Birkhoff Normal Form and Long Time Existence for Periodic Gravity Water Waves

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Abstract

We consider the gravity water waves system with a periodic one-dimensional interface in infinite depth and give a rigorous proof of a conjecture of Dyachenko-Zakharov [16] concerning the approximate integrability of these equations. More precisely, we prove a rigorous reduction of the water waves equations to its integrable Birkhoff normal form up to order 4. As a consequence, we also obtain a long-time stability result: periodic perturbations of a flat interface that are initially of size ε remain regular and small up to times of order ε^{-3}. This time scale is expected to be optimal. © 2022 The Authors. *Communications on Pure and Applied Mathematics* published by Wiley Periodicals LLC.

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1 Introduction

We consider an incompressible and irrotational perfect fluid, under the action of gravity, occupying at time t a two-dimensional domain with infinite depth, periodic in the horizontal variable, given by

(1.1) $$\mathcal{D}_\eta := \{(x, y) \in \mathbb{T} \times \mathbb{R} \colon -\infty < y < \eta(t, x)\big\}, \quad \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}),$$

where η is a regular enough function. The velocity field in the time-dependent domain $\mathcal{D}_\eta$ is the gradient of a harmonic function Φ, called the velocity potential. The time evolution of the fluid is determined by a system of equations for the two functions $(t, x) \to \eta(t, x)$, $(t, x, y) \to \Phi(t, x, y)$. Following Zakharov [36], given $\eta(t, x)$ and the restriction $\psi(t, x) := \Phi(t, x, \eta(t, x))$ of the velocity potential at the top boundary, one can recover $\Phi(t, x, y)$ as the unique solution of the elliptic problem

(1.2) $$\Delta\Phi = 0 \text{ in } \mathcal{D}_\eta, \quad \partial_y \Phi \to 0 \text{ as } y \to -\infty, \quad \Phi = \psi \text{ on } \{y = \eta(t, x)\}.$$

The (η, ψ) variables then satisfy the gravity water waves system

(1.3) $$\begin{cases} \partial_t \eta = G(\eta)\psi \\ \partial_t \psi = -g\eta - \dfrac{1}{2}\psi_x^2 + \dfrac{1}{2}\dfrac{(\eta_x \psi_x + G(\eta)\psi)^2}{1 + \eta_x^2} \end{cases}$$

where $G(\eta)\psi$ is the Dirichlet-Neumann operator

(1.4) $$G(\eta)\psi := \sqrt{1 + \eta_x^2}(\partial_n \Phi)\Big|_{y=\eta(t,x)} = (\partial_y \Phi - \eta_x \partial_x \Phi)(t, x, \eta(t, x)),$$

and n is the outward unit normal at the free interface $y = \eta(t, x)$. Without loss of generality, we set the gravity constant to $g = 1$.

It was first observed by Zakharov [36] that (1.3) is the Hamiltonian system

(1.5) $$\partial_t \eta = \nabla_\psi H(\eta, \psi), \quad \partial_t \psi = -\nabla_\eta H(\eta, \psi),$$

where ∇ denotes the L^2-gradient, with Hamiltonian

$$H(\eta, \psi) := \frac{1}{2} \int_{\mathbb{T}} \psi \, G(\eta)\psi \, dx + \frac{1}{2} \int_{\mathbb{T}} \eta^2 \, dx \tag{1.6}$$

given by the sum of the kinetic and potential energy of the fluid. Note that the mass $\int_{\mathbb{T}} \eta \, dx$ is a prime integral of (1.3) and, with no loss of generality, we can fix it to zero by shifting the y-coordinate. Moreover, (1.3) is invariant under spatial translations and Noether's theorem implies that the momentum $\int_{\mathbb{T}} \eta_x(x)\psi(x)dx$ is a prime integral of (1.5).

Let $H^s(\mathbb{T}) := H^s$, $s \in \mathbb{R}$, be the Sobolev spaces of 2π-periodic functions of x. It is natural to consider η in the subspace of zero average functions $H_0^s(\mathbb{T}) \subset H^s(\mathbb{T})$, and ψ in the standard homogeneous Sobolev space $\dot{H}^s(\mathbb{T})$.[1] Moreover, since the space averages $\widehat{\eta}_0(t) := \frac{1}{2\pi} \int_{\mathbb{T}} \eta(t, x)dx$, $\widehat{\psi}_0(t) := \frac{1}{2\pi} \int_{\mathbb{T}} \psi(t, x)dx$ evolve according to the decoupled equations[2] $\partial_t \widehat{\eta}_0(t) = 0$, $\partial_t \widehat{\psi}_0(t) = -g\widehat{\eta}_0(t)$, we may restrict, with no loss of generality, to the invariant subspace with $\widehat{\eta}_0(t) = \widehat{\psi}_0(t) = 0$.

The main result of this paper (Theorem 1.1) proves a conjecture of Dyachenko-Zakharov [16], supported by Craig-Worfolk [12] on the approximate integrability of the water waves system (1.3). More precisely, we prove that (1.3) can be conjugated, via a bounded and invertible transformation in a neighborhood of the origin, to its Hamiltonian Birkhoff normal form, up to order 4. This latter—in the PDE literature sometimes referred to as the "resonant system"—was formally computed in [12, 16] (see also [11]) and, remarkably, shown to be integrable. Despite several attempts, the formal approach in [12, 16] has never been translated into a rigorous result. The proof we give in this paper is actually based on a completely different approach to the Birkhoff normal form reduction, which we describe at the end of this introduction. As a consequence of Theorem 1.1, we also obtain a long-time stability result (Theorem 1.2): periodic perturbations that are initially ε-close to the flat equilibrium lead to solutions that remain regular and small for times of order ε^{-3}. This time scale is expected to be optimal. These results have been announced in [8].

While in recent years several results have been obtained for quasilinear equations with initial data that decay sufficiently fast at infinity, fewer results are available in the periodic setting. In this context, the achievement of Birkhoff normal forms reductions is a key step to provide an accurate description of the long-term dynamics of evolution PDEs like (1.3). We also remark that the stability result in Theorem 1.2 is obtained by completely different mechanisms compared to most

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[1] The spaces $\dot{H}^s(\mathbb{T})$ and $H_0^s(\mathbb{T})$ are isometric. Thus we will conveniently identify ψ with a zero average function.

[2] Since the domain $\mathcal{D}_\eta$ has infinite depth, if Φ solves (1.2), then $\Phi_c(x, y) := \Phi(x, y - c)$, $\forall c \in \mathbb{R}$, solves the same problem in $\mathcal{D}_{\eta+c}$ assuming the Dirichlet datum ψ at the free boundary $\eta + c$. Therefore $G(\eta + c) = G(\eta)$ and $\int_{\mathbb{T}} \nabla_\eta K \, dx = 0$ where $K := \frac{1}{2} \int_{\mathbb{T}} \psi G(\eta)\psi \, dx$ denotes the kinetic energy.

recent works; see, for example, [21,23,26,34], which obtain a shorter ε^{-2} stability time in the absence of cubic resonances (see (1.25)). Indeed, we deduce Theorem 1.2 by the *complete conjugation* of the water waves equations (1.3) to its integrable Birkhoff normal form.

1.1 Main results

We denote the horizontal and vertical components of the velocity field at the free interface by

$$V := V(\eta, \psi) := \psi_x - \eta_x B, \quad B := B(\eta, \psi) := \frac{G(\eta)\psi + \eta_x \psi_x}{1 + \eta_x^2}, \tag{1.7}$$

and the "good unknown" of Alinhac

$$\omega := \psi - Op^{\mathrm{BW}}(B(\eta, \psi))\eta, \tag{1.8}$$

as introduced in [3] (see Definition 2.4 for the definition of the paradifferential operator Op^{BW}).

To state our first main result concerning the rigorous reduction to Birkhoff normal form of the system (1.3), let us assume that, for N large enough and some $T > 0$, we have a classical solution

$$(\eta, \psi) \in C^0([-T, T]; H^{N+\frac{1}{4}} \times H^{N+\frac{1}{4}}) \tag{1.9}$$

of the Cauchy problem for (1.3) with the initial height satisfying

$$\int_{\mathbb{T}} \eta(0, x) dx = 0. \tag{1.10}$$

The existence of such a solution for small enough T is guaranteed by the local well-posedness theory (see, for example, Theorem 1.3) under the regularity assumption $(\eta, \psi, V, B)_{|t=0} \in X^{N-(1/4)}$ where we denote

$$X^s := H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s. \tag{1.11}$$

Defining the complex scalar unknown

$$u := \tfrac{1}{\sqrt{2}}|D|^{-\frac{1}{4}}\eta + \tfrac{\mathrm{i}}{\sqrt{2}}|D|^{\frac{1}{4}}\omega, \tag{1.12}$$

we deduce, by (1.9), that $u \in C^0([-T, T]; H^N)$, and u solves an equation of the form $\partial_t u + \mathrm{i}|D|^{1/2}u = M_{\geq 2}(u, \overline{u})$ where $M_{\geq 2}(u, \overline{u})$ is a fully nonlinear vector field that contains up to first-order derivatives of u. Moreover, since the zero average condition (1.10) is preserved by the flow of (1.3), it follows that

$$\int_{\mathbb{T}} u(t, x) dx = 0 \quad \forall t \in [-T, T]. \tag{1.13}$$

This is our first main result.

THEOREM 1.1 (Birkhoff normal form). *Let u be defined as in* (1.12)*, with ω as in* (1.8)*, for* (η, ψ) *solution of* (1.3) *satisfying* (1.9)–(1.10)*. There exist $N \gg K \gg 1$ and $0 < \overline{\varepsilon} \ll 1$ such that, if*

$$\sup_{t \in [-T,T]} \sum_{k=0}^{K} \left\| \partial_t^k u(t) \right\|_{\dot{H}^{N-k}} \leq \overline{\varepsilon}, \tag{1.14}$$

then there exist a bounded and invertible transformation $\mathfrak{B} = \mathfrak{B}(u)$ of $\dot{H}^N$, which depends (*nonlinearly*) *on u, and a constant $C := C(N) > 0$ such that*

$$\|\mathfrak{B}(u)\|_{\mathcal{L}(\dot{H}^N, \dot{H}^N)} + \|(\mathfrak{B}(u))^{-1}\|_{\mathcal{L}(\dot{H}^N, \dot{H}^N)} \leq 1 + C\|u\|_{\dot{H}^N}, \tag{1.15}$$

and the variable $z := \mathfrak{B}(u)u$ satisfies the equation

$$\partial_t z = -\mathrm{i}\partial_{\overline{z}} H_{ZD}(z, \overline{z}) + \mathcal{X}_{\geq 4}^{+} \tag{1.16}$$

where

(1) *the Hamiltonian H_{ZD} has the form*

$$H_{ZD} = H_{ZD}^{(2)} + H_{ZD}^{(4)}, \quad H_{ZD}^{(2)}(z, \overline{z}) := \frac{1}{2} \int_{\mathbb{T}} \left| |D|^{\frac{1}{4}} z \right|^2 dx, \tag{1.17}$$

with

$$\begin{aligned} H_{ZD}^{(4)}(z, \overline{z}) &:= \frac{1}{4\pi} \sum_{k \in \mathbb{Z}} |k|^3 \big(|z_k|^4 - 2|z_k|^2 |z_{-k}|^2 \big) \\ &+ \frac{1}{\pi} \sum_{\substack{k_1, k_2 \in \mathbb{Z}, \\ \mathrm{sign}(k_1) = \mathrm{sign}(k_2), \\ |k_2| < |k_1|}} |k_1||k_2|^2 \big(-|z_{-k_1}|^2 |z_{k_2}|^2 + |z_{k_1}|^2 |z_{k_2}|^2 \big) \end{aligned} \tag{1.18}$$

where z_k denotes the k^{th} Fourier coefficient of the function z.

(2) $\mathcal{X}_{\geq 4}^{+} := \mathcal{X}_{\geq 4}^{+}(u, \overline{u}, z, \overline{z})$ *is a quartic nonlinear term satisfying, for some $C := C(N) > 0$, the "energy estimate"*

$$\mathrm{Re} \int_{\mathbb{T}} |D|^N \mathcal{X}_{\geq 4}^{+} \cdot \overline{|D|^N z} \, dx \leq C \|z\|_{\dot{H}^N}^5. \tag{1.19}$$

The main point of Theorem 1.1 is the construction of the *bounded* and *invertible* transformation $\mathfrak{B}(u)$ in (1.15) which recasts the water waves system (1.3) into the equation (1.16)–(1.19). Theorem 1.1 rigorously relates the flow of the full water waves system (1.3) to the flow of the system (1.16), which is made by the explicit Hamiltonian term $-\mathrm{i}\partial_{\overline{z}} H_{ZD}$ plus remainders of higher homogeneity. These remainders are under full control thanks to the energy estimates (1.19). The Hamiltonian H_{ZD} is *integrable*, as observed in [12, 16], and its flow preserves all the Sobolev norms; see Theorem 1.4. Thus, relying on Theorem 1.1, we can prove the following result:

THEOREM 1.2 (Long-time existence). *There exists*[3] $s_0 > 0$ *such that, for all* $s \geq s_0$*, there is* $\varepsilon_0 > 0$ *such that, for any initial data* (η_0, ψ_0) *satisfying* (*recall* (1.11))

$$\|(\eta_0, \psi_0, V_0, B_0)\|_{X^s} \leq \varepsilon \leq \varepsilon_0, \quad \int_{\mathbb{T}} \eta_0(x)dx = 0, \tag{1.20}$$

where $V_0 := V(\eta_0, \psi_0)$*,* $B_0 := B(\eta_0, \psi_0)$ *are defined by* (1.7)*, the following holds*: *there exist constants* $c > 0$ *and* $C > 0$*, and a unique classical solution* $(\eta, \psi, V, B) \in C^0([-T_\varepsilon, T_\varepsilon], X^s)$ *of the water waves system* (1.3) *with initial condition* $(\eta, \psi)_{|t=0} = (\eta_0, \psi_0)$ *with*

$$T_\varepsilon \geq c\varepsilon^{-3}, \tag{1.21}$$

satisfying

$$\begin{aligned} \sup_{[-T_\varepsilon, T_\varepsilon]} \big(\|(\eta, \psi)\|_{H^s \times H^s} + \|(V, B)\|_{H^{s-1} \times H^{s-1}}\big) \leq C\varepsilon, \\ \int_{\mathbb{T}} \eta(t, x)dx = 0. \end{aligned} \tag{1.22}$$

Let us briefly describe some of the key points of the above results:

(1) To our knowledge, Theorem 1.2 is the first normal form ε^{-3} existence result for dispersive PDEs with a quadratic nonlinearity in the *absence of external parameters* (and excluding equations admitting conserved quantities that control high Sobolev norms). One of the main difficulties is that (1.3) presents a family of nontrivial quartic resonances, the Benjamin-Feir resonances (1.27), which are potentially a strong obstruction to controlling the dynamics for times of order ε^{-3}. For parameter-depending PDEs with external parameters one can avoid such nontrivial resonances by modulating the dispersion relation, cf. paragraph "Parameters" below Theorem 1.4.

(2) The stability time $\sim \varepsilon^{-3}$ in Theorem 1.2 is expected to be *optimal* in view of the presence of quintic resonances as exhibited by Craig-Worfolk [12] and Dyachenko-Lvov-Zakharov [15]. In other words, one cannot expect a stability time $\sim \varepsilon^{-4}$ for all initial data.

(3) We develop a general method to justify the formal/heuristic calculations of the Hamiltonian Birkhoff normal form of any Hamiltonian PDE. Applying several nonlinear flow conjugation maps (generated by paradifferential or smoothing operators) we transform (1.3) in Poincaré-Birkhoff normal form (see (1.29)–(1.30)), which is not a priori explicit. Then, a key step in the proof of Theorem 1.1 is a *normal form uniqueness argument* to identify the cubic Poincaré-Birkhoff resonant system with the Hamiltonian equations associated to the Hamiltonian H_{ZD} computed by a formal expansion in [11, 12, 15, 16] (see (1.17)–(1.18)). The uniqueness

[3] We did not try to optimize the regularity index s_0. With a more careful analysis one could likely pick some $s_0 \leq 30$. In any case, the Sobolev regularity is an unimportant aspect in the study of the long-time behavior of classical solutions to quasilinear problems.

of the normal form is based on the absence of cubic resonances. An inspiration for this identification argument is the famous Moser's indirect proof of the convergence of the Lindsted power series to the KAM quasi-periodic solutions [30].

We also make a couple of technical comments about the rigorous conjugation of (1.3) to its cubic Poincaré-Birkhoff normal form.

- Besides the resonant interactions, one also needs to pay attention to *near resonances*, which can prevent the boundedness of Poincaré-Birkhoff normal form transformations. We overcome this issue by performing an iterative reduction of the water waves equations (1.3) to *constant integrable* coefficients, modulo smoothing remainders; see (1.28). In this process we identify and exploit *specific algebraic cancellations* of (1.3) in infinite depth.
- Since the dispersion relation $\sqrt{|k|}$ is sublinear, our reduction procedure substantially differs from the recent work of Berti-Delort [6], where the dispersion relation $\sim |k|^{3/2}$ is superlinear. Moreover, in contrast to [6] we have to deal with nontrivial resonances (the Benjamin-Feir resonances) that we cannot eliminate modulating the surface tension parameter as in [6], and we do not restrict to even initial data. However, we still employ the paradifferential framework of [6] as it readily provides us with a convenient paralinearization of the Dirichlet-Neumann operator (1.4).

We have chosen to formulate the long-time existence result of Therem 1.2 using the original symplectic variables (η, ψ) as well as the velocity components (V, B) in (1.7) consistently with the following local existence result.

THEOREM 1.3 (Local existence [1]). *Let $s > 3/2$ and consider (η_0, ψ_0) such that $(\eta_0, \psi_0, V_0, B_0)$ is in X^s; see* (1.11).

Then there exists $T_{\rm loc} > 0$ such that the Cauchy problem for (1.3) *with initial data (η_0, ψ_0) has a unique solution $(\eta, \psi) \in C^0([0, T_{\rm loc}], H^{s+(1/2)} \times H^{s+(1/2)})$ with $(V, B) \in C^0([0, T_{\rm loc}], H^s \times H^s)$. Moreover, denoting by T_* the maximal time of existence of (η, ψ), if, for some $T_0 > 0$,*

$$\sup_{[0,T_0]} \|(\eta, \psi, V, B)(t)\|_{X^5} < +\infty, \tag{1.23}$$

then $T_0 < T_*$ and $\sup_{[0,T_0]} \|(\eta, \psi, V, B)(t)\|_{X^s} < +\infty$.

Theorem 1.3 is essentially the local existence result [1, theorem 1.2], stated in the case of the torus $\mathbb{T}$, for a fluid in infinite depth. The result is based on energy methods for hyperbolic symmetrizable quasi-linear systems, which are the same in $\mathbb{T}^d$ and in $\mathbb{R}^d$. By time reversibility, the solutions of (1.3) are defined in a symmetric interval $[-T, T]$. In agreement with Theorem 1.3, at any time t the solution (η, ψ, V, B) of Theorem 1.2 belongs to the same space X^s as the initial datum (see (1.20)), but in (1.22) we control only a weaker norm of the solution. This is a well-known phenomenon of the pure gravity water waves equations (see, for instance, [1, 2]): in the variables (η, ω) the Sobolev regularity of the solution is preserved along the flow, but there is a loss of derivatives in passing to the unknowns (η, ψ). The weaker bound (1.22) is still more than sufficient to apply the continuation criterion of Theorem 1.3.

1.2 Some literature, the Dyachenko-Zakharov conjecture, and some ideas of the proof

The local well-posedness of the water waves and free boundary Euler equations has been addressed by many authors (see, for example, [1, 10, 31, 33]), and it is presently well understood; we refer to the review [27, sec. 2] for an extensive list of references. In particular, for smooth enough initial data that are of size ε, the solutions exist and stay regular for times of order ε^{-1}. When the horizontal variable $x \in \mathbb{R}^d$, for sufficiently small and spatially localized initial data, it is possible to construct global-in-time solutions exploiting the dispersive properties of the flow. Results for (1.3) have been proved in [2, 22, 24, 32, 34] and in [20, 35] for the 3-D case. We refer again to [27] and to the introduction of [14], for a more detailed presentation of these results.

Long-time existence on tori. When the horizontal variable $x \in \mathbb{T}^d$, there are no dispersive effects that control solutions for long times, and a tool to extend the lifespan of solutions is normal form theory. To explain the idea, let us consider a generic evolution equation of the form

$$\partial_t u + \mathrm{i}\omega(D)u = Q(u, \overline{u}), \quad u(t=0) = u_0, \quad \|u_0\|_{H^N} \le \varepsilon, \tag{1.24}$$

where $\omega(D)$ is a real Fourier multiplier, and Q is a quadratic nonlinearity that depends on $(u, \overline{u})$ and their derivatives in a quasi-linear way. In the case of (1.3) the dispersion relation is $\omega(k) = \sqrt{|k|}$. An energy estimate for (1.24) of the form $\frac{d}{dt}E(t) \lesssim \|u(t)\|_{H^N} E(t)$, where $E(t) \approx \|u(t)\|^2_{H^N}$, allows the construction of local solutions on time scales of $O(\varepsilon^{-1})$. In order to prove existence for times of $O(\varepsilon^{-2})$ one can try to obtain a quartic energy inequality of the form $\frac{d}{dt}E(t) \lesssim \|u(t)\|^2_{H^N} E(t)$. For (1.3) inequalities of this type have been proven in [2, 21, 24, 34]; see also [23, 26] for capillary waves, and [7] for gravity-capillary water waves (relying on methods developed in this paper). Although some delicate analysis is needed due to the quasilinearity of the PDE, the possibility of proving such quartic energy estimates ultimately relies on the absence of *3-waves resonances*, that is, nonzero integers (n_1, n_2, n_3) solving, for some $\sigma_j \in \{+, -\}$,

$$\sigma_1\omega(n_1) + \sigma_2\omega(n_2) + \sigma_3\omega(n_3) = 0, \quad \sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3 = 0. \tag{1.25}$$

The Dyachenko-Zakharov conjecture. In order to extend the lifespan of solutions of (1.3) up to times of order ε^{-3} one may try to obtain a quintic energy estimate like $\frac{d}{dt}E(t) \lesssim \|u(t)\|^3_{H^N} E(t)$. At a formal level, this would be possible in the absence of nontrivial *4-waves resonances*, namely integer solutions of

$$\begin{aligned} &\sigma_1\omega(n_1)+\sigma_2\omega(n_2)+\sigma_3\omega(n_3)+\sigma_4\omega(n_4)=0,\\ &\sigma_1 n_1+\sigma_2 n_2+\sigma_3 n_3+\sigma_4 n_4=0, \end{aligned} \tag{1.26}$$

which do not appear in pairs with corresponding opposite signs. This property is not satisfied by the gravity water waves system (1.3). Indeed, as shown in [16], there are many solutions to (1.26). For example, if $\sigma_1=\sigma_3=1=-\sigma_2=-\sigma_4$, in addition to the trivial solutions $(n_1,n_2,n_3,n_4)=(k,k,j,j)$, there is the two-parameter family of solutions, called *Benjamin-Feir resonances*,

$$\bigcup_{\lambda\in\mathbb{Z}\setminus\{0\},b\in\mathbb{N}} \begin{aligned} &\{n_1=-\lambda b^2,\ n_2=\lambda(b+1)^2,\\ &\ n_3=\lambda(b^2+b+1)^2,\ n_4=\lambda(b+1)^2b^2\}. \end{aligned} \tag{1.27}$$

We then perform a diagonalization of the paralinearized system (3.7) up to smoothing remainders, obtaining system (3.33); see Proposition 3.10.

Applying a purely formal reduction to Birkhoff normal form up to order 4, the trivial resonances give rise to benign integrable monomials of the form $|z_k|^2|z_j|^2$, whereas the Benjamin-Feir resonances could give nonintegrable monomials of the form $z_{-\lambda b^2}\overline{z_{\lambda(b+1)^2}}z_{\lambda(b^2+b+1)^2}\overline{z_{\lambda(b+1)^2b^2}}$ + c.c. We refer to Section 6.1 for more details. A striking property proved in [16] (see also [11, 12]), is that the coefficients of the formal Birkhoff Hamiltonian that are supported on (1.27) are actually zero. In particular, one has the following:

THEOREM 1.4 (Formal integrability at order 4 [11,12,15,16]). *There exists a formal transformation Φ such that the truncation of $H\circ\Phi$ at order* 4 *of homogeneity is given by H_{ZD} as in* (1.18)*. Moreover, H_{ZD} is integrable* (*can be written in action-angle variables as* (6.18)) *and possesses the actions $|z_n|^2$, $n\in\mathbb{Z}\setminus\{0\}$, as prime integrals. In particular, its flow preserves all Sobolev norms.*

This result is a purely formal calculation, and no actual relation is established between the flow of H (which is well-posed for short times) and that of $H\circ\Phi$ or H_{ZD}. This is the goal of Theorem 1.1. Before describing some ideas for the proof of Theorem 1.1 we recall some other normal form results when the dispersion relation $\omega(k)$ in (1.24) depends on additional *parameters*.

Parameters. Under suitable nondegeneracy conditions one could prove that, for most values of the parameters, there are no N-waves resonances, that is, integer solutions of $\sum_{j=1}^N\sigma_j\omega(n_j)=0$, $\sum_{j=1}^N\sigma_j n_j=0$, except the trivial resonances. In this direction we mention the normal form results [5, 13] for Hamiltonian semilinear, resp., quasi-linear, Klein-Gordon equations. For 1-D, resp., 2-D, gravity-capillary water waves, the first ε^{-N}, resp., $\varepsilon^{-5/3+}$, existence result was proved in [6], resp., [28], for almost all values of the surface tension. See also [19] for fully nonlinear 1-D Schrödinger equations with an external convolution potential used as a parameter. We finally mention that time quasi-periodic, even in x, solutions have been constructed in [9], resp., [4], for 1-D gravity-capillary, resp., gravity, water waves using the surface tension, resp., the depth, as a parameter. We remark that a key point of Theorems 1.1 and 1.2 is the absence of external parameters.

1.3 Ideas of the proof of Theorem 1.1.

Step 1. Diagonalization up to smoothing remainders.

We begin our analysis by paralinearizing the water waves system (1.3), expressed in the complex variable $U := \begin{bmatrix} u \\ \overline{u} \end{bmatrix}$ introduced in (1.12); see Propositions 3.1 and 3.3.

Step 2. Reduction to constant, integrable coefficients and Poincaré-Birkhoff normal forms. In Section 4 we reduce all the paradifferential operators in the diagonalized system (3.33) to constant-in-x coefficients, which are "integrable" in the sense of Definition 4.1, up to smoothing remainders of homogeneity 2 and 3, and higher-order "admissible" contributions satisfying energy estimates of the form (1.19) (see Proposition 4.4). The most delicate reductions concern the highest-order fully nonlinear transport term $\mathrm{i}Op^{\mathrm{BW}}(V\xi)w$ and the quasilinear dispersive term $\mathrm{i}Op^{\mathrm{BW}}((1+a^{(0)})|\xi|^{\frac{1}{2}})w$ in the right-hand side of (3.33).

Let us briefly describe how to deal with the transport term. At the highest order, system (3.33) looks like $\partial_t w = -\mathrm{i}Op^{\mathrm{BW}}(V\xi)w + \cdots$ where $V = \mathtt{V}_1(u;x) + \mathtt{V}_2(u;x)$ and the functions $\mathtt{V}_1, \mathtt{V}_2$ are, respectively, linear and quadratic in u. In Sections 4.1 and 4.1 we construct a bounded and invertible map Φ^θ as the flow of the paradifferential operator $\mathrm{i}Op^{\mathrm{BW}}(b(u;\theta,x)\xi)$ where

$$b(u;\theta,x) = \frac{\beta(u;\theta,x)}{1+\theta\beta_x(u;\theta,x)} \quad \text{and} \quad \beta(u;x) = \beta_1(u;x) + \beta_2(u;x)$$

is a real-valued function to be determined. Here $\beta_1(u;x)$, $\beta_2(u;x)$ are functions respectively linear and quadratic in u. Setting $v = \Phi^{\theta=1}u$ we obtain $\partial_t v = -\mathrm{i}Op^{\mathrm{BW}}\big((V(u;x) + \partial_t\beta(u;x) + Q(\mathtt{V}_1,\beta_1))\xi\big)v + \cdots$ where $Q(\mathtt{V}_1,\beta_1)$ is a real function, quadratic in $(\mathtt{V}_1,\beta_1)$, and "$\cdots$" denote paradifferential operators of order less than 1, or admissible terms satisfying (1.19). Then we look for β solving $V(u;x) + \partial_t\beta(u;x) + Q(\mathtt{V}_1,\beta_1) = \zeta(u) + O(u^3)$ where $\zeta(u)$ is constant-in-x. However, in general, one can only obtain

$$\begin{aligned} &\partial_t\beta(u) + V(u) + Q(\mathtt{V}_1,\beta_1) \\ &\quad = \sum_{n\in\mathbb{Z}\setminus\{0\}} \big(\mathtt{V}_2^{(1)}\big)_{n,n}^{+-}|u_n|^2 + \sum_{n\in\mathbb{Z}\setminus\{0\}} \big(\mathtt{V}_2^{(1)}\big)_{n,-n}^{+-} u_n\overline{u_{-n}}e^{\mathrm{i}2nx} + O(u^3), \end{aligned}$$

where $(\mathtt{V}_2^{(1)})^{+-}_{n_1 n_2}$ are some coefficients depending on the function V. We then verify the *essential*[4] *cancellation* $(\mathtt{V}_2^{(1)})^{+-}_{n,-n} \equiv 0$, thus obtaining

$$\partial_t v = -\mathrm{i} Op^{\mathrm{BW}}\big((\zeta(u) + O(u^3))\xi\big)v + \cdots, \quad \zeta(u) = \frac{1}{\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}} n|n||u_n|^2.$$

Note that this equation reads, in Fourier,

$$\dot{v}_n = -\frac{\mathrm{i}}{\pi}\Big(\sum_{j\in\mathbb{Z}\setminus\{0\}} j|j||v_j|^2\Big) n v_n,$$

up to higher-order admissible terms, where the cubic vector field contains only Poincaré-Birkhoff resonant cubic monomials, namely of the form (1.30), which are integrable, i.e., of the special form $c_{j,n}|v_j|^2 v_n$.

Similar arguments allow us to reduce to constant coefficients—and in Poincaré-Birkhoff normal form—the modified dispersive term $\mathrm{i}(1 + \mathtt{a}_2)|\xi|^{1/2}$ and all other lower-order operators. We then obtain a system of the form

$$\partial_t z = -\zeta(z)\partial_x z - \mathrm{i}|D|^{\frac{1}{2}} z + r_{-1/2}(z; D)[z] + R(z) + \mathcal{X}_{\geq 4} \tag{1.28}$$

where $r_{-1/2}$ is a constant-coefficient integrable symbol of order $-1/2$, $R(z)$ a very regular nonlinear term, and $\mathcal{X}_{\geq 4}$ an admissible remainder satisfying (1.19). Note that the cubic integrable vector field $-\zeta(z)\partial_x z + r_{-1/2}(z; D)[z]$ in (1.28) is already in Poincaré-Birkhoff normal form.

Step 3. Poincaré-Birkhoff transformations and normal form identification.

