$

Two plots of g, for given values of $p > 4$ and $n \geq 2$, are displayed in Figure 1.

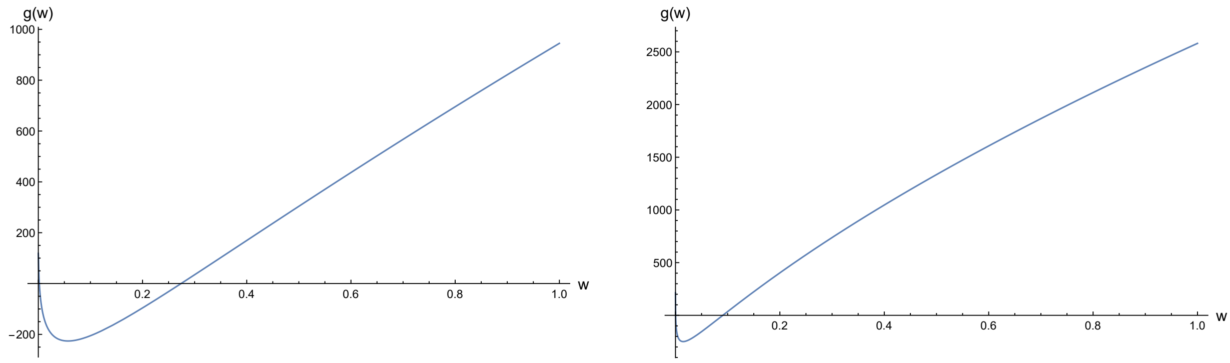


Figure 1. Plot of g on the interval $[0, 1]$ for $(p, n) = (4.5, 3)$ (left) and $(p, n) = (4.2, 6)$ (right).

Since $m = 2$, the Eq (1.2) in radial coordinates becomes

$$\begin{cases} u^{(4)}(r) + \dfrac{2(n-1)}{r}u'''(r) + \dfrac{(n-1)(n-3)}{r^2}u''(r) - \dfrac{(n-1)(n-3)}{r^3}u'(r) = g(u(r)) & \forall r \in (0,1) \\ u'(0) = u'''(0) = u(1) = u'(1) = 0\,, \end{cases} \tag{2.6}$$

where the conditions $u'(0) = u'''(0) = 0$ are needed to ensure the smoothness of u in $\mathbf{B}$, see [8]. By putting $u(r) \doteq (1 - r^2)^p$, for $r \in [0, 1]$, one sees that such u satisfies $u(1) = u'(1) = 0$ (fulfilling the Dirichlet boundary conditions) and also the equation in (2.6), where $g(u(r))$ is as in (2.4). Hence,

$$u(x) = (1 - |x|^2)^p \qquad \text{for } x \in \mathbf{B}$$

is a $C^4(\overline{\mathbf{B}})$-positive, radial and strictly radially decreasing solution ($u'(r) < 0$ in $(0, 1)$) of the problem

$$\begin{cases} \Delta^2 u = g(u) & \text{in} \quad \mathbf{B} \\ u = \dfrac{\partial u}{\partial r} = 0 & \text{on} \quad \partial\mathbf{B}. \end{cases}$$

In order to obtain the statement, we need some rescaling. First we extend u to all $\mathbb{R}^n$ by setting

$$u(x) = \begin{cases} (1 - |x|^2)^p & \text{if } x \in \mathbf{B} \\ 0 & \text{if } x \notin \mathbf{B}, \end{cases} \tag{2.7}$$

so that $u \in C^4(\mathbb{R}^n)$ is nonnegative and nontrivial (but not strictly positive), radial and radially decreasing (but not strictly decreasing). We now argue as in [10, p. 220]: take any point $x_0 \in \mathbb{R}^n$ such that $|x_0| = 3$ and consider the function $\hat{u} \in C^4(\mathbb{R}^n)$ defined as

$$\hat{u}(x) = u(x) + u(x - x_0) = \begin{cases} (1 - |x|^2)^p & \text{if } \ |x| < 1 \\ (1 - |x - x_0|^2)^p & \text{if } \ |x - x_0| < 1 \\ 0 & \text{otherwise,} \end{cases} \tag{2.8}$$

which is a nonnegative solution of the problem

$$\begin{cases} \Delta^2 \hat{u} = g(\hat{u}) & \text{in} \quad \mathbf{B}_5 \\ \hat{u} = \dfrac{\partial \hat{u}}{\partial r} = 0 & \text{on} \quad \partial \mathbf{B}_5, \end{cases}$$

where $\mathbf{B}_5 \subset \mathbb{R}^n$ is the (open) ball of radius 5 centered at the origin of $\mathbb{R}^n$, and g is given by (2.4). Nevertheless, $\hat{u}$ is not radially symmetric!

Then we rescale the problem as follows. For $u = u(r)$ as in (2.7) and for some $\gamma > 0$, define $v(r) \doteq u(5r)/\gamma$ for $r \in [0, 1]$, so that

$$v(r) = \begin{cases} \dfrac{1}{\gamma}(1 - 25r^2)^p & \text{if} \ \ 0 \le r < \dfrac{1}{5} \\ 0 & \text{if} \ \ r \ge \dfrac{1}{5}, \end{cases}$$

and v solves

$$\begin{cases} \Delta^2 v = g^\gamma(v) & \text{in} \quad \mathbf{B} \\ v = \dfrac{\partial v}{\partial r} = 0 & \text{on} \quad \partial \mathbf{B} \end{cases} \qquad \text{where} \quad g^\gamma(w) \doteq \frac{625}{\gamma} g(\gamma w) \qquad \forall w \ge 0.$$

Finally, fix $\varepsilon > 0$ and let us construct the function g_ε in the statement. Choose $\gamma_0 > 0$ such that

$$\gamma_0 > \frac{1}{\varepsilon} \max\{\Gamma_1, 625M\} \quad \Longrightarrow \quad \frac{\Gamma_2}{\gamma_0} < \frac{\Gamma_1}{\gamma_0} < \varepsilon \ \text{ and } \ \frac{625M}{\gamma_0} < \varepsilon.$$

Then $g_\varepsilon \doteq g^{\gamma_0}$ satisfies all the assumptions of the statement and the function v is a radial solution, nonnegative but not strictly positive (since it vanishes in $\mathbf{B} \setminus \mathbf{B}_{1/5}$), and radially decreasing but not strictly radially decreasing (since it is constant in $\mathbf{B} \setminus \mathbf{B}_{1/5}$). Moreover, after rescaling the function $\hat{u}$ in (2.8), we obtain infinitely many nonnegative nontrivial solutions which are not radially symmetric, one for each x_0 and, possibly with multiple bumps. □

The next result shows that the monotonicity assumption in (1.5) is *not necessary* to obtain the statement of Theorem 1.2. For $n \ge 2$, consider the function

$$f_n(w) \doteq 1632 + 576(n-1) + 32(n-1)(n-3) - [1680 + 672(n-1) + 48(n-1)(n-3)]\sqrt{w} \quad \forall w \ge 0, \tag{2.9}$$

which is strictly decreasing and sign-changing, see Figure 2 below.

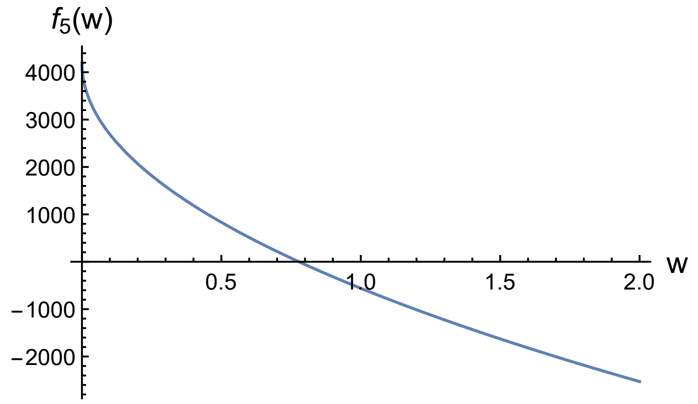


Figure 2. Plot of f_n on the interval $[0, 2]$ for $n = 5$ (right).

However, we can prove

Proposition 2.4. *Let f_n be as in* (2.9). *The unique strong solution of the problem*

$$\begin{cases} \Delta^2 u = f_n(u) & in \quad \mathbf{B} \\ u = \dfrac{\partial u}{\partial r} = 0 & on \quad \partial\mathbf{B}, \end{cases} \tag{2.10}$$

is given by

$$u(x) = (1 - |x|^4)^2 \qquad \forall x \in \mathbf{B}\,,$$

and, hence, it is positive, radially symmetric and strictly decreasing in the radial variable.

Proof. The function f_n defined in (2.9) satisfies the assumption (2.2) so that, by arguing as in the proof of Theorem 2.1, we infer that (2.10) admits at most one strong solution, which is necessarily radially symmetric.