In Section 5 we apply transformations to eliminate all nonresonant quadratic and cubic nonlinear terms in $R(z)$. Here, potential losses from *small divisors* created by near-resonances (see Proposition 5.3) are compensated by the smoothing properties of R. We then obtain a new system that is in Poincaré-Birkhoff normal form (Proposition 5.2)

$$\partial_t z = -\zeta(z)\partial_x z - \mathrm{i}|D|^{\frac{1}{2}} z + r_{-1/2}(z; D)[z] + R^{\mathrm{res}}(z) + \mathcal{X}_{\geq 4} \tag{1.29}$$

$$R^{\mathrm{res}}(z) := \sum_{\substack{\sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3 = n \\ \sigma_1\omega(n_1) + \sigma_2\omega(n_2) + \sigma_3\omega(n_3) = \omega(n)}} c^{\sigma_1,\sigma_2,\sigma_3}_{n_1,n_2,n_3} z^{\sigma_1}_{n_1} z^{\sigma_2}_{n_2} z^{\sigma_3}_{n_3} e^{\mathrm{i}nx}, \quad c^{\sigma_1,\sigma_2\sigma_3}_{n_1,n_2,n_3} \in \mathbb{C}. \tag{1.30}$$

At this stage we do not know if the equation (1.29)–(1.30) is Hamiltonian since we have performed nonsymplectic transformations. This is why we call (1.29)–(1.30) the cubic Poincaré-Birkhoff normal form of (1.3), and not its (Hamiltonian)

[4] While we do verify explicitly several key cancellations, some, but not all, of them can be derived by the following invariance properties: (i) the water waves vector field $X(\eta, \psi)$ in the right-hand side of (1.3) is reversible with respect to the involution

$$S : \begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} \mapsto \begin{bmatrix} \eta(-x) \\ -\psi(-x) \end{bmatrix},$$

i.e., $X \circ S = -S \circ X$. (ii) X maps even functions into even functions.

Birkhoff normal form. The coefficients $c^{\sigma_1,\sigma_2,\sigma_3}_{n_1,n_2,n_3}$ are in principle computable, but their explicit expression is definitively very involved. Then, the last main step in Section 6.2 is an *identification argument* to prove that the cubic Poincaré-Birkhoff terms in (1.29)–(1.30) are uniquely determined and coincide with the Hamiltonian system generated by the fourth-order Birkhoff normal form Hamiltonian H_{ZD} in (1.18), namely

$$-\zeta(z)\partial_x z + r_{-1/2}(z;D)[z] + R^{\rm res}(z) = -\mathrm{i}\partial_{\bar z} H^{(4)}_{ZD}.$$

The uniqueness of the normal form is based on the absence of cubic resonances. A related argument in the context of linear KAM norm form is given in [17].

2 Functional Setting and Paradifferential Calculus

In this section we introduce our notation and recall several results on paradifferential calculus, mostly following chapter 3 of [6]. We find convenient the use of this setup to obtain our initial paralinearization of the water waves equations (1.3) with multilinear expansions, as stated in Proposition 3.1, and several tools for conjugations via paradifferential flows, which are contained in Appendix A.2.

Given an interval $I \subset \mathbb{R}$ symmetric with respect to $t = 0$ and $s \in \mathbb{R}$ we define the space

$$C^K_*\big(I, \dot H^s(\mathbb{T}, \mathbb{C}^2)\big) := \bigcap_{k=0}^{K} C^k\big(I; \dot H^{s-k}(\mathbb{T}; \mathbb{C}^2)\big),$$

endowed with the norm

$$\sup_{t\in I} \|U(t,\cdot)\|_{K,s} \quad \text{where } \|U(t,\cdot)\|_{K,s} := \sum_{k=0}^{K} \big\|\partial_t^k U(t,\cdot)\big\|_{\dot H^{s-k}}. \tag{2.1}$$

We denote by $C^K_{*\mathbb{R}}(I, \dot H^s(\mathbb{T}, \mathbb{C}^2))$ the space of functions U in $C^K_*(I, \dot H^s(\mathbb{T}, \mathbb{C}^2))$ such that $U = [\begin{smallmatrix} u \\ \bar u \end{smallmatrix}]$. Given $r > 0$ we set

$$B^K_s(I;r) := \big\{U \in C^K_*(I, \dot H^s(\mathbb{T};\mathbb{C}^2)) : \sup_{t\in I} \|U(t,\cdot)\|_{K,s} < r\big\}. \tag{2.2}$$

With similar meaning we denote $C^K_*(I; \dot H^s(\mathbb{T};\mathbb{C}))$. We expand a 2π-periodic function $u(x)$, with zero average in x (which is identified with u in the homogeneous space), in Fourier series as

$$u(x) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \hat u(n) \frac{e^{\mathrm{i}nx}}{\sqrt{2\pi}}, \qquad \hat u(n) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} u(x) e^{-\mathrm{i}nx}\, dx. \tag{2.3}$$

We also use the notation $u_n^+ := u_n := \hat u(n)$ and $u_n^- := \overline{u_n} := \overline{\hat u(n)}$. For $n \in \mathbb{N}^* := \mathbb{N}\setminus\{0\}$ we denote by Π_n the orthogonal projector from $L^2(\mathbb{T};\mathbb{C})$ to the subspace spanned by $\{e^{\mathrm{i}nx}, e^{-\mathrm{i}nx}\}$, i.e.,

$$(\Pi_n u)(x) := \hat u(n) \frac{e^{\mathrm{i}nx}}{\sqrt{2\pi}} + \hat u(-n) \frac{e^{-\mathrm{i}nx}}{\sqrt{2\pi}},$$

and we also denote by Π_n the corresponding projector in $L^2(\mathbb{T},\mathbb{C}^2)$. If $\mathcal{U} = (U_1,\dots,U_p)$ is a p-tuple of functions, $\vec{n} = (n_1,\dots,n_p) \in (\mathbb{N}^*)^p$, we set

$$\Pi_{\vec{n}}\mathcal{U} := (\Pi_{n_1}U_1,\dots,\Pi_{n_p}U_p). \tag{2.4}$$

We deal with vector fields X that satisfy the *x-translation invariance* property

$$X \circ \tau_\theta = \tau_\theta \circ X \quad \forall \theta \in \mathbb{R} \quad \text{where } \tau_\theta : u(x) \mapsto (\tau_\theta u)(x) := u(x+\theta). \tag{2.5}$$

Paradifferential operators. We first give the definition of the classes of symbols that we are going to use, collecting Definitions 3.1, 3.2, and 3.4 in [6].

DEFINITION 2.1 (CLASSES OF SYMBOLS). Let $m \in \mathbb{R}$, $p, N \in \mathbb{N}$ with $p \le N$, $K' \le K$ in $\mathbb{N}$, $r > 0$.

(i) *p-homogeneous symbols.* We denote by $\widetilde{\Gamma}^m_p$ the space of symmetric p-linear maps from $(\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p$ to the space of C^∞ functions of $(x,\xi) \in \mathbb{T}\times\mathbb{R}$, $\mathcal{U} \to ((x,\xi) \to a(\mathcal{U};x,\xi))$, satisfying the following. There is $\mu > 0$ and, for any $\alpha,\beta \in \mathbb{N}$, there is $C > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a(\Pi_{\vec{n}}\mathcal{U};x,\xi)\right| \le C|\vec{n}|^{\mu+\alpha}\langle\xi\rangle^{m-\beta}\prod_{j=1}^{p}\|\Pi_{n_j}U_j\|_{L^2} \tag{2.6}$$

for any $\mathcal{U} = (U_1,\dots,U_p)$ in $(\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p$, and $\vec{n} = (n_1,\dots,n_p) \in (\mathbb{N}^*)^p$.

Moreover, we assume that, if for some $(n_0,\dots,n_p) \in \mathbb{N}\times(\mathbb{N}^*)^p$, we have $\Pi_{n_0}a(\Pi_{n_1}U_1,\dots,\Pi_{n_p}U_p;\cdot) \ne 0$, then there exists a choice of signs $\sigma_0,\dots,\sigma_p \in \{-1,1\}$ such that $\sum_{j=0}^p \sigma_j n_j = 0$. This condition is automatically satisfied by requiring the translation invariance property

$$a(\tau_\theta\mathcal{U};x,\xi) = a(\mathcal{U};x+\theta,\xi) \quad \forall\theta\in\mathbb{R}. \tag{2.7}$$

For $p = 0$ we denote by $\widetilde{\Gamma}^m_0$ the space of constant coefficients symbols $\xi \mapsto a(\xi)$ that satisfy (2.6) with $\alpha = 0$, and the right-hand side replaced by $C\langle\xi\rangle^{m-\beta}$.

(ii) *Non-homogeneous symbols.* Let $p \ge 1$. We denote by $\Gamma^m_{K,K',p}[r]$ the space of functions $(U;t,x,\xi) \mapsto a(U;t,x,\xi)$ defined for $U \in B^K_{s_0}(I;r)$ for some large enough s_0, with complex values such that for any $0 \le k \le K-K'$ and any $\sigma \ge s_0$, there are $C > 0$, $0 < r(\sigma) < r$, and for any $U \in B^K_{s_0}(I;r(\sigma)) \cap C^{k+K'}_*(I,\dot{H}^\sigma(\mathbb{T};\mathbb{C}^2))$ and any $\alpha,\beta\in\mathbb{N}$, with $\alpha \le \sigma - s_0$

$$\left|\partial_t^k\partial_x^\alpha\partial_\xi^\beta a(U;t,x,\xi)\right| \le C\langle\xi\rangle^{m-\beta}\|U\|^{p-1}_{k+K',s_0}\|U\|_{k+K',\sigma}. \tag{2.8}$$

(iii) *Symbols.* We denote by $\Sigma\Gamma^m_{K,K',p}[r,N]$ the space of functions

$$(U,t,x,\xi) \to a(U;t,x,\xi)$$

such that there are homogeneous symbols $a_q \in \widetilde{\Gamma}^m_q$ for $q = p, \dots, N-1$ and a nonhomogeneous symbol $a_N \in \Gamma^m_{K,K',N}[r]$ such that

$$a(U; t, x, \xi) = \sum_{q=p}^{N-1} a_q(U, \dots, U; x, \xi) + a_N(U; t, x, \xi). \tag{2.9}$$

We denote by $\Sigma\Gamma^m_{K,K',p}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ the space of 2×2 matrices whose entries are symbols in $\Sigma\Gamma^m_{K,K',p}[r, N]$.

Remark 2.2. The property (2.7) means that the dependence with respect to the variable x of the symbol $a(\mathcal{U}; x, \xi)$ enters only through the function $\mathcal{U}(x)$. It implies the more general assumption made in [6]: if $\Pi_{n_0} a(\Pi_{n_1} U_1, \dots, \Pi_{n_p} U_p; \cdot) \neq 0$, then there is a choice of signs $\sigma_0, \dots, \sigma_p \in \{-1, 1\}$ such that $\sum_{j=0}^p \sigma_j n_j = 0$. We mention this condition to be consistent with the notation of [6].

Note that

$$a \in \widetilde{\Gamma}^m_p,\ b \in \widetilde{\Gamma}^{m'}_q \Rightarrow ab \in \widetilde{\Gamma}^{m+m'}_{p+q},\ \partial_x a \in \widetilde{\Gamma}^m_p,\ \partial_\xi a \in \widetilde{\Gamma}^{m-1}_p;$$

$$\begin{aligned} & a \in \Gamma^m_{K,K',p}[r],\ K'+1 \le K \\ & \quad \Rightarrow \partial_t a \in \Gamma^m_{K,K'+1,p}[r],\ \partial_x a \in \Gamma^m_{K,K',p}[r],\ \partial_\xi a \in \Gamma^{m-1}_{K,K',p}[r]; \\ & a \in \Gamma^m_{K,K',p}[r], b \in \Gamma^{m'}_{K,K',q}[r] \Rightarrow ab \in \Gamma^{m+m'}_{K,K',p+q}[r]; \\ & a(\mathcal{U}; \cdot) \in \widetilde{\Gamma}^m_p \Rightarrow a(U, \dots, U; \cdot) \in \Gamma^m_{K,0,p}[r] \quad \forall r > 0. \end{aligned} \tag{2.10}$$

Throughout this paper we will systematically use the following expansions, which are a consequence of (2.7) and $u \in \dot{H}^\infty(\mathbb{T}; \mathbb{C})$. If $\mathtt{a}_1 \in \widetilde{\Gamma}^m_1$, then

$$\mathtt{a}_1(U; x, \xi) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}\setminus\{0\}, \sigma = \pm} (\mathtt{a}_1)^\sigma_n(\xi) u^\sigma_n e^{\mathrm{i}\sigma n x}, \tag{2.11}$$

for some $(\mathtt{a}_1)^\sigma_n(\xi) \in \mathbb{C}$, and if $\mathtt{a}_2 \in \widetilde{\Gamma}^m_2$, then

$$\begin{aligned} \mathtt{a}_2(U, U; x, \xi) = & \sum_{\substack{n_1, n_2 \in \mathbb{Z}\setminus\{0\} \\ \sigma = \pm}} (\mathtt{a}_2)^{\sigma\sigma}_{n_1,n_2}(\xi) u^\sigma_{n_1} u^\sigma_{n_2} \frac{e^{\mathrm{i}\sigma(n_1+n_2)x}}{2\pi} \\ & + \sum_{n_1, n_2 \in \mathbb{Z}\setminus\{0\}} (\mathtt{a}_2)^{+-}_{n_1,n_2}(\xi) u_{n_1} \overline{u_{n_2}} \frac{e^{\mathrm{i}(n_1-n_2)x}}{2\pi} \end{aligned} \tag{2.12}$$

for some $(\mathtt{a}_2)^{\sigma\sigma'}_{n_1,n_2}(\xi) \in \mathbb{C}$ with $\sigma, \sigma' = \pm$. In the sequel for simplicity we may also write $\mathtt{a}_2(U; x, \xi)$ instead of $\mathtt{a}_2(U, U; x, \xi)$.

We also define the following classes of functions in analogy with our classes of symbols.

DEFINITION 2.3 (*Functions*). Fix $N \in \mathbb{N}$, $p \in \mathbb{N}$ with $p \le N$, $K, K' \in \mathbb{N}$ with $K' \le K$, $r > 0$. We denote by $\widetilde{\mathcal{F}}_p$, resp., $\mathcal{F}_{K,K',p}[r]$, $\Sigma\mathcal{F}_p[r,N]$, the subspace of $\widetilde{\Gamma}^0_p$, resp., $\Gamma^0_p[r]$, $\Sigma\Gamma^0_p[r,N]$, made of those symbols that are independent of ξ. We write $\widetilde{\mathcal{F}}^{\mathbb{R}}_p$, resp., $\mathcal{F}^{\mathbb{R}}_{K,K',p}[r]$, $\Sigma\mathcal{F}^{\mathbb{R}}_p[r,N]$, to denote functions in $\widetilde{\mathcal{F}}_p$, resp., $\mathcal{F}_{K,K',p}[r]$, $\Sigma\mathcal{F}_p[r,N]$, which are real-valued.

Note that functions $\mathtt{a}_1 \in \widetilde{\mathcal{F}}_1$, $\mathtt{a}_2 \in \widetilde{\mathcal{F}}_2$ expanded as in (2.11), (2.12) are real-valued if and only if

$$\overline{(\mathtt{a}_1)^+_n} = (\mathtt{a}_1)^-_n, \quad \overline{(\mathtt{a}_2)^{++}_{n_1,n_2}} = (\mathtt{a}_2)^{--}_{n_1,n_2}, \quad \overline{(\mathtt{a}_2)^{+-}_{n_1,n_2}} = (\mathtt{a}_2)^{+-}_{n_2,n_1}. \tag{2.13}$$

Paradifferential quantization. Given $p \in \mathbb{N}$ we consider smooth functions $\chi_p \in C^\infty(\mathbb{R}^p \times \mathbb{R}; \mathbb{R})$ and $\chi \in C^\infty(\mathbb{R} \times \mathbb{R}; \mathbb{R})$, even with respect to each of their arguments, satisfying, for some $0 < \delta \ll 1$,

$$\operatorname{supp} \chi_p \subset \{(\xi', \xi) \in \mathbb{R}^p \times \mathbb{R}; |\xi'| \le \delta\langle\xi\rangle\}, \ \chi_p(\xi', \xi) \equiv 1 \text{ for } |\xi'| \le \delta\langle\xi\rangle/2,$$
$$\operatorname{supp} \chi \subset \{(\xi', \xi) \in \mathbb{R} \times \mathbb{R}; |\xi'| \le \delta\langle\xi\rangle\}, \ \chi(\xi', \xi) \equiv 1 \text{ for } |\xi'| \le \delta\langle\xi\rangle/2.$$

For $p = 0$ we set $\chi_0 \equiv 1$. We assume also that

$$\left|\partial_\xi^\alpha \partial_{\xi'}^\beta \chi_p(\xi', \xi)\right| \le C_{\alpha,\beta}\langle\xi\rangle^{-\alpha-|\beta|} \quad \forall \alpha \in \mathbb{N}, \ \beta \in \mathbb{N}^p,$$

and

$$\left|\partial_\xi^\alpha \partial_{\xi'}^\beta \chi(\xi', \xi)\right| \le C_{\alpha,\beta}\langle\xi\rangle^{-\alpha-\beta} \quad \forall \alpha, \beta \in \mathbb{N}.$$

A function satisfying the above condition is $\chi(\xi', \xi) := \widetilde{\chi}(\xi'/\langle\xi\rangle)$ where $\widetilde{\chi}$ is a function in $C_0^\infty(\mathbb{R}; \mathbb{R})$ having a small enough support and equal to 1 in a neighborhood of 0.

DEFINITION 2.4 (*Bony-Weyl quantization*). If a is a symbol in $\widetilde{\Gamma}^m_p$, resp., in $\Gamma^m_{K,K',p}[r]$, we define its *Weyl* quantization as the operator acting on a 2π-periodic function $u(x)$ (written as in (2.3)) as

$$Op^W(a)u = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} \widehat{a}\Big(k - j, \frac{k+j}{2}\Big) \widehat{u}(j) \right) \frac{e^{\mathrm{i}kx}}{\sqrt{2\pi}} \tag{2.14}$$

where $\widehat{a}(k, \xi)$ is the k^{th}-Fourier coefficient of the 2π-periodic function $x \mapsto a(x, \xi)$. We set, using notation (2.4),

$$a_{\chi_p}(\mathcal{U}; x, \xi) := \sum_{\vec{n} \in \mathbb{N}^p} \chi_p(\vec{n}, \xi) a(\Pi_{\vec{n}}\mathcal{U}; x, \xi),$$
$$a_\chi(U; t, x, \xi) := \frac{1}{2\pi} \int_{\mathbb{R}} \chi\big(\xi', \xi\big) \widehat{a}(U; t, \xi', \xi) e^{\mathrm{i}\xi' x} \, d\xi',$$

where in the last equality $\widehat{a}$ stands for the Fourier transform with respect to the x-variable. Then we define the *Bony-Weyl* quantization of a as

$$Op^{\mathrm{BW}}(a(\mathcal{U}; \cdot)) = Op^W(a_{\chi_p}(\mathcal{U}; \cdot)), \quad Op^{\mathrm{BW}}(a(U; t, \cdot)) = Op^W(a_\chi(U; t, \cdot)). \tag{2.15}$$

If a is a symbol in $\Sigma\Gamma^{m}_{K,K',p}[r,N]$, that we decompose as in (2.9), we define its *Bony-Weyl* quantization

$$Op^{\mathrm{BW}}(a(U;t,\cdot)) = \sum_{q=p}^{N-1} Op^{\mathrm{BW}}(a_q(U,\dots,U;\cdot)) + Op^{\mathrm{BW}}(a_N(U;t,\cdot)).$$

• By the translation invariance property (2.7), we have

$$\begin{aligned} &Op^{\mathrm{BW}}(a_q(\tau_\theta U,\dots,\tau_\theta U;\cdot,\xi))[\tau_\theta V] \\ &\quad = \tau_\theta\big(Op^{\mathrm{BW}}(a_q(U,\dots,U;\cdot,\xi))[V]\big). \end{aligned} \tag{2.16}$$

• The operator $Op^{\mathrm{BW}}(a)$ acts on homogeneous spaces of functions; see Proposition 2.6.

• The action of $Op^{\mathrm{BW}}(a)$ on homogeneous spaces only depends on the values of the symbol $a = a(U;t,x,\xi)$ (or $a(\mathcal{U};t,x,\xi)$) for $|\xi| \geq 1$. Therefore, we may identify two symbols $a(U;t,x,\xi)$ and $b(U;t,x,\xi)$ if they agree for $|\xi| \geq 1/2$. In particular, whenever we encounter a symbol that is not smooth at $\xi = 0$, such as, for example, $a = g(x)|\xi|^m$ for $m \in \mathbb{R}\setminus\{0\}$, or $\mathrm{sign}(\xi)$, we will consider its smoothed out version $\chi(\xi)a$, where $\chi \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$ is an even and positive cutoff function satisfying

$$\chi(\xi) = 0 \ \text{ if } |\xi| \leq \tfrac{1}{8}, \quad \chi(\xi) = 1 \ \text{ if } |\xi| > \tfrac{1}{4}, \quad \partial_\xi \chi(\xi) > 0 \quad \forall \xi \in \big(\tfrac{1}{8},\tfrac{1}{4}\big). \tag{2.17}$$

• If a is a homogeneous symbol, the two definitions of quantization in (2.15), differ by a smoothing operator that we introduce in Definition 2.5 below.

Definition 2.4 is independent of the cutoff functions χ_p, χ up to smoothing operators that we define below (see definition 3.7 in [6]). Given $(n_1,\dots,n_{p+1}) \in \mathbb{N}^{p+1}$ we denote by $\max_2(n_1,\dots,n_{p+1})$ the second largest among the integers $n_1,\dots,n_{p+1}$.

DEFINITION 2.5. Let $K' \leq K \in \mathbb{N}$, $N \in \mathbb{N}$, $N \geq 1$, $\mu \in \mathbb{R}$, $\rho \geq 0$, and $r > 0$.

(i) *p-homogeneous smoothing operators.* We denote by $\widetilde{\mathcal{R}}^{-\rho}_p$ the space of $(p+1)$-linear maps R from the space $(\dot H^\infty(\mathbb{T};\mathbb{C}^2))^p \times \dot H^\infty(\mathbb{T};\mathbb{C})$ to the space $\dot H^\infty(\mathbb{T};\mathbb{C})$ symmetric in $(U_1,\dots,U_p)$, of the form

$$(U_1,\dots,U_{p+1}) \to R(U_1,\dots,U_p)U_{p+1}$$

that satisfy the following. There are $\mu \geq 0$, $C > 0$ such that

$$\|\Pi_{n_0} R(\Pi_{\vec n}\mathcal{U})\Pi_{n_{p+1}} U_{p+1}\|_{L^2} \leq C\frac{\max_2(n_1,\dots,n_{p+1})^{\rho+\mu}}{\max(n_1,\dots,n_{p+1})^{\rho}} \prod_{j=1}^{p+1} \|\Pi_{n_j} U_j\|_{L^2}$$

for any $\mathcal{U} = (U_1,\dots,U_p) \in (\dot H^\infty(\mathbb{T};\mathbb{C}^2))^p$, any $U_{p+1} \in \dot H^\infty(\mathbb{T};\mathbb{C})$, any vector $\vec n = (n_1,\dots,n_p) \in (\mathbb{N}^*)^p$, any $n_0, n_{p+1} \in \mathbb{N}^*$. Moreover, if

$$\Pi_{n_0} R(\Pi_{n_1} U_1,\dots,\Pi_{n_p} U_p)\Pi_{n_{p+1}} U_{p+1} \neq 0, \tag{2.18}$$

then there is a choice of signs $\sigma_0, \dots, \sigma_{p+1} \in \{\pm 1\}$ such that $\sum_{j=0}^{p+1} \sigma_j n_j = 0$. In addition, we require the translation invariance property

$$R(\tau_\theta \mathcal{U})[\tau_\theta U_{p+1}] = \tau_\theta (R(\mathcal{U}) U_{p+1}) \quad \forall \theta \in \mathbb{R}. \tag{2.19}$$

(ii) *Nonhomogeneous smoothing operators.* We denote by $\mathcal{R}^{-\rho}_{K,K',N}[r]$ the space of maps $(V, t, U) \mapsto R(V)U$ defined on

$$B^K_{s_0}(I; r) \times I \times C^K_*(I, \dot{H}^{s_0}(\mathbb{T}, \mathbb{C})) \quad \text{for some } s_0 > 0,$$

which are linear in the variable U and such that the following holds true. For any $s \geq s_0$ there exist a constant $C > 0$ and $r(s) \in]0, r[$ such that for any $V \in B^K_{s_0}(I; r) \cap C^K_*(I, \dot{H}^s(\mathbb{T}, \mathbb{C}^2))$, any $U \in C^K_*(I, \dot{H}^s(\mathbb{T}, \mathbb{C}))$, any $0 \leq k \leq K - K'$, and any $t \in I$, we have

$$\begin{aligned} &\|\partial_t^k (R(V; t)U)(t, \cdot)\|_{\dot{H}^{s-k+\rho}} \\ &\leq \sum_{k'+k''=k} C\big(\|U\|_{k'',s} \|V\|^N_{k'+K',s_0} \\ &\qquad + \|U\|_{k'',s_0} \|V\|^{N-1}_{k'+K',s_0} \|V\|_{k'+K',s}\big). \end{aligned} \tag{2.20}$$

(iii) *Smoothing operators.* We denote by $\Sigma\mathcal{R}^{-\rho}_{K,K',p}[r, N]$ the space of maps $(V, t, U) \to R(V, t)U$ that may be written as

$$R(V; t)U = \sum_{q=p}^{N-1} R_q(V, \dots, V)U + R_N(V; t)U$$

for some R_q in $\widetilde{\mathcal{R}}^{-\rho}_q$, $q = p, \dots, N-1$, and R_N in $\mathcal{R}^{-\rho}_{K,K',N}[r]$.

We denote by $\Sigma\mathcal{R}^{-\rho}_{K,K',p}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ the space of 2×2 matrices whose entries are in $\Sigma\mathcal{R}^{-\rho}_{K,K',p}[r, N]$.

• If R is in $\widetilde{\mathcal{R}}^{-\rho}_p$, then $(V, U) \mapsto R(V, \dots, V)U$ is in $\mathcal{R}^{-\rho}_{K,0,p}[r]$, i.e., (2.20) holds with $N \rightsquigarrow p$, $K' = 0$.

• If $R_i \in \Sigma\mathcal{R}^{-\rho}_{K,K',p_i}[r, N]$, $i = 1, 2$, then the composition operator $R_1 \circ R_2$ is in $\Sigma\mathcal{R}^{-\rho}_{K,K',p_1+p_2}[r, N]$.

The next proposition states boundedness properties on Sobolev spaces of the paradifferential operators (see proposition 3.8 in [6]).

PROPOSITION 2.6 (*Action of paradifferential operator*). *Let $r > 0$, $m \in \mathbb{R}$, $p \in \mathbb{N}$, $K' \leq K \in \mathbb{N}$. Then*:

(i) *There is $s_0 > 0$ such that for any symbol $a \in \widetilde{\Gamma}^m_p$, there is a constant $C > 0$, depending only on s and on* (2.6) *with $\alpha = \beta = 0$ such that for any $\mathcal{U} = (U_1, \dots, U_p)$*

$$\|Op^{\mathrm{BW}}(a(\mathcal{U}; \cdot))U_{p+1}\|_{\dot{H}^{s-m}} \leq C \prod_{j=1}^{p} \|U_j\|_{\dot{H}^{s_0}} \|U_{p+1}\|_{\dot{H}^s} \tag{2.21}$$

for $p \geq 1$, while for $p = 0$ (2.21) holds by replacing the right-hand side with $C\|U_{p+1}\|_{\dot H^s}$.

(ii) *There is $s_0 > 0$ such that for any symbol $a \in \Gamma^m_{K,K',p}[r]$ there is a constant $C > 0$, depending only on s, r, and* (2.8) *with $0 \leq \alpha \leq 2$, $\beta = 0$, such that, for any $t \in I$, any $0 \leq k \leq K - K'$,*

$$\| Op^{\mathrm{BW}}\big(\partial_t^k a(U;t,\cdot)\big)\|_{\mathcal{L}(\dot H^s,\dot H^{s-m})} \leq C\|U\|^p_{k+K',s_0}.$$

• If $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$ with $m \leq 0$ and $p \geq 1$, then $Op^{\mathrm{BW}}(a(V;t,\cdot))U$ is in $\Sigma\mathcal{R}^m_{K,K',p}[r,N]$.

Below we deal with classes of operators without keeping track of the number of lost derivatives in a precise way (see definition 3.9 in [6]). The class $\widetilde{\mathcal{M}}^m_p$ denotes multilinear maps that lose m derivatives and are p-homogeneous in U, while the class $\mathcal{M}^m_{K,K',p}$ contains nonhomogeneous maps that lose m derivatives, vanish at degree at least p in U, and are $(K-K')$-times differentiable in t.

DEFINITION 2.7. Let $p, N \in \mathbb{N}$, with $p \leq N$, $N \geq 1$, $K, K' \in \mathbb{N}$ with $K' \leq K$, and $m \geq 0$.

(i) *p-homogeneous maps.* We denote by $\widetilde{\mathcal{M}}^m_p$ the space of $(p+1)$-linear maps M from the space $(\dot H^\infty(\mathbb{T};\mathbb{C}^2))^p \times \dot H^\infty(\mathbb{T};\mathbb{C})$ to the space $\dot H^\infty(\mathbb{T};\mathbb{C})$, which are symmetric in $(U_1,\dots,U_p)$, are of the form $(U_1,\dots,U_{p+1}) \to M(U_1,\dots,U_p)U_{p+1}$, and satisfy the following. There is a $C > 0$ such that

$$\|\Pi_{n_0} M(\Pi_{\vec n}\mathcal{U})\Pi_{n_{p+1}}U_{p+1}\|_{L^2} \leq C(n_0+n_1+\cdots+n_{p+1})^m \prod_{j=1}^{p+1}\|\Pi_{n_j}U_j\|_{L^2}$$

for any $\mathcal{U} = (U_1,\dots,U_p) \in (\dot H^\infty(\mathbb{T};\mathbb{C}^2))^p$, any $U_{p+1} \in \dot H^\infty(\mathbb{T};\mathbb{C})$, any vector $\vec n = (n_1,\dots,n_p) \in (\mathbb{N}^*)^p$, and any $n_0, n_{p+1} \in \mathbb{N}^*$. Moreover, the properties (2.18)–(2.19) hold.

(ii) *Nonhomogeneous maps.* We denote by $\mathcal{M}^m_{K,K',N}[r]$ the space of functions $(V,t,U) \mapsto M(V;t)U$ defined on $B^K_{s_0}(I;r) \times I \times C^K_*(I,\dot H^{s_0}(\mathbb{T},\mathbb{C}))$ for some $s_0 > 0$ that are linear in the variable U and such that the following holds true. For any $s \geq s_0$ there exist a constant $C > 0$ and $r(s) \in\,]0,r[$ such that for any $V \in B^K_{s_0}(I;r) \cap C^K_*(I,\dot H^s(\mathbb{T},\mathbb{C}^2))$, any $U \in C^K_*(I,\dot H^s(\mathbb{T},\mathbb{C}))$, any $0 \leq k \leq K-K'$, and any $t \in I$, we have $\|\partial_t^k(M(V;t)U)(t,\cdot)\|_{\dot H^{s-k-m}}$ is bounded by the right-hand side of (2.20).