In radial coordinates, when $m = 2$, the Eq (1.2) becomes (2.6). The fact that $u(r) = (1 - r^4)^2$, for $r \in [0, 1]$, is a (radial) solution follows by noticing that $u(1) = u'(1) = 0$ (fulfilling the Dirichlet boundary conditions), and then, by inserting the expressions of $u(r)$ and $f_n(u(r))$ into (2.6). □

3. A reflection property in conformal dimensions

We introduce here the Green function of the polyharmonic operator $(-\Delta)^m$ under Dirichlet boundary conditions in $\mathbf{B} \subset \mathbb{R}^n$ and we determine some of its properties. Let us firstly define

$$\theta(x, y) \doteq (1 - |x|^2)(1 - |y|^2) \qquad \forall x, y \in \mathbf{B}\,.$$

Then for $x, y \in \mathbf{B}$, with $x \neq y$, Boggio [5, p.126] (see also [9, Section 2.6]) gave the following explicit representation of the Green function:

$$\begin{aligned} G(x, y) &= k_n^m |x - y|^{2m-n} \int_1^{\left(\frac{\theta(x,y)}{|x-y|^2}+1\right)^{1/2}} \frac{(z^2 - 1)^{m-1}}{z^{n-1}}\, dz \\ &= \frac{k_n^m}{2} |x - y|^{2m-n} \int_0^{\frac{\theta(x,y)}{|x-y|^2}} \frac{z^{m-1}}{(z + 1)^{n/2}}\, dz = \frac{k_n^m}{2} H(|x - y|^2, \theta(x, y)), \end{aligned} \tag{3.1}$$

where k_n^m is a positive constant defined by

$$k_n^m = \frac{\Gamma\left(1 + \dfrac{n}{2}\right)}{n\, \pi^{n/2}\, 4^{m-1}((m - 1)!)^2}\,, \tag{3.2}$$

see [9, Lemma 2.27], and $H : (0, \infty) \times [0, \infty) \to \mathbb{R}$ is defined as

$$H(s, t) = s^{m-\frac{n}{2}} \int_0^{\frac{t}{s}} \frac{z^{m-1}}{(z + 1)^{n/2}}\, dz \qquad \forall s > 0,\ t \geq 0. \tag{3.3}$$

The following statement is a direct consequence of Boggio's work [5], elliptic regularity (see [9, Section 2.5]) and the estimates in [12].

Proposition 3.1. *Let $h \in L^\infty(\mathbf{B})$, and let $u \in H_0^m(\mathbf{B})$ be a weak solution of $(-\Delta)^m u = h$ in $\mathbf{B}$ under Dirichlet boundary conditions, that is*

$$\langle u, v\rangle_m = \int_{\mathbf{B}} hv \qquad \text{for all } v \in H_0^m(\mathbf{B}). \tag{3.4}$$

Then $u \in C^{2m-1,\alpha}(\overline{\Omega})$ and it satisfies

$$D^k u(x) = \int_{\mathbf{B}} D_x^k G(x,y)h(y)\,dy \qquad \forall x \in \overline{\mathbf{B}}\,, \tag{3.5}$$

where D^k stands for any partial derivative of order $|k| < 2m$. In particular, u is a strong solution and $D^k u \equiv 0$ on $\partial\mathbf{B}$ for $|k| \le m-1$.

In conformal dimensions $n = 2m$, the Green function admits a simpler representation. Indeed, the change of variables $\xi = \frac{1}{z+1}$ yields

$$\begin{aligned} H(s,t) &= \int_0^{\frac{t}{s}} \frac{z^{m-1}}{(z+1)^m}\,dz = \int_0^{\frac{t}{s}} \left(1 - \frac{1}{z+1}\right)^{m-1} \frac{1}{z+1}\,dz = \int_{\frac{s}{s+t}}^1 \frac{1}{\xi}(1-\xi)^{m-1}\,d\xi \\ &= \int_{\frac{s}{s+t}}^1 \left[\frac{1}{\xi} + \sum_{k=1}^{m-1} (-1)^k \binom{m-1}{k} \xi^{k-1}\right] d\xi \\ &= \log\left(1 + \frac{t}{s}\right) + \sum_{k=1}^{m-1} \frac{(-1)^k}{k}\binom{m-1}{k}\left(1 - \left(\frac{s}{s+t}\right)^k\right) \qquad \forall s > 0,\ t \ge 0. \end{aligned}$$

Therefore, by setting $\kappa_m \doteq k_{2m}^m$, for every $x, y \in \mathbf{B}$ we have

$$G(x,y) = \frac{\kappa_m}{2}\left[\log\left(1 + \frac{\theta(x,y)}{|x-y|^2}\right) + \sum_{k=1}^{m-1} \frac{(-1)^k}{k}\binom{m-1}{k}\left(1 - \left(\frac{|x-y|^2}{|x-y|^2 + \theta(x,y)}\right)^k\right)\right]. \tag{3.6}$$

The main result of this section (Theorem 3.2) shows that, in conformal dimensions, the Green function satisfies an elegant reflection property. For this, given the point $x = (x_1, x') \in \mathbf{B}$ with $x_1 > 0$, we define

$$R_x \doteq \frac{1-|x|^2}{2x_1}\,, \qquad y_x \doteq (x_1 + R_x, x')\,, \qquad C_x \doteq \{y \in \mathbf{B}\ ;\ |y - y_x| > R_x\ \}\,, \tag{3.7}$$

and the set

$$\mathcal{S}_x \doteq \left\{y \in \mathbb{R}^n\ ;\ (1-|x|^2)(x_1 - y_1) + x_1|x-y|^2 = 0\right\}, \tag{3.8}$$

which is a sphere with center at y_x and radius R_x (see the proof of Theorem 3.2 below and also Figure 4). We then denote by $\mathcal{P}_x : \mathbb{R}^n \setminus \{y_x\} \longrightarrow \mathbb{R}^n$ the *inversion in $\mathcal{S}_x$*, given by the expression

$$\mathcal{P}_x(y) = y_x + R_x^2 \frac{y - y_x}{|y - y_x|^2} \qquad \forall y \in \mathbb{R}^n \setminus \{y_x\}\,.$$

Finally, the *Kelvin transform* of a function $h : C_x \longrightarrow \mathbb{R}$ with respect to $\mathcal{S}_x$ is defined as

$$\mathcal{K}_x(h)(y) = h(\mathcal{P}_x(y)) \qquad \forall y \in C_x\,.$$

Then we prove

Theorem 3.2. *For any $x = (x_1, x') \in \mathbf{B} \subset \mathbb{R}^{2m}$ with $x_1 > 0$ and any $u \in H^{2m}(\mathbf{B}) \cap H_0^m(\mathbf{B})$ we have*

$$\frac{1-|x|^2}{\kappa_m\, x_1}\frac{\partial u}{\partial x_1}(x) = \int_{C_x} \frac{|y-y_x|^2 - R_x^2}{|x-y|^2}\left(\frac{\theta(x,y)}{|x-y|^2+\theta(x,y)}\right)^m (-\Delta_y)^m \left[\mathcal{K}_x(u)(y) - u(y)\right] dy\,. \tag{3.9}$$

Proof. The following identities hold for the function H defined in (3.3): for all $s, t > 0$ we have

$$\frac{\partial H}{\partial t}(s,t) = \frac{t^{m-1}}{(t+s)^m}\,, \qquad \frac{\partial H}{\partial s}(s,t) = -\frac{t^m}{s(t+s)^m}\,, \qquad \frac{\partial^2 H}{\partial s \partial t}(s,t) = -\frac{mt^{m-1}}{(t+s)^{m+1}}\,, \tag{3.10}$$

see also [3]. Moreover, for every $x, y \in \mathbf{B}$ we have

$$\frac{\partial G}{\partial x_1}(x,y) = \kappa_m\left[(x_1-y_1)\frac{\partial H}{\partial s}\left(|x-y|^2, \theta(x,y)\right) - x_1(1-|y|^2)\frac{\partial H}{\partial t}\left(|x-y|^2, \theta(x,y)\right)\right]. \tag{3.11}$$

We then prove the announced symmetry property, namely that the zero level sets of $\partial_{x_1} G(x,\cdot)$ are spherical caps. Let us denote the zero level set by

$$\mathcal{L}_x \doteq \left\{ y \in \mathbf{B}\ ;\ \frac{\partial G}{\partial x_1}(x,y) = 0 \right\}.$$

By (3.10) and by (3.11) we have

$$\begin{aligned} -\frac{1}{\kappa_m}\frac{\partial G}{\partial x_1}(x,y) &= x_1(1-|y|^2)\frac{\partial H}{\partial t}(|x-y|^2, \theta(x,y)) - (x_1-y_1)\frac{\partial H}{\partial s}(|x-y|^2, \theta(x,y)) \\ &= x_1(1-|y|^2)\frac{\theta(x,y)^{m-1}}{(|x-y|^2+\theta(x,y))^m} + (x_1-y_1)\frac{\theta(x,y)^m}{|x-y|^2(|x-y|^2+\theta(x,y))^m} \qquad (3.12) \\ &= \frac{\theta(x,y)^{m-1}}{|x-y|^2(|x-y|^2+\theta(x,y))^m}(1-|y|^2)\left[x_1|x-y|^2 + (1-|x|^2)(x_1-y_1)\right]. \end{aligned}$$

Therefore, $\partial_{x_1} G(x,y)$ vanishes if and only if $y \in \mathbf{B}$ belongs to the set $\mathcal{S}_x$ defined in (3.8). Then we notice that $y \in \mathbf{B} \cap \mathcal{S}_x$ if and only if

$$\begin{aligned} 0 = \frac{1}{x_1}\left[(1-|x|^2)(x_1-y_1) + x_1|x-y|^2\right] &= \frac{1-|x|^2}{x_1}(x_1-y_1) + (x_1-y_1)^2 + |x'-y'|^2 \\ &= 2R_x(x_1-y_1) + (x_1-y_1)^2 + |x'-y'|^2 \\ &= (R_x + x_1 - y_1)^2 + |x'-y'|^2 - R_x^2 \\ &= |y_x - y|^2 - R_x^2\,. \end{aligned} \tag{3.13}$$

Hence, $\mathcal{S}_x$ is a sphere with center at y_x and radius R_x and the set of negativity of $\partial_{x_1} G(x,y)$ is the cap C_x defined in (3.7).

Next, we notice that, by definition of y_x and R_x, we have

$$|y_x|^2 = \left(x_1 + \frac{1-|x|^2}{2x_1}\right)^2 + |x'|^2 = x_1^2 + (1-|x|^2) + R_x^2 + |x'|^2 = 1 + R_x^2\,.$$

Then we compute

$$
\begin{aligned}
|\mathcal{P}_x(y)|^2 &= |\mathcal{P}_x(y) - y_x + y_x|^2 = |\mathcal{P}_x(y) - y_x|^2 + 2(\mathcal{P}_x(y) - y_x)\cdot y_x + |y_x|^2 \\
&= \frac{R_x^4}{|y-y_x|^2} + 2R_x^2\frac{y-y_x}{|y-y_x|^2}\cdot y_x + |y_x|^2 = \frac{R_x^2}{|y-y_x|^2}\left(R_x^2 + 2(y-y_x)\cdot y_x + |y_x|^2\frac{|y-y_x|^2}{R_x^2}\right) \\
&= \frac{R_x^2}{|y-y_x|^2}\left((R_x^2 - |y-y_x|^2) + (|y-y_x|^2 + 2(y-y_x)\cdot y_x + |y_x|^2) - |y_x|^2 + |y_x|^2\frac{|y-y_x|^2}{R_x^2}\right) \\
&= \frac{R_x^2}{|y-y_x|^2}\left((R_x^2 - |y-y_x|^2) + |y|^2 - |y_x|^2 + |y_x|^2\frac{|y-y_x|^2}{R_x^2}\right) \\
&= \frac{R_x^2}{|y-y_x|^2}\left((R_x^2 + |y|^2 - |y_x|^2) + \frac{(|y_x|^2 - R_x^2)|y-y_x|^2}{R_x^2}\right) \\
&= \frac{R_x^2}{|y-y_x|^2}\left((|y|^2 - 1) + \frac{|y-y_x|^2}{R_x^2}\right) = 1 - \frac{R_x^2}{|y-y_x|^2}(1-|y|^2)\,.
\end{aligned}
$$

This computation has two main consequences. First, that the unit sphere $\partial\mathbf{B}$ is invariant under the inversion in $\mathcal{S}_x$, that is,

$$
y \in \partial\mathbf{B} \iff \mathcal{P}_x(y) \in \partial\mathbf{B}\,.
$$

Second, that

$$
\theta(x, \mathcal{P}_x(y)) = \frac{R_x^2}{|y_x - y|^2}\,\theta(x,y) \qquad \forall y \in \mathbf{B}\,. \tag{3.14}
$$

We next claim that if C_x is the cap defined in (3.7), then $\mathbf{B} = C_x \cup \mathcal{P}_x(C_x) \cup \mathcal{L}_x$. To this end, it suffices to show that

$$
\mathcal{P}_x(C_x) = \{y \in \mathbf{B}\ ;\ |y - y_x| < R_x\}\,.
$$

Indeed, let $y \in C_x$, so that $\mathcal{P}_x(y) \in \mathcal{P}_x(C_x)$ and $|y - y_x| > R_x$. By definition of $\mathcal{P}_x$ we have

$$
|\mathcal{P}_x(y) - y_x| = \frac{R_x^2}{|y - y_x|} < R_x,
$$

thus proving that $\mathcal{P}_x(C_x) \subset \{y \in \mathbf{B}\ ;\ |y - y_x| < R_x\}$. On the other hand, let $y \in \mathbf{B}$ be such that $|y-y_x| < R_x$. Since the function $\mathcal{P}_x$ is bijective, there exists a point $y_0 \in \mathbb{R}^n$ such that $\mathcal{P}_x(y_0) = y$. By definition, we then have

$$
|y_0 - y_x| = \frac{R_x^2}{|y - y_x|} > R_x,
$$

implying $y_0 \in C_x$, and subsequently, that $y \in \mathcal{P}_x(C_x)$ and $\{y \in \mathbf{B}\ ;\ |y - y_x| < R_x\} \subset \mathcal{P}_x(C_x)$.

Since $\mathbf{B} = C_x \cup \mathcal{P}_x(C_x) \cup \mathcal{L}_x$, by Proposition 3.1, we have

$$
\begin{aligned}
\frac{\partial u}{\partial x_1}(x) &= \int_{C_x} \frac{\partial G}{\partial x_1}(x,y)(-\Delta_y)^m u(y)\,dy + \int_{\mathcal{P}_x(C_x)} \frac{\partial G}{\partial x_1}(x,y)(-\Delta_y)^m u(y)\,dy \\
&= \int_{C_x} \frac{\partial G}{\partial x_1}(x,y)(-\Delta_y)^m u(y)\,dy + \int_{C_x}\left(\frac{R_x}{|y-y_x|}\right)^{2n} \frac{\partial G}{\partial x_1}(x,\mathcal{P}_x(y))((-\Delta_y)^m u)(\mathcal{P}_x(y))\,dy \\
&= \int_{C_x} \frac{\partial G}{\partial x_1}(x,y)(-\Delta_y)^m u(y)\,dy + \int_{C_x}\left(\frac{R_x}{|y-y_x|}\right)^{4m} \frac{\partial G}{\partial x_1}(x,\mathcal{P}_x(y))((-\Delta_y)^m u)(\mathcal{P}_x(y))\,dy,
\end{aligned}
$$

where we used that the Jacobian matrix $\mathcal{P}'_x$ satisfies

$$\left|\det(\mathcal{P}'_x(y))\right| = \left(\frac{R_x}{|y-y_x|}\right)^{2n} \qquad \forall y \in \mathbf{B}\,.$$

In view of (3.12) and (3.13) we can write

$$-\frac{1}{\kappa_m}\frac{1-|x|^2}{x_1}\frac{\partial G}{\partial x_1}(x,y) = \frac{|y_x-y|^2-R_x^2}{|x-y|^2}\left(\frac{\theta(x,y)}{\theta(x,y)+|x-y|^2}\right)^m \qquad \forall y \in \mathbf{B}\,, \tag{3.15}$$

so that

$$-\frac{1}{\kappa_m}\frac{1-|x|^2}{x_1}\frac{\partial u}{\partial x_1}(x) = I_1(x) + I_2(x)\,, \tag{3.16}$$

where

$$\begin{cases} I_1(x) \doteq \displaystyle\int_{C_x} \frac{|y_x-y|^2-R_x^2}{|x-y|^2}\left(\frac{\theta(x,y)}{\theta(x,y)+|x-y|^2}\right)^m (-\Delta_y)^m u(y)\, dy\,, \\ I_2(x) \doteq \displaystyle\int_{C_x} \frac{|y_x-\mathcal{P}_x(y)|^2-R_x^2}{|x-\mathcal{P}_x(y)|^2}\left(\frac{\theta(x,\mathcal{P}_x(y))}{\theta(x,\mathcal{P}_x(y))+|x-\mathcal{P}_x(y)|^2}\right)^m \left(\frac{R_x}{|y-y_x|}\right)^{4m} ((-\Delta_y)^m u)(\mathcal{P}_x(y))\, dy\,. \end{cases}$$