(iii) *Maps.* We denote by $\Sigma\mathcal{M}^m_{K,K',p}[r,N]$ the space of maps $(V,t,U) \to M(V,t)U$ that may be written as

$$M(V;t)U = \sum_{q=p}^{N-1} M_q(V,\dots,V)U + M_N(V;t)U$$

for some M_q in $\widetilde{\mathcal{M}}^m_q$, $q = p, \dots, N-1$, and M_N in $\mathcal{M}^m_{K,K',N}[r]$. Finally, we set

$$\widetilde{\mathcal{M}}_p := \bigcup_{m\geq 0} \widetilde{\mathcal{M}}^m_p, \quad \mathcal{M}_{K,K',p}[r] := \bigcup_{m\geq 0} \mathcal{M}^m_{K,K',p}[r]$$

and $\Sigma\mathcal{M}_{K,K',p}[r,N] := \bigcup_{m\geq 0} \Sigma\mathcal{M}^m_{K,K',p}[r]$.

We denote by $\Sigma\mathcal{M}^m_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ the space of 2×2 matrices whose entries are maps in the class $\Sigma\mathcal{M}^m_{K,K',p}[r,N]$. We also set $\Sigma\mathcal{M}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C}) = \bigcup_{m\in\mathbb{R}} \Sigma\mathcal{M}^m_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$.

- If M is in $\widetilde{\mathcal{M}}^m_p$, $p \geq N$, then $(V,U) \to M(V,\dots,V)U$ is in $\mathcal{M}^m_{K,0,N}[r]$.
- If $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$ for $p \geq 1$, then the map $(V,U) \to Op^{\mathrm{BW}}(a(V;t,\cdot))U$ is in $\Sigma\mathcal{M}^{m'}_{K,K',p}[r,N]$ for some $m' \geq m$.
- Any $R \in \Sigma\mathcal{R}^{-\rho}_{K,K',p}[r,N]$ defines an element of $\Sigma\mathcal{M}^m_{K,K',p}[r,N]$ for some $m \geq 0$.
- If $M \in \Sigma\mathcal{M}_{K,K'_1,p}[r,N]$ and $\widetilde{M} \in \Sigma\mathcal{M}_{K,K'_2,1}[r,N-p]$, then the map $(V,t,U) \to M(V+\widetilde{M}(V;t)V;t)[U]$ is in $\Sigma\mathcal{M}_{K,K'_1+K'_2,p}[r,N]$.
- If $M \in \Sigma\mathcal{M}^m_{K,K',p}[r,N]$ and $\widetilde{M} \in \Sigma\mathcal{M}^{m'}_{K,K',q}[r,N]$, then the map $M(U;t)\circ \widetilde{M}(U;t)$ is in $\Sigma\mathcal{M}^{m+m'}_{K,K',p+q}[r,N]$.

Note that, given $M_1 \in \widetilde{\mathcal{M}}_1$, the property (2.19) implies that

$$M_1(U)U = \frac{1}{2\pi}\sum_{\substack{n_1,n_2\in\mathbb{Z}\setminus\{0\},\\ \sigma=\pm}} (M_2)^{\sigma\sigma}_{n_1,n_2} u^\sigma_{n_1} u^\sigma_{n_2} e^{\mathrm{i}\sigma(n_1+n_2)x} + \frac{1}{2\pi}\sum_{n_1,n_2\in\mathbb{Z}\setminus\{0\}} (M_2)^{+-}_{n_1,n_2} u_{n_1}\overline{u_{n_2}} e^{\mathrm{i}(n_1-n_2)x} \tag{2.22}$$

for some coefficients $(M_2)^{\sigma\sigma'}_{n_1,n_2} \in \mathbb{C}$ with $\sigma,\sigma' = \pm$ and $n_1, n_2 \in \mathbb{Z}\setminus\{0\}$.

Composition theorems. Let $\varsigma(D_x, D_\xi, D_y, D_\eta) := D_\xi D_y - D_x D_\eta$ where $D_x := \frac{1}{\mathrm{i}}\partial_x$ and D_ξ, D_y, D_η are similarly defined.

DEFINITION 2.8. [*Asymptotic expansion of composition symbol*] Let $K' \leq K$, ρ, p, q be in $\mathbb{N}$, $m, m' \in \mathbb{R}$, $r > 0$. Consider $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$ and $b \in \Sigma\Gamma^{m'}_{K,K',q}[r,N]$. For U in $B^K_\sigma(I;r)$ we define, for $\rho < \sigma - s_0$, the symbol

$$(a\#_\rho b)(U;t,x,\xi) := \sum_{k=0}^{\rho} \frac{1}{k!}\left(\frac{\mathrm{i}}{2}\varsigma(D_x, D_\xi, D_y, D_\eta)\right)^k [a(U;t,x,\xi)b(U;t,y,\eta)]\Big|_{x=y,\xi=\eta} \tag{2.23}$$

modulo symbols in $\Sigma\Gamma^{m+m'-\rho}_{K,K',p+q}[r,N]$.

• By (2.10) the symbol $a\#_\rho b$ belongs to $\Sigma\Gamma^{m+m'}_{K,K',p+q}[r,N]$.
• We have the expansion

$$a\#_\rho b = ab + \frac{1}{2\mathrm{i}}\{a,b\} + \cdots ,$$

up to a symbol in $\Sigma\Gamma^{m+m'-2}_{K,K',p+q}[r,N]$, where $\{a,b\} := \partial_\xi a \partial_x b - \partial_x a \partial_\xi b$ denotes the Poisson bracket.
• Note that the terms of even (resp., odd) rank in the asymptotic expansion (2.23) in the Weyl quantization are symmetric (resp., antisymmetric) in (a,b). Consequently, the terms of even rank vanish in the symbol of the commutator $[Op^{\mathrm{BW}}(a), Op^{\mathrm{BW}}(b)]$.

PROPOSITION 2.9. *[Composition of Bony-Weyl operators] Let* $K' \le K$, ρ, p, q *be in* $\mathbb{N}$, $m, m' \in \mathbb{R}$, $r > 0$. *Consider* $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$ *and* $b \in \Sigma\Gamma^{m'}_{K,K',q}[r,N]$. *Then*

$$R(U) := Op^{\mathrm{BW}}(a(U;t,x,\xi)) \circ Op^{\mathrm{BW}}(b(U;t,x,\xi)) - Op^{\mathrm{BW}}\big((a\#_\rho b)(U;t,x,\xi)\big)$$

is a nonhomogeneous smoothing remainder in $\Sigma\mathcal{R}^{-\rho+m+m'}_{K,K',p+q}[r,N]$.

PROOF. See propositions 3.12 and 3.15 in [6]. The homogeneous components of the symbols a and b satisfy (2.7). Using (2.16) and (2.23) one can check that the homogeneous components of $R(U)$ satisfy (2.19). ☐

PROPOSITION 2.10. *[Compositions] Let* $m, m', m'' \in \mathbb{R}$, $K, K', N, p_1, p_2, p_3, \rho \in \mathbb{N}$ *with* $K' \le K$, $p_1 + p_2 < N$, $\rho \ge 0$, *and* $r > 0$. *Let* $a \in \Sigma\Gamma^m_{K,K',p_1}[r,N]$, $R \in \Sigma\mathcal{R}^{-\rho}_{K,K',p_2}[r,N]$, *and* $M \in \Sigma\mathcal{M}^{m''}_{K,K',p_3}[r,N]$. *Then*

(i) $R(U;t) \circ Op^{\mathrm{BW}}(a(U;t,x,\xi))$, $Op^{\mathrm{BW}}(a(U;t,x,\xi)) \circ R(U;t)$ *are in* $\Sigma\mathcal{R}^{-\rho+m}_{K,K',p_1+p_2}[r,N]$.

(ii) $R(U;t) \circ M(U;t)$ *is a smoothing operator in* $\Sigma\mathcal{R}^{-\rho+m''}_{K,K',p_2+p_3}[r,N]$.

(iii) *If* $R_2 \in \widetilde{\mathcal{R}}^{-\rho}_{p_2}$, *then* $R_2(U,\dots,U,M(U;t)U)$ *belongs to*

$$\Sigma\mathcal{R}^{-\rho+m''}_{K,K',p_2+p_3}[r,N].$$

(iv) *Let* $c \in \widetilde{\Gamma}^m_p$, $p \in \mathbb{N}$. *Then the symbol*

$$U \to c_M(U;t,x,\xi) := c(U,\dots,U,M(U;t)U;t,x,\xi)$$

is in $\Sigma\Gamma^m_{K,K',p+p_3}[r,N]$. *If* $c \in \widetilde{\mathcal{F}}_p$ *then* $c_M \in \Sigma\mathcal{F}_{K,K',p+p_3}[r,N]$. *Moreover if* $c \in \Gamma^m_{K,K',N}[r]$ *then* $c_M \in \Gamma^m_{K,K',N}[r]$.

(v)

$$Op^{\mathrm{BW}}(c(U,\dots,U,W;t,x,\xi))_{|W=M(U;t)U} = Op^{\mathrm{BW}}(b(U;t,x,\xi)) + R(U;t)$$

where $b(U;t,x,\xi) := c(U,\dots,U,M(U;t)U;t,x,\xi)$ *and* $R(U;t)$ *is in* $\Sigma\mathcal{R}^{-\rho}_{K,K',p+p_1}[r,N]$.

PROOF. See propositions 3.16, 3.17, and 3.18 in [6]. The translation invariance properties for the composed operators and symbols in items (i)–(v) follow as in the proof of Proposition 2.9. □

Real-to-real operators. Given a linear operator $R(U)[\cdot]$ acting on $\mathbb{C}$ (it may be a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$ or a map in $\Sigma\mathcal{M}_{K,K',1}$) we associate the linear operator defined by the relation

$$\overline{R}(U)[V] := \overline{R(U)[\overline{V}]} \quad \forall V \in \mathbb{C}. \tag{2.24}$$

We say that a matrix of operators acting in $\mathbb{C}^2$ is *real-to-real*, if it has the form

$$R(U) = \begin{bmatrix} R_1(U) & R_2(U) \\ \overline{R_2}(U) & \overline{R_1}(U) \end{bmatrix}. \tag{2.25}$$

Note that

• if $R(U)$ is a real-to-real matrix of operators, for $V = \left[\begin{smallmatrix} v \\ \overline{v} \end{smallmatrix}\right]$, then we have

$$R(U)[V] =: Z = \left[\begin{smallmatrix} z \\ \overline{z} \end{smallmatrix}\right].$$

• If a matrix of symbols $A(U; x, \xi)$, in some class $\Sigma\Gamma^m_{K,K',1} \otimes \mathcal{M}_2(\mathbb{C})$, has the form

$$A(U; x, \xi) = \begin{bmatrix} a(U;x,\xi) & b(U;x,\xi) \\ \overline{b(U;x,-\xi)} & \overline{a(U;x,-\xi)} \end{bmatrix},$$

then the matrix of operators $Op^{\mathrm{BW}}(A(U; x, \xi))$ is real-to-real.

Notation.

- To simplify the notation, we will often omit the dependence on the time t from the symbols, smoothing remainders, and maps. Moreover, given a symbol in $\Sigma\Gamma^m_{K,K',p}$ we may omit to write its dependence on U when this does not cause confusion.
- Since in the rest of the paper we only need to control expansions in degrees of homogeneity of symbols, smoothing operators and maps, up to cubic terms $O(u^3)$, we fix once and for all $N = 3$. We will omit the dependence on r and $N = 3$ in the class of symbols, writing $\Sigma\Gamma^m_{K,K',p}$, instead of $\Sigma\Gamma^m_{K,K',p}[r, 3]$, and similarly for smoothing operators and maps.
- $A \lesssim_s B$ means $A \le C(s)B$ where $C(s) > 0$ is a constant depending on $s \in \mathbb{R}$.

3 Paralinearization and Block Diagonalization

3.1 Complex form of the water waves equations

Following [1, 2], we begin by writing the water waves system (1.3) using the good-unknown $\omega = \psi - Op^{\mathrm{BW}}(B(\eta, \psi))\eta$; see (1.7)–(1.8). The water waves equations (1.3), written in the new coordinates

$$\begin{bmatrix} \eta \\ \omega \end{bmatrix} = \mathcal{G}\begin{bmatrix} \eta \\ \psi \end{bmatrix} := \begin{bmatrix} \eta \\ \psi - Op^{\mathrm{BW}}(B(\eta, \psi))\eta \end{bmatrix}, \tag{3.1}$$

assume the following paralinearized form derived in [6]:

PROPOSITION 3.1. *[Water-waves equations in (η, ω) variables] Let $I = [-T, T]$ with $T > 0$. Let $K \in \mathbb{N}^*$ and $\rho \gg 1$. There exists $s_0 > 0$ such that, for any $s \geq s_0$, for all $0 < r \leq r_0(s)$ small enough, if $(\eta, \psi) \in B_s^K(I; r)$ solves* (1.3)*, then*

$$\partial_t \eta = |D|\omega + Op^{\mathrm{BW}}\big(-\mathrm{i}V\xi - \tfrac{V_x}{2}\big)\eta + Op^{\mathrm{BW}}(b_{-1}(\eta; \cdot))\omega + R_1(\eta, \omega)\omega + R_1'(\eta, \omega)\eta \tag{3.2}$$

$$\partial_t \omega = -\eta + Op^{\mathrm{BW}}(-\mathrm{i}V\xi + \tfrac{V_x}{2})\omega - Op^{\mathrm{BW}}(\partial_t B + VB_x)\eta + R_2'(\eta, \omega)\omega + R_2''(\eta, \omega)\eta \tag{3.3}$$

where the functions V, B defined in (1.7) *are in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}$, the symbol $b_{-1}(\eta; \cdot)$ belongs to $\Sigma\Gamma^{-1}_{K,0,1}$, and the smoothing operators R_1, R_1', R_2, R_2' are in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}$. The vector field in the right-hand side of* (3.2)–(3.3) *is x-translation invariant, i.e.,* (2.5) *holds.*

PROOF. The proof follows by the computations in [6] in the absence of capillarity, specified in the case of infinite depth, in particular by propositions 7.5 and 7.6 and chapter 8.2 in [6]. The right-hand side in (3.2) is the paralinearization of the Dirichlet-Neumann operator in [6]. The approach in [6] does not make use of a variational method to study the Dirichlet-Neumann boundary value problem as in [1, 3], but uses a paradifferential parametrix à la Boutet de Monvel, introducing classes of para-Poisson operators whose symbols have a decomposition in multilinear terms. Moreover, $G(\tau_\theta\eta)[\tau_\theta\psi] = \tau_\theta G(\eta)[\psi]$, where τ_θ is the translation operator in (2.5). Hence the functions V, B satisfy the property (2.7), and the map $\mathcal{G}$ in (3.1) satisfies $\mathcal{G} \circ \tau_\theta = \tau_\theta \circ \mathcal{G}$. In conclusion, the whole vector field in the r.h.s. of (3.2)–(3.3) satisfies the x-invariance property, and the smoothing remainders satisfy (2.19) by difference. □

In Section 3.1 we will provide explicit expansions for the symbols of nonnegative order in (3.2)–(3.3) in linear and quadratic degrees of homogeneity.

Remark 3.2. [*Expansion of the Dirichlet-Neumann operator*]

(i) Substituting (3.1) in the right-hand side of (3.2), which is equal to $G(\eta)\psi$, we have, using the remarks under Definition. 2.7 and the fact that $B(\eta, \psi) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}$ is linear in ψ, that $G(\eta) - |D|$ is a map in $\Sigma\mathcal{M}_{K,0,1}$ and

$$G(\eta)\psi = |D|\psi + \widetilde{M}_1(\eta)\psi + \widetilde{M}_2(\eta)\psi + \widetilde{M}_{\geq 3}(\eta)\psi \tag{3.4}$$

for some maps $\widetilde{M}_1 \in \widetilde{\mathcal{M}}_1$, $\widetilde{M}_2 \in \widetilde{\mathcal{M}}_2$, and $\widetilde{M}_{\geq 3} \in \mathcal{M}_{K,0,3}$.

(ii) The Dirichlet-Neumann operator admits a Taylor expansion (see, e.g., formula (2.5) of [11]) of the form

$$G(\eta)\psi = |D|\psi + G_1(\eta)\psi + G_2(\eta)\psi + G_{\geq 3}(\eta)\psi, \quad D := -\mathrm{i}\partial_x, \tag{3.5}$$

$$G_1(\eta) := -\partial_x\eta\partial_x - |D|\eta|D|, \qquad G_2(\eta) := -\tfrac{1}{2}\big(D^2\eta^2|D| + |D|\eta^2 D^2 - 2|D|\eta|D|\eta|D|\big), \tag{3.6}$$

and where $G_{\geq 3}$ collects all the terms with homogeneity in η greater than 2. The notation above $|D|\eta|D|$, resp. $|D|\eta|D|\eta|D|$, means the composition operator $|D|\circ\eta\circ|D|$, resp., $|D|\circ\eta\circ|D|\circ\eta\circ|D|$, of the Fourier multiplier $|D|$ and the multiplication operator for the function η. We then see that the quadratic and cubic components of the expansions (3.5) and (3.4) coincide, namely, $G_1 = \widetilde{M}_1$ and $G_2 = \widetilde{M}_2$. It follows that $G_{\geq 3}$ is in $\mathcal{M}_{K,0,3}$.

We now write the equations (3.2)–(3.3) in terms of the complex variable u defined in (1.12).

PROPOSITION 3.3 (*Water-waves equations in complex variables*). *Let* $K \in \mathbb{N}^*$ *and* $\rho \gg 1$. *There exists* $s_0 > 0$ *such that, for any* $s \geq s_0$, *for all* $0 < r \leq r_0(s)$ *small enough, if* (η,ω) *solves* (3.2)–(3.3) *and* $U := \left[\begin{smallmatrix} u \\ \overline{u}\end{smallmatrix}\right]$ *with* u *defined in* (1.12) *belongs to* $B^K_s(I;r)$, *then* U *solves*

$$\partial_t U = Op^{\mathrm{BW}}\big(\mathrm{i}A_1(U;x)\xi + \mathrm{i}A_{1/2}(U;x)|\xi|^{\frac{1}{2}} + A_0(U;x) + A_{-1}(U;x,\xi)\big)U + R(U)U \tag{3.7}$$

where

$$A_1(U;x) := -\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right]V(U;x) \tag{3.8}$$

$$A_{1/2}(U;x) := \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1\end{smallmatrix}\right] + \left[\begin{smallmatrix} -1 & -1 \\ 1 & 1\end{smallmatrix}\right]a(U;x), \quad a := \tfrac{1}{2}(\partial_t B + VB_x), \tag{3.9}$$

$$A_0(U;x) := -\tfrac{1}{4}\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right]V_x(U;x), \tag{3.10}$$

A_{-1} *is a matrix of symbols in* $\Sigma\Gamma^{-1}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$, *and* $R(U)$ *is a matrix of smoothing operators belonging to* $\Sigma\mathcal{R}^{-\rho}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$. *The vector field in the right-hand side of* (3.7) *is* x-invariant *and it is* real-to-real *according to* (2.25).

PROOF. We first rewrite (3.2)–(3.3) as the system

$$\partial_t\begin{bmatrix}\eta\\ \omega\end{bmatrix} = Op^{\mathrm{BW}}\left(\left[\begin{smallmatrix} -\mathrm{i}V\xi - \frac{V_x}{2} & |\xi| + b_{-1} \\ -(1+a_0) & -\mathrm{i}V\xi + \frac{V_x}{2}\end{smallmatrix}\right]\right)\begin{bmatrix}\eta\\ \omega\end{bmatrix} + R(\eta,\omega)\begin{bmatrix}\eta\\ \omega\end{bmatrix}, \tag{3.11}$$

where $R \in \Sigma\mathcal{R}^{-\rho}_{K,0,1}\otimes\mathcal{M}_2(\mathbb{C})$ and the function $a_0 := \partial_t B + VB_x$ is in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,1,1}$. We now symmetrize (3.11) at the highest order, applying the change of variable

$$\begin{bmatrix}\eta\\ \omega\end{bmatrix} := \left[\begin{smallmatrix} |D|^{1/4} & 0 \\ 0 & |D|^{-1/4}\end{smallmatrix}\right]\begin{bmatrix}\widetilde{\eta}\\ \widetilde{\omega}\end{bmatrix}. \tag{3.12}$$

The conjugated system is, by Propositions 2.9 and 2.10,

$$\partial_t\begin{bmatrix}\widetilde{\eta}\\ \widetilde{\omega}\end{bmatrix} = Op^{\mathrm{BW}}\Big(\left[\begin{smallmatrix} |\xi|^{-1/4} & 0 \\ 0 & |\xi|^{1/4}\end{smallmatrix}\right]\#_\rho\left[\begin{smallmatrix} -\mathrm{i}V\xi - \frac{V_x}{2} & |\xi| + b_{-1} \\ -(1+a_0) & -\mathrm{i}V\xi + \frac{V_x}{2}\end{smallmatrix}\right]\#_\rho\left[\begin{smallmatrix} |\xi|^{1/4} & 0 \\ 0 & |\xi|^{-1/4}\end{smallmatrix}\right]\Big)\begin{bmatrix}\widetilde{\eta}\\ \widetilde{\omega}\end{bmatrix} + R(\widetilde{\eta},\widetilde{\omega})\begin{bmatrix}\widetilde{\eta}\\ \widetilde{\omega}\end{bmatrix} \tag{3.13}$$

for a new smoothing remainder R in $\Sigma\mathcal{R}^{-\rho+1}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$. Recalling (2.23) we expand in decreasing orders the symbols in (3.13).

DIAGONAL SYMBOLS. Up to a symbol in $\Sigma\Gamma^{-1}_{K,0,1}$ we have (using Proposition 2.9 and formula (2.23))

$$|\xi|^{\pm 1/4}\#_\rho(-\mathrm{i}V\xi \pm \tfrac{V_x}{2})\#_\rho|\xi|^{\mp 1/4} = -\mathrm{i}V\xi \pm \tfrac{V_x}{4}.$$

OFF-DIAGONAL SYMBOLS. Up to a symbol in $\Sigma\Gamma^{-3/2}_{K,0,1}$ we get (using Proposition 2.9 and formula (2.23)) $|\xi|^{-1/4}\#_\rho(|\xi|+b_{-1})\#_\rho|\xi|^{-1/4} = |\xi|^{1/2}$ (recall that b_{-1} is in $\Sigma\Gamma^{-1}_{K,0,1}$) and, up to a symbol in $\Sigma\Gamma^{-3/2}_{K,1,1}$, we have $-|\xi|^{1/4}\#_\rho(1+a_0)\#_\rho|\xi|^{1/4} = -(1+a_0)|\xi|^{1/2}$. The expansions above imply that the system (3.13) has the form

$$\partial_t\begin{bmatrix}\widetilde{\eta}\\ \widetilde{\omega}\end{bmatrix} = Op^{\mathrm{BW}}\left(\begin{bmatrix} -\mathrm{i}V\xi-\frac{V_x}{4} & |\xi|^{1/2}\\ -(1+a_0)|\xi|^{1/2} & -\mathrm{i}V\xi+\frac{V_x}{4}\end{bmatrix} + A_{-1}\right)\begin{bmatrix}\widetilde{\eta}\\ \widetilde{\omega}\end{bmatrix} + R(\widetilde{\eta},\widetilde{\omega})\begin{bmatrix}\widetilde{\eta}\\ \widetilde{\omega}\end{bmatrix} \tag{3.14}$$

where A_{-1} is a matrix of symbols in $\Sigma\Gamma^{-1}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$ and R is in $\Sigma\mathcal{R}^{-\rho+1}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$.

Finally, we write (3.14) in the complex variable (1.12) (recall (3.12)), and we deduce (3.7) with matrices as in (3.8), (3.9), (3.10), and a new matrix of symbols A_{-1} in $\Sigma\Gamma^{-1}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$ and a new smoothing operator $R(U)$ in $\Sigma\mathcal{R}^{-\rho}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$, renaming $\rho-1$ as ρ. Finally, since the Fourier multiplier transformation (1.12) trivially commutes with the translation operators τ_θ, the water waves vector field in (3.7) is x-invariant as the water waves vector field (3.2)–(3.3). □

In some instances we will write the water waves system (3.7) as

$$\partial_t U = -\mathrm{i}\Omega U + \mathbf{M}(U)[U], \quad \Omega := |D|^{\frac{1}{2}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \tag{3.15}$$

where $\mathbf{M}(U)$ is a real-to-real matrix of maps in $\Sigma\mathcal{M}^{m_1}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$ for some $m_1>0$; see the remarks after Definition 2.7. We will also write system (3.15) in Fourier basis as

$$\dot{u}_n = -\mathrm{i}\omega_n u_n + \mathrm{i}(F_2(U)+F_{\geq 3}(U))_n, \quad n\in\mathbb{Z}\setminus\{0\}, \quad \omega_n := \sqrt{|n|} \tag{3.16}$$

where $F_2(U) = M_1(U)[U]$ is the quadratic component of the water waves vector field and $F_{\geq 3}(U)$ collects all the cubic terms (the second equation of (3.15) for $\overline{u}$ is just the complex conjugated of the one for u). Using the x-invariance property, the vector field $F_2(U)$ can be expanded as

$$\begin{aligned} F_2(U) = & \sum_{n_1,n_2\in\mathbb{Z}\setminus\{0\},\sigma=\pm} (F_2)^{\sigma\sigma}_{n_1,n_2} u^\sigma_{n_1} u^\sigma_{n_2}\frac{e^{\mathrm{i}\sigma(n_1+n_2)x}}{2\pi}\\ & + \sum_{n_1,n_2\in\mathbb{Z}\setminus\{0\},} (F_2)^{+-}_{n_1,n_2} u_{n_1}\overline{u_{n_2}}\frac{e^{\mathrm{i}(n_1-n_2)x}}{2\pi} \end{aligned} \tag{3.17}$$

with coefficients $(F_2)^{\sigma\sigma'}_{n_1,n_2}$ in $\mathbb{C}$. We provide the explicit expression of $\mathrm{i}F_2(U)$ in Lemma 3.9.

Homogeneity expansions

By the expansion of the Dirichlet-Neumann operator in Remark 3.2, we get the quadratic approximation of the water waves equations (1.3),

$$\begin{aligned}\partial_t \eta &= |D|\psi - \partial_x(\eta\partial_x\psi) - |D|(\eta|D|\psi),\\ \partial_t\psi &= -\eta - \tfrac{1}{2}\psi_x^2 + \tfrac{1}{2}(|D|\psi)^2,\end{aligned} \tag{3.18}$$

up to functions in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. In this section, using this expansion, we compute explicitly the quadratic vector field $\mathrm{i}F_2(U)$ in (3.16), and the homogeneous expansions up to cubic terms of the functions V and a appearing in (3.8)–(3.10). We write

$$V = \mathtt{V}_1 + \mathtt{V}_2 + \mathtt{V}_{\geq 3}, \quad \mathtt{V}_j \in \widetilde{\mathcal{F}}^{\mathbb{R}}_j,\ j = 1,2,\ \ \mathtt{V}_{\geq 3} \in \mathcal{F}^{\mathbb{R}}_{K,0,3}, \tag{3.19}$$

$$a = \mathtt{a}_1 + \mathtt{a}_2 + \mathtt{a}_{\geq 3} \quad \mathtt{a}_j \in \widetilde{\mathcal{F}}^{\mathbb{R}}_j,\ j = 1,2,\ \ \mathtt{a}_{\geq 3} \in \mathcal{F}^{\mathbb{R}}_{K,1,3}. \tag{3.20}$$

In the following it is useful to note that the relation (1.12) has inverse

$$\eta := \tfrac{1}{\sqrt{2}}|D|^{\frac{1}{4}}(u+\overline{u}), \quad \omega = \tfrac{1}{\mathrm{i}\sqrt{2}}|D|^{-\frac{1}{4}}(u-\overline{u}). \tag{3.21}$$

LEMMA 3.4. *[Expansion of V] The function V defined in* (1.7) *admits the expansion*

$$V = \omega_x + \partial_x\big(Op^{\mathrm{BW}}(|D|\omega)\eta\big) - (|D|\omega)\eta_x + \mathtt{V}_{\geq 3} \tag{3.22}$$

where $\mathtt{V}_{\geq 3}$ is a function in $\mathcal{F}^{\mathbb{R}}_{K,0,3}$. Thus, in the complex variable u in (1.12), (3.21)*, we have*

$$\mathtt{V}_1 = \frac{1}{\mathrm{i}\sqrt{2}}\partial_x|D|^{-\frac{1}{4}}(u-\overline{u}), \tag{3.23}$$

$$\begin{aligned}\mathtt{V}_2 = &\frac{1}{2\mathrm{i}}\partial_x\Big(Op^{\mathrm{BW}}\big(|D|^{\frac{3}{4}}(u-\overline{u})\big)\big[|D|^{\frac{1}{4}}(u+\overline{u})\big]\Big)\\ &- \frac{1}{2\mathrm{i}}\big(|D|^{\frac{3}{4}}(u-\overline{u})\big)\big(\partial_x|D|^{\frac{1}{4}}(u+\overline{u})\big).\end{aligned} \tag{3.24}$$

PROOF. By (1.7) and using the expansion (3.5), we deduce $B = |D|\psi$ up to a quadratic function in $\mathcal{F}^{\mathbb{R}}_{K,0,2}$. As a consequence, by (1.7) and (3.1), we have

$$\begin{aligned}V = \psi_x - B\eta_x &= (\omega + Op^{\mathrm{BW}}(B)\eta)_x - B\eta_x\\ &= \omega_x + \partial_x\big(Op^{\mathrm{BW}}(|D|\psi)\eta\big) - (|D|\psi)\eta_x\end{aligned}$$

up to a function in $\mathcal{F}^{\mathbb{R}}_{K,0,3}$. Since $\psi = \omega$ plus a quadratic function in $\mathcal{F}^{\mathbb{R}}_{K,0,2}$ (see (3.1)) we get (3.22). □

LEMMA 3.5. *[Expansion of $\partial_t B$] Let B be the function defined in* (1.7)*. Then*

$$\begin{aligned}\partial_t B = &-|D|\eta - \eta|D|^2\eta + |D|(\eta|D|\eta) + |D|\big(-\tfrac{1}{2}\omega_x^2 - \tfrac{1}{2}(|D|\omega)^2\big)\\ &+ (|D|\omega)(|D|^2\omega)\end{aligned} \tag{3.25}$$

plus a cubic function in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$.