To compute $I_2(x)$, we first notice that

$$|y_x-\mathcal{P}_x(y)|^2-R_x^2 = \frac{R_x^4}{|y_x-y|^2}-R_x^2 = -\frac{R_x^2}{|y_x-y|^2}(|y_x-y|^2-R_x^2) < 0 \qquad \forall y \in C_x\,, \tag{3.17}$$

whereas, for every $y \in \mathbf{B}$ we have

$$\begin{aligned} |x-\mathcal{P}_x(y)|^2 &= |x-y_x|^2 + 2(x-y_x)\cdot(y_x-\mathcal{P}_x(y)) + |y_x-\mathcal{P}_x(y)|^2 \\ &= |x-y_x|^2 + \frac{2R_x^2}{|y_x-y|^2}(x-y_x)\cdot(y_x-y) + \frac{R_x^4}{|y-y_x|^2} \\ &= \frac{R_x^2}{|y_x-y|^2}\left(\frac{|y_x-y|^2}{R_x^2}|x-y_x|^2 + 2(x-y_x)\cdot(y_x-y) + R_x^2\right) \\ &= \frac{R_x^2}{|y_x-y|^2}\left(|y_x-y|^2 + 2(x-y_x)\cdot(y_x-y) + |x-y_x|^2\right) = \frac{R_x^2}{|y_x-y|^2}\,|x-y|^2\,. \end{aligned} \tag{3.18}$$

After plugging (3.14), (3.17) and (3.18) into $I_2(x)$ we get

$$I_2(x) = -\int_{C_x} \frac{|y-y_x|^2-R_x^2}{|x-y|^2}\left(\frac{\theta(x,y)}{|x-y|^2+\theta(x,y)}\right)^m \left(\frac{R_x}{|y-y_x|}\right)^{4m} ((-\Delta_y)^m u)(\mathcal{P}_x(y))dy$$

and, after recalling (3.16), we deduce

$$\begin{aligned} &-\frac{1}{\kappa_m}\frac{1-|x|^2}{x_1}\frac{\partial u}{\partial x_1}(x) \\ &= \int_{C_x} \frac{|y-y_x|^2-R_x^2}{|x-y|^2}\left(\frac{\theta(x,y)}{|x-y|^2+\theta(x,y)}\right)^m \left[(-\Delta_y)^m u(y) - \left(\frac{R_x}{|y-y_x|}\right)^{4m} ((-\Delta_y)^m u)(\mathcal{P}_x(y))\right] dy\,. \end{aligned}$$

A direct computation (see [13, Proposition 7.3]) shows that

$$(-\Delta_y)^m \mathcal{K}_x(u)(y) = \left(\frac{R_x}{|y - y_x|}\right)^{4m} ((-\Delta_y)^m u)(\mathcal{P}_x(y)) \qquad \forall y \in C_x\,, \tag{3.19}$$

which concludes the proof after changing the sign. □

Problem 3.3. *It is not clear how to derive an expression similar to* (3.9) *in non-conformal dimensions $n \geq 2$, $n \neq 2m$. This can be seen after computing*

$$\frac{\partial H}{\partial s}(s,t) = \left(n - \frac{m}{2}\right) s^{m-\frac{n}{2}-1} \int_0^{\frac{t}{s}} \frac{z^{m-1}}{(z+1)^{n/2}}\, dz - \frac{t^m}{s(t+s)^{n/2}} \qquad \forall s > 0,\ t \geq 0\,,$$

so that the determination of the level sets of $\partial_{x_1} G(x,\cdot)$ as in (3.12) *becomes a delicate task. Is it possible to obtain a representation formula similar to* (3.9) *in any dimension $n \geq 2$?*

Clearly, a representation formula similar to (3.9) holds for any other directional derivative of u. Note that (3.7) may be rewritten as $y_x = x + (R_x, 0)$, which implies that y_x is an "horizontal translation" of $x \in \mathbf{B}$. Moreover, $R_x \to 0$ if $x \to \partial\mathbf{B}$ while $R_x \to +\infty$ if $x_1 \to 0$: the limit of the map $x \mapsto R_x$ does not exist when x approaches the equator $\partial\mathbf{B} \cap \{x_1 = 0\}$. The level surfaces of $x \mapsto R_x$ are spheres:

$$R_x = k \in (0,\infty) \implies (x_1 + k)^2 + |x'|^2 = 1 + k^2\,.$$

The center of the level-surface sphere is then $(x_1, x') = (-k, 0)$ and its radius is $\sqrt{1+k^2}$. For a fixed $k > 0$, when x runs over the level surface $R_x = k$, the corresponding y_x, defined by (3.7), runs over a portion of the sphere centered in the origin with the same radius $\sqrt{1+k^2}$, see Figure 3.

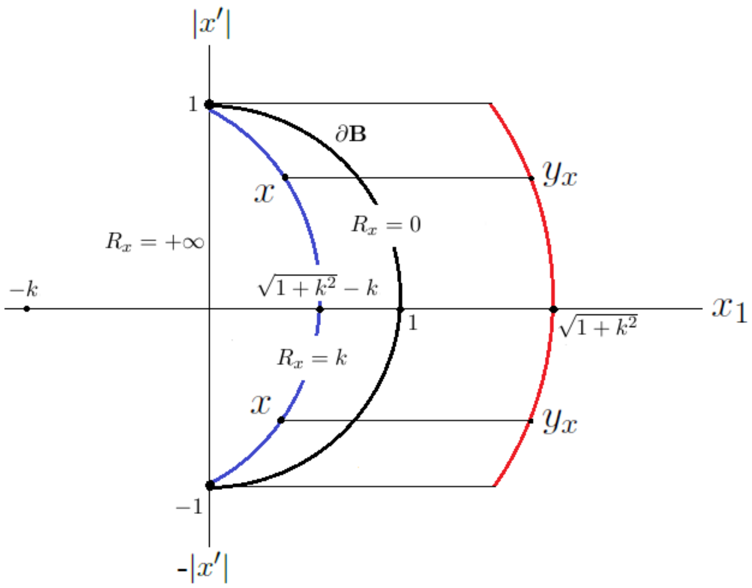


Figure 3. Value of R_x, positions of x and y_x, see (3.7). In blue: the level surface $R_x = k > 0$.

Note also that C_x, the domain of negativity of $\partial_{x_1} G(x,y)$, is the intersection between two balls. In Figure 4 we sketch the mutual position of the sphere $\mathcal{S}_x$ and the ball $\mathbf{B}$. In particular, we emphasize

that
$$|y_x|^2 = \left(x_1 + \frac{1-|x|^2}{2x_1}\right)^2 + |x'|^2 = 1 + \frac{(1-|x|^2)^2}{4x_1^2} > 1 \qquad \forall x \in \mathbf{B} \quad (x_1 > 0)$$
which, again, tells us that y_x is exterior to $\mathbf{B}$ and it approaches $\partial\mathbf{B}$ as x approaches $\partial\mathbf{B}$, while it goes to infinity if x approaches the plane $x_1 = 0$.

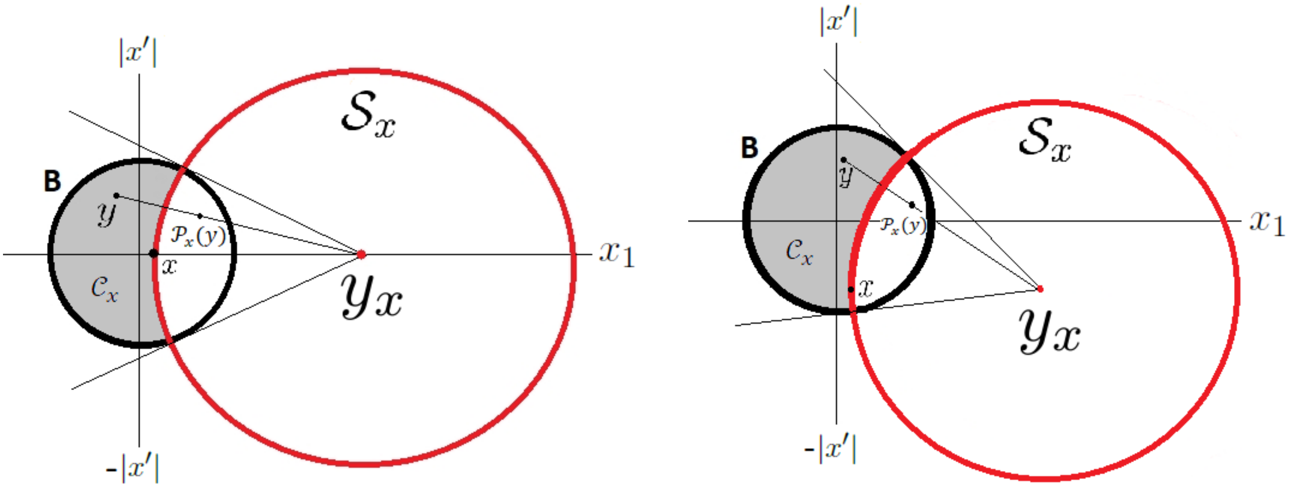


Figure 4. Representation of the sphere $\mathcal{S}_x$ for different positions of $x \in \mathbf{B}$.

Theorem 3.2 has several relevant consequences. Firstly, we deduce

Corollary 3.4. *Let $m \geq 1$. For every $x \in \mathbf{B} \subset \mathbb{R}^{2m}$ such that $x_1 > 0$, the following identities hold*
$$(1-|x|^2)^m = \frac{1}{\pi^m}\frac{(2m-1)!}{(m-1)!}\int_{\mathcal{C}_x} \frac{|y-y_x|^2 - R_x^2}{|x-y|^2}\left(\frac{\theta(x,y)}{|x-y|^2+\theta(x,y)}\right)^m \left[1 - \left(\frac{R_x}{|y-y_x|}\right)^{4m}\right] dy\,, \tag{3.20}$$
$$\mathbb{K}_m \doteq \pi^m \frac{(m-1)!}{(2m-1)!} = \int_{\mathcal{C}_x} \frac{|y-y_x|^2 - R_x^2}{|x-y|^2}\left(\frac{1-|y|^2}{|x-y|^2+\theta(x,y)}\right)^m \left[1 - \left(\frac{R_x}{|y-y_x|}\right)^{4m}\right] dy\,. \tag{3.21}$$

Proof. We start by noticing that
$$\Delta r^{2m} = 2m(2m+n-2)r^{2m-2}\,, \qquad \Delta^2 r^{2m} = 2m(2m-2)(2m+n-2)(2m+n-4)r^{2m-4}\,,$$
$$\Delta^3 r^{2m} = 2m(2m-2)(2m-4)(2m+n-2)(2m+n-4)(2m+n-6)r^{2m-6}\,,$$
where $n = 2m$. By induction we then obtain
$$\Delta^m r^{2m} = (2m)!!\,\frac{(2m+n-2)!!}{(n-2)!!} = (2m)!!\,\frac{(4m-2)!!}{(2m-2)!!} = 2^{2m-1}(2m)!\,.$$
Consider the polyharmonic version of the original problem (1.1) proposed by Serrin:
$$\begin{cases} (-\Delta)^m u = 1 & \text{in } \mathbf{B} \\ u = \dfrac{\partial u}{\partial r} = \cdots = \dfrac{\partial^{m-1} u}{\partial r^{m-1}} = 0 & \text{on } \partial\mathbf{B}. \end{cases} \tag{3.22}$$

A direct application of the binomial expansion yields

$$(-\Delta)^m(1-r^2)^m = (-\Delta)^m\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^k r^{2k}\right] = \Delta^m r^{2m} = 2^{2m-1}(2m)!\,,$$

that is

$$(-\Delta)^m U_m(r) = 1 \qquad \text{with} \qquad U_m(r) \doteq \frac{(1-r^2)^m}{2^{2m-1}(2m)!} \qquad \forall r \in [0,1]. \tag{3.23}$$

Since U_m satisfies the Dirichlet boundary conditions in $(3.22)_2$, it is a strong $C^\infty(\overline{\mathbf{B}})$-solution of (3.22). Applying identity (3.9) to the function U_m and recalling (3.2) (with $n = 2m$), we derive equality (3.20). The identity (3.21) is a straightforward consequence of (3.20), after replacing $\theta(x,y)$. □

It is remarkable that the right hand side of (3.21) *does not depend on* $x \in \mathbf{B}$. As a further consequence of Theorem 3.2, we give an alternative formula to compute the partial derivatives of the solutions of (1.2), which should be compared with Proposition 3.1.

Corollary 3.5. *Let* $\mathbf{B} \subset \mathbb{R}^{2m}$ *be the unit ball. Suppose that* $f \in C(\mathbb{R};\mathbb{R})$ *and that* $u \in H_0^m(\mathbf{B}) \cap L^\infty(\mathbf{B})$ *is a strong solution of* (1.2). *Then, for every* $x \in \mathbf{B}$ *such that* $x_1 > 0$*, the following formula holds:*

$$\frac{|x|^2-1}{\kappa_m\, x_1}\frac{\partial u}{\partial x_1}(x) = \int_{C_x} \frac{|y-y_x|^2 - R_x^2}{|x-y|^2}\left[\frac{\theta(x,y)}{|x-y|^2+\theta(x,y)}\right]^m \left[f(u(y)) - \left(\frac{R_x}{|y-y_x|}\right)^{4m} f(u(\mathcal{P}_x(y)))\right]dy. \tag{3.24}$$

Formula (3.24) follows directly from (3.9)–(3.19) and suggests the problem:

Problem 3.6. *Since the moving plane procedure developed in [3, Section 3] requires that*

$$\frac{\partial u}{\partial x_1}(x) < 0 \qquad \textit{for all } x \in \mathbf{B} \ \textit{ with } \ x_1 > 0\,,$$

is it possible to use (3.24) *to prove this inequality and to relax assumption* (1.5)*, but still ensuring the statement of Theorem 1.2?*

In connection with Problem 3.6 we notice that, by looking at (3.24), it is clear that the overall sign of the integral over C_x is determined by the behavior of the quantity

$$\Phi_x(y) \doteq f(u(y)) - \left(\frac{R_x}{|y-y_x|}\right)^{4m} f(u(\mathcal{P}_x(y))) \qquad \forall y \in C_x\,.$$

Fixing $x \in \mathbf{B}$ such that $x_1 > 0$ and $u \in H_0^m(\mathbf{B})$, notice that $\Phi_x(y) = 0$ for all $y \in \partial C_x \cap \mathbf{B}$, while for $y \in \partial C_x \cap \partial\mathbf{B}$ (so that $\mathcal{P}_x(y) \in \partial\mathbf{B}$) we have that $\Phi_x(y) = 0$ if $f(0) = 0$, and $\Phi_x(y) > 0$ if $f(0) > 0$.

Under suitable assumptions on the source f, in the next result we give an upper bound for the partial derivative of u (recall that $|x| < 1$!), which shows that $\partial_{x_1}u(x) < 0$ for all $x \in \mathbf{B}$ located far away from $\partial\mathbf{B}$, with $x_1 \geq \varepsilon$ for some $\varepsilon > 0$, see Figure 5 for a schematic representation. Therefore, in order to solve Problem 3.6, one should mainly focus the attention on the complement of this region.

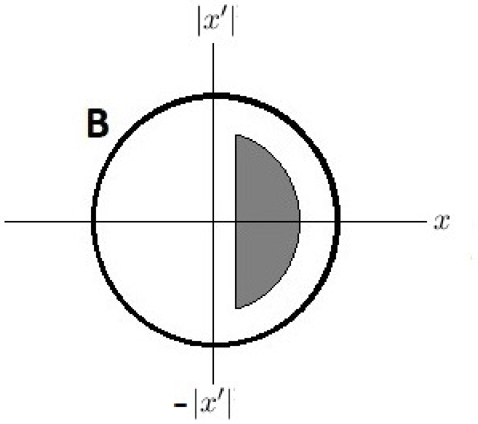


Figure 5. In gray: subset of **B** in which the strict negativity of $\partial_{x_1} u(x)$ holds.

To this end, we define

$$\frac{1}{\Gamma} \doteq \inf_{v \in C^{2m}(\overline{\mathbf{B}}) \cap H_0^m(\mathbf{B}) \setminus \{0\}} \frac{\|\Delta^m v\|_{L^\infty(\mathbf{B})}}{\|\nabla v\|_{L^\infty(\mathbf{B})}},$$

and we state:

Proposition 3.7. *Assume that $f \in W^{1,\infty}(\mathbb{R}_+;\mathbb{R}_+)$ satisfies*

$$f(s) \geq M \qquad \forall s \geq 0\,, \ \textit{for some} \ \ M > 0, \tag{3.25}$$

and let $u \in H_0^m(\mathbf{B}) \cap L^\infty(\mathbf{B})$ be a strong solution of (1.2)*. For any point $x = (x_1, x') \in \mathbf{B}$ with $x_1 > 0$ we have*

$$\frac{|x|^2 - 1}{\kappa_m\, x_1} \frac{\partial u}{\partial x_1}(x) \geq \left(M - 2\Gamma \|f\|_{L^\infty(\mathbb{R}_+)} \|f'\|_{L^\infty(\mathbb{R}_+)}\right) \mathbb{K}_m (1 - |x|^2)^m$$

$$- 2\Gamma \|f\|_{L^\infty(\mathbb{R}_+)} \|f'\|_{L^\infty(\mathbb{R}_+)} \frac{1 - |x|^2}{\kappa_m\, x_1} \int_{C_x} \left(\frac{R_x}{|y - y_x|}\right)^{4m} \frac{\partial G}{\partial x_1}(x, \mathcal{P}_x(y))\, dy\,,$$

where $\mathbb{K}_m > 0$ is defined in (3.21).