PROOF. Recalling (1.7), and using (3.5), we have to compute the expansion of

$$
\begin{aligned}
\partial_t B &= \frac{\partial_t(G(\eta)\psi + \eta_x\psi_x)}{1+\eta_x^2} - \frac{(G(\eta)\psi + \eta_x\psi_x)2\eta_x(\eta_t)_x}{(1+\eta_x^2)^2} \\
&= \partial_t(G(\eta)\psi) + (\partial_t\eta)_x\psi_x + \eta_x(\partial_t\psi)_x
\end{aligned} \tag{3.26}
$$

plus a cubic function in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. For the first term in (3.26) we use the "shape derivative" formula (see [29])

$$
G'(\eta)[\widehat{\eta}]\psi = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\{G(\eta+\epsilon\widehat{\eta})\psi - G(\eta)\psi\} = -G(\eta)(B\widehat{\eta}) - \partial_x(V\widehat{\eta}) \tag{3.27}
$$

where $V = \psi_x - B\eta_x$ is in (1.7). Then, using (3.26), (3.27), and (3.18), we obtain, after simplification,

$$
\begin{aligned}
\partial_t B &= -|D|\eta - \frac{1}{2}|D|((|D|\psi)^2) - \frac{1}{2}|D|\psi_x^2 + |D|(\eta|D|\eta) \\
&\quad + \eta\eta_{xx} - \psi_{xx}|D|\psi
\end{aligned} \tag{3.28}
$$

plus a cubic function $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. Since $\partial_{xx} = -|D|^2$ and $\psi = \omega$ plus a quadratic function in $\mathcal{F}^{\mathbb{R}}_{K,1,2}$, we have that (3.28) implies (3.25). □

We now expand the function $a = \frac{1}{2}(\partial_t B + VB_x)$ that appears in (3.9).

LEMMA 3.6 (*Expansion of a*). *We have*

$$
\begin{aligned}
2a = &-|D|\eta - \eta(|D|^2\eta) + |D|(\eta|D|\eta) - \tfrac{1}{2}|D|(\omega_x^2 + (|D|\omega)^2) \\
&+ (|D|\omega)(|D|^2\omega) + \omega_x(\partial_x|D|\omega)
\end{aligned}
$$

plus a cubic function in $\mathcal{F}^{\mathbb{R}}_{K,0,3}$.

PROOF. By (3.22) and (1.7) we have that $a = \frac{1}{2}(\partial_t B + VB_x) = \frac{1}{2}\partial_t B + \frac{1}{2}\omega_x(|D|\psi_x)$ plus a cubic function in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. Hence (3.25) implies the lemma. □

We Fourier develop the functions $\mathtt{a}_1, \mathtt{V}_1, \mathtt{a}_2, \mathtt{V}_2$, as in (2.11), (2.12).

LEMMA 3.7. *[Coefficients of $\mathtt{V}_1$ and $\mathtt{V}_2$] The coefficients of $\mathtt{V}_1$ and $\mathtt{V}_2$ in* (3.23)–(3.24) *are, for any $n \in \mathbb{Z}\setminus\{0\}$*

$$
(\mathtt{V}_1)_n^+ = (\mathtt{V}_1)_n^- = \tfrac{1}{\sqrt{2}}n|n|^{-1/4}, \quad (\mathtt{V}_2)_{n,n}^{+-} = n|n|, \quad (\mathtt{V}_2)_{n,-n}^{+-} = 0. \tag{3.29}
$$

PROOF. It follows by explicit computation using (3.23), (3.24), recalling (2.3), and using Definition 2.4 of the Bony-Weyl quantitation (and (2.14)). □

We now compute the coefficients of the linear and quadratic component of a in (3.9).

LEMMA 3.8. *[Coefficients of $\mathtt{a}_1$ and $\mathtt{a}_2$] The coefficients of $\mathtt{a}_1$ and $\mathtt{a}_2$ in* (3.20) *satisfy*

$$
(\mathtt{a}_1)_n^+ = (\mathtt{a}_1)_n^- = -\tfrac{1}{2\sqrt{2}}|n|^{5/4}, \quad (\mathtt{a}_2)_{n,n}^{+-} = \tfrac{1}{2}|n|^{5/2} \quad \forall n \in \mathbb{Z}\setminus\{0\}. \tag{3.30}
$$

PROOF. It follows by the explicit expression in Lemma 3.6 and passing to the variables in (1.12). □

It turns out that $(\mathtt{a}_2)^{+-}_{n,-n} = |n|^{5/2}$, but we do not use this information in the paper.

LEMMA 3.9 (*Quadratic water waves vector field* $\mathrm{i}F_2(U)$). *The coefficients* $(F_2)^{+-}_{n_1,n_2}$ *defined in* (3.17) *of the quadratic water waves vector field* $\mathrm{i}F_2(U)$ *in* (3.16) *satisfy*

$$(F_2)^{+-}_{n,-n} = (F_2)^{+-}_{-n,n} = 2^{-\frac{1}{4}}|n|^{\frac{7}{4}}. \tag{3.31}$$

PROOF. It follows by direct computatiup to ons using equations (3.18), passing to the variables $u, \overline{u}$ defined in (1.12) and recalling that, by (1.7) and (3.5), we have the approximate identity $\omega = \psi - Op^{\mathrm{BW}}(|D|\psi)\eta$. □

3.2 Block-diagonalization

The goal of this section is to transform the water waves system (3.7) into the system (3.33) below, which is block-diagonal in the variables $(u, \overline{u})$ modulo a smoothing operator $R(U)$.

PROPOSITION 3.10 (*Block-diagonalization*). *Let* $\rho \gg 1$ *and* $K \geq K' := 2\rho + 2$. *There exists* $s_0 > 0$ *such that, for any* $s \geq s_0$*, for all* $0 < r \leq r_0(s)$ *small enough, and any solution* $U \in B^K_s(I; r)$ *of* (3.7)*, the following holds*:

(i) *there is a map* $\Psi^\theta_{\mathrm{diag}}(U)$*,* $\theta \in [0, 1]$*, satisfying, for some* $C = C(s, r, K) > 0$

$$\begin{aligned} &\left\|\partial_t^k \Psi^\theta_{\mathrm{diag}}(U)[V]\right\|_{\dot H^{s-k}} + \left\|\partial_t^k (\Psi^\theta_{\mathrm{diag}}(U))^{-1}[V]\right\|_{\dot H^{s-k}} \\ &\quad \leq (1 + C\|U\|_{K,s_0})\|V\|_{k,s}, \end{aligned} \tag{3.32}$$

for any $0 \leq k \leq K - K'$ *and any* $V = \left[\begin{smallmatrix} v \\ \overline{v}\end{smallmatrix}\right]$ *in* $C^{K-K'}_{*\mathbb{R}}(I, \dot H^s(\mathbb{T}; \mathbb{C}^2))$*,* $\theta \in [0, 1]$*;*

(ii) *the function* $W := (\Psi^\theta_{\mathrm{diag}}(U)U)_{|\theta=1}$ *solves the system*

$$\partial_t W = Op^{\mathrm{BW}}\left(\begin{bmatrix} d(U;x,\xi)+r_{-1/2}(U;x,\xi) & 0 \\ 0 & \overline{d(U;x,-\xi)+r_{-1/2}(U;x,-\xi)}\end{bmatrix}\right)W + R(U)[W] \tag{3.33}$$

where $d(U; x, \xi)$ *is a symbol of the form*

$$d(U; x, \xi) := -\mathrm{i}V(U; x)\xi - \mathrm{i}(1 + a^{(0)}(U; x))|\xi|^{1/2} \tag{3.34}$$

where $a^{(0)}$ *is a function that is in* $\Sigma\mathcal{F}^{\mathbb{R}}_{K,1,1}$*,* $r_{-1/2}(U; x, \xi)$ *is a symbol in* $\Sigma\Gamma^{-1/2}_{K,2\rho+2,1}$*, and* $R(U)$ *is a real-to-real matrix of smoothing operators in* $\Sigma\mathcal{R}^{-\rho}_{K,2\rho+2,1} \otimes \mathcal{M}_2(\mathbb{C})$*. The function* $a^{(0)}$ *has the expansion*

$$a^{(0)} = \mathtt{a}_1 + \mathtt{a}^{(0)}_2 + \mathtt{a}^{(0)}_{\geq 3}, \quad \mathtt{a}^{(0)}_2 := \mathtt{a}_2 - \tfrac{1}{2}\mathtt{a}_1^2 \in \widetilde{\mathcal{F}}^{\mathbb{R}}_2, \tag{3.35}$$

where $\mathtt{a}_1$ *and* $\mathtt{a}_2$ *are defined in* (3.20).

Proposition 3.10 is proved by applying a sequence a transformations that iteratively block-diagonalize (3.7) in decreasing orders. In Section 3.2 we block-diagonalize (3.7) at the order $1/2$, and in Section 3.2 we perform the block-diagonalization until the negative order $-\rho$.

Block-Diagonalization at order 1/2

In this subsection we aim to diagonalize the matrix of symbols $A_{1/2}(U;x)|\xi|^{1/2}$ in (3.7), up to a matrix of symbols of order 0. We apply a parametrix argument conjugating the system (3.7) with a paradifferential operator whose principal matrix symbol is

$$C := \begin{bmatrix} f & g \\ g & f \end{bmatrix}, \quad f(U;x) := \frac{1+a+\lambda_+}{\sqrt{(1+a+\lambda_+)^2 - a^2}}, \qquad g(U;x) := \frac{-a}{\sqrt{(1+a+\lambda_+)^2 - a^2}}, \tag{3.36}$$

where

$$\lambda_\pm = \lambda_\pm(U;x) := \pm\sqrt{(1+a)^2 - a^2} \tag{3.37}$$

are the eigenvalues of $A_{1/2}$. We have

$$\det(C) = f^2 - g^2 = 1, \quad C^{-1} = \begin{bmatrix} f & -g \\ -g & f \end{bmatrix}, \tag{3.38}$$

and

$$C^{-1}A_{1/2}C = \lambda_+\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -(1+a^{(0)}) & 0 \\ 0 & 1+a^{(0)} \end{bmatrix}, \qquad a^{(0)} := \lambda_+ - 1 \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,1,1}. \tag{3.39}$$

LEMMA 3.11. *There exists a function $m_{-1}(U;x)$ in $\Sigma\mathcal{F}_{K,1,1}$ such that the flow*

$$\partial_\theta \Psi^\theta_{-1}(U) = Op^{\mathrm{BW}}(M_{-1})\Psi^\theta_{-1}(U), \quad \Psi^0_{-1}(U) = \mathrm{Id}, \qquad M_{-1} := \begin{bmatrix} 0 & m_{-1}(U;x) \\ \overline{m_{-1}(U;x)} & 0 \end{bmatrix}, \tag{3.40}$$

has the form

$$(\Psi^\theta_{-1}(U))^\pm_{|\theta=1} = Op^{\mathrm{BW}}(C^{\mp 1}) + R^\pm(U), \qquad R^\pm(U) \in \Sigma\mathcal{R}^{-\rho}_{K,1,1} \otimes \mathcal{M}_2(\mathbb{C}). \tag{3.41}$$

Moreover, if U solves (3.7)*, then the function*

$$W_0 := (\Psi^\theta_{-1}(U))_{|\theta=1} U \tag{3.42}$$

solves the system

$$\partial_t W_0 = Op^{\mathrm{BW}}\Big(\begin{bmatrix} d(U;x,\xi) & 0 \\ 0 & \overline{d(U;x,-\xi)} \end{bmatrix} + A^{(0)}\Big) W_0 + R^{(0)}(U)W_0 \tag{3.43}$$

where $d(U;x,\xi)$ is the symbol in (3.34) *with $a^{(0)}(U;x)$ defined in* (3.39)*, a matrix of symbols*

$$A^{(0)} := \begin{pmatrix} c_0(U;x,\xi) & b_0(U;x,\xi) \\ \overline{b_0(U;x,-\xi)} & \overline{c_0(U;x,-\xi)} \end{pmatrix}, \quad c_0 \in \Sigma\Gamma^{-\frac{1}{2}}_{K,2,1}, \ \ b_0 \in \Sigma\Gamma^{0}_{K,2,1}, \tag{3.44}$$

and a real-to-real matrix of smoothing operators $R^{(0)}(U)$ in $\Sigma\mathcal{R}^{-\rho}_{K,2,1} \otimes \mathcal{M}_2(\mathbb{C})$. Moreover, the function $a^{(0)}$ has the expansion (3.35).

PROOF. Formulæ (3.41) follow by reasoning as in proposition 3.6 and corollary 3.1 in [18]. We conjugate (3.7) with the flow $(\Psi^{\theta}_{-1}(U))_{|\theta=1}$ using formula (A.2) in Lemma A.1. By Proposition 2.10 we deduce that, if U solves (3.7), then

$$\begin{aligned}\partial_t W_0 &\overset{(3.41)}{=} \partial_t Op^{\mathrm{BW}}(C^{-1}) Op^{\mathrm{BW}}(C) W_0 \\ &\quad + Op^{\mathrm{BW}}(C^{-1}) Op^{\mathrm{BW}}\big(\mathrm{i}A_1\xi + \mathrm{i}A_{1/2}|\xi|^{\frac{1}{2}} + A_0 + A_{-1}\big) Op^{\mathrm{BW}}(C) W_0\end{aligned}$$

up to a matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho+1}_{K,2,1} \otimes \mathcal{M}_2(\mathbb{C})$ acting on W_0. Moreover, Proposition 2.9 implies that

$$\partial_t W_0 = Op^{\mathrm{BW}}\Big(\partial_t C^{-1} \#_\rho C + C^{-1} \#_\rho \big(\mathrm{i}A_1\xi + \mathrm{i}A_{1/2}|\xi|^{\frac{1}{2}} + A_0 + A_{-1}\big) \#_\rho C\Big) W_0 \tag{3.45}$$

up to terms in $\Sigma\mathcal{R}^{-\rho+1}_{K,2,1} \otimes \mathcal{M}_2(\mathbb{C})$. By (3.36), (3.38) we have $(\partial_t C^{-1})\#_\rho C = ((\partial_t f)g - (\partial_t g)f)\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ because differentiating $f^2 - g^2 = 1$ we get $(\partial_t f)f - (\partial_t g)g = 0$.

By (3.8), using symbolic calculus and $f^2 - g^2 = 1$ (see (3.38)), we obtain the exact expansion

$$C^{-1}\#_\rho(\mathrm{i}A_1\xi)\#_\rho C \quad = -\mathrm{i}V\xi\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] + V(f_x g - g_x f)\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right].$$

By (3.39) we have $C^{-1}\#_\rho(\mathrm{i}A_{1/2}|\xi|^{\frac{1}{2}})\#_\rho C = \mathrm{i}(1 + a^{(0)})|\xi|^{\frac{1}{2}}\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ modulo a matrix of symbols $\Sigma\Gamma^{-\frac{1}{2}}_{K,1,1} \otimes \mathcal{M}_2(\mathbb{C})$. Moreover, recalling (3.10), we have the paraproduct expansion $C^{-1}\#_\rho A_0 \#_\rho C = A_0 = -\frac{V_x}{4}\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and finally, since A_{-1} is in $\Sigma\Gamma^{-1}_{K,1,1} \otimes \mathcal{M}_2(\mathbb{C})$ we deduce $C^{-1}\#_\rho A_{-1}\#_\rho C \in \Sigma\Gamma^{-1}_{K,1,1} \otimes \mathcal{M}_2(\mathbb{C})$. The discussion above imply (3.43), (3.44), with a remainder $R^{(0)}(U)$ in $\Sigma\mathcal{R}^{-\rho}_{K,2,1} \otimes \mathcal{M}_2(\mathbb{C})$, renaming $\rho - 1$ as ρ. Finally, by (3.39), (3.37) and (3.20) we get the expansion (3.35). □

Block-Diagonalization at negative orders

The aim of this subsection is to block-diagonalize the system (3.43) (which is yet block-diagonal at the orders 1 and $1/2$) into (3.33).

LEMMA 3.12. *For $j = 0, \ldots, 2\rho$, there are*

- *paradifferential operators of the form*

$$\mathcal{Y}^{(j)}(U) := Op^{\mathrm{BW}}\Big(\Big[\begin{smallmatrix} d(U;x,\xi) & 0 \\ 0 & \overline{d(U;x,-\xi)} \end{smallmatrix}\Big]\Big) + Op^{\mathrm{BW}}(A^{(j)}) \tag{3.46}$$

where $d(U;x,\xi)$ is the symbol defined in Lemma 3.11*,* $A^{(j)}$ *is a matrix of symbols of the form*

$$A^{(j)} = \begin{pmatrix} c_j(U;x,\xi) & b_j(U;x,\xi) \\ \overline{b_j(U;x,-\xi)} & \overline{c_j(U;x,-\xi)} \end{pmatrix}, \qquad c_j \in \Sigma\Gamma^{-\frac{1}{2}}_{K,j+2,1}, \quad b_j \in \Sigma\Gamma^{-\frac{j}{2}}_{K,j+2,1}, \tag{3.47}$$

• *a real-to-real matrix of smoothing operators* $R^{(j)}(U)$ *in* $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1} \otimes \mathcal{M}_2(\mathbb{C})$ *such that, if* W_j*,* $j = 0, \dots, 2\rho - 1$*, solves*

$$\partial_t W_j = \big(\mathcal{Y}^{(j)}(U) + R^{(j)}(U)\big) W_j, \quad W_j := \begin{bmatrix} w_j \\ \overline{w}_j \end{bmatrix}, \tag{3.48}$$

then

$$W_{j+1} := \big(\Psi^{\theta}_j(U) W_j\big)_{|\theta=1} \tag{3.49}$$

where $\Psi^{\theta}_j(U)$ *is the flow at time* $\theta \in [0,1]$ *of*

$$\partial_\theta \Psi^{\theta}_j(U) = \mathrm{i} Op^{\mathrm{BW}}(M_j(U;x,\xi)) \Psi^{\theta}_j(U), \quad \Psi^{0}_j(U) = \mathrm{Id}, \tag{3.50}$$

with

$$M_j(U;x,\xi) := \begin{bmatrix} 0 & -\mathrm{i} m_j(U;x,\xi) \\ \overline{\mathrm{i} m_j(U;x,-\xi)} & 0, \end{bmatrix} \qquad m_j = \frac{-\chi(\xi) b_j(U;x,\xi)}{2\mathrm{i}(1 + a^{(0)}(U;x))|\xi|^{\frac{1}{2}}} \in \Sigma\Gamma^{-\frac{j+1}{2}}_{K,j+2,1}, \tag{3.51}$$

and χ *defined in* (2.17) *satisfies a system of the form* (3.48) *with* $j+1$ *instead of* j*.*

PROOF. The proof proceeds by induction.

Initialization. System (3.43) is (3.48) for $j = 0$ where the paradifferential operator $\mathcal{Y}^{(0)}(U)$ has the form (3.46) with the matrix of symbols $A^{(0)}$ defined in Lemma 3.11.

Iteration. We now argue by induction. Suppose that W_j solves system (3.48) with operators $\mathcal{Y}^{(j)}(U)$ of the form (3.46)–(3.47) and smoothing operators $R^{(j)}(U)$ in $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1} \otimes \mathcal{M}_2(\mathbb{C})$. Let us study the system solved by the function W_{j+1} defined in (3.49). Note that the symbols of the matrix M_j defined in (3.51) have negative order for any $j \geq 0$. By formula (A.2) the conjugated system has the form

$$\partial_t W_{j+1} = Op^{\mathrm{BW}}\big((\partial_t \Psi^1_j(U))\Psi^{-1}_j(U) + \Psi^1_j(U)\mathcal{Y}^{(j)}(U)\Psi^{-1}_j(U)\big) W_{j+1} \tag{3.52}$$

up to a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1} \otimes \mathcal{M}_2(\mathbb{C})$.

Moreover, the operator $(\partial_t \Psi^1_j(U))\Psi^{-1}_j(U)$ admits the Lie expansion in (A.4) specified for $\mathbf{A} := Op^{\mathrm{BW}}(M_j(U))$. We recall (see (2.23)) that

$$M_j \#_\rho \partial_t M_j - \partial_t M_j \#_\rho M_j = \{M_j, \partial_t M_j\} \in \Sigma\Gamma^{-(j+1)-1}_{K,j+3,2}$$

up to a symbol in $\Sigma\Gamma^{-(j+1)-3}_{K,j+3,2}$.

By Proposition 2.9 we have that $\mathrm{Ad}_{\mathrm{i}Op^{\mathrm{BW}}(M_j)}[\mathrm{i}Op^{\mathrm{BW}}(\partial_t M_j)]$ is a paradifferential operator with symbol in $\Sigma\Gamma^{-(j+1)-1}_{K,j+3,2}\otimes\mathcal{M}_2(\mathbb{C})$ plus a smoothing remainder in $\Sigma\mathcal{R}^{-\rho}_{K,j+3,2}\otimes\mathcal{M}_2(\mathbb{C})$. As a consequence, we deduce, for $k\geq 2$,

$$\mathrm{Ad}^k_{\mathrm{i}Op^{\mathrm{BW}}(M_j)}\big[\mathrm{i}Op^{\mathrm{BW}}(\partial_t M_j)\big]=Op^{\mathrm{BW}}(B_k)+R_k,$$
$$B_k\in\Gamma^{-\frac{j+1}{2}(k+1)-k}_{K,j+3,k+1}\otimes\mathcal{M}_2(\mathbb{C}),$$

and $R_k\in\mathcal{R}^{-\rho}_{K,j+3,k+1}\otimes\mathcal{M}_2(\mathbb{C})$. By taking L large enough with respect to ρ, we get that $(\partial_t\Psi^1_j(U))\Psi^{-1}_j(U)$ is a paradifferential operator with symbol in

$$\Sigma\Gamma^{-\frac{j+1}{2}}_{K,j+3,1}\otimes\mathcal{M}_2(\mathbb{C})$$

plus a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,j+3,1}\otimes\mathcal{M}_2(\mathbb{C})$. We now want to apply the expansion (A.3) with $\mathbf{A}:=Op^{\mathrm{BW}}(M_j(U))$ and $X:=\mathcal{Y}^{(j)}$ in order to study the second summand in (3.52). We claim that

$$\begin{aligned}&\Psi^1_j(U)\mathcal{Y}^{(j)}(U)\Psi^{-1}_j(U)\\&\quad=Op^{\mathrm{BW}}(\mathcal{Y}^{(j)}(U))+\big[Op^{\mathrm{BW}}(\mathrm{i}M_j(U)),\mathcal{Y}^{(j)}(U)\big]\end{aligned}\tag{3.53}$$

plus a paradifferential operator with symbol in $\Sigma\Gamma^{-(j+1)/2}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$ and a smoothing operator belonging to $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$. We first give the expansion of $[Op^{\mathrm{BW}}(\mathrm{i}M_j(U)),\mathcal{Y}^{(j)}(U)]$ using the expression of $\mathcal{Y}^{(j)}(U)$ in (3.46). We have

$$\begin{aligned}&\Big[Op^{\mathrm{BW}}(\mathrm{i}M_j(U)),Op^{\mathrm{BW}}\Big(\begin{bmatrix}d(U;x,\xi)&0\\0&\overline{d(U;x,-\xi)}\end{bmatrix}\Big)\Big]\\&\quad:=Op^{\mathrm{BW}}\Big(\begin{bmatrix}0&p_j(U;x,\xi)\\\overline{p_j(U;x,-\xi)}&0\end{bmatrix}\Big)\\&\quad p_j:=2\mathrm{i}m_j(U;x,\xi)(1+a^{(0)}(U;x))|\xi|^{\frac12}\end{aligned}\tag{3.54}$$

up to a symbol in $\Sigma\Gamma^{-(j+1)/2}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$. Moreover, since $A^{(j)}$ is a matrix of symbols of order $-1/2$, for $j\geq 1$, resp., 0 for $j=0$ (see (3.47)), we have that $\big[Op^{\mathrm{BW}}(\mathrm{i}M_j),Op^{\mathrm{BW}}(A^{(j)})\big]$ belongs to $\Sigma\Gamma^{-\frac{j+2}{2}}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$ for $j\geq 1$ and to $\Sigma\Gamma^{-\frac12}_{K,2,1}\otimes\mathcal{M}_2(\mathbb{C})$ for $j=0$ up to a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$. It follows that the off-diagonal symbols of order $-j/2$ in (3.53) are of the form $\big[\begin{smallmatrix}0&q_j(U;x,\xi)\\\overline{q_j(U;x,-\xi)}&0\end{smallmatrix}\big]$ with

$$q_j(U;x,\xi)\overset{(3.54)}{:=}b_j(U;x,\xi)+2\mathrm{i}m_j(U;x,\xi)(1+a^{(0)})|\xi|^{\frac12}.\tag{3.55}$$

By the definition of χ in (2.17) and the remark under Definition 2.6, the operator $Op^{\mathrm{BW}}((1-\chi(\xi))b_j(U;x,\xi))$ is in $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1}\otimes\mathcal{M}_2(\mathbb{C})$ for any $\rho\geq 0$. Moreover, by the choice of $m_j(U;x,\xi)$ in (3.51), we have that

$$\chi(\xi)b_j(U;x,\xi) + 2\mathrm{i}m_j(U;x,\xi)(1+a^{(0)})|\xi|^{\frac{1}{2}} = 0.$$

This implies that $[\mathrm{i}Op^{\mathrm{BW}}(M_j), \mathcal{Y}^{(j)}(U)]$ is a paradifferential operator with symbol in $\Sigma\Gamma^{-\frac{j+1}{2}}_{K,j+2,1} \otimes \mathcal{M}_2(\mathbb{C})$ plus a remainder in $\Sigma\mathcal{R}^{-\rho}_{K,j+2,1} \otimes \mathcal{M}_2(\mathbb{C})$. Now, using Proposition 2.9, we deduce, for $k \geq 2$,

$$\mathrm{Ad}^k_{\mathrm{i}Op^{\mathrm{BW}}(M_j)}[\mathcal{Y}^{(j)}(U)] = Op^{\mathrm{BW}}(\widetilde{B}_k) + \widetilde{R}_k, \quad \widetilde{B}_k \in \Gamma^{-\frac{j+1}{2}k}_{K,j+2,k+1} \otimes \mathcal{M}_2(\mathbb{C}),$$

where $\widetilde{R}_k$ is in $\mathcal{R}^{-\rho}_{K,j+2,k+1} \otimes \mathcal{M}_2(\mathbb{C})$. Using formula (A.3) with L large enough and the estimates of flow in (3.50) (see Lemma A.2) one obtains the claim in (3.53). We conclude that (3.49) solves a system of the form 3.46–(3.48) with $j \rightsquigarrow j+1$. □

Proof of Proposition 3.10. For $\theta \in [0,1]$ we define

$$\Psi^{\theta}_{\mathrm{diag}}(U) := \Psi^{\theta}_{2\rho-1}(U) \circ \cdots \circ \Psi^{\theta}_0(U) \circ \Psi^{\theta}_{-1}(U) \tag{3.56}$$

where the maps $\Psi^{\theta}_{-1}(U)$ and $\Psi^{\theta}_j(U)$, $j = 0,1,\dots,2\rho-1$ are defined respectively in (3.42), (3.49). The bound (3.32) follows by Lemma A.2. Lemmata 3.11, 3.12 imply that if U solves (3.7) then the function $W := W_{2\rho} = (\Psi^{\theta}_{\mathrm{diag}}(U)U)_{|\theta=1}$ solves the system (3.48) with $j = 2\rho$ which is (3.33) with $r_{-1/2} := c_{2\rho}$ and

$$R(U) := Op^{\mathrm{BW}}\Big(\Big[\begin{smallmatrix} 0 & b_{2\rho}(U;x,\xi) \\ \overline{b_{2\rho}(U;x,-\xi)} & 0 \end{smallmatrix}\Big]\Big) + R^{(2\rho)}(U), \quad b_{2\rho} \in \Sigma\Gamma^{-\rho}_{K,2\rho+2,1},$$

which is a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,2\rho+2,1} \otimes \mathcal{M}_2(\mathbb{C})$ by the remark below Proposition 2.6. The expansion (3.35) is proved in Lemma 3.11. □

4 Reductions to Constant Integrable Coefficients

The aim of this section is to conjugate (3.33) to a system in which the symbols of the paradifferential operators are constant in the spatial variable x and are "*integrable*" according to Definition 4.1 below, up to symbols which are "*admissible*" according to Definition 4.2.

DEFINITION 4.1 (*Integrable symbol*). A homogeneous symbol f in $\widetilde{\Gamma}^m_2$ is *integrable* if it is independent of x and it has the form

$$f(U;x,\xi) = f(U;\xi) := \frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}} f^{+-}_{n,n}(\xi)|u_n|^2, \qquad f^{+-}_{n,n}(\xi) \in \mathbb{C},\ n \in \mathbb{Z}\setminus\{0\}. \tag{4.1}$$

DEFINITION 4.2 (*Admissible symbol*). A nonhomogeneous symbol $H_{\geq 3}$ in $\Gamma^1_{K,K',3}$ is *admissible* if it has the form

$$H_{\geq 3}(U;x,\xi) := \mathrm{i}\alpha_{\geq 3}(U;x)\xi + \mathrm{i}\beta_{\geq 3}(U;x)|\xi|^{\frac{1}{2}} + \gamma_{\geq 3}(U;x,\xi) \tag{4.2}$$

with real-valued functions $\alpha_{\geq 3}(U;x)$, $\beta_{\geq 3}(U;x)$ in $\mathcal{F}^{\mathbb{R}}_{K,K',3}$ and a symbol $\gamma_{\geq 3}(U;x,\xi)$ in $\Gamma^0_{K,K',3}$. A matrix of symbols $\mathbf{H}_{\geq 3}$ in $\Gamma^1_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C})$ is admissible if it has the form

$$\mathbf{H}_{\geq 3}(U;x,\xi) = \begin{bmatrix} H_{\geq 3}(U;x,\xi) & 0 \\ 0 & \overline{H_{\geq 3}(U;x,-\xi)} \end{bmatrix} \tag{4.3}$$

for a scalar admissible symbol $H_{\geq 3}$.

The relevance of Definition 4.2 is explained in the next remark.

Remark 4.3. An equation of the form $\partial_t v = Op^{\mathrm{BW}}(H_{\geq 3}(U;x,\xi))[v]$, where $H_{\geq 3}(U;x,\xi)$ is an admissible symbol in $\Gamma^1_{K,K',3}$, admits an energy estimate of the form

$$\partial_t \|v(t,\cdot)\|^2_{\dot H^s} \lesssim_s \|U(t,\cdot)\|^3_{K,s_0} \|v(t,\cdot)\|^2_{\dot H^s}$$

for $s \geq s_0 \gg 1$; see Lemma 6.4. For this reason vector fields of this form are "admissible" to prove existence of solutions up to times $O(\varepsilon^{-3})$.

The main result of this section is the following.