Proof. In view of Corollary 3.4, by the Maximum Principle and assumption (3.25), any strong solution $u \in H_0^m(\mathbf{B}) \cap L^\infty(\mathbf{B})$ of (1.2) satisfies

$$u(x) \geq \frac{M}{2^{2m-1}(2m)!}(1 - |x|^2)^m \qquad \forall x \in \mathbf{B}\,.$$

By embedding theorems and elliptic regularity we also have that

$$\|\nabla u\|_{L^\infty(\mathbf{B})} \leq \Gamma \|f\|_{L^\infty(\mathbb{R}_+)}\,.$$

Then, by the Mean Value Theorem, we obtain the inequality

$$|u(\mathcal{P}_x(y)) - u(y)| \leq \Gamma \|f\|_{L^\infty(\mathbb{R}_+)} |\mathcal{P}_x(y) - y| \qquad \forall y \in C_x\,.$$

From this we deduce that

$$|f(u(\mathcal{P}_x(y))) - f(u(y))| \le \Gamma \|f\|_{L^\infty(\mathbb{R}_+)} \|f'\|_{L^\infty(\mathbb{R}_+)} \, |\mathcal{P}_x(y) - y| \qquad \forall y \in C_x\,,$$

thus yielding

$$f(u(\mathcal{P}_x(y))) \le f(u(y)) + \Gamma \|f\|_{L^\infty(\mathbb{R}_+)} \|f'\|_{L^\infty(\mathbb{R}_+)} \, |\mathcal{P}_x(y) - y| \qquad \forall y \in C_x$$

and, by using (3.25), we finally obtain

$$\begin{aligned} f(u(y)) - \left(\frac{R_x}{|y-y_x|}\right)^{4m} f(u(\mathcal{P}_x(y))) \ge \left[1 - \left(\frac{R_x}{|y-y_x|}\right)^{4m}\right] M \\ - \left(\frac{R_x}{|y-y_x|}\right)^{4m} \Gamma \|f\|_{L^\infty(\mathbb{R}_+)} \|f'\|_{L^\infty(\mathbb{R}_+)} \, |\mathcal{P}_x(y) - y|\,. \end{aligned}$$

By inserting this bound within (3.24), we get

$$\frac{|x|^2 - 1}{\kappa_m \, x_1} \frac{\partial u}{\partial x_1}(x) \ge I_1(x) - I_2(x)\,, \tag{3.26}$$

where

$$\begin{cases} I_1(x) \doteq M \displaystyle\int_{C_x} \frac{|y-y_x|^2 - R_x^2}{|x-y|^2} \left[\frac{\theta(x,y)}{|x-y|^2 + \theta(x,y)}\right]^m \left[1 - \left(\frac{R_x}{|y-y_x|}\right)^{4m}\right] dy \\ I_2(x) \doteq \Gamma \|f\|_{L^\infty(\mathbb{R}_+)} \|f'\|_{L^\infty(\mathbb{R}_+)} \displaystyle\int_{C_x} \frac{|y-y_x|^2 - R_x^2}{|x-y|^2} \left[\frac{\theta(x,y)}{|x-y|^2 + \theta(x,y)}\right]^m \left(\frac{R_x}{|y-y_x|}\right)^{4m} |\mathcal{P}_x(y) - y| \, dy. \end{cases}$$

From Corollary 3.4 we infer that

$$I_1(x) = M \mathbb{K}_m (1 - |x|^2)^m\,. \tag{3.27}$$

In order to bound $I_2(x)$, we notice that

$$|\mathcal{P}_x(y) - y| < 2 \quad \text{and} \quad \frac{R_x}{|y-y_x|} < 1 \qquad \forall y \in C_x\,.$$

Hence,

$$I_2(x) \le 2\Gamma \|f\|_{L^\infty(\mathbb{R}_+)} \|f'\|_{L^\infty(\mathbb{R}_+)} \int_{C_x} \frac{|y-y_x|^2 - R_x^2}{|x-y|^2} \left[\frac{\theta(x,y)}{|x-y|^2 + \theta(x,y)}\right]^m dy\,. \tag{3.28}$$

From (3.15) and Proposition 3.1, and by proceeding as in the proof of Corollary 3.4, we obtain

$$\begin{aligned} &\int_{C_x} \frac{|y-y_x|^2 - R_x^2}{|x-y|^2} \left[\frac{\theta(x,y)}{|x-y|^2 + \theta(x,y)}\right]^m dy = -\frac{1}{\kappa_m} \frac{1 - |x|^2}{x_1} \int_{C_x} \frac{\partial G}{\partial x_1}(x,y) \, dy \\ &= -\frac{1}{\kappa_m} \frac{1 - |x|^2}{x_1} \left(\int_{\mathbf{B}} \frac{\partial G}{\partial x_1}(x,y) \, dy - \int_{\mathcal{P}_x(C_x)} \frac{\partial G}{\partial x_1}(x,y) \, dy\right) \\ &= -\frac{1}{\kappa_m} \frac{1 - |x|^2}{x_1} \left[\frac{1}{2^{2m-1}(2m)!} \frac{\partial}{\partial x_1} \left((1 - |x|^2)^m\right) - \int_{C_x} \left(\frac{R_x}{|y-y_x|}\right)^{4m} \frac{\partial G}{\partial x_1}(x, \mathcal{P}_x(y)) \, dy\right] \\ &= \mathbb{K}_m (1 - |x|^2)^m + \frac{1 - |x|^2}{\kappa_m \, x_1} \int_{C_x} \left(\frac{R_x}{|y-y_x|}\right)^{4m} \frac{\partial G}{\partial x_1}(x, \mathcal{P}_x(y)) \, dy\,. \end{aligned}$$

After plugging this into (3.28) we deduce that

$$I_2(x) \leq 2\Gamma \|f\|_{L^\infty(\mathbb{R}_+)} \|f'\|_{L^\infty(\mathbb{R}_+)} \left[\mathbb{K}_m (1-|x|^2)^m + \frac{1-|x|^2}{\kappa_m \, x_1} \int_{C_x} \left(\frac{R_x}{|y-y_x|} \right)^{4m} \frac{\partial G}{\partial x_1}(x, \mathcal{P}_x(y)) \, dy \right]. \qquad (3.29)$$

The conclusion is reached after inserting (3.27) and (3.29) into (3.26). □

Proposition 3.7 suggests that the radial symmetry and monotonicity of the solutions of (1.2) can be ensured if $\|f'\|_{L^\infty(\mathbb{R}_+)}$ is small. This assumption is not needed if $m = 1$ (see Theorem 1.1), while for $m \geq 2$, the required condition is that f' be nonnegative (see Theorem 1.2). Therefore, one is led to analyze the cases where the negative part of f' is small, that is, $\|(f')_-\|_{L^\infty(\mathbb{R}_+)}$ small. This issue is tackled numerically in the next section, for the biharmonic equation in $\mathbf{B} \subset \mathbb{R}^4$.

4. On a counterexample by Guido Sweers

As a direct consequence of the Hopf-type lemma by Grunau-Sweers [11, Theorem 3.2] we obtain

Proposition 4.1. *Assume that* $f \in W^{1,\infty}(\mathbb{R}_+; \mathbb{R}_+)$ *satisfies* (3.25)*, and let* $u \in H_0^m(\mathbf{B}) \cap L^\infty(\mathbf{B})$ *be a strong solution of* (1.2)*. Then, there exists* $\gamma_f > 0$ *such that*

$$x \cdot \nabla u(x) < 0 \qquad \text{for all } x \in \mathbf{B} \text{ such that } \gamma_f < |x| < 1\,.$$

Guido Sweers [16] provided an explicit example of a non-decreasing radial solution of a linear biharmonic problem in $\mathbf{B} \subset \mathbb{R}^2$ (equation (1.2) with $m = 2$). In this section we take advantage of his example and give numerical evidence to claim that Proposition 4.1 might be complemented with the statement that γ_f can be made arbitrarily small (possibly zero) provided that $\|(f')_-\|_{L^\infty(\mathbb{R}_+)}$ is sufficiently small.

Consider the following radial function defined in $\mathbb{R}^4$ (the conformal dimension for the biharmonic operator):

$$v(x) = \sum_{k=0}^{\infty} a_k \, r^{4k+2} \doteq \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{4k+2}} \frac{r^{4k+2}}{(2k+2)!(2k+1)!} \qquad \forall x \in \mathbb{R}^4\,,$$

whose plot in the interval $[-10, 10]$ is displayed in Figure 6 below.

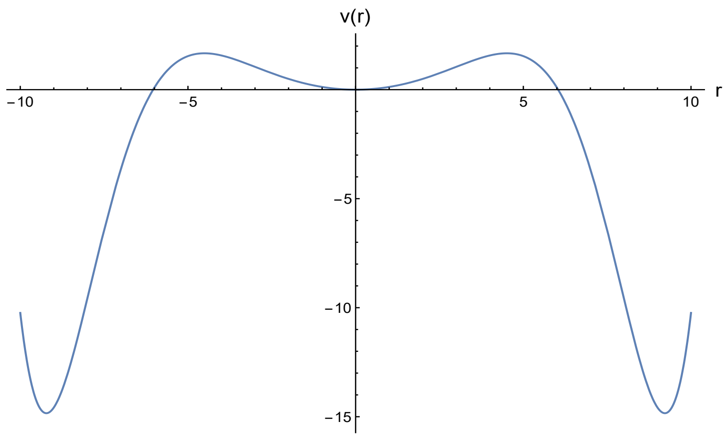


Figure 6. Plot of v in the interval $[-10, 10]$.

Since $m = 2$ and $n = 4$, the biharmonic operator in radial coordinates becomes

$$\Delta^2\psi(r) = \psi^{(4)}(r) + \frac{6}{r}\psi'''(r) + \frac{3}{r^2}\psi''(r) - \frac{3}{r^3}\psi'(r) \qquad \forall r \in (0, 1).$$

After noticing that

$$a_{k+1} = -\frac{1}{64(k+2)(k+1)(2k+3)^2}\, a_k \qquad \forall k \in \mathbb{N}\,,$$

one readily sees that

$$\Delta^2 v = -v \quad \text{in} \quad \mathbb{R}^4\,.$$

Let $r_0 \approx 9.2218$ be the first nonzero local minimum of v; numerically, we find that $v(r_0) \approx -14.8388$. Then define the function

$$u(r) \doteq v(r_0 r) - v(r_0) \qquad \forall r \in [0, 1]\,, \tag{4.1}$$

which is radially symmetric and strictly positive in $[0, 1)$, *but not decreasing*, in the interval $[0, 1]$; see the left picture in Figure 7 below.

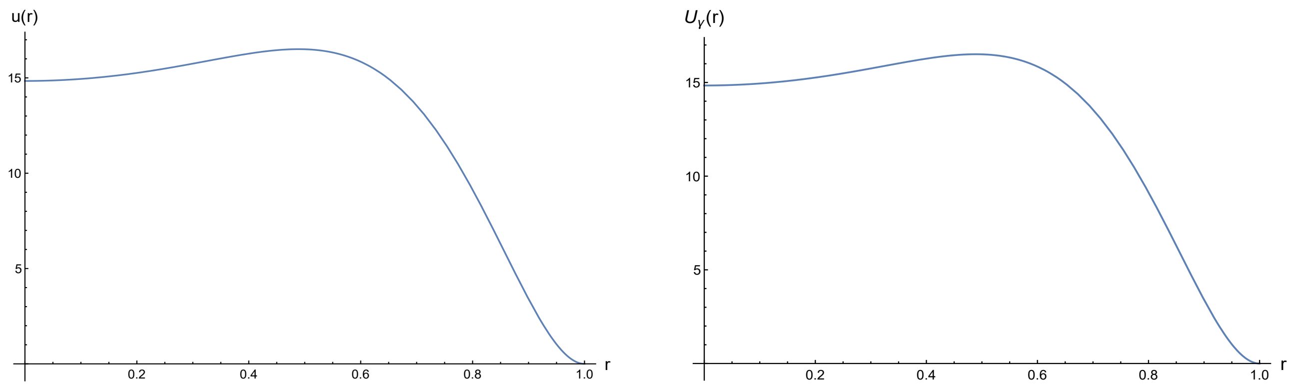


Figure 7. Left: graph of u in (4.1). Right: graph of the numerical solution of (4.2), for $\gamma = 1$.

By defining

$$f(w) = r_0^4(|v(r_0)| - w) \qquad \forall w \geq 0\,,$$

we observe that $f'(w) \equiv -r_0^4 < 0$, so that f is decreasing and (2.2) is satisfied. By Theorem 2.1 we then know that there exists at most one strong solution of the following problem:

$$\begin{cases} \Delta^2 u = f(u) & \text{in} \quad \mathbf{B} \\ u = \dfrac{\partial u}{\partial r} = 0 & \text{on} \quad \partial\mathbf{B}. \end{cases}$$

The solution is radially symmetric and a simple computation shows that it coincides with u defined in (4.1). Note that $u(0) = |v(r_0)|$ and $u(0.5) > 16.5 > |v(r_0)|$, so that $f(u)$ is sign-changing in $[0, 1]$.

We now modify f in order to reduce $\|(f')_-\|_{L^\infty(\mathbb{R}_+)}$. We take

$$f_\gamma(w) \doteq r_0^4(|v(r_0)| - \gamma w) \qquad \forall w \geq 0\,,$$

being $\gamma > 0$ a variable parameter. Then we consider the problem

$$\begin{cases} \Delta^2 U = f_\gamma(U) & \text{in} \quad \mathbf{B} \\ U = \dfrac{\partial U}{\partial r} = 0 & \text{on} \quad \partial\mathbf{B}, \end{cases} \tag{4.2}$$

and prove the following result:

Proposition 4.2. *For every $\gamma > 0$, the unique strong solution $U_\gamma \in H_0^2(\mathbf{B}) \cap L^\infty(\mathbf{B})$ of (4.2) (which is radially symmetric) is analytic and can be written as*

$$U_\gamma(r) = \sum_{k=0}^{\infty} A_k(\gamma) r^{2k} = A_0(\gamma) + A_1(\gamma)r^2 + A_2(\gamma)r^4 + A_3(\gamma)r^6 + ... \qquad \forall r \in [0,1], \tag{4.3}$$

for some coefficients $\{A_k(\gamma)\}_{k\in\mathbb{N}} \subset \mathbb{R}$ that satisfy the following properties:

$$\begin{cases} A_0(\gamma) = \displaystyle\sum_{k=2}^{\infty} (k-1)A_k(\gamma)\,, & A_1(\gamma) = -\displaystyle\sum_{k=2}^{\infty} kA_k(\gamma)\,, \\ A_2(\gamma) = \dfrac{r_0^4}{192}(|v(r_0)| - \gamma A_0(\gamma))\,, & A_k(\gamma) = -\dfrac{\gamma r_0^4}{16k^2(k^2-1)} A_{k-2}(\gamma) \qquad \forall k \geq 3\,. \end{cases}$$

Moreover, there exists $\gamma_0 > 0$ such that, for every $\gamma \in (0, \gamma_0)$, the function U_γ is positive in $[0, 1)$ and strictly decreasing in the radial variable.

Proof. For every $\gamma > 0$ we notice that the condition (2.2) is satisfied. Then, Theorem 2.1 guarantees the existence of at most one strong solution $U_\gamma \in H_0^2(\mathbf{B}) \cap L^\infty(\mathbf{B})$ to problem (4.2) which is, moreover, radially symmetric and analytic in $\overline{\mathbf{B}}$. Problem (4.2) in radial coordinates reads:

$$\begin{cases} U_\gamma^{(4)}(r) + \dfrac{6}{r} U_\gamma'''(r) + \dfrac{3}{r^2} U_\gamma''(r) - \dfrac{3}{r^3} U_\gamma'(r) = r_0^4 \Big(|v(r_0)| - \gamma U_\gamma(r)\Big) & \forall r \in (0,1) \\ U_\gamma(1) = U_\gamma'(1) = 0\,. \end{cases} \tag{4.4}$$

Upon substitution of (4.3) into $(4.4)_1$ we obtain

$$192 A_2(\gamma) + 16 \sum_{k=3}^{\infty} A_k(\gamma) k^2 (k^2-1) r^{2k-4} = r_0^4(|v(r_0)| - \gamma A_0(\gamma)) - \gamma r_0^4 \sum_{k=3}^{\infty} A_{k-2}(\gamma) r^{2k-4} \qquad \forall r \in (0,1),$$

which yields the identities

$$A_2(\gamma) = \frac{r_0^4}{192}(|v(r_0)| - \gamma A_0(\gamma))\,, \qquad A_k(\gamma) = -\frac{\gamma r_0^4}{16k^2(k^2-1)} A_{k-2}(\gamma) \qquad \forall k \geq 3\,. \tag{4.5}$$

Moreover, the boundary conditions in $(4.4)_2$ imply that

$$\sum_{k=0}^{\infty} A_k(\gamma) = 0\,, \ \sum_{k=1}^{\infty} kA_k(\gamma) = 0 \implies A_1(\gamma) = -\sum_{k=2}^{\infty} kA_k(\gamma)\,, \ A_0(\gamma) = \sum_{k=2}^{\infty} (k-1)A_k(\gamma)\,.$$

In connection with (3.23) we define

$$U_*(r) \doteq \sigma_0(1-r^2)^2 \doteq \frac{r_0^4|v(r_0)|}{64}(1-r^2)^2 \qquad \forall r \in [0,1].$$