PROPOSITION 4.4 (Integrability of water waves at cubic degree up to smoothing remainders). *Fix $\rho > 0$ arbitrary and $K \geq K' := 2\rho + 2$. There exists $s_0 > 0$ such that, for any $s \geq s_0$, for all $0 < r \leq r_0(s)$ small enough, and any solution $U \in B^K_s(I;r)$ of* (3.7)*, there is a family of nonlinear maps $\mathbf{F}^\theta(U)$, $\theta \in [0,1]$, such that the function $Z := \mathbf{F}^1(U)$ solves the system*

$$\partial_t Z = -\mathrm{i}\Omega Z + Op^{\mathrm{BW}}\big(-\mathrm{i}\mathtt{D}(U;\xi) + \mathtt{H}_{\geq 3}\big)Z + \mathtt{R}(U)[Z] \tag{4.4}$$

where Ω is defined in (3.15) *and*

- *The symbol $\mathtt{D}(U;\xi)$ has the form*

$$\begin{aligned} \mathtt{D}(U;\xi) &:= \begin{bmatrix} \zeta(U)\xi + \mathcal{D}_{-1/2}(U;\xi) & 0 \\ 0 & \zeta(U)\xi - \overline{\mathcal{D}_{-1/2}(U;-\xi)} \end{bmatrix}, \\ \zeta(U) &:= \frac{1}{\pi} \sum_{n \in \mathbb{Z}\setminus\{0\}} n|n||u_n|^2, \end{aligned} \tag{4.5}$$

  with an integrable symbol $\mathcal{D}_{-1/2}(U;\xi) \in \widetilde{\Gamma}_2^{-\frac{1}{2}}$ (see Definition 4.1*).*
- *The matrix of symbols $\mathtt{H}_{\geq 3} \in \Gamma^1_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C})$ is admissible (see Definition* 4.2*).*
- *$\mathtt{R}(U)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho+4m}_{K,K',1} \otimes \mathcal{M}_2(\mathbb{C})$ for some $m > 0$.*
- *The family of transformations has the form*

$$\mathbf{F}^\theta(U) := \mathfrak{F}^\theta(U)[U] \tag{4.6}$$

  with $\mathfrak{F}^\theta(U)$ real-to-real, bounded, and invertible, and there is a constant $C = C(s,r,K)$ such that, $\forall\, 0 \leq k \leq K - K'$, for any

$$V \in C^{K-K'}_{*\mathbb{R}}(I;\dot H^s(\mathbb{T};\mathbb{C}^2)),$$

one has

$$\begin{aligned}\left\|\partial_t^k \mathfrak{F}^\theta(U)[V]\right\|_{\dot H^{s-k}} + \left\|\partial_t^k (\mathfrak{F}^\theta(U))^{-1}[V]\right\|_{\dot H^{s-k}} \\ \le \|V\|_{k,s}(1 + C\|U\|_{K,s_0}),\end{aligned} \tag{4.7}$$

uniformly in $\theta \in [0,1]$.

The proof of Proposition 4.4 above is divided into several steps in Sections 4.1–4.3 below. We combine these steps in Section 4.4.

4.1 Integrability at order 1

By Proposition 3.10 we have obtained, writing only the first line of the system (3.33)–(3.34),

$$\begin{aligned}\partial_t w = {} & Op^{\mathrm{BW}}\big(-\mathrm{i}V(U;x)\xi - \mathrm{i}(1 + a^{(0)}(U;x))|\xi|^{1/2} + r_{-1/2}\big)w \\ & + R(U)[W]\end{aligned} \tag{4.8}$$

where $R(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$ with $K' = 2\rho + 2$ and $W = [\begin{smallmatrix} w \\ \overline{w} \end{smallmatrix}]$. The second component of system (3.33) is the complex conjugated of the first one. Expanding in degrees of homogeneity the symbol

$$r_{-1/2} = \mathtt{r}_1 + \mathtt{r}_2 + \mathtt{r}_{\ge 3}, \quad \mathtt{r}_1 \in \widetilde{\Gamma}_1^{-\frac12},\ \mathtt{r}_2 \in \widetilde{\Gamma}_2^{-\frac12},\ \mathtt{r}_{\ge 3} \in \Gamma^{-\frac12}_{K,K',3},$$

recalling (3.19) and item (ii) in Proposition 3.10, we rewrite (4.8) as

$$\begin{aligned}\partial_t w = {} & Op^{\mathrm{BW}}\big(-\mathrm{i}(\mathtt{V}_1 + \mathtt{V}_2)\xi - \mathrm{i}\big(1 + \mathtt{a}_1 + \mathtt{a}_2^{(0)}\big)|\xi|^{1/2} + \mathtt{r}_1 + \mathtt{r}_2 + H_{\ge 3}\big)w \\ & + R(U)[W]\end{aligned} \tag{4.9}$$

where $H_{\ge 3}$ is an *admissible* symbol according to Definition 4.2.

Elimination of the linear symbol of the transport

The goal of this subsection is to eliminate the transport operator $Op^{\mathrm{BW}}(-\mathrm{i}\mathtt{V}_1\xi)$ in (4.9). With this aim we conjugate the equation (4.9) under the flow

$$\begin{aligned}&\partial_\theta \Phi_1^\theta(U) = \mathrm{i}Op^{\mathrm{BW}}(b(U;\theta,x)\xi)\Phi_1^\theta(U), \\ &\Phi_1^0(U) = \mathrm{Id}, \quad b(U;\theta,x) := \frac{\beta(U;x)}{1 + \theta\beta_x(U;x)},\end{aligned} \tag{4.10}$$

where $\beta(U;x)$ is a real-valued function in $\widetilde{\mathcal{F}}_1^{\mathbb{R}}$ of the same form as $\mathtt{V}_1(U;x)$, i.e.,

$$\beta(U;x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}\setminus\{0\}} \beta_n^+ u_n e^{\mathrm{i}nx} + \beta_n^- \overline{u_n} e^{-\mathrm{i}nx}. \tag{4.11}$$

The function $\beta(U;x)$ is real if a condition like (2.13) holds, i.e.,

$$\overline{\beta_n^+} = \beta_n^-. \tag{4.12}$$

The flow of the transport equation (4.10) is well-posed by Lemma A.2. We introduce the new variable

$$V_1 := \begin{bmatrix} v_1 \\ \overline{v_1} \end{bmatrix} = \big(\vec{\Phi}_1^\theta(U)[W]\big)_{|\theta=1} = \big(\Phi_1^\theta(U)[w], \overline{\Phi_1^\theta(U)}[\overline{w}]\big)^T_{|\theta=1}, \tag{4.13}$$

where the operator $\overline{\Phi_1^\theta(U)}[\cdot]$ is defined as in (2.24).

LEMMA 4.5. *Define* $\beta \in \widetilde{\mathcal{F}}_1^{\mathbb{R}}$ *in* (4.11) *with coefficients*

$$\beta_n^+ := -\frac{(\mathtt{V}_1)_n^+}{\mathrm{i}\omega_n} = \frac{\mathrm{i}n}{\sqrt{2}|n|^{\frac{3}{4}}}, \quad \beta_n^- := \frac{(\mathtt{V}_1)_n^-}{\mathrm{i}\omega_n} = -\frac{\mathrm{i}n}{\sqrt{2}|n|^{\frac{3}{4}}}, \quad n \neq 0, \tag{4.14}$$

and $(\beta)_0^\sigma := 0$, $\sigma = \pm$. *Then, if* w *solves* (4.9)*, the function* v_1 *defined in* (4.13) *solves*

$$\begin{aligned} \partial_t v_1 = {} & Op^{\mathrm{BW}}\Big(-\mathrm{i}\mathtt{V}_2^{(1)}\xi - \mathrm{i}\big(1 + \mathtt{a}_2^{(1)}\big)|\xi|^{\frac{1}{2}} + \mathtt{r}_1^{(1)} + \mathtt{r}_2^{(1)} + H_{\geq 3}^{(1)}\Big)v_1 \\ & + R^{(1)}(U)[V_1] \end{aligned} \tag{4.15}$$

where

- $\mathtt{V}_2^{(1)} \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ *and its coefficients* (*according to the expansion* (2.12)) *satisfy*

$$\big(\mathtt{V}_2^{(1)}\big)_{n,n}^{+-} = 2n|n|, \quad \big(\mathtt{V}_2^{(1)}\big)_{n,-n}^{+-} = 0; \tag{4.16}$$

- $\mathtt{a}_2^{(1)} \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ *and its coefficients satisfy*

$$\big(\mathtt{a}_2^{(1)}\big)_{n,n}^{+-} = 0; \tag{4.17}$$

- $\mathtt{r}_1^{(1)} \in \widetilde{\Gamma}_1^{-\frac{1}{2}}$, $\mathtt{r}_2^{(1)} \in \widetilde{\Gamma}_2^{-\frac{1}{2}}$, $H_{\geq 3}^{(1)} \in \Gamma_{K,K',3}^1$ *is an admissible symbol, and* $R^{(1)}(U) \in \Sigma\mathcal{R}_{K,K',1}^{-\rho}$.

Note that the procedure that eliminates the linear term of the transport in (4.15), that is, the contribution with degree of homogeneity 1 to the coefficient of ξ, automatically also eliminates the contribution with degree of homogeneity 1 to the coefficient of the symbol of order $1/2$.

PROOF OF LEMMA 4.5. *Conjugation under the flow in* (4.10)*.* We use Lemmata A.4 and A.5.

Step 1. We apply Lemma A.5 with β in $\widetilde{\mathcal{F}}_1^{\mathbb{R}} \subset \mathcal{F}_{K,0,1}^{\mathbb{R}}$ by the fourth remark in (2.10). Then

$$\partial_t \Phi_1^1(U)\big(\Phi_1^1(U)\big)^{-1} = Op^{\mathrm{BW}}(\mathrm{i}(\beta_t - \beta_x\beta_t)\xi + H_{\geq 3}) + R(U)$$

where $H_{\geq 3} := \mathrm{i}g_{\geq 3}\xi$ is an admissible symbol in $\Gamma_{K,1,3}^1$ and $R(U)$ belongs to $\Sigma\mathcal{R}_{K,1,1}^{-\rho}$.

Step 2. We apply Lemma A.4 with $a = -\mathrm{i}V\xi$. Thus by (A.14)–(A.17) we deduce

$$\begin{aligned}&\Phi_1^1(U)Op^{\mathrm{BW}}(-\mathrm{i}V\xi)(\Phi_1^1(U))^{-1}\\&\quad= Op^{\mathrm{BW}}\big(-\mathrm{i}(\mathtt{V}_1+\mathtt{V}_2)\xi+\mathrm{i}(\mathtt{V}_1\beta_x-(\mathtt{V}_1)_x\beta)\xi+H_{\geq 3}\big)+R(U)\end{aligned}$$

where $H_{\geq 3}\in\Gamma^1_{K,K',3}$ is an admissible symbol and $R(U)$ belongs to $\Sigma\mathcal{R}^{-\rho+1}_{K,K',2}$.

Step 3. Using Lemma A.4 (see (A.16), (A.17)) we have the expansion

$$\begin{aligned}&-\Phi_1^1(U)Op^{\mathrm{BW}}\big(\mathrm{i}\big(1+\mathtt{a}_1+\mathtt{a}_2^{(0)}\big)|\xi|^{1/2}\big)(\Phi_1^1(U))^{-1}\\&\quad= Op^{\mathrm{BW}}\bigg(-\mathrm{i}\bigg(1+\mathtt{a}_1-\frac{\beta_x}{2}+\mathtt{a}_2^{(0)}+(\mathtt{a}_1)_x\beta-\frac{1}{2}\beta_x\mathtt{a}_1+\frac{3}{8}\beta_x^2\bigg)|\xi|^{1/2}\\&\qquad\qquad+r+H_{\geq 3}\bigg)+R(U)\end{aligned}$$

where $r\in\Sigma\Gamma^{-\frac{3}{2}}_{K,K',1}$, $H_{\geq 3}\in\Gamma^{\frac{1}{2}}_{K,K',3}$ is an admissible symbol, and $R(U)$ is in $\Sigma\mathcal{R}^{-\rho+\frac{1}{2}}_{K,K',1}$.

Step 4. By Lemma A.4 the conjugated operator

$$\begin{aligned}&\Phi_1^1(U)Op^{\mathrm{BW}}(\mathtt{r}_1+\mathtt{r}_2+H_{\geq 3})(\Phi_1^1(U))^{-1}\\&\quad= Op^{\mathrm{BW}}\big(\mathtt{r}_1^{(1)}+\mathtt{r}_2^{(1)}+H'_{\geq 3}\big)+R(U)\end{aligned}$$

where $\mathtt{r}_1^{(1)}\in\widetilde{\Gamma}_1^{-\frac{1}{2}}$, $\mathtt{r}_2^{(1)}\in\widetilde{\Gamma}_2^{-\frac{1}{2}}$, a new admissible symbol $H'_{\geq 3}\in\Gamma^1_{K,K',3}$, and a smoothing remainder $R(U)$ in $\Sigma\mathcal{R}^{-\rho+1}_{K,K',1}$.

Step 5. Since also the conjugated operator $\phi_1^1(U)R(U)(\phi_1^1(U))^{-1}$ of $R(U)$ is a smoothing remainder, in conclusion, we get that if w solves (4.9), then v_1 defined in (4.13) satisfies

$$\begin{aligned}\partial_t v_1 &= \mathrm{i}Op^{\mathrm{BW}}\big((-\mathtt{V}_1+\partial_t\beta)\xi+\big(-\mathtt{V}_2+(\mathtt{V}_1\beta_x-(\mathtt{V}_1)_x\beta)-\beta_x\beta_t\big)\xi\big)v_1\\&\quad+\mathrm{i}Op^{\mathrm{BW}}\bigg(-|\xi|^{\frac{1}{2}}-\bigg(\mathtt{a}_1-\frac{\beta_x}{2}\bigg)|\xi|^{\frac{1}{2}}\\&\qquad-\bigg(\mathtt{a}_2^{(0)}+(\mathtt{a}_1)_x\beta-\frac{1}{2}\beta_x\mathtt{a}_1+\frac{3}{8}\beta_x^2\bigg)|\xi|^{\frac{1}{2}}\bigg)v_1\\&\quad+Op^{\mathrm{BW}}\big(\mathtt{r}_1^{(1)}+\mathtt{r}_2^{(1)}\big)v_1+Op^{\mathrm{BW}}(H_{\geq 3})v_1+R^{(1)}(U)[V_1]\end{aligned}\tag{4.18}$$

where $\mathtt{r}_1^{(1)}\in\widetilde{\Gamma}_1^{-1/2}$, $\mathtt{r}_2^{(1)}\in\widetilde{\Gamma}_2^{-1/2}$, $H_{\geq 3}\in\Gamma^1_{K,K',3}$ is admissible according to Definition 4.2, and $R^{(1)}(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$ (renaming ρ).

Choice of β. Recall that the coefficients $\beta_n^{\pm}$ defined in (4.14) satisfy (4.12) and the function $\beta(U;x)$ is real. Using (3.16) we get

$$\begin{aligned}&\partial_t\beta(U;x)\\ &=\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}\setminus\{0\}}(-\mathrm{i}\omega_n)\beta_n^+e^{\mathrm{i}nx}u_n+\mathrm{i}\omega_n\beta_n^-e^{-\mathrm{i}nx}\overline{u_n}+\mathtt{h}_2+\mathtt{h}_{\geq 3}\end{aligned} \tag{4.19}$$

where $\mathtt{h}_2$, $\mathtt{h}_{\geq 3}$ are defined as

$$\mathtt{h}_p:=\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\beta_n^+\mathrm{i}[F_p(U)]_ne^{\mathrm{i}nx}-\beta_n^-\mathrm{i}\overline{[F_p(U)]_n}e^{-\mathrm{i}nx}, \tag{4.20}$$

with $p=2$ or $p\geq 3$. By (4.19) and (4.14) we deduce that

$$-\mathtt{V}_1+\partial_t\beta=\mathtt{h}_2+\mathtt{h}_{\geq 3}. \tag{4.21}$$

By (4.12) the functions $\mathtt{h}_2$ and $\mathtt{h}_{\geq 3}$ are real. Moreover $\mathtt{h}_2\in\widetilde{\mathcal{F}}_2^{\mathbb{R}}$ and $\mathtt{h}_{\geq 3}\in\mathcal{F}_{K,1,3}^{\mathbb{R}}$ by item (iv) of Proposition 2.10 and the fact that $F_2(U)+F_{\geq 3}(U)=M(U)[U]$ for some M in $\Sigma\mathcal{M}_{K,1,1}$, see (3.15).

The new equation. From (4.21) and the first line of (4.18) we deduce that $\mathtt{V}_2^{(1)}$ in (4.15) is given by

$$-\mathtt{V}_2^{(1)}:=\mathtt{h}_2-\mathtt{V}_2-(\mathtt{V}_1)_x\beta, \tag{4.22}$$

having used $(\mathtt{V}_1-\partial_t\beta)\beta_x\in\mathcal{F}_{K,1,3}^{\mathbb{R}}$. By the second line of (4.18) we deduce that $\mathtt{a}_2^{(1)}$ in (4.15) is given by

$$\mathtt{a}_2^{(1)}:=\mathtt{a}_2^{(0)}+(\mathtt{a}_1)_x\beta-\tfrac{1}{2}\beta_x\mathtt{a}_1+\tfrac{3}{8}\beta_x^2\in\widetilde{\mathcal{F}}_2^{\mathbb{R}} \tag{4.23}$$

having noted that the function $\mathtt{a}_1-\frac{\beta_x}{2}=0$ by (3.30) and (4.14).

Let us prove (4.16). By (4.22) we have

$$\begin{aligned}((\mathtt{V}_2)^{(1)})_{n_1,n_2}^{+-}&=-(\mathtt{h}_2)_{n_1,n_2}^{+-}+(\mathtt{V}_2)_{n_1,n_2}^{+-}\\ &\quad+\mathrm{i}\big((\mathtt{V}_1)_{n_1}^{+}\beta_{n_2}^{-}n_1-(\mathtt{V}_1)_{n_2}^{-}\beta_{n_1}^{+}n_2\big).\end{aligned} \tag{4.24}$$

The coefficients $(\mathtt{h}_2)_{n_1,n_2}^{+-}$ associated to $\mathtt{h}_2$ defined in (4.20) are

$$(\mathtt{h}_2)_{n_1,n_2}^{+-}=\mathrm{i}\beta_{n_1-n_2}^{+}(F_2)_{n_1,n_2}^{+-}-\mathrm{i}\beta_{-(n_1-n_2)}^{-}\overline{(F_2)_{n_2,n_1}^{+-}}$$

with $(F_2)_{n_1,n_2}^{+-}$ defined by (3.16)–(3.17). We claim that

$$(\mathtt{h}_2)_{n,n}^{+-}=0,\quad (\mathtt{h}_2)_{n,-n}^{+-}=0. \tag{4.25}$$

The first identity in (4.25) is trivial since the coefficients β_n^{σ} in (4.14) are zero for $n=0$. To prove the second identity in (4.25) we compute by (4.20) and (3.31) $(\mathtt{h}_2)_{n,-n}^{+-}=\mathrm{i}(F_2)_{n,-n}^{+-}\big(\beta_{2n}^{+}-\beta_{-2n}^{-}\big)=0$ in view of (4.14). By (4.24), (4.25), (4.14), and (3.29) we get $(\mathtt{V}_2^{(1)})_{n,n}^{+-}=2n|n|$ and $(\mathtt{V}_2^{(1)})_{n,-n}^{+-}=0$.

To conclude we prove (4.17). From (4.23) we calculate

$$\begin{aligned}\big(\mathtt{a}_2^{(1)}\big)_{n,n}^{+-} &= \big(\mathtt{a}_2^{(0)}\big)_{n,n}^{+-} + \mathrm{i}n(\mathtt{a}_1)_n^{+}\beta_n^{-} - \mathrm{i}n(\mathtt{a}_1)_n^{-}\beta_n^{+} \\ &\quad - \tfrac{1}{2}\mathrm{i}n\big(\beta_n^{+}(\mathtt{a}_1)_n^{-} - \beta_n^{-}(\mathtt{a}_1)_n^{+}\big) + \tfrac{3}{4}\beta_n^{+}\beta_n^{-}n^2\end{aligned}$$

where β_n^{σ} are defined in (4.14).

By (3.35) we have $(\mathtt{a}_2^{(0)})_{n_1,n_2}^{+-} = (\mathtt{a}_2)_{n_1,n_2}^{+-} - (\mathtt{a}_1)_{n_1}^{+}(\mathtt{a}_1)_{n_2}^{-}$ so that, using (3.30), we calculate $(\mathtt{a}_2^{(0)})_{n,n}^{+-} = \frac{3}{8}|n|^{5/2}$. Furthermore, one can check directly using the formulas (3.30) and (4.14), that $(\mathtt{a}_2^{(1)})_{n,n}^{+-} = 0$. □

Reduction of the quadratic symbol of the transport

The aim of this section is to reduce the transport operator $-\mathrm{i}Op^{\mathrm{BW}}(\mathtt{V}_2^{(1)}(U;x)\xi)$ in (4.15) into the "integrable" one $-\mathrm{i}Op^{\mathrm{BW}}(\zeta(U)\xi)$ where $\zeta(U)$ is the function, constant in x, defined in (4.5). To do this we conjugate the equation (4.15) under the flow of the transport equation

$$\partial_\theta \Phi_2^{\theta}(U) = \mathrm{i}Op^{\mathrm{BW}}(b_2(U;\theta,x)\xi)\Phi_2^{\theta}(U), \quad \Phi_2^{0}(U) = \mathrm{Id}, \tag{4.26}$$

where b_2 is defined as in (4.10) in terms of a real-valued function $\beta_2(U;x) \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$. The flow in (4.26) is well-posed by Lemma A.2. We then define the new variable

$$V_2 := \begin{bmatrix} v_2 \\ \overline{v_2}\end{bmatrix} = \big(\vec{\Phi}_2^{\theta}(U)[V_1]\big)_{|\theta=1} := \big(\Phi_2^{\theta}(U)[v_1], \overline{\Phi_2^{\theta}(U)}[\overline{v_1}]\big)_{|\theta=1}^{T} \tag{4.27}$$

where $\overline{\Phi_2^{\theta}(U)}$ is defined as in (2.24).

LEMMA 4.6. *Define $\beta_2 \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ with coefficients for $n_1, n_2 \in \mathbb{Z}\setminus\{0\}$,*

$$\begin{aligned}(\beta_2)_{n_1,n_2}^{\sigma\sigma} &:= \frac{-(\mathtt{V}_2^{(1)})_{n_1,n_2}^{\sigma\sigma}}{\mathrm{i}\sigma(\omega_{n_1}+\omega_{n_2})}, \quad \sigma = \pm, \\ (\beta_2)_{n_1,n_2}^{+-} &:= \frac{-(\mathtt{V}_2^{(1)})_{n_1,n_2}^{+-}}{\mathrm{i}(\omega_{n_1}-\omega_{n_2})}, \quad n_1 \neq \pm n_2,\end{aligned} \tag{4.28}$$

and $(\beta_2)_{0,0}^{\sigma\sigma} := 0$, $(\beta_2)_{n,\sigma n}^{+-} := 0$, $\sigma = \pm$, where $\mathtt{V}_2^{(1)}$ is the real-valued function defined in Lemma 4.5. *If v_1 solves* (4.15)*, then the function v_2 in* (4.27) *solves*

$$\begin{aligned}\partial_t v_2 = {}& Op^{\mathrm{BW}}\big(-\mathrm{i}\zeta(U)\xi - \mathrm{i}\big(1+\mathtt{a}_2^{(2)}\big)|\xi|^{\frac{1}{2}} + \mathtt{r}_1^{(1)} + \mathtt{r}_2^{(2)} + H_{\geq 3}^{(2)}\big)v_2 \\ &+ R^{(2)}(U)[V_2]\end{aligned} \tag{4.29}$$

where

- *$\zeta(U) \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ is the integrable function defined in* (4.5);
- *$\mathtt{a}_2^{(2)} \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ satisfies*

$$\mathtt{a}_2^{(2)} := \mathtt{a}_2^{(1)} - \frac{1}{2}(\beta_2)_x, \quad (\mathtt{a}_2^{(2)})_{n,n}^{+-} = 0; \tag{4.30}$$

- $\mathtt{r}_1^{(1)} \in \widetilde{\Gamma}_1^{-\frac{1}{2}}$ *is the same symbol in* (4.15)*, and* $\mathtt{r}_2^{(2)} \in \widetilde{\Gamma}_2^{-\frac{1}{2}}$;
- $H_{\geq 3}^{(2)} \in \Gamma_{K,K',3}^1$ *is admissible, and* $R^{(2)}(U)$ *is a* 1×2 *matrix of smoothing operators in* $\Sigma\mathcal{R}_{K,K',1}^{-\rho}$.

PROOF. The function β_2 is real-valued since the coefficients $(\mathtt{V}_2)_{n_1,n_2}^{\sigma\sigma'}$ of the real function $\mathtt{V}_2^{(1)}$ in (4.22) satisfy (2.13). In order to conjugate (4.15) under the map Φ_2^θ in (4.27) we apply Lemmata A.4 and A.5. By (A.17) and (A.20), and since β_2 is quadratic in u, the only quadratic contributions are $Op^{\mathrm{BW}}(\{\beta_2\xi, -\mathrm{i}|\xi|^{\frac{1}{2}}\})v_2 + \mathrm{i}Op^{\mathrm{BW}}(\partial_t\beta_2\xi)v_2$, implying

$$\begin{aligned}\partial_t v_2 = Op^{\mathrm{BW}}\big(&\mathrm{i}\big(-\mathtt{V}_2^{(1)} + \partial_t\beta_2\big)\xi + \tfrac{\mathrm{i}}{2}(\beta_2)_x|\xi|^{\frac{1}{2}} - \mathrm{i}\big(1 + \mathtt{a}_2^{(1)}\big)|\xi|^{\frac{1}{2}} \\ &+ \mathtt{r}_1^{(1)} + \widetilde{\mathtt{r}}_2^{(1)} + H_{\geq 3}^{(1)}\big)v_2 + R(U)[V_2]\end{aligned} \tag{4.31}$$

where $\widetilde{\mathtt{r}}_2^{(1)}$ is a symbol in $\widetilde{\Gamma}_2^{-\frac{1}{2}}$, $H_{\geq 3}^{(1)} \in \Gamma_{K,K',3}^1$ is a new admissible symbol, and $R(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}_{K,K',1}^{-\rho}$ (by renaming ρ). By the choice of β_2 in (4.28), using (3.16), reasoning as in the proof of Lemma 4.5, and using (4.16) we have

$$-\mathtt{V}_2^{(1)} + \partial_t\beta_2 = -\zeta(U) + f_{\geq 3} \tag{4.32}$$

with $\zeta(U)$ defined in (4.5) and where $f_{\geq 3}$ is in $\mathcal{F}_{K,1,3}^{\mathbb{R}}$. System (4.31) and (4.32) imply (4.29) where $\mathtt{a}_2^{(2)}$ is the function defined in (4.30). Recalling (4.17) we deduce that $(\mathtt{a}_2^{(2)})_{n,n}^{+-} = 0$. □

4.2 Integrability at order 1/2 and 0

The first aim of this section is to reduce the operator $-\mathrm{i}Op^{\mathrm{BW}}\big(\mathtt{a}_2^{(2)}(U;x)|\xi|^{1/2}\big)$ in (4.29) to an integrable one. It actually turns out that, thanks to (4.30), we reduce it to the Fourier multiplier $-\mathrm{i}|D|^{1/2}$; see (4.45). This is done in two steps. In 4.2 we apply a transformation that is a paradifferential "semi-Fourier integral operator," generated as the flow of (4.33). Then, in Section 4.2 we apply the paradifferential version of a torus diffeomorphism that is "almost" time independent; see (4.41)–(4.42). Eventually we deal with the operators of order 0 in Section 4.2.

Elimination of the time dependence at order $1/2$ up to $O(u^3)$

We conjugate (4.29) under the flow

$$\partial_\theta\Phi_3^\theta(U) = \mathrm{i}Op^{\mathrm{BW}}\big(\beta_3(U;x)|\xi|^{\frac{1}{2}}\big)\Phi_3^\theta(U), \quad \Phi_3^0(U) = \mathrm{Id}, \tag{4.33}$$

where $\beta_3(U;x) \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ is a real-valued function. We set

$$V_3 := \begin{bmatrix} v_3 \\ \overline{v_3} \end{bmatrix} = \big(\vec{\Phi}_3^\theta(U)[V_2]\big)_{|\theta=1} = \big(\Phi_3^\theta(U)[v_2], \overline{\Phi_3^\theta(U)}[\overline{v_2}]\big)_{|\theta=1}^T \tag{4.34}$$

where $\overline{\Phi_3^\theta(U)}$ is defined as in (2.24).

LEMMA 4.7. *Define* $\beta_3 \in \widetilde{\mathcal{F}}_2^{\mathbb{R}}$ *with coefficients*

$$
(\beta_3)_{n_1,n_2}^{\sigma\sigma} := \frac{-(\mathtt{a}_2^{(2)})_{n_1,n_2}^{\sigma\sigma}}{\mathrm{i}\sigma(\omega_{n_1}+\omega_{n_2})}, \quad \sigma = \pm,
\qquad
(\beta_3)_{n_1,n_2}^{+-} := \frac{-(\mathtt{a}_2^{(2)})_{n_1,n_2}^{+-}}{\mathrm{i}(\omega_{n_1}-\omega_{n_2})}, \quad n_1 \neq \pm n_2, \tag{4.35}
$$

and $(\beta_3)_{0,0}^{\sigma\sigma} := 0$, $(\beta_3)_{n,\sigma n}^{+-} := 0$, $\sigma = \pm$, *where* $\mathtt{a}_2^{(2)}$ *is defined in* (4.30). *If* v_2 *solves* (4.29), *then*

$$
\partial_t v_3 = Op^{\mathrm{BW}}\big(-\mathrm{i}\zeta(U)\xi - \mathrm{i}(1+\mathtt{a}_2^{(3)})|\xi|^{\frac12} + \mathrm{i}\mathtt{b}_2^{(3)}\mathrm{sign}(\xi) + \mathtt{r}_1^{(1)} + \mathtt{r}_2^{(3)} + H_{\geq 3}^{(3)}\big)v_3 + R^{(3)}(U)[V_3] \tag{4.36}
$$

where

$$
\mathtt{a}_2^{(3)} := \frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}} \big(\mathtt{a}_2^{(2)}\big)_{n,-n}^{+-} u_n \overline{u_{-n}} e^{\mathrm{i}2nx}, \quad \mathtt{b}_2^{(3)} := \frac12 (\beta_3)_x, \tag{4.37}
$$

$\mathtt{r}_1^{(1)} \in \widetilde{\Gamma}_1^{-\frac12}$ *is the same symbol in* (4.15), $\mathtt{r}_2^{(3)} \in \widetilde{\Gamma}_2^{-\frac12}$, $H_{\geq 3}^{(3)} \in \Gamma_{K,K',3}^{1}$ *is admissible, and* $R^{(3)}(U)$ *is a* 1×2 *matrix of smoothing operators in* $\Sigma\mathcal{R}_{K,K',1}^{-\rho}$. *Moreover,*

$$
\big(\mathtt{b}_2^{(3)}\big)_{n,n}^{+-} = \big(\mathtt{b}_2^{(3)}\big)_{n,-n}^{+-} = 0. \tag{4.38}
$$

PROOF. By (4.35) and (2.13) we deduce that β_3 is a real function. To conjugate system (4.29) we apply Lemmata A.6 and A.7 with $m \rightsquigarrow 1/2$ and $m' \rightsquigarrow 1$. The only new contributions at quadratic degree of homogeneity and positive order are $Op^{\mathrm{BW}}(\{\beta_3|\xi|^{1/2}, -\mathrm{i}|\xi|^{\frac12}\})$ and $\mathrm{i}Op^{\mathrm{BW}}(\partial_t\beta_3|\xi|^{1/2})$. Then we have

$$
\partial_t v_3 = Op^{\mathrm{BW}}\big(-\mathrm{i}\zeta(U)\xi - \mathrm{i}(1+\mathtt{a}_2^{(2)} - \partial_t\beta_3)|\xi|^{\frac12} + \mathrm{i}\tfrac{(\beta_3)_x}{2}\mathrm{sign}(\xi) + \mathtt{r}_1^{(1)} + \widetilde{\mathtt{r}}_2^{(3)} + H_{\geq 3}\big)v_3 + R(U)[V_3]
$$

where $\widetilde{\mathtt{r}}_2^{(3)} \in \widetilde{\Gamma}_2^{-1/2}$, the symbol $H_{\geq 3} \in \Gamma_{K,K',3}^{1}$ is admissible and $R(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}_{K,K',1}^{-\rho}$. By (4.35) and (3.16) we have

$$
-\mathrm{i}\mathtt{a}_2^{(2)} + \mathrm{i}\partial_t\beta_3 = -\mathrm{i}\sum_{n\in\mathbb{Z}\setminus\{0\}} \big(\mathtt{a}_2^{(2)}\big)_{n,n}^{+-}|u_n|^2 + \big(\mathtt{a}_2^{(2)}\big)_{n,-n}^{+-} u_n\overline{u_{-n}}e^{\mathrm{i}2nx} \tag{4.39}
$$

up to a function $f_{\geq 3}$ in $\mathcal{F}_{K,1,3}^{\mathbb{R}}$. The conjugation of the remainder $R^{(2)}(U)$ in (4.29) is another smoothing operator. In conclusion, (4.39) and the vanishing of the coefficients (4.30) imply (4.36)–(4.37). Finally, (4.38) follows from $(\mathtt{b}_2^{(3)})_{n_1,n_2}^{+-} = \frac12(\beta_3)_{n_1,n_2}^{+-}(\mathrm{i}n_1 - \mathrm{i}n_2)$. □

Elimination of the x-dependence at order $1/2$ up to $O(u^3)$

The aim of this section is to cancel out the operator

$$-\,\mathrm{i}Op^{\mathrm{BW}}\Big(\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\big(\mathtt{a}_2^{(2)}\big)_{n,-n}^{+-}u_n\overline{u_{-n}}e^{\mathrm{i}2nx}\Big) \tag{4.40}$$

arising by the nonintegrable part of the function $\mathtt{a}_2^{(3)}(U;x)$ in (4.37). Note that the symbol in (4.40) is a prime integral up to cubic terms $O(u^3)$. We conjugate (4.36) under the flow

$$\partial_\theta\Phi_4^\theta(U)=\mathrm{i}Op^{\mathrm{BW}}(b_4(U;\theta,x)\xi)\Phi_4^\theta(U),\quad \Phi_4^0(U)=\mathrm{Id}, \tag{4.41}$$

where b_4 is defined as in (4.10) in terms of a real-valued function $\beta_4(U;x)\in\widetilde{\mathcal{F}}_2^{\mathbb{R}}$ of the same form of the symbol in (4.40), i.e.,

$$\beta_4(U;x)=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}}(\beta_4)_{n,-n}^{+-}u_n\overline{u_{-n}}e^{\mathrm{i}2nx}. \tag{4.42}$$

The flow in (4.41) is well-posed by Lemma A.2. We set

$$V_4:=\begin{bmatrix}v_4\\ \overline{v_4}\end{bmatrix}=\big(\vec{\Phi}_4^\theta(U)[V_3]\big)_{|\theta=1}=\big(\Phi_4^\theta(U)[v_3],\overline{\Phi_4^\theta(U)}[\overline{v_3}]\big)^T_{|\theta=1} \tag{4.43}$$

where $\overline{\Phi_4^\theta(U)}$ is defined as in (2.24).

LEMMA 4.8. *Define the function $\beta_4\in\widetilde{\mathcal{F}}_2^{\mathbb{R}}$ as in* (4.42) *with coefficients*

$$(\beta_4)_{n,-n}^{+-}:=\frac{(\mathtt{a}_2^{(2)})_{n,-n}^{+-}}{\mathrm{i}n},\ \ n\neq 0,\quad (\beta_4)_{0,0}^{+-}:=0. \tag{4.44}$$

If v_3 solves (4.36)*, then*

(4.45)
$$\begin{aligned}\partial_t v_4 &= Op^{\mathrm{BW}}\big(-\mathrm{i}\zeta(U)\xi-\mathrm{i}|\xi|^{\frac12}+\mathrm{i}\mathtt{b}_2^{(3)}\mathrm{sign}(\xi)+\mathtt{r}_1^{(1)}+\mathtt{r}_2^{(3)}+H_{\geq 3}^{(4)}\big)v_4\\ &\quad + R^{(4)}(U)[V_4]\end{aligned}$$

where the symbols $\mathtt{b}_2^{(3)}$, $\mathtt{r}_1^{(1)}$, $\mathtt{r}_2^{(3)}$ are the same as in equation (4.36)*, the symbol $H_{\geq 3}^{(4)}\in\Gamma^1_{K,K',3}$ is admissible, and $R^{(4)}(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$.*

PROOF. In order to conjugate (4.36) we apply Lemmata A.4 and A.5. The contribution coming from the conjugation of ∂_t is $\mathrm{i}Op^{\mathrm{BW}}\big((\partial_t\beta_4)\xi\big)v_4$ plus a paradifferential operator with symbol $\mathrm{i}(-(\beta_4)_x(\beta_4)_t+g_{\geq 3})\xi$ (see (A.20)), which is admissible, and a smoothing remainder in $\Sigma\mathcal{R}^{-\rho}_{K,1,1}$. One has $-\mathrm{i}\omega_n u_n\overline{u_{-n}}+u_n\mathrm{i}\omega_{-n}\overline{u_{-n}}$. Hence, recalling (3.16), we have

$$\frac{d}{dt}\sum_{n\in\mathbb{Z}\setminus\{0\}}(\beta_4)_{n,-n}^{+-}u_n\overline{u_{-n}}e^{\mathrm{i}2nx}=h_{\geq 3} \tag{4.46}$$

because $\omega_{-n} = \omega_n$ and where, arguing as in the proof of Lemma 4.5, $h_{\geq 3}$ is a function in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$. This implies that the function $\partial_t \beta_4$ is in $\mathcal{F}^{\mathbb{R}}_{K,1,3}$ and therefore $\mathrm{i}(\partial_t \beta_4)\xi$ is an admissible symbol.

Lemma A.4 implies that the conjugation of the spatial operator in (4.36) is a paradifferential operator with symbol

$$-\mathrm{i}\zeta(U)\xi - \mathrm{i}\big(1 + \mathtt{a}_2^{(3)}\big)|\xi|^{\frac{1}{2}} + \{\beta_4\xi, -\mathrm{i}|\xi|^{\frac{1}{2}}\} + \mathrm{i}\mathtt{b}_2^{(3)}\mathrm{sign}(\xi) + \mathtt{r}_1^{(1)} + \mathtt{r}_2^{(3)} \tag{4.47}$$

plus a symbol in $\Sigma\Gamma^{-3/2}_{K,K',1}$, an admissible symbol and a smoothing operator in the class $\Sigma\mathcal{R}^{-\rho+1}_{K,1,1}$. Note that $\{\beta_4\xi, -\mathrm{i}|\xi|^{1/2}\} = \frac{\mathrm{i}}{2}(\beta_4)_x|\xi|^{1/2}$ and that this equals $\mathtt{a}_2^{(3)}\mathrm{i}|\xi|^{1/2}$ in view of the definitions of β_4 in (4.42) and (4.44), and of $\mathtt{a}_2^{(3)}$ in (4.37). It follows that the symbol in (4.47) reduces to

$$-\mathrm{i}\zeta(U)\xi - \mathrm{i}|\xi|^{1/2} + \mathrm{i}\mathtt{b}_2^{(3)}\mathrm{sign}(\xi) + \mathtt{r}_1^{(1)} + \mathtt{r}_2^{(3)}.$$

We have therefore obtained (4.45) (after slightly redefining ρ) as desired. □

Integrability at order 0

Our aim here is to eliminate in (4.45) the zeroth-order paradifferential operator $Op^{\mathrm{BW}}\big(\mathrm{i}\mathtt{b}_2^{(3)}\mathrm{sign}(\xi)\big)$. We conjugate (4.45) with the flow

$$\partial_\theta \Phi_5^\theta(U) = Op^{\mathrm{BW}}(\mathrm{i}\beta_5(U;x)\,\mathrm{sign}(\xi))\Phi_5^\theta(U), \quad \Phi_5^0(U) = \mathrm{Id}, \tag{4.48}$$

where $\beta_5(U;x) \in \widetilde{\mathcal{F}}^{\mathbb{R}}_2$ is a real-valued function. We introduce the variable

$$V_5 := \begin{bmatrix} v_5 \\ \overline{v_5} \end{bmatrix} = \big(\vec{\Phi}_5^\theta(U)[V_4]\big)_{|\theta=1} = \big(\Phi_5^\theta(U)[v_4], \overline{\Phi_5^\theta(U)}[\overline{v_4}]\big)^T_{|\theta=1} \tag{4.49}$$

where $\overline{\Phi_5^\theta(U)}$ is defined as in (2.24).

LEMMA 4.9. *Define $\beta_5 \in \widetilde{\mathcal{F}}^{\mathbb{R}}_2$ (of the form (2.12)) with*

$$\begin{aligned} (\beta_5)^{\sigma\sigma}_{n_1,n_2} &:= \frac{(\mathtt{b}_2^{(3)})^{\sigma\sigma}_{n_1,n_2}}{\mathrm{i}\sigma(\omega_{n_1} + \omega_{n_2})}, \quad \sigma = \pm, \\ (\beta_5)^{+-}_{n_1,n_2} &:= \frac{(\mathtt{b}_2^{(3)})^{+-}_{n_1,n_2}}{\mathrm{i}(\omega_{n_1} - \omega_{n_2})}, \quad n_1 \neq \pm n_2, \end{aligned} \tag{4.50}$$

and $(\beta_5)^{\sigma\sigma}_{0,0} := 0$, $(\beta_5)^{+-}_{n,\sigma n} := 0$, $\sigma = \pm$. If v_4 solves (4.45), then

$$\begin{aligned} \partial_t v_5 = {} & Op^{\mathrm{BW}}\big(-\mathrm{i}\zeta(U)\xi - \mathrm{i}|\xi|^{\frac{1}{2}} + \mathtt{r}_1^{(5)} + \mathtt{r}_2^{(5)} + H^{(5)}_{\geq 3}\big)v_5 \\ & + R^{(5)}(U)[V_5] \end{aligned} \tag{4.51}$$

where $\mathtt{r}_1^{(5)} \in \widetilde{\Gamma}^{-1/2}_1$, $\mathtt{r}_2^{(5)} \in \widetilde{\Gamma}^{-1/2}_2$, the symbol $H^{(5)}_{\geq 3} \in \Gamma^1_{K,K',3}$ is admissible, and $R^{(5)}(U)$ is a 1×2 matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$.

PROOF. To conjugate (4.45) we apply Lemmata A.6 and A.7. By (4.50) we get

$$Op^{\mathrm{BW}}\big(\mathrm{i}\big(\mathtt{b}_2^{(3)}+\partial_t\beta_5\big)\mathrm{sign}(\xi)\big)=\mathrm{i}Op^{\mathrm{BW}}\big(\mathtt{b}_2^{(5)}\mathrm{sign}(\xi)\big),$$

up to symbols with degree of homogeneity greater than 3, and where

$$\mathtt{b}_2^{(5)}(U;x):=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}}\big(\mathtt{b}_2^{(3)}\big)_{n,n}^{+-}|u_n|^2+\big(\mathtt{b}_2^{(3)}\big)_{n,-n}^{+-}u_n\overline{u_{-n}}e^{\mathrm{i}2nx}\overset{(4.38)}{=}0.$$

The lemma is proved. □

In the following subsection we will be dealing with negative order operators, and will not need additional algebraic information about the coefficients and their vanishing.

4.3 Integrability at negative orders

In this section we algorithmically reduce the linear and quadratic symbols $\mathtt{r}_1^{(5)}+\mathtt{r}_2^{(5)}$ of order $-1/2$ in (4.51) into an integrable 1, plus an admissible symbol.

PROPOSITION 4.10. *For any $j=0,\dots,2\rho-1$, there exist*

- *integrable symbols $\mathtt{p}_2^{(j)}\in\widetilde{\Gamma}_2^{-\frac12}$ (Definition 4.1), symbols $\mathtt{q}^{(j)}(U;x,\xi)\in\Sigma\Gamma_{K,K',1}^{-m_j}$ with $m_j:=\frac{j+1}{2}$, admissible symbols $H_{\geq3}^{(j)}$ in $\Gamma_{K,K',3}^{1}$, and a 1×2 matrix of smoothing operators $R^{(j)}(U)$ in $\Sigma\mathcal{R}_{K,K',1}^{-\rho}$,*
- *bounded maps $\Upsilon_{j+1}^{\theta}(U)$, $\theta\in[0,1]$, defined as the compositions of three flows generated by paradifferential operators with symbols of order ≤ 0 (see (4.68) and (4.55), (4.59) and (4.65))*

such that: if z_j solves

$$\begin{aligned}\partial_t z_j&=Op^{\mathrm{BW}}\big(-\mathrm{i}\zeta(U)\xi-\mathrm{i}|\xi|^{\frac12}+\mathtt{p}_2^{(j)}(U;\xi)+\mathtt{q}^{(j)}(U;x,\xi)+H_{\geq3}^{(j)}\big)z_j\\&\quad+R^{(j)}(U)[Z_j],\end{aligned}\tag{4.52}$$

then the first component z_{j+1} of the vector defined by

$$Z_{j+1}=\begin{bmatrix}z_{j+1}\\\overline{z_{j+1}}\end{bmatrix}:=\big(\Upsilon_{j+1}^{\theta}(U)\big)_{\theta=1}Z_j\tag{4.53}$$

solves an equation of the form (4.52) with $j+1$ instead of j.

The proof proceeds by induction.

Initialization. Note that equation (4.51) has the form (4.52) with $j=0$, denoting $z_0:=v_5$, $\mathtt{p}_2^{(0)}:=0$, $\mathtt{q}^{(0)}:=\mathtt{r}_1^{(5)}+\mathtt{r}_2^{(5)}\in\Sigma\Gamma_{K,K',1}^{-1/2}$, and renaming $H_{\geq3}^{(0)}$ the admissible symbol $H_{\geq3}^{(5)}$ in (4.51) and $R^{(0)}(U)$ the smoothing operator $R^{(5)}(U)$.

We remark that the *integrable* corrections $\mathtt{p}_2^{(j)}$ in (4.52) (initially $\mathtt{p}_2^{(0)}=0$) are generated by the reductions on quadratic symbols made in Lemma 4.12 below.

Iteration. The aim of the iterative procedure is to cancel out the symbol $\mathtt{q}^{(j)}$ up to a symbol of order $-m_j-1/2$. This is done in two steps.

Step 1. *Elimination of the linear symbols of negative order.* We expand the symbol $\mathtt{q}^{(j)} = \mathtt{q}_1^{(j)} + \mathtt{q}_2^{(j)} + \cdots$ with $\mathtt{q}_l^{(j)} \in \Gamma_l^{-m_j}$, $l = 1, 2$. In order to eliminate the operator $Op^{\mathrm{BW}}(\mathtt{q}_1^{(j)}(U; x, \xi))$ in (4.52) we conjugate it by the flow

$$\partial_\theta \Phi^\theta_{\gamma^{(1)}_{j+1}}(U) = Op^{\mathrm{BW}}\big(\gamma^{(1)}_{j+1}(U; x, \xi)\big)\Phi^\theta_{\gamma^{(1)}_{j+1}}(U), \quad \Phi^0_{\gamma^{(1)}_{j+1}}(U) = \mathrm{Id}, \tag{4.54}$$

where $\gamma^{(1)}_{j+1}(U; x; \xi)$ is a symbol in $\widetilde{\Gamma}_1^{-m_j}$. The flow (4.54) is well-posed because the order of $\gamma^{(1)}_{j+1}$ is negative. We introduce the new variable

$$\begin{aligned} \widetilde{Z}_{j+1} := \begin{bmatrix} \widetilde{z}_{j+1} \\ \overline{\widetilde{z}_{j+1}} \end{bmatrix} &= \big(\mathcal{A}^\theta_{j+1,1}(U)[Z_j]\big)_{|\theta=1} \\ &= \Big(\Phi^\theta_{\gamma^{(1)}_{j+1}}(U)[z_j], \overline{\Phi^\theta_{\gamma^{(1)}_{j+1}}(U)}[\overline{z_j}]\Big)^T_{|\theta=1} \end{aligned} \tag{4.55}$$

where the map $\overline{\Phi^\theta_{\gamma^{(1)}_{j+1}}(U)}$ is defined as in (2.24).

LEMMA 4.11. *Define* $\gamma^{(1)}_{j+1} \in \widetilde{\Gamma}_1^{-m_j}$ *with coefficients*

$$\begin{gathered} \big(\gamma^{(1)}_{j+1}\big)^+_n := \frac{(\mathtt{q}_1^{(j)})^+_n}{\mathrm{i}\omega_n}, \quad (\gamma_{j+1})^-_n := \frac{-(\mathtt{q}_1^{(j)})^-_n}{\mathrm{i}\omega_n}, \ \ n \neq 0, \\ (\gamma^{(1)}_{j+1})^\sigma_0 := 0, \ \ \sigma = \pm. \end{gathered} \tag{4.56}$$

If z_j *solves* (4.52)*, then*

$$\begin{aligned} \partial_t \widetilde{z}_{j+1} = Op^{\mathrm{BW}}(&-\mathrm{i}\zeta(U)\xi - \mathrm{i}|\xi|^{\frac{1}{2}} + \mathtt{p}_2^{(j)}(U; \xi) + \widetilde{\mathtt{q}}_2^{(j)}(U; x, \xi) \\ &+ \widetilde{\mathtt{k}}_1^{(j)}(U; x, \xi) + \widetilde{\mathtt{k}}_2^{(j)}(U; x, \xi)\big)\widetilde{z}_{j+1} + Op^{\mathrm{BW}}\big(H^{(j)}_{\geq 3}\big)\widetilde{z}_{j+1} \\ &+ R^{(j)}(U)[\widetilde{Z}_{j+1}] \end{aligned} \tag{4.57}$$

where $\mathtt{p}_2^{(j)}(U; \xi) \in \widetilde{\Gamma}_2^{-1/2}$ *is the same of* (4.52)*,* $\widetilde{\mathtt{q}}_2^{(j)} \in \widetilde{\Gamma}_2^{-m_j}$*,* $\widetilde{\mathtt{k}}_1^{(j)} \in \widetilde{\Gamma}_1^{-m_j - \frac{1}{2}}$*,* $\widetilde{\mathtt{k}}_2^{(j)} \in \widetilde{\Gamma}_2^{-m_j - \frac{1}{2}}$*,* $H^{(j)}_{\geq 3} \in \Gamma^1_{K, K'+1, 3}$ *is admissible and* $R^{(j)}(U)$ *is a* 1×2 *matrix of smoothing operators in* $\Sigma\mathcal{R}^{-\rho}_{K, K', 1}$.

PROOF. In order to conjugate (4.52) we apply Lemmata A.6 and A.7. The only contributions at homogeneity degree 1 and order $-m_j$ are given by

$$Op^{\mathrm{BW}}\big(\mathtt{q}_1^{(j)} + \partial_t \gamma^{(1)}_{j+1}\big)$$

up to smoothing remainders. From the time contribution, a symbol that has homogeneity 2 and order less than or equal to $-m_j - 1/2$ appears (see the term r_1 in (A.26) of Lemma A.7). By (4.56) and (3.16) we have that

$$\mathtt{q}_1^{(j)} + \partial_t \gamma^{(1)}_{j+1} = \mathtt{q}_{2,j} + \mathtt{q}_{\geq 3}, \quad \mathtt{q}_{2,j} \in \widetilde{\Gamma}_2^{-m_j}, \quad \mathtt{q}_{\geq 3} \in \Gamma^{-m_j}_{K,1,3},$$

and we set $\widetilde{\mathrm{q}}_2^{(j)} := \mathrm{q}_2^{(j)} + \mathrm{q}_{2,j}$, and absorb $\mathrm{q}_{\geq 3}$ in the admissible symbol $H^{(j)}_{\geq 3}$. The contributions in (4.57) at order less than or equal to $-m_j - \frac{1}{2}$, and homogeneity ≤ 2 come from the conjugation of $-\mathrm{i}|\xi|^{1/2}$. In particular, by formula (A.24), we can set $\widetilde{\mathrm{k}}_1^{(j)} := -\frac{\mathrm{i}}{2}(\gamma^{(1)}_{j+1})_x |\xi|^{-\frac{1}{2}} \mathrm{sign}(\xi)$ and get (4.57) with some $\widetilde{\mathrm{k}}_2^{(j)}$ in $\Gamma_2^{-m_j-1/2}$. □

Step 2. *Reduction of the quadratic symbols of negative order.* We now cancel out the symbol $\widetilde{\mathrm{q}}_2^{(j)}$ in (4.57), up to an integrable one and a lower-order symbol. We use two different transformations.

ELIMINATION OF THE TIME DEPENDENCE UP TO $O(u^3)$. We consider the flow generated by

$$\partial_\theta \Phi^\theta_{\gamma^{(2)}_{j+1}}(U) = Op^{\mathrm{BW}}(\gamma^{(2)}_{j+1}(U; x, \xi))\Phi^\theta_{\gamma^{(2)}_{j+1}}(U), \quad \Phi^0_{\gamma^{(2)}_{j+1}}(U) = \mathrm{Id}, \tag{4.58}$$

where $\gamma^{(2)}_{j+1}(U; x; \xi)$ is a symbol in $\widetilde{\Gamma}_2^{-m_j}$. We introduce the new variable

$$\begin{aligned} \check{Z}_{j+1} := \begin{bmatrix} \check{z}_{j+1} \\ \overline{\check{z}_{j+1}} \end{bmatrix} &= \big(\mathcal{A}^\theta_{j+1,2}(U)[\widetilde{Z}_j]\big)_{|\theta=1} \\ &= \Big(\Phi^\theta_{\gamma^{(1)}_{j+1}}(U)[\widetilde{z}_j], \overline{\Phi^\theta_{\gamma^{(1)}_{j+1}}(U)}[\overline{\widetilde{z}_j}]\Big)^T_{|\theta=1} \end{aligned} \tag{4.59}$$

where the map $\overline{\Phi^\theta_{\gamma^{(2)}_{j+1}}(U)}$ is defined as in (2.24).

LEMMA 4.12. *Let* $\gamma^{(2)}_{j+1}(U; x; \xi)$ *be a symbol in* $\widetilde{\Gamma}_2^{-m_j}$ *of the form* (2.12) *with coefficients*

$$\begin{aligned} (\gamma^{(2)}_{j+1})^{\sigma\sigma}_{n_1,n_2} &:= \frac{(\widetilde{\mathrm{q}}_2^{(j)})^{\sigma\sigma}_{n_1,n_2}}{\mathrm{i}\sigma(\omega_{n_1} + \omega_{n_2})}, \quad \sigma = \pm, \\ (\gamma^{(2)}_{j+1})^{+-}_{n_1,n_2} &:= \frac{(\widetilde{\mathrm{q}}_2^{(j)})^{+-}_{n_1,n_2}}{\mathrm{i}(\omega_{n_1} - \omega_{n_2})}, \quad n_1 \neq \pm n_2. \end{aligned} \tag{4.60}$$

If $\widetilde{z}_j$ *solves* (4.57), *then*

$$\begin{aligned} \partial_t \check{z}_{j+1} &= Op^{\mathrm{BW}}\big(-\mathrm{i}|\xi|^{\frac{1}{2}} - \mathrm{i}\zeta(U)\xi + \mathrm{p}_2^{(j)}(U; \xi)\big)\check{z}_{j+1} \\ &\quad + Op^{\mathrm{BW}}\Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} (\widetilde{\mathrm{q}}_2^{(j)})^{+-}_{n,n}(\xi)|u_n|^2 + (\widetilde{\mathrm{q}}_2^{(j)})^{+-}_{n,-n}(\xi) u_n \overline{u_{-n}} e^{\mathrm{i}2nx}\Big)\check{z}_{j+1} \\ &\quad + Op^{\mathrm{BW}}\big(\check{\mathrm{k}}_1^{(j)}(U; x, \xi) + \check{\mathrm{k}}_2^{(j)}(U; x, \xi) + H^{(j)}_{\geq 3}\big)\check{z}_{j+1} + R^{(j)}(U)[\check{Z}_{j+1}] \end{aligned} \tag{4.61}$$

where $\check{\mathrm{k}}_1^{(j)} \in \widetilde{\Gamma}_1^{-m_j-1/2}$, $\check{\mathrm{k}}_2^{(j)} \in \widetilde{\Gamma}_2^{-m_j-1/2}$, *the symbol* $H^{(j)}_{\geq 3} \in \Gamma^1_{K,K',3}$ *is admissible, and* $R^{(j)}(U)$ *is a* 1×2 *matrix of smoothing operators in* $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$.

PROOF. In order to conjugate (4.57) we apply Lemmata A.6 and A.7. The contributions at order $-m_j$ and degree 2 are given by $Op^{\mathrm{BW}}\big(\widetilde{\mathrm{q}}_2^{(j)} + \partial_t \gamma^{(2)}_{j+1}\big)$.

All the other contributions have homogeneity greater than or equal to 3 and are admissible. By the choice of $\gamma_{j+1}^{(2)}$ in (4.60) we have

$$\widetilde{\mathtt{q}}_2^{(j)} + \partial_t \gamma_{j+1}^{(2)} = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}\setminus\{0\}} \big(\widetilde{\mathtt{q}}_2^{(j)}\big)_{n,n}^{+-}(\xi)|u_n|^2 + \big(\widetilde{\mathtt{q}}_2^{(j)}\big)_{n,-n}^{+-}(\xi) u_n \overline{u_{-n}} e^{\mathrm{i}2nx}$$

up to a symbol in $\Gamma_{K,1,3}^{-m_j}$. □

ELIMINATION OF THE x-DEPENDENCE UP TO $O(u^3)$. In order to eliminate the nonintegrable symbol

$$\frac{1}{2\pi} \sum_{n\in\mathbb{Z}\setminus\{0\}} (\widetilde{\mathtt{q}}_2^{(j)})_{n,-n}^{+-}(\xi) u_n \overline{u_{-n}} e^{\mathrm{i}2nx} \tag{4.62}$$

in (4.61) we follow the same strategy used in Section 4.2. We conjugate (4.61) by the flow

$$\partial_\theta \Phi^\theta_{\gamma^{(3)}_{j+1}}(U) = \mathrm{i} Op^{\mathrm{BW}}\big(\gamma_{j+1}^{(3)}(U;x,\xi)\big)\Phi^\theta_{\gamma^{(3)}_{j+1}}(U), \quad \Phi^0_{\gamma^{(3)}_{j+1}}(U) = \mathrm{Id}, \tag{4.63}$$

where $\gamma_{j+1}^{(3)}(U;x,\xi)$ is a symbol in $\widetilde{\Gamma}_2^{-m_j+1/2}$ of the same form (4.62), i.e.,

$$\gamma_{j+1}^{(3)}(U;x,\xi) := \frac{1}{2\pi} \sum_{n\in\mathbb{Z}\setminus\{0\}} (\gamma_{j+1}^{(3)})_{n,-n}^{+-}(\xi) u_n \overline{u_{-n}} e^{\mathrm{i}2nx}. \tag{4.64}$$

We introduce the new variable

$$\begin{aligned} Z_{j+1} := \begin{bmatrix} z_{j+1} \\ \overline{z}_{j+1} \end{bmatrix} &= \big(\mathcal{A}^\theta_{j+1,3}(U)[\check{Z}_{j+1}]\big)_{|\theta=1} \\ &= \big(\Phi^\theta_{\gamma^{(3)}_{j+1}}(U)[\check{z}_{j+1}], \overline{\Phi^\theta_{\gamma^{(3)}_{j+1}}(U)}[\overline{\check{z}_{j+1}}]\big)^T_{|\theta=1} \end{aligned} \tag{4.65}$$

where the map $\overline{\Phi^\theta_{\gamma^{(3)}_{j+1}}(U)}$ is defined as in (2.24).

LEMMA 4.13. *Define $\gamma_{j+1}^{(3)}$ in $\widetilde{\Gamma}_2^{-m_j+\frac{1}{2}}$ as in* (4.64) *with coefficients*

$$\big(\gamma_{j+1}^{(3)}\big)_{n,-n}^{+-}(\xi) := |\xi|^{\frac{1}{2}} \mathrm{sign}(\xi) \frac{1}{n} \big(\widetilde{\mathtt{q}}_2^{(j)}\big)_{n,-n}^{+-}(\xi), \quad n \neq 0. \tag{4.66}$$

If $\check{z}_j$ solves (4.61)*, then*

$$\begin{aligned} \partial_t z_{j+1} = Op^{\mathrm{BW}}\big(&-\mathrm{i}\zeta(U)\xi - \mathrm{i}|\xi|^{\frac{1}{2}} + \mathtt{p}_2^{(j+1)}(U;\xi) \\ &+ \mathtt{q}^{(j+1)}(U;x,\xi) + H_{\geq 3}^{(j+1)}\big) z_{j+1} \\ &+ R^{(j+1)}(U)[Z_{j+1}] \end{aligned} \tag{4.67}$$

where $\mathtt{p}_2^{(j+1)}(U;\xi)$ *is an integrable symbol in* $\widetilde{\Gamma}_2^{-1/2}$, $\mathtt{q}^{(j+1)}(U;x,\xi)$ *is in* $\Sigma\Gamma^{-m_{j+1}}_{K,K',1}$, *the symbol* $H_{\geq 3}^{(j+1)} \in \Gamma^1_{K,K',3}$ *is admissible, and* $R^{(j+1)}(U)$ *is a* 1×2 *matrix of smoothing operators in* $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$.

PROOF. Reasoning as in (4.46), we have $\frac{d}{dt}\gamma^{(3)}_{j+1}(U;x,\xi) = 0$ up to a cubic symbol in $\Gamma^{-m_j+1/2}_{K,1,3}$. In order to conjugate (4.61) we apply Lemmata A.6–A.7. The contributions with homogeneity 2 and order $-m_j$ are

$$Op^{\mathrm{BW}}\Big(\frac{\mathrm{i}}{2}\big(\gamma^{(3)}_{j+1}\big)_x|\xi|^{-\frac12}\mathrm{sign}(\xi) + \frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}}\big(\widetilde{\mathtt{q}}_2^{(j)}\big)^{+-}_{n,n}(\xi)|u_n|^2 + \big(\widetilde{\mathtt{q}}_2^{(j)}\big)^{+-}_{n,-n}(\xi)u_n\overline{u_{-n}}e^{\mathrm{i}2nx}\Big).$$

Then, by the choice of $\gamma^{(3)}_{j+1}$ in (4.64), (4.66), we have that (4.67) follows with the new integrable symbol $\mathtt{p}_2^{(j+1)}(U;\xi) := \mathtt{p}_2^{(j)}(U;\xi) + \sum_{n\in\mathbb{Z}\setminus\{0\}}(\widetilde{\mathtt{q}}_2^{(j)})^{+-}_{n,n}(\xi)|u_n|^2$ and a symbol $\mathtt{q}^{(j+1)}(U;x,\xi)$ in $\Sigma\Gamma^{-m_{j+1}}_{K,K',1}$ where $m_{j+1} = m_j + 1/2$. □

Lemmata 4.11, 4.12, 4.13 imply Proposition 4.10 by defining the map

$$\Upsilon^\theta_{j+1}(U) := \mathcal{A}^\theta_{j+1,3}(U)\circ\mathcal{A}^\theta_{j+1,2}(U)\circ\mathcal{A}^\theta_{j+1,1}(U) \tag{4.68}$$

where $\mathcal{A}^\theta_{j+1,k}(U)$, for $k = 1,2,3$, are defined, resp., in (4.55), (4.59), and (4.65).

4.4 Proof of Proposition 4.4

We set

$$\mathfrak{F}^\theta(U) := \Upsilon^\theta_{fin}(U)\circ\vec{\Phi}^\theta_5(U)\circ\cdots\vec{\Phi}^\theta_1(U)\circ\Psi^\theta_{\mathrm{diag}}(U) \tag{4.69}$$

and $\mathbf{F}^\theta(U) := \mathfrak{F}^\theta(U)[U]$ as in (4.6), where $\Psi^\theta_{\mathrm{diag}}(U)$ is defined in Proposition 3.10, the maps $\vec{\Phi}^\theta_j(U)$, $j = 1,\dots,5$, are given, resp., in (4.13), (4.27), (4.34), (4.43), (4.49), and $\Upsilon^\theta_{fin}(U) := \Upsilon^\theta_{2\rho}(U)\circ\cdots\circ\Upsilon^\theta_1(U)$ where $\Upsilon^\theta_{j+1}(U)$, $j = 0,\dots,2\rho-1$, are defined in (4.68). Then, by the construction in Sections 4.1–4.3, we have that $Z := (\mathbf{F}^\theta(U))_{\theta=1}$ solves the system (4.52) with $j = 2\rho-1$, which has the form (4.4) with $\mathcal{D}_{-1/2}(U;\xi) \rightsquigarrow \mathtt{p}_2^{(2\rho-1)}(U;\xi)$, $\mathtt{H}_{\geq 3} \rightsquigarrow H^{(2\rho-1)}_{\geq 3}$, and $\mathtt{R}(U) \rightsquigarrow R^{(2\rho-1)}(U)$. The bounds (4.7) follow since $\mathfrak{F}^\theta(U)$ is the composition of maps constructed using Lemma A.2 (see bounds (A.10)).

5 Poincaré-Birkhoff Normal Forms

The aim of this section is to eliminate all the terms of the system (4.4) up to cubic degree of homogeneity that are not yet in Poincaré-Birkhoff normal form.

Such terms appear only in the smoothing remainder $\mathtt{R}(U)[Z]$ that we write as

$$\mathtt{R}(U) = \mathtt{R}_1(U) + \mathtt{R}_2(U) + \mathtt{R}_{\geq 3}(U), \quad \mathtt{R}_{\geq 3}(U) \in \mathcal{R}^{-\rho}_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C}), \tag{5.1}$$

$$\begin{gathered} \mathtt{R}_i(U) = \begin{bmatrix} \mathtt{R}_i(U))^+_+ & \mathtt{R}_i(U))^-_+ \\ \mathtt{R}_i(U))^+_- & \mathtt{R}_i(U))^-_- \end{bmatrix}, \\ (\mathtt{R}_i(U))^{\sigma'}_{\sigma} \in \widetilde{\mathcal{R}}^{-\rho}_i, \quad (\mathtt{R}_i(U))^{\sigma'}_{\sigma} = \overline{(\mathtt{R}_i(U))^{-\sigma'}_{-\sigma}}, \end{gathered} \tag{5.2}$$

for $\sigma, \sigma' = \pm$ and $i = 1, 2$. The third identity in (5.2) means that the matrix of operators $\mathtt{R}(U)$ is *real-to-real* (see (2.25)). For any $\sigma, \sigma' = \pm$ we expand

$$(\mathtt{R}_1(U))^{\sigma'}_{\sigma} = \sum_{\epsilon=\pm} (\mathtt{R}_{1,\epsilon}(U))^{\sigma'}_{\sigma}, \quad (\mathtt{R}_2(U))^{\sigma'}_{\sigma} = \sum_{\epsilon=\pm} (\mathtt{R}_{2,\epsilon,\epsilon}(U))^{\sigma'}_{\sigma} + (\mathtt{R}_{2,+,-}(U))^{\sigma'}_{\sigma}, \tag{5.3}$$

where $(\mathtt{R}_{1,\epsilon}(U))^{\sigma'}_{\sigma} \in \widetilde{\mathcal{R}}^{-\rho}_1$, $(\mathtt{R}_{2,\epsilon,\epsilon'}(U))^{\sigma'}_{\sigma} \in \widetilde{\mathcal{R}}^{-\rho}_2$ with $\epsilon, \epsilon' = \pm$, are the homogeneous smoothing operators

$$(\mathtt{R}_{1,\epsilon}(U))^{\sigma'}_{\sigma} z^{\sigma'} = \frac{1}{\sqrt{2\pi}} \sum_{j \in \mathbb{Z}\setminus\{0\}} \Big(\sum_{k \in \mathbb{Z}\setminus\{0\}} (\mathtt{R}_{1,\epsilon}(U))^{\sigma',k}_{\sigma,j} z^{\sigma'}_k \Big) e^{\mathrm{i}\sigma j x} \tag{5.4}$$

with entries

$$(\mathtt{R}_{1,\epsilon}(U))^{\sigma',k}_{\sigma,j} := \frac{1}{\sqrt{2\pi}} \sum_{\epsilon n + \sigma' k = \sigma j, n \in \mathbb{Z}\setminus\{0\}} (\mathtt{r}_{1,\epsilon})^{\sigma,\sigma'}_{n,k} u^{\epsilon}_n, \quad j, k \in \mathbb{Z} \setminus \{0\}, \tag{5.5}$$

for suitable scalar coefficients $(\mathtt{r}_{1,\epsilon})^{\sigma,\sigma'}_{n,k} \in \mathbb{C}$, and

$$(\mathtt{R}_{2,\epsilon,\epsilon'}(U))^{\sigma'}_{\sigma} z^{\sigma'} = \frac{1}{\sqrt{2\pi}} \sum_{j \in \mathbb{Z}\setminus\{0\}} \Big(\sum_{k \in \mathbb{Z}\setminus\{0\}} (\mathtt{R}_{2,\epsilon,\epsilon'}(U))^{\sigma',k}_{\sigma,j} z^{\sigma'}_k \Big) e^{\mathrm{i}\sigma j x} \tag{5.6}$$

with entries

$$\begin{aligned} &(\mathtt{R}_{2,\epsilon,\epsilon'}(U))^{\sigma',k}_{\sigma,j} \\ &:= \frac{1}{2\pi} \sum_{\substack{n_1, n_2 \in \mathbb{Z}\setminus\{0\} \\ \epsilon n_1 + \epsilon' n_2 + \sigma' k = \sigma j}} (\mathtt{r}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k} u^{\epsilon}_{n_1} u^{\epsilon'}_{n_2}, \quad j, k \in \mathbb{Z} \setminus \{0\}, \end{aligned} \tag{5.7}$$

and suitable scalar coefficients $(\mathtt{r}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k} \in \mathbb{C}$.

DEFINITION 5.1 (*Poincaré-Birkhoff Resonant smoothing operator*). Let $\mathtt{R}(U)$ be a real-to-real smoothing operator in $\widetilde{\mathcal{R}}^{-\rho}_2 \otimes \mathcal{M}_2(\mathbb{C})$ with $\rho \geq 0$ and scalar coefficients $(\mathtt{r}_{\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k} \in \mathbb{C}$ defined as in (5.7). We denote by $\mathtt{R}^{\mathrm{res}}(U)$ the real-to-real smoothing operator in $\widetilde{\mathcal{R}}^{-\rho}_2 \otimes \mathcal{M}_2(\mathbb{C})$ with coefficients

$$\begin{aligned} &(\mathtt{R}^{\mathrm{res}}_{\epsilon,\epsilon'}(U))^{\sigma',k}_{\sigma,j} \\ &:= \sum_{\substack{n_1, n_2 \in \mathbb{Z}\setminus\{0\},\ \epsilon n_1 + \epsilon' n_2 + \sigma' k - \sigma j = 0 \\ \epsilon\omega(n_1) + \epsilon'\omega(n_2) + \sigma'\omega(k) - \sigma\omega(j) = 0}} (\mathtt{r}_{\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k} u^{\epsilon}_{n_1} u^{\epsilon'}_{n_2}, \ j, k \in \mathbb{Z} \setminus \{0\}, \end{aligned} \tag{5.8}$$

where we recall that $\omega(j) = |j|^{1/2}$.

In Sections 5.2 and 5.2 we will reduce the remainder $\mathtt{R}(U)$ in (5.1) to its Poincaré-Birkhoff resonant component. The key result of this section is the following.

PROPOSITION 5.2 (*Poincaré-Birkhoff normal form of the water waves at cubic degree*). *There exists $\rho_0 > 0$ such that, for all $\rho \geq \rho_0$, $K \geq K' = 2\rho + 2$, there exists $s_0 > 0$ such that, for any $s \geq s_0$, for all $0 < r \leq r_0(s)$ small enough, and any solution $U \in B_s^K(I;r)$ of the water waves system* (3.7)*, there is a nonlinear map $\mathbf{F}_T^\theta(U)$, $\theta \in [0,1]$, of the form*

$$\mathbf{F}_T^\theta(U) := \mathfrak{C}^\theta(U)[U] \tag{5.9}$$

where $\mathfrak{C}^\theta(U)$ is a real-to-real, bounded, and invertible operator such that $Y := \left[\begin{smallmatrix} y \\ \bar{y} \end{smallmatrix}\right] = \mathbf{F}_T^1(U)$ solves

$$\partial_t Y = -\mathrm{i}\Omega Y - \mathrm{i}Op^{\mathrm{BW}}(\mathtt{D}(Y;\xi))[Y] + \widetilde{\mathtt{R}}^{\mathrm{res}}(Y)[Y] + \mathcal{X}_{\geq 4}(U,Y) \tag{5.10}$$

where:

- Ω *is the diagonal matrix of Fourier multipliers defined in* (3.15)*, and* $\mathtt{D}(Y;\xi)$ *is the diagonal matrix of integrable symbols $\widetilde{\Gamma}_2^1 \otimes \mathcal{M}_2(\mathbb{C})$ defined in* (4.5)*;*
- *the smoothing operator $\widetilde{\mathtt{R}}^{\mathrm{res}}(Y) \in \widetilde{\mathcal{R}}_2^{-(\rho-\rho_0)} \otimes \mathcal{M}_2(\mathbb{C})$ is Poincaré-Birkhoff resonant (Definition* 5.1*);*
- $\mathcal{X}_{\geq 4}(U,Y)$ *has the form*

$$\mathcal{X}_{\geq 4}(U,Y) = Op^{\mathrm{BW}}(\mathfrak{H}_{\geq 3}(U;x,\xi))[Y] + \mathfrak{R}_{\geq 3}(U)[Y] \tag{5.11}$$

where $\mathfrak{H}_{\geq 3}(U;x,\xi) \in \Gamma^1_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C})$ is an admissible matrix of symbols (*Definition* 4.2) *and $\mathfrak{R}_{\geq 3}(U)$ is a matrix of real-to-real smoothing operators in $\mathcal{R}^{-(\rho-\rho_0)}_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C})$.*

Furthermore, the map $\mathbf{F}_T^\theta(U)$ defined in (5.9) *satisfies the following properties*:

(i) *There is a constant C depending on s, r, and K such that, for $s \geq s_0$,*

$$\begin{aligned} &\|\partial_t^k \mathfrak{C}^\theta(U)[V]\|_{\dot{H}^{s-k}} + \|\partial_t^k (\mathfrak{C}^\theta(U))^{-1}[V]\|_{\dot{H}^{s-k}} \\ &\quad \leq \|V\|_{k,s}(1 + C\|U\|_{K,s_0}) + C\|V\|_{k,s_0}\|U\|_{K,s}, \end{aligned} \tag{5.12}$$

for any $0 \leq k \leq K - K'$, $V \in C^{K-K'}_{*\mathbb{R}}(I;\dot{H}^s(\mathbb{T};\mathbb{C}^2))$ and uniformly in $\theta \in [0,1]$.

(ii) *The function $Y = \mathbf{F}_T^\theta(U)_{|\theta=1}$ satisfies*

$$C^{-1}\|U\|_{\dot{H}^s} \leq \|Y\|_{\dot{H}^s} \leq C\|U\|_{\dot{H}^s}. \tag{5.13}$$

(iii) *The map $\mathbf{F}_T^\theta(U)$ admits an expansion as*

$$\mathbf{F}_T^\theta(U) = U + \theta\big(M_1(U)[U] + M_2^{(1)}(U)[U]\big) + \theta^2 M_2^{(2)}(U)[U] + M_{\geq 3}(\theta;U)[U],$$

where $M_1(U)$ is in $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$, the maps $M_2^{(1)}(U), M_2^{(2)}(U)$ are in $\widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$, and $M_{\geq 3}(\theta; U)$ is in $\mathcal{M}_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C})$ with estimates uniform in $\theta \in [0, 1]$.

In the following subsection we provide lower bounds on the "small divisors" that appear in the Poincaré-Birkhoff reduction procedure. Then, in Section 5.2, we prove Proposition 5.2.

5.1 Cubic and quartic wave interactions

We study in this section the cubic and quartic resonances among the linear frequencies $\omega(n) = |n|^{\frac{1}{2}}$.

PROPOSITION 5.3 (*Nonresonance conditions*). *There are constants $\mathtt{c} > 0$ and $N_0 > 0$ such that*

- **(cubic resonances)** *for any $\sigma, \sigma' = \pm$ and $n_1, n_2, n_3 \in \mathbb{Z} \setminus \{0\}$ satisfying $n_1 + \sigma n_2 + \sigma' n_3 = 0$ we have*

$$|\omega(n_1) + \sigma\omega(n_2) + \sigma'\omega(n_3)| \geq \mathtt{c}\,. \tag{5.14}$$

- **(quartic resonances)** *For any $\sigma, \sigma', \sigma'' = \pm$ and $n_1, n_2, n_3, n_4 \in \mathbb{Z} \setminus \{0\}$ such that*

$$n_1+\sigma n_2+\sigma' n_3+\sigma'' n_4 = 0\,,\ \omega(n_1)+\sigma\omega(n_2)+\sigma'\omega(n_3)+\sigma''\omega(n_4) \neq 0\,, \tag{5.15}$$

we have

$$\begin{aligned}&|\omega(n_1) + \sigma\omega(n_2) + \sigma'\omega(n_3) + \sigma''\omega(n_4)| \\ &\qquad \geq \mathtt{c} \max\{|n_1|, |n_2|, |n_3|, |n_4|\}^{-N_0}\,.\end{aligned} \tag{5.16}$$

PROOF. We first consider the cubic and then the quartic resonances.

PROOF OF (5.14). If $\sigma = \sigma' = +$, then the bound (5.14) is trivial. Assume $\sigma = +$ and $\sigma' = -$. By $n_1 + n_2 - n_3 = 0$ we have that $|n_3| \leq |n_1| + |n_2|$ and therefore

$$|\sqrt{|n_1|} + \sqrt{|n_2|} - \sqrt{|n_3|}| = \frac{||n_1| + |n_2| - |n_3| + 2\sqrt{|n_1||n_2|}|}{\sqrt{|n_1|} + \sqrt{|n_2|} + \sqrt{|n_3|}} \geq \frac{2}{2 + \sqrt{2}}$$

since $|n_1|, |n_2| \geq 1$. The bound (5.14) in the case $\sigma = -$ and $\sigma' = +$ is the same.

PROOF OF (5.16). The case $\sigma = \sigma' = \sigma'' = +$ is trivial. Assume $\sigma = \sigma' = +$ and $\sigma'' = -$. We have

$$\begin{aligned}&|\omega(n_1) + \omega(n_2) + \omega(n_3) - \omega(n_4)| \\ &\quad = \frac{||n_1| + |n_2| + |n_3| - |n_4| + 2\sqrt{|n_1 n_2|} + 2\sqrt{|n_2 n_3|} + 2\sqrt{|n_1 n_3|}|}{\omega(n_1) + \omega(n_2) + \omega(n_3) + \omega(n_4)}\,.\end{aligned}$$

The first (momentum) condition in (5.15) implies that $|n_1| + |n_2| + |n_3| - |n_4| \geq 0$ and hence (5.16) follows (actually with $N_0 = 0$). It remains the case that $\sigma =$

$\sigma'' = -$ and $\sigma' = +$; i.e., we have to prove that the phase

$$\psi(n_1, n_2, n_3, n_4) := \frac{|n_1| - |n_2| + |n_3| - |n_4| + 2\sqrt{|n_1 n_3|} - 2\sqrt{|n_2 n_4|}}{|n_1|^{1/2} + |n_2|^{1/2} + |n_3|^{1/2} + |n_4|^{1/2}} \tag{5.17}$$

satisfies (5.16). Note that the first (momentum) equality in (5.15) becomes

$$n_1 - n_2 + n_3 - n_4 = 0\,. \tag{5.18}$$

Let $|n_1| := \max\{|n_1|, |n_2|, |n_3|, |n_4|\}$ and assume, without loss of generality, that $n_1 > 0$ and $|n_2| \geq |n_4|$ (the phase (5.17) is symmetric in $|n_2|, |n_4|$). We consider different cases.

Case a. Assume that $n_1 = |n_2|$. Then by (5.17) we have that

$$|\psi(n_1, n_2, n_3, n_4)| = ||n_3| - |n_4||/(|n_3|^{1/2} + |n_4|^{1/2}).$$

Since $\psi \neq 0$, then $|n_3| - |n_4|$ is a nonzero integer and we get (5.16). Thus in the sequel we suppose

$$n_1 > |n_2| \geq |n_4|. \tag{5.19}$$

Case b. Assume that $|n_3| \geq |n_4|$. Then by (5.17), (5.19) we get

$$\psi(n_1, n_2, n_3, n_4) \geq (|n_1|^{\frac{1}{2}} + |n_2|^{\frac{1}{2}})^{-1},$$

which implies (5.16). Thus in the sequel we suppose, in addition to (5.19), that

$$n_1 > |n_2| \geq |n_4| > |n_3|. \tag{5.20}$$

The case $n_2 < 0$ is not possible. Indeed, if $n_2 < 0$, then (5.18) implies $n_4 = n_1 + |n_2| + n_3 > n_1$ by (5.20), which is in contradiction with $n_1 > |n_4|$. Hence from now on we assume that

$$n_1 > n_2 \geq |n_4| > |n_3| > 0. \tag{5.21}$$

Case c1. Assume that all the frequencies have all the same sign, i.e., $n_1 > n_2 \geq n_4 > n_3 > 0$. In this case, by (5.17)–(5.18), we get

$$\begin{aligned} |\psi(n_1, n_2, n_3, n_4)| &= \frac{|2\sqrt{n_1 n_3} - 2\sqrt{n_2 n_4}|}{|n_1|^{1/2} + |n_2|^{1/2} + |n_3|^{1/2} + |n_4|^{1/2}} \\ &\geq \frac{2}{\sum_{i=1}^4 |n_i|^{1/2}} \frac{|n_1 n_3 - n_2 n_4|}{\sqrt{n_1 n_3} + \sqrt{n_2 n_4}}. \end{aligned}$$

Since $\psi \neq 0$ we have $n_1 n_3 \neq n_2 n_4$, and therefore (5.16) follows.

Case c2. Assume now that two frequencies are positive and two are negative, i.e., $n_4 < n_3 < 0 < n_2 < n_1$. The momentum condition (5.18) becomes $n_1 - n_2 = -|n_4| + |n_3|$ and, since $n_1 > n_2$, then $|n_3| > |n_4|$ contradicting (5.21).

Case c3. Assume that three frequencies have the same sign and one has the opposite sign. By (5.18) and (5.21) we then have $n_1 > n_2 > n_4 > 0 > n_3$, $n_4 > |n_3|$. Hence by (5.17) we get

$$\psi(n_1, n_2, n_3, n_4) \overset{(5.18)}{=} \frac{2}{\sum_{i=1}^{4} |n_i|^{\frac{1}{2}}} \frac{n_3^2 + n_1|n_3| - n_2 n_4 + 2|n_3|\sqrt{n_1|n_3|}}{|n_3| + \sqrt{n_1|n_3|} + \sqrt{n_2 n_4}}. \tag{5.22}$$

If $n_2 n_4 \leq n_1|n_3|$, then (5.22) implies the bound (5.16). If instead $n_2 n_4 > n_1|n_3|$, we reason as follows. Note that

$$\begin{aligned} B := n_3^2 + n_1|n_3| - n_2 n_4 - 2|n_3|\sqrt{n_1|n_3|} &\leq n_3^2 - 2|n_3|\sqrt{n_1|n_3|} \\ &\leq -|n_3|\sqrt{n_1|n_3|} \leq -1, \end{aligned}$$

in particular $B \neq 0$. Then we rationalize again (5.22) to obtain $\psi(n_1, n_2, n_3, n_4) = C \cdot A \cdot B^{-1}$ where

$$\begin{aligned} A &:= (n_3^2 + n_1|n_3| - n_2 n_4)^2 - 4|n_3|^3 n_1, \\ C &:= \frac{2}{\sum_{i=1}^{4} |n_i|^{1/2}} \frac{1}{|n_3| + \sqrt{n_1|n_3|} + \sqrt{n_2 n_4}}. \end{aligned}$$

Since $\psi \neq 0$, then A is a nonzero integer and so $|\psi| \geq C|B|^{-1}$. Moreover, $|B| \leq c n_1^2$, for some constant $c > 0$, and (5.16) follows. □

5.2 Poincaré-Birkhoff reductions

The proof of Proposition 5.2 is divided into two steps: in the first (Section 5.2) we eliminate all the quadratic terms in (4.4); in the second one (Section 5.2) we eliminate all the nonresonant cubic terms.

Elimination of the quadratic vector field

In this section we cancel out the smoothing term $\mathtt{R}_1(U)$ in (5.1) of system (4.4). We conjugate (4.4) with the flow

$$\partial_\theta \mathcal{B}_1^\theta(U) = \mathtt{Q}_1(U)\mathcal{B}_1^\theta(U), \quad \mathcal{B}_1^0(U) = \mathrm{Id}, \tag{5.23}$$

with $\mathtt{Q}_1(U) \in \widetilde{\mathcal{R}}_1^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$ of the same form of $\mathtt{R}_1(U)$ in (5.2)–(5.5), to be determined. We introduce the new variable $Y_1 := \left[\begin{smallmatrix} y_1 \\ \overline{y_1} \end{smallmatrix}\right] = \left(\mathcal{B}_1^\theta(U)[Z]\right)_{|\theta=1}$.

LEMMA 5.4 (*First Poincaré-Birkhoff step*). *Assume that* $\mathtt{Q}_1(U) \in \widetilde{\mathcal{R}}_1^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$ *solves the homological equation*

$$\mathtt{Q}_1(-\mathrm{i}\Omega U) + \left[\mathtt{Q}_1(U), -\mathrm{i}\Omega\right] + \mathtt{R}_1(U) = 0. \tag{5.24}$$

Then

$$\begin{aligned} \partial_t Y_1 = -\mathrm{i}\Omega Y_1 &+ Op^{\mathrm{BW}}(-\mathrm{i}\mathtt{D}(U;\xi) + \mathtt{H}_{\geq 3})[Y_1] \\ &+ \left(\mathtt{R}_2^+(U) + \mathtt{R}_{\geq 3}^+(U)\right)[Y_1] \end{aligned} \tag{5.25}$$

where Ω *is defined in* (3.15)*,* $\mathtt{D}(U;\xi)$ *in* (4.5)*,* $\mathtt{H}_{\geq 3}$ *is an admissible symbol in* $\Gamma^{1}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$*, and* $\mathtt{R}^{+}_{2}(U)\in\widetilde{\mathcal{R}}^{-\rho+m_1}_{2}\otimes\mathcal{M}_2(\mathbb{C})$*,* $\mathtt{R}^{+}_{\geq 3}(U)\in\mathcal{R}^{-\rho+m_1}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$*, with* $m_1\geq 1$ *as in* (3.15)*.*

PROOF. To conjugate (4.4) we apply Lemma A.1 with $\mathtt{Q}_1(U)=\mathrm{i}\mathbf{A}(U)$. By (A.3) with $L=1$ we have

$$
\begin{aligned}
&-\mathrm{i}\mathcal{B}^1_1(U)\Omega(\mathcal{B}^1_1(U))^{-1}\\
&\quad=-\mathrm{i}\Omega+\big[\mathtt{Q}_1(U),-\mathrm{i}\Omega\big]\\
&\qquad+\int_0^1(1-\theta)\mathcal{B}^{\theta}_1(U)\big[\mathtt{Q}_1(U),\big[\mathtt{Q}_1(U),-\mathrm{i}\Omega\big]\big]\big(\mathcal{B}^{\theta}_1(U)\big)^{-1}\,d\theta.
\end{aligned}
\tag{5.26}
$$

Using that $\mathtt{Q}_1(U)$ belongs to $\widetilde{\mathcal{R}}^{-\rho}_{1}\otimes\mathcal{M}_2(\mathbb{C})$ and applying Proposition 2.10, and Lemma A.3, the term in (5.26) is a smoothing operator in $\Sigma\mathcal{R}^{-\rho+1/2}_{K,K',2}\otimes\mathcal{M}_2(\mathbb{C})$. Similarly we obtain

$$
-\mathrm{i}\mathcal{B}^1_1(U)Op^{\mathrm{BW}}(\mathtt{D}(U;\xi))\big(\mathcal{B}^1_1(U)\big)^{-1}=-\mathrm{i}Op^{\mathrm{BW}}(\mathtt{D}(U;\xi))
\tag{5.27}
$$

up to a term in $\Sigma\mathcal{R}^{-\rho+1}_{K,K',2}\otimes\mathcal{M}_2(\mathbb{C})$, and

$$
\begin{aligned}
&\mathcal{B}^1_1(U)\big(Op^{\mathrm{BW}}(\mathtt{H}_{\geq 3})+\mathtt{R}_1(U)+\mathtt{R}_2(U)+\mathtt{R}_{\geq 3}(U)\big)\big(\mathcal{B}^1_1(U)\big)^{-1}\\
&\quad=Op^{\mathrm{BW}}(\mathtt{H}_{\geq 3})+\mathtt{R}_1(U)
\end{aligned}
\tag{5.28}
$$

up to a matrix of smoothing operators in $\Sigma\mathcal{R}^{-\rho+1}_{K,K',2}\otimes\mathcal{M}_2(\mathbb{C})$. Next we consider the contribution coming from the conjugation of ∂_t. Applying formula (A.4) with $L=2$, we get

$$
\begin{aligned}
\partial_t\mathcal{B}^1_1(U)\big(\mathcal{B}^1_1(U)\big)^{-1}&=\partial_t\mathtt{Q}_1(U)+\frac{1}{2}\big[\mathtt{Q}_1(U),\partial_t\mathtt{Q}_1(U)\big]\\
&+\frac{1}{2}\int_0^1(1-\theta)^2\mathcal{B}^{\theta}_1(U)\big[\mathtt{Q}_1(U),\big[\mathtt{Q}_1(U),\partial_t\mathtt{Q}_1(U)\big]\big]\big(\mathcal{B}^{\theta}_1(U)\big)^{-1}\,d\theta.
\end{aligned}
\tag{5.29}
$$

Recalling (3.15) we have $\partial_t\mathtt{Q}_1(U)=\mathtt{Q}_1(-\mathrm{i}\Omega U+\mathbf{M}(U)[U])=\mathtt{Q}_1(-\mathrm{i}\Omega U)$ up to a term in $\Sigma\mathcal{R}^{-\rho+m_1}_{K,K',2}\otimes\mathcal{M}_2(\mathbb{C})$, where we used item (iii) of Proposition 2.10. By the fact that $\mathtt{Q}_1(-\mathrm{i}\Omega U)$ is in $\widetilde{\mathcal{R}}^{-\rho+1/2}_{1}\otimes\mathcal{M}_2(\mathbb{C})$ we have that the second line (5.29) belongs to $\Sigma\mathcal{R}^{-\rho+m_1}_{K,K',2}\otimes\mathcal{M}_2(\mathbb{C})$. In conclusion, by (5.26), (5.27), (5.28), (5.29) and the assumption that $\mathtt{Q}_1$ solves (5.24) we deduce (5.25). ☐

Notation. Given $p\in\mathbb{N}$ we denote $\max_2(|n_1|,\dots,|n_p|)$ and $\max(|n_1|,\dots,|n_p|)$, resp., the second largest and the largest among $|n_1|,\dots,|n_p|$.

The following lemma is deduced by the definition of smoothing homogeneous operators in Definition 2.5.

LEMMA 5.5. *An operator* $\mathtt{R}_1(U)$ *of the form* (5.2)–(5.5) *is in* $\widetilde{\mathcal{R}}_1^{-\rho}\otimes\mathcal{M}_2(\mathbb{C})$ *if and only if, for some* $\mu>0$,

$$\big|(\mathtt{r}_{1,\epsilon})_{n,k}^{\sigma,\sigma'}\big|\le\frac{\max_2(|n|,|k|)^{\rho+\mu}}{\max(|n|,|k|)^{\rho}},\quad \forall\,\epsilon,\sigma,\sigma'=\pm,\ \ n,k\in\mathbb{Z}\setminus\{0\}. \tag{5.30}$$

An operator $\mathtt{R}_2(U)$ *of the form* (5.2)–(5.3) *as in* (5.6)–(5.7) *belongs to* $\widetilde{\mathcal{R}}_2^{-\rho}\otimes\mathcal{M}_2(\mathbb{C})$ *if and only if, for some* $\mu>0$,

$$\begin{gathered}\big|(\mathtt{r}_{2,\epsilon,\epsilon'})_{n_1,n_2,k}^{\sigma,\sigma'}\big|\le\frac{\max_2(|n_1|,|n_2|,|k|)^{\rho+\mu}}{\max(|n_1|,|n_2|,|k|)^{\rho}}\\ \forall\epsilon,\epsilon',\sigma,\sigma'=\pm,\ \ n_1,n_2,k\in\mathbb{Z}\setminus\{0\}.\end{gathered} \tag{5.31}$$

We now solve the homological equation (5.24).

LEMMA 5.6 (*First homological equation*). *The operator* $\mathtt{Q}_1$ *of the form* (5.2)–(5.5) *with coefficients*

$$(\mathtt{q}_{1,\epsilon})_{n,k}^{\sigma,\sigma'}:=\frac{-(\mathtt{r}_{1,\epsilon})_{n,k}^{\sigma,\sigma'}}{\mathrm{i}\big(\sigma|j|^{1/2}-\sigma'|k|^{1/2}-\epsilon|n|^{1/2}\big)},\quad \sigma j-\sigma'k-\epsilon n=0, \tag{5.32}$$

with $\sigma,\sigma',\epsilon=\pm$, $j,n,k\in\mathbb{Z}\setminus\{0\}$ *solves the homological equation* (5.24) *and* $\mathtt{Q}_1$ *is in* $\widetilde{\mathcal{R}}_1^{-\rho}\otimes\mathcal{M}_2(\mathbb{C})$.

PROOF. First note that the coefficients in (5.32) are well-defined since

$$\sigma|j|^{\frac12}-\sigma'|k|^{1/2}-\epsilon|n|^{1/2}\neq 0$$

for any $\sigma,\sigma',\epsilon=\pm$, $n,k\in\mathbb{Z}\setminus\{0\}$, by Proposition 5.3. Moreover, by (5.14) and Lemma 5.5 we have

$$|(\mathtt{q}_{1,\epsilon})_{n,k}^{\sigma,\sigma'}|\le\max_2(|n|,|k|)^{\rho+\mu}\max(|n|,|k|)^{-\rho},$$

and therefore the operator $\mathtt{Q}_1$ is in $\widetilde{\mathcal{R}}_1^{-\rho}\otimes\mathcal{M}_2(\mathbb{C})$.

Next, recalling (5.2) and (3.15), the homological equation (5.24) amounts to the equations, $\forall\sigma,\sigma'=\pm$,

$$(\mathtt{Q}_1(-\mathrm{i}\Omega U))_\sigma^{\sigma'}-(\mathtt{Q}_1(U))_\sigma^{\sigma'}\sigma'\mathrm{i}|D|^{\frac12}+\sigma\mathrm{i}|D|^{\frac12}(\mathtt{Q}_1(U))_\sigma^{\sigma'}+(\mathtt{R}_1(U))_\sigma^{\sigma'}=0,$$

and expanding $(\mathtt{Q}_1(U))_\sigma^{\sigma'}$ as in (5.3)–(5.5) with entries

$$(\mathtt{Q}_{1,\epsilon}(U))_{\sigma,j}^{\sigma',k}=\frac{1}{\sqrt{2\pi}}\sum_{\epsilon n+\sigma'k=\sigma j,n\in\mathbb{Z}\setminus\{0\}}(\mathtt{q}_{1,\epsilon})_{n,k}^{\sigma,\sigma'}u_n^{\epsilon},\quad j,k\in\mathbb{Z}\setminus\{0\}, \tag{5.33}$$

to the equations, for any $j,k\in\mathbb{Z}\setminus\{0\}$, $\epsilon=\pm$,

$$\begin{aligned}(\mathtt{Q}_{1,\epsilon}(-\mathrm{i}\Omega U))_{\sigma,j}^{\sigma',k}+(\mathtt{Q}_{1,\epsilon}(U))_{\sigma,j}^{\sigma',k}\big(\sigma\mathrm{i}|j|^{\frac12}-\sigma'\mathrm{i}|k|^{\frac12}\big)&\\ +(\mathtt{R}_{1,\epsilon}(U))_{\sigma,j}^{\sigma',k}=0.&\end{aligned} \tag{5.34}$$

By (5.33) and (3.15) we have

$$(\mathtt{Q}_{1,\epsilon}(-\mathrm{i}\Omega U))_{\sigma,j}^{\sigma',k} = \frac{1}{\sqrt{2\pi}} \sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ \epsilon n+\sigma' k=\sigma j}} (\mathtt{q}_{1,\epsilon})_{n,k}^{\sigma,\sigma'}(-\mathrm{i}\epsilon|n|^{\frac{1}{2}})u_n^{\epsilon}.$$

Then one checks that (5.34) is solved by the coefficients $(\mathtt{q}_{1,\epsilon})_{n,k}^{\sigma,\sigma'}$ in (5.32). □

Elimination of the cubic vector field

In this section we reduce to Poincaré-Birkhoff normal form the smoothing term $\mathtt{R}_2^{+}(U) \in \widetilde{\mathcal{R}}_2^{-\rho+m_1} \otimes \mathcal{M}_2(\mathbb{C})$ in (5.25). We conjugate (5.25) with the flow

$$\partial_\theta \mathcal{B}_2^{\theta}(U) = \mathtt{Q}_2(U)\mathcal{B}_2^{\theta}(U), \quad \mathcal{B}_2^{0}(U) = \mathrm{Id}, \tag{5.35}$$

where $\mathtt{Q}_2(U)$ is a matrix of smoothing operators in $\widetilde{\mathcal{R}}_2^{-\rho+N_0+m_1} \otimes \mathcal{M}_2(\mathbb{C})$ of the same form of $\mathtt{R}_2^{+}(U)$ to be determined. We introduce the new variable $Y_2 := \left[\begin{smallmatrix} y_2 \\ \overline{y_2}\end{smallmatrix}\right] = \big(\mathcal{B}_2^{\theta}(U)[Y_1]\big)_{|\theta=1}$.

Notation. Given the operator $\mathtt{Q}_2(U)$ in (5.35), we denote by $\mathtt{Q}_2(-\mathrm{i}\Omega U)$ the operator of the form (5.2), (5.3), (5.6)–(5.7) with coefficients defined as

$$\begin{aligned}(\mathtt{Q}_{2,\epsilon,\epsilon'}&(-\mathrm{i}\Omega U))_{\sigma,j}^{\sigma',k}\\ &= \frac{1}{2\pi} \sum_{\substack{n_1,n_2\in\mathbb{Z}\setminus\{0\}\\ \epsilon n_1+\epsilon' n_2+\sigma' k=\sigma j}} (\mathtt{q}_{2,\epsilon,\epsilon'})_{n_1,n_2,k}^{\sigma,\sigma'}\big(-\mathrm{i}\epsilon|n_1|^{\frac{1}{2}} - \mathrm{i}\epsilon'|n_2|^{\frac{1}{2}}\big)u_{n_1}^{\epsilon}u_{n_2}^{\epsilon'}.\end{aligned} \tag{5.36}$$

LEMMA 5.7 (*Second Poincaré-Birkhoff step*). *Assume that*

$$\mathtt{Q}_2(U) \in \widetilde{\mathcal{R}}_2^{-\rho+N_0+m_1} \otimes \mathcal{M}_2(\mathbb{C})$$

solves the homological equation

$$\mathtt{Q}_2(-\mathrm{i}\Omega U) + [\mathtt{Q}_2(U), -\mathrm{i}\Omega] + \mathtt{R}_2^{+}(U) = (\mathtt{R}_2^{+})^{\mathrm{res}}(U). \tag{5.37}$$

Then

$$\begin{aligned}\partial_t Y_2 = -\mathrm{i}\Omega Y_2 &+ Op^{\mathrm{BW}}(-\mathrm{i}\mathtt{D}(U;\xi) + \mathtt{H}_{\geq 3})[Y_2]\\ &+ \big((\mathtt{R}_2^{+})^{\mathrm{res}}(U) + \mathtt{R}'_{\geq 3}(U)\big)[Y_2]\end{aligned} \tag{5.38}$$

where Ω is defined in (3.15) *and* $\mathtt{D}(U;\xi)$ *in* (4.5)*,* $\mathtt{H}_{\geq 3}$ *is an admissible symbol in* $\Gamma^1_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C})$*,* $(\mathtt{R}_2^{+})^{\mathrm{res}}(U)$ *is a Poincaré-Birkhoff resonant smoothing operator* (*cf. Definition* 5.1) *in* $\widetilde{\mathcal{R}}_2^{-\rho+m_1} \otimes \mathcal{M}_2(\mathbb{C})$*, and* $\mathtt{R}'_{\geq 3}(U)$ *is a matrix of smoothing operators in* $\mathcal{R}^{-\rho+N_0+2m_1}_{K,K',3} \otimes \mathcal{M}_2(\mathbb{C})$ *with* $m_1 \geq 1$ *as in* (3.15).

PROOF. To conjugate system (5.25) we apply Lemma A.1 with $\mathtt{Q}_2(U) = \mathrm{i}\mathbf{A}(U)$. Applying formula (A.3) with $L = 1$, the fact that $\mathtt{Q}_2(U)$ is a smoothing operator in $\widetilde{\mathcal{R}}_2^{-\rho+N_0+m_1} \otimes \mathcal{M}_2(\mathbb{C})$, Proposition 2.10, and Lemma A.3, we have that

$$\mathcal{B}_2^{1}(U)(-\mathrm{i}\Omega)\big(\mathcal{B}_2^{1}(U)\big)^{-1} = -\mathrm{i}\Omega + [\mathtt{Q}_2(U), -\mathrm{i}\Omega]$$

plus a smoothing operator in $\mathcal{R}^{-\rho+N_0+m_1+1}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$. Similarly, the conjugate of $Op^{\mathrm{BW}}\big(-\mathrm{i}\mathtt{D}(U;\xi)+\mathtt{H}_{\geq 3}\big)+\mathtt{R}_2^+(U)+\mathtt{R}^+_{\geq 3}(U)$ remains the same up to a smoothing operator in $\mathcal{R}^{-\rho+N_0+m_1+1}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$.

Next we consider the contribution coming from the conjugation of ∂_t. First, note that, using equation (3.15), $\partial_t\mathtt{Q}_2(U)=\mathtt{Q}_2(\partial_t U)=\mathtt{Q}_2(-\mathrm{i}\Omega U)$ (defined in (5.36)) up to a smoothing operator in $\mathcal{R}^{-\rho+N_0+2m_1}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$. The operator $\mathtt{Q}_2(-\mathrm{i}\Omega U)$ is in $\widetilde{\mathcal{R}}_2^{-\rho+N_0+m_1+(1/2)}\otimes\mathcal{M}_2(\mathbb{C})$. Then, applying formula (A.4) with $L=2$ we have $\partial_t\mathcal{B}^1_2(U)(\mathcal{B}^1_2(U))^{-1}=\mathtt{Q}_2(-\mathrm{i}\Omega U)$ up to a smoothing operator in $\mathcal{R}^{-\rho+N_0+2m_1}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$.

In conclusion, $\mathtt{Q}_2(-\mathrm{i}\Omega U)+[\mathtt{Q}_2(U),-\mathrm{i}\Omega]+\mathtt{R}^+_2(U)$ collects all the nonintegrable terms quadratic in U in the transformed system. Since $\mathtt{Q}_2$ solves (5.37) we conclude that Y_2 solves (5.38). □

We now solve the homological equation (5.37).

LEMMA 5.8 (Second homological equation). *The operator* $\mathtt{Q}_2$ *of the form* (5.2)–(5.3)*,* (5.6)–(5.7) *with coefficients*

$$
(\mathtt{q}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}:=
\begin{cases}\dfrac{-(\mathtt{r}^+_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}}{\mathrm{i}(\sigma|j|^{1/2}-\sigma'|k|^{1/2}-\epsilon|n_1|^{1/2}-\epsilon'|n_2|^{1/2})}, & \sigma|j|^{1/2}-\sigma'|k|^{1/2}-\epsilon|n_1|^{1/2}-\epsilon'|n_2|^{1/2}\neq 0,\\ 0, & \sigma|j|^{1/2}-\sigma'|k|^{1/2}-\epsilon|n_1|^{1/2}-\epsilon'|n_2|^{1/2}=0,\end{cases}
\tag{5.39}
$$

with $\sigma,\sigma',\epsilon,\epsilon'=\pm$*,* $n_1,n_2,k\in\mathbb{Z}\setminus\{0\}$*, satisfying* $\sigma j-\sigma' k-\epsilon n_1-\epsilon' n_2=0$*, solves the homological equation* (5.37)*. We have that* $\mathtt{Q}_2$ *is in* $\widetilde{\mathcal{R}}_2^{-\rho+N_0+m_1}\otimes\mathcal{M}_2(\mathbb{C})$*.*

PROOF. First note that the coefficients in (5.39) are well-defined thanks to Proposition 5.3, in particular (5.16), and satisfy, using also $|j|\leq|k|+|n_1|+|n_2|$,

$$
\begin{aligned}
|(\mathtt{q}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}| &\leq C|(\mathtt{r}^+_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}|\max(|n_1|,|n_2|,|k|)^{N_0}\\
&\leq C\frac{\max_2(|n_1|,|n_2|,|k|)^{\rho-m_1-N_0+\mu'}}{\max(|n_1|,|n_2|,|k|)^{\rho-m_1-N_0}}
\end{aligned}
\tag{5.40}
$$

with $\mu'=\mu+N_0$, because $(\mathtt{r}^+_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}$ are the coefficients of a remainder in $\widetilde{\mathcal{R}}_2^{-\rho+m_1}\otimes\mathcal{M}_2(\mathbb{C})$, and so they satisfy the bound (5.31) with $\rho\rightsquigarrow\rho-m_1$. The estimate (5.40) and Lemma 5.5 imply that $\mathtt{Q}_2(U)$ belongs to the class $\widetilde{\mathcal{R}}_2^{-\rho+m_1+N_0}\otimes\mathcal{M}_2(\mathbb{C})$.

Next, the homological equation (5.37) amounts to, for any $\sigma,\sigma',\epsilon,\epsilon'=\pm$,

$$
\begin{aligned}
&(\mathtt{Q}_{2,\epsilon,\epsilon'}(-\mathrm{i}\Omega U))^{\sigma',k}_{\sigma,j}+(\mathtt{Q}_{2,\epsilon,\epsilon'}(U))^{\sigma',k}_{\sigma,j}\big(\sigma\mathrm{i}|j|^{\frac12}-\sigma'\mathrm{i}|k|^{\frac12}\big)\\
&\quad+(\mathtt{R}^+_{2,\epsilon,\epsilon'}(U))^{\sigma',k}_{\sigma,j}=\big((\mathtt{R}^+_{2,\epsilon,\epsilon'})^{\mathrm{res}}(U)\big)^{\sigma',k}_{\sigma,j}
\end{aligned}
\tag{5.41}
$$

for any $j,k \in \mathbb{Z}\setminus\{0\}$. Recalling (5.36) and (5.8), the left-hand side of (5.41) is given by

$$(\mathtt{q}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}\mathrm{i}\big(\sigma|j|^{\frac12}-\sigma'|k|^{\frac12}-\epsilon|n_1|^{\frac12}-\epsilon'|n_2|^{\frac12}\big)+(\mathtt{r}^{+}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k},$$

for $j,k,n_1,n_2 \in \mathbb{Z}\setminus\{0\}$, $\sigma,\sigma',\epsilon,\epsilon' = \pm$, and $\epsilon n_1+\epsilon' n_2+\sigma' k=\sigma j$. Thus, recalling Definition 5.1, the operator $\mathtt{Q}_2$ with coefficients $(\mathtt{q}_{2,\epsilon,\epsilon'})^{\sigma,\sigma'}_{n_1,n_2,k}$ defined in (5.39) solves the homological equation (5.37). □

We can now prove the main result of this section.

PROOF OF PROPOSITION 5.2. Let Z be the function given by Proposition 4.4. We define $Y := (\mathcal{B}^{\theta}(U))_{\theta=1}[Z]$ where $\mathcal{B}^{\theta}(U) := \mathcal{B}^{\theta}_2(U)\circ\mathcal{B}^{\theta}_1(U)$, $\theta\in[0,1]$, and $\mathcal{B}^{\theta}_i(U)$, $i=1,2$, are the flow maps defined, resp., in (5.23), (5.35) (see also Lemmata 5.4, and 5.7). Then Y solves (recall (5.38))

$$\partial_t Y = -\mathrm{i}\Omega Y + Op^{\mathrm{BW}}(-\mathrm{i}\mathtt{D}(U;\xi)+\mathtt{H}_{\geq3})[Y]+\widetilde{\mathtt{R}}^{\mathrm{res}}(U)[Y]+\mathtt{R}'_{\geq3}(U)[Y] \tag{5.42}$$

where Ω and $\mathtt{D}(U;\xi)$ are defined, resp., in (3.15) and (4.5), the operator $\widetilde{\mathtt{R}}^{\mathrm{res}}(U) := (\mathtt{R}^{+}_2)^{\mathrm{res}}(U)$ in $\widetilde{\mathcal{R}}^{-\rho+m_1}_2\otimes\mathcal{M}_2(\mathbb{C})$ (being $m_1\geq1$ the loss in (3.15)), $\mathtt{H}_{\geq3}\in\Gamma^1_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$ is admissible, and $\mathtt{R}'_{\geq3}(U)$ is in $\mathcal{R}^{-\rho+N_0+2m_1}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$ where the constant N_0 is defined by Proposition 5.3. We define $\mathbf{F}^{\theta}_T(U) := \mathfrak{C}^{\theta}(U)[U] := \mathcal{B}^{\theta}(U)\circ\mathfrak{F}^{\theta}(U)[U]$ as in (5.9) where $\mathfrak{F}^{\theta}(U)$ in (4.6). By Lemma A.3 the maps $\mathcal{B}^{\theta}_i$, $i=1,2$, satisfy the bounds (A.13), (A.11) and recall that $\mathfrak{F}^{\theta}(U)$ satisfies (4.7). Then $\mathfrak{C}^{\theta}(U)$ satisfies (5.12) and (5.13). By Lemmata A.2 and A.3 applied, resp., to $\mathfrak{F}^{\theta}(U)$ and $\mathcal{B}^{\theta}(U)$, we have that the map $\mathbf{F}^{\theta}_T(U)$ admits a multilinear expansion like (A.12), implying item (iii) of Proposition 5.2. Moreover,

$$\begin{gathered} Y = \big(\mathbf{F}^{\theta}_T(U)\big)_{|\theta=1} = U+\mathtt{M}(U)[U] \\ \text{where } \mathtt{M}(U)\in\Sigma\mathcal{M}_{K,K',1}\otimes\mathcal{M}_2(\mathbb{C}). \end{gathered} \tag{5.43}$$

Then, substituting (5.43) in (5.42), we obtain (5.10)–(5.11) with

$$\mathfrak{H}_{\geq3}(U;x,\xi) := -\mathrm{i}\big(\mathtt{D}(U;\xi)-\mathtt{D}(U+\mathtt{M}(U)[U];\xi)\big)+\mathtt{H}_{\geq3}(U;x,\xi), \tag{5.44}$$

$$\mathfrak{R}_{\geq3}(U) := \widetilde{\mathtt{R}}^{\mathrm{res}}(U)-\widetilde{\mathtt{R}}^{\mathrm{res}}(U+\mathtt{M}(U)[U])+\mathtt{R}'_{\geq3}(U). \tag{5.45}$$

Since the integrable symbol $\mathtt{D}(U;\xi)$ in (4.5) is homogeneous of degree 2 and $\widetilde{\mathtt{R}}^{\mathrm{res}}(U)\in\widetilde{\mathcal{R}}^{-\rho+m_1}_2\otimes\mathcal{M}_2(\mathbb{C})$, we have that the quadratic terms in the r.h.s. of (5.44) and (5.45) cancel out. Then, by (5.43) and items (iii) and (iv) of Proposition 2.10, we deduce that $\mathfrak{H}_{\geq3}(U;x,\xi)\in\Gamma^1_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$ is an admissible symbol and $\mathfrak{R}_{\geq3}(U)\in\Sigma\mathcal{R}^{-(\rho-\rho_0)}_{K,K',3}\otimes\mathcal{M}_2(\mathbb{C})$ where $\rho_0 := N_0+2m_1$. □

6 Long Time Existence

The system

$$\partial_t Y = -\mathrm{i}\Omega Y - \mathrm{i}Op^{\mathrm{BW}}\big(\mathtt{D}(Y;\xi)\big)[Y]+\widetilde{\mathtt{R}}^{\mathrm{res}}(Y)[Y], \tag{6.1}$$

obtained retaining only the vector fields in (5.10) up to degree 3 of homogeneity, is in Poincaré-Birkhoff normal form. In Section 6.2 we will actually prove that this is uniquely determined and that (6.1) coincides with the Hamiltonian system generated by the fourth-order Birkhoff normal form Hamiltonian H_{ZD} computed by a formal expansion in [11, 12, 15, 16]; see Section 6.1. Such normal form is integrable and its corresponding Hamiltonian system preserves all Sobolev norms; see Theorem 1.4. The key new relevant information in Proposition 5.2 is that the quartic remainder in (5.11) satisfies energy estimates (see Lemma 6.4). This allows us to prove in Section 6.3 energy estimates for the whole system (5.10) and thus the long time existence result of Theorem 1.2.

6.1 The formal Birkhoff normal form

We introduce, as in formula (2.7) in [11], the complex symplectic variable

$$\begin{aligned}\begin{pmatrix} w \\ \overline{w} \end{pmatrix} &= \Lambda \begin{pmatrix} \eta \\ \psi \end{pmatrix} := \frac{1}{\sqrt{2}} \begin{pmatrix} |D|^{-\frac{1}{4}}\eta + \mathrm{i}|D|^{\frac{1}{4}}\psi \\ |D|^{-\frac{1}{4}}\eta - \mathrm{i}|D|^{\frac{1}{4}}\psi \end{pmatrix} \begin{pmatrix} \eta \\ \psi \end{pmatrix} \\ &= \Lambda^{-1} \begin{pmatrix} w \\ \overline{w} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} |D|^{\frac{1}{4}}(w+\overline{w}) \\ -\mathrm{i}|D|^{-\frac{1}{4}}(w-\overline{w}) \end{pmatrix}.\end{aligned} \tag{6.2}$$

Compare this formula with (1.12) and recall that, in view of (1.13), we may disregard the zero frequency in what follows. In the new complex variables $(w, \overline{w})$, a vector field $X(\eta, \psi)$ becomes

$$X^{\mathbb{C}} := \Lambda^{\star} X := \Lambda X \Lambda^{-1}. \tag{6.3}$$

The push-forward acts naturally on the commutator of nonlinear vector fields (A.32), namely $\Lambda^{\star}[\![X, Y]\!] = [\![\Lambda^{\star}X, \Lambda^{\star}Y]\!] = [\![X^{\mathbb{C}}, Y^{\mathbb{C}}]\!]$.

The Poisson bracket assumes the form

$$\{F, H\} = \frac{1}{\mathrm{i}} \sum_{k\in\mathbb{Z}\setminus\{0\}} \big(\partial_{w_k} H \partial_{\overline{w_k}} F - \partial_{\overline{w_k}} H \partial_{w_k} F\big).$$

Given a Hamiltonian $F(\eta, \psi)$ we denote by $F_{\mathbb{C}} := F \circ \Lambda^{-1}$ the same Hamiltonian expressed in terms of the complex variables $(w, \overline{w})$. The associated Hamiltonian vector field $X_{F_{\mathbb{C}}}$ is

$$X_{F_{\mathbb{C}}} = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}\setminus\{0\}} \begin{pmatrix} -\mathrm{i}\partial_{\overline{w_k}} F_{\mathbb{C}}\, e^{\mathrm{i}kx} \\ \mathrm{i}\partial_{w_k} F_{\mathbb{C}}\, e^{-\mathrm{i}kx} \end{pmatrix}, \tag{6.4}$$

which we also identify, using the standard vector field notation, with

$$X_{F_{\mathbb{C}}} = \sum_{k\in\mathbb{Z}\setminus\{0\},\sigma=\pm} -\mathrm{i}\sigma\partial_{w_k^{-\sigma}} F_{\mathbb{C}}\, \partial_{w_k^{\sigma}}. \tag{6.5}$$

Note that, if X_F is the Hamiltonian vector field of F in the real variables, then, using (6.3), we have

$$X_F^{\mathbb{C}} := \Lambda^{\star} X_F = X_{F_{\mathbb{C}}}, \quad F_{\mathbb{C}} := F \circ \Lambda^{-1}, \tag{6.6}$$

and

$$\llbracket X_H^{\mathbb{C}}, X_K^{\mathbb{C}} \rrbracket = X_{\{H,K\}}^{\mathbb{C}} = X_{\{H_{\mathbb{C}}, K_{\mathbb{C}}\}}. \tag{6.7}$$

We now describe the formal Birkhoff normal form procedure performed in [11, 12, 15, 16]. One first expands the water waves Hamiltonian (1.6), written in the complex variables $(w, \bar{w})$, in degrees of homogeneity

$$H_{\mathbb{C}} := H \circ \Lambda^{-1} = H_{\mathbb{C}}^{(2)} + H_{\mathbb{C}}^{(3)} + H_{\mathbb{C}}^{(4)} + H_{\mathbb{C}}^{(\geq 5)}, \tag{6.8}$$

where

$$H_{\mathbb{C}}^{(2)} = \sum_{j \in \mathbb{Z} \setminus \{0\}} \omega_j w_j \bar{w}_j, \quad \omega_j := \sqrt{|j|}, \tag{6.9}$$

$$H_{\mathbb{C}}^{(3)} = \sum_{\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 = 0} H_{j_1, j_2, j_3}^{\sigma_1, \sigma_2, \sigma_3} w_{j_1}^{\sigma_1} w_{j_2}^{\sigma_2} w_{j_3}^{\sigma_3}, \tag{6.10}$$

$$H_{\mathbb{C}}^{(4)} = \sum_{\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0} H_{j_1, j_2, j_3, j_4}^{\sigma_1, \sigma_2, \sigma_3, \sigma_4} w_{j_1}^{\sigma_1} w_{j_2}^{\sigma_2} w_{j_3}^{\sigma_3} w_{j_4}^{\sigma_4}, \tag{6.11}$$

can be explicitly computed, and $H_{\mathbb{C}}^{(\geq 5)}$ collects all the monomials of homogeneity greater or equal 5.

Step 1. Elimination of the cubic Hamiltonian. One looks for a symplectic transformation $\Phi^{(3)}$ as the (formal) time 1 flow generated by a cubic real Hamiltonian $F_{\mathbb{C}}^{(3)}$ of the form (6.10). A Lie expansion gives

$$\begin{aligned} H_{\mathbb{C}} \circ \Phi^{(3)} = {} & H_{\mathbb{C}}^{(2)} + \{F_{\mathbb{C}}^{(3)}, H_{\mathbb{C}}^{(2)}\} + H_{\mathbb{C}}^{(3)} \\ & + H_{\mathbb{C}}^{(4)} + \frac{1}{2}\{F_{\mathbb{C}}^{(3)}, \{F_{\mathbb{C}}^{(3)}, H_{\mathbb{C}}^{(2)}\}\} + \{F_{\mathbb{C}}^{(3)}, H_{\mathbb{C}}^{(3)}\} + \cdots \end{aligned} \tag{6.12}$$

up to terms of quintic degree. The cohomological equation

$$H_{\mathbb{C}}^{(3)} + \{F_{\mathbb{C}}^{(3)}, H_{\mathbb{C}}^{(2)}\} = 0 \tag{6.13}$$

has a unique solution since

$$\{w_{j_1}^{\sigma_1} w_{j_2}^{\sigma_2} w_{j_3}^{\sigma_3}, H_{\mathbb{C}}^{(2)}\} = \mathrm{i}(\sigma_1 \omega(j_1) + \sigma_2 \omega(j_2) + \sigma_3 \omega(j_3)) w_{j_1}^{\sigma_1} w_{j_2}^{\sigma_2} w_{j_3}^{\sigma_3},$$

and the system

$$\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 = 0, \quad \sigma_1 \omega(j_1) + \sigma_2 \omega(j_2) + \sigma_3 \omega(j_3) = 0, \tag{6.14}$$

has no integer solutions; see Proposition 5.3. Hence, defining $F_{\mathbb{C}}^{(3)}$ as the solution of (6.13), the Hamiltonian in (6.12) reduces to

$$H_{\mathbb{C}} \circ \Phi^{(3)} = H_{\mathbb{C}}^{(2)} + H_{\mathbb{C}}^{(4)} + \frac{1}{2}\{F_{\mathbb{C}}^{(3)}, H_{\mathbb{C}}^{(3)}\} + \text{quintic terms}.$$

Step 2. NORMALIZATION OF THE QUARTIC HAMILTONIAN. Similarly, one can find a symplectic transformation $\Phi^{(4)}$, defined as the (formal) time 1 flow generated by a real quartic Hamiltonian $F^{(4)}_{\mathbb{C}}$ of the form (6.11) such that

$$H_{\mathbb{C}}\circ\Phi^{(3)}\circ\Phi^{(4)} = H^{(2)}_{\mathbb{C}}+\Pi_{\ker}\Big(H^{(4)}_{\mathbb{C}}+\frac{1}{2}\{F^{(3)}_{\mathbb{C}},H^{(3)}_{\mathbb{C}}\}\Big)+\text{quintic terms,} \tag{6.15}$$

where, given a quartic monomial $w^{\sigma_1}_{j_1}w^{\sigma_2}_{j_2}w^{\sigma_3}_{j_3}w^{\sigma_4}_{j_4}$ satisfying $\sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3+\sigma_4 j_4 = 0$, we define

$$\Pi_{\ker}\Big(w^{\sigma_1}_{j_1}w^{\sigma_2}_{j_2}w^{\sigma_3}_{j_3}w^{\sigma_4}_{j_4}\Big) := \begin{cases} w^{\sigma_1}_{j_1}w^{\sigma_2}_{j_2}w^{\sigma_3}_{j_3}w^{\sigma_4}_{j_4} & \text{if } \sigma_1\omega(j_1)+\sigma_2\omega(j_2)+\sigma_3\omega(j_3)+\sigma_4\omega(j_4)=0\\ 0 & \text{otherwise.}\end{cases} \tag{6.16}$$

The fourth-order formal Birkhoff normal form Hamiltonian in (6.15), that is,

$$\begin{aligned} H_{ZD} &= H^{(2)}_{ZD}+H^{(4)}_{ZD}, \quad H^{(2)}_{ZD} := H^{(2)}_{\mathbb{C}},\\ H^{(4)}_{ZD} &:= \Pi_{\ker}\Big(H^{(4)}_{\mathbb{C}}+\frac{1}{2}\{F^{(3)}_{\mathbb{C}},H^{(3)}_{\mathbb{C}}\}\Big), \end{aligned} \tag{6.17}$$

has been computed explicitly in [11, 12, 15, 16], and it is completely integrable. In [12] this is expressed as

$$\begin{aligned} H_{ZD} &= \sum_{k>0}\Big(2\omega_k I_1(k) - \frac{k^3}{2\pi}(I_1^2(k)-3I_2^2(k))\Big)\\ &\quad + \frac{4}{\pi}\sum_{0<k<l} k^2 l I_2(k)I_2(l) \end{aligned} \tag{6.18}$$

with actions

$$I_1(k) := \frac{w_k\overline{w_k}+w_{-k}\overline{w_{-k}}}{2}, \quad I_2(k) := \frac{w_k\overline{w_k}-\overline{w_{-k}}w_{-k}}{2}, \tag{6.19}$$

where w_k denote the Fourier coefficients of w defined in (6.2). The Hamiltonian H_{ZD} is given by (1.17)–(1.18) with w_k replaced by z_k. Note in particular that $|z_n|^2$ are prime integrals, as stated in Theorem 1.4.

Remark 6.1 (Comparison with (6.1)). By a direct calculation, the Hamiltonian equations associated to $H_{ZD}(z,\overline{z})$ can be written in the form

$$\dot z_n = -\mathrm{i}\omega_n z_n - \frac{\mathrm{i}}{\pi}\Big(\sum_{|j|<\epsilon|n|} j|j||z_j|^2\Big)nz_n + [\mathfrak{R}(z)]_n \tag{6.20}$$

where $0<\epsilon<1$ and $\mathfrak{R}(z)$ is a smoothing vector field satisfying $\|\mathfrak{R}(z)\|_{s+\rho} \le C(s)\|z\|_s^3$, for any $0\le\rho\le 2s-3$. For a sequence $a=\{a_j\}_{j\in\mathbb{Z}}$ we denoted $\|a\|_s^2 := \sum_{j\in\mathbb{Z}}\langle j\rangle^{2s}|a_j|^2$. Note that the second term in the right-hand side in (6.20) exactly correspondence to the paradifferential transport in (5.10) and (4.5).

Moreover, (6.20) does not contain paradifferential operators at nonnegative orders, in agreement with the cubic Poincaré-Birkhoff normal form (6.1).

6.2 Normal form identification

In Sections 3–5 we have transformed the water waves system (1.3) into (5.10), whose cubic component (6.1) is in Poincaré-Birkhoff normal form. All the conjugation maps that we have used have an expansion in homogeneous components up to degree 4. In this section we identify the cubic monomials left in the Poincaré-Birkhoff normal form (6.1). The main result is the following.

PROPOSITION 6.2 (*Identification of normal forms*). *The cubic vector field component in* (5.10)*, i.e.,*

$$\mathcal