By Corollary (3.4) and by continuous dependence, we have that

$$U_\gamma \to U_* \quad \text{in } C^3(\overline{\mathbf{B}}) \text{ as } \gamma \to 0^+ ;$$

in fact, the convergence holds in a stronger norm. Notice that

$$U_*''(0) = -4\sigma_0 < 0\,, \qquad U_*''(1) = 8\sigma_0 > 0\,.$$

Thus, by uniform convergence of U_γ'' and U_γ''', there exist $\delta_0 \in (0,1)$ and $\gamma_0 > 0$ such that:

$$U_\gamma''(r) \le -2\sigma_0 \quad \text{if} \quad r \in [0,\delta_0], \qquad U_\gamma''(r) \ge 4\sigma_0 \quad \text{if} \quad r \in [1-\delta_0,1], \qquad \forall \gamma \in (0,\gamma_0).$$

Since $U_\gamma'(0) = U_\gamma'(1) = 0$ for every $\gamma > 0$, this implies that

$$U_\gamma'(r) \le -2\sigma_0 r < 0 \quad \text{if} \quad r \in (0,\delta_0], \qquad U_\gamma'(r) \le 4\sigma_0(r-1) < 0 \quad \text{if} \quad r \in [1-\delta_0,1), \qquad \forall \gamma \in (0,\gamma_0).$$

Moreover, by uniform convergence of U_γ', there exists $\gamma_1 > 0$ such that $U_\gamma'(r) < 0$ if $r \in [\delta_0, 1-\delta_0]$, for every $\gamma \in (0,\gamma_1)$. Therefore $U_\gamma' < 0$ in $(0,1)$ whenever $\gamma < \min\{\gamma_0, \gamma_1\}$, which also implies that U_γ is strictly positive in $[0,1)$, since $U_\gamma(1) = 0$. □

In fact, the function U_γ is also unique within the class of weak solutions. Proposition 4.2 does not come unexpected, since the solution of (4.4) is closely related to a Bessel function, see [16]. Therefore, the number of critical points of U_γ is increasing with respect to γ and becomes arbitrarily large as $\gamma \to \infty$, whereas there are no critical points for any sufficiently small γ. By taking $\gamma = 1$, we numerically obtain the solution displayed in the right picture of Figure 7, which represents the function u in (4.1). Several numerical simulations were performed in order to yield a radially decreasing solution of problem (4.2). The obtained results may be summarized as follows:

- if $\gamma \le 0.317$, then the associated solution U_γ of (4.2) is radially symmetric and strictly decreasing in the radial variable in the interval $[0,1]$ (and therefore, positive), see Figure 8;
- if $\gamma \ge 0.318$, then the associated solution U_γ of (4.2) is radially symmetric but not strictly decreasing in the interval $[0,1]$, see Figure 9.

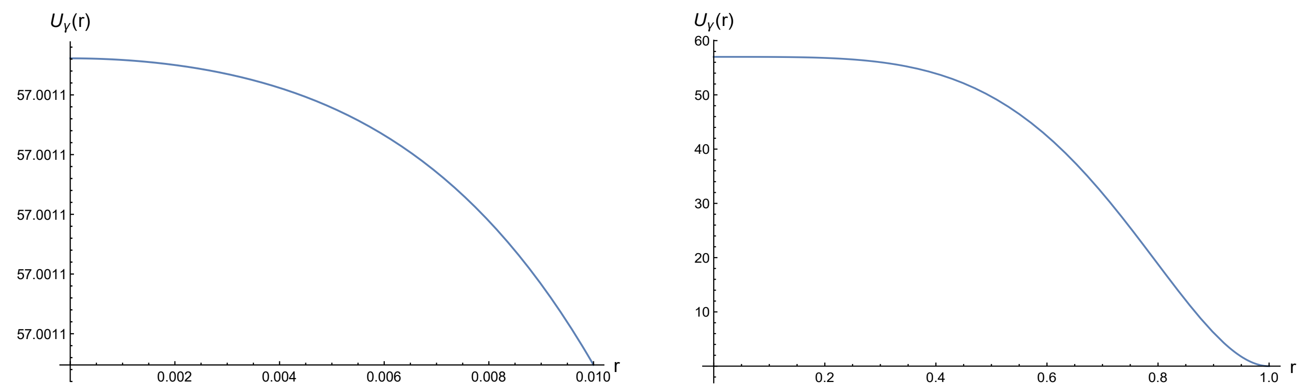


Figure 8. Plots of U_γ in the intervals $[0, 0.01]$ and $[0, 1]$, for $\gamma = 0.317$.

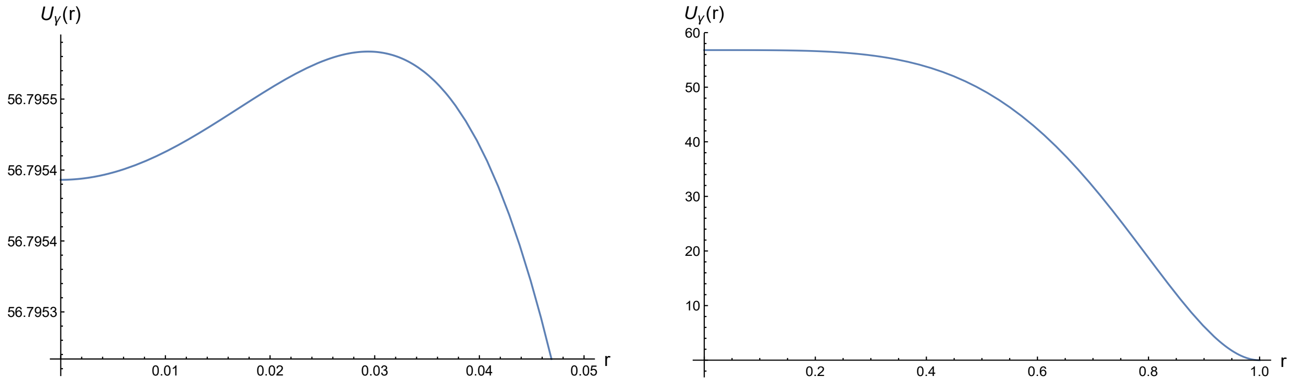


Figure 9. Plots of U_γ in the intervals $[0, 0.05]$ and $[0, 1]$, for $\gamma = 0.318$.

We numerically determined the position of the maximum point $r_\gamma \in [0, 1]$ of U_γ as a function of $\gamma \in [0, 1]$, see the Table and Figure below. From these data we deduce that the map $\gamma \in [0, 1] \mapsto r_\gamma$ is increasing (but not continuous!), giving thus consistency to the discussion following Proposition 4.1. Finally, in connection with the condition (3.25) given in Propositions 3.7 and 4.1, we observe that, for the unique radial solution $U_\gamma \in H_0^2(\mathbf{B})$ of problem (4.2), the following inequality holds:

$$f_\gamma(U_\gamma(r)) \geq r_0^4 \left[|v(r_0)| - \gamma \max_{t \in [0,1]} |U_\gamma(t)| \right] \qquad \forall r \in [0, 1].$$

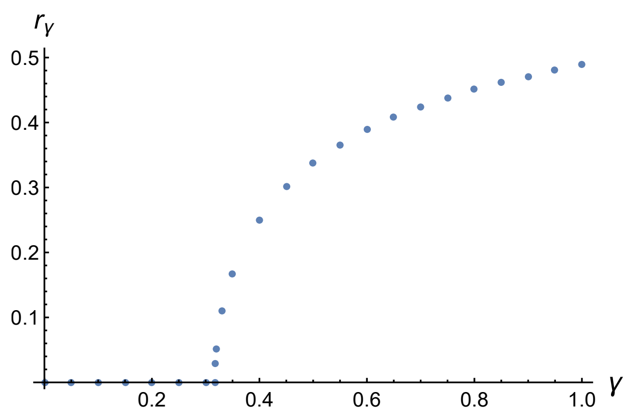
From the Table and Figure displayed below we further deduce that

$$f_\gamma(U_\gamma(r)) \geq r_0^4 \left(|v(r_0)| - \gamma U_\gamma(0) \right) \qquad \forall r \in [0, 1], \quad \forall \gamma \in [0, 0.317]; \tag{4.6}$$

recall that U_γ is nonnegative in $[0, 1]$ for every $\gamma \in [0, 0.317]$. Numerical experiments yield the value of $\gamma = 0.057$ as a threshold for ensuring the positivity of the right-hand side of (4.6), in the sense that

$$f_{0.057}(U_{0.057}(r)) \geq 0.014 \qquad \text{and} \qquad f_{0.058}(U_{0.058}(r)) \geq -0.092 \qquad \forall r \in [0, 1].$$

Value of γ	Maximum point r_γ
≤ 0.317	0
0.318	0.029
0.32	0.052
0.35	0.167
0.4	0.25
0.45	0.301
0.5	0.338
0.55	0.366
0.6	0.389
0.65	0.408
0.7	0.424
0.75	0.438
0.8	0.451
0.85	0.462
0.9	0.471
0.95	0.481
1	0.489



Position of the maximum point $r_\gamma \in [0, 1]$ of U_γ as a function of $\gamma \in [0, 1]$.

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Conflict of interest

The authors declare no conflict of interest.

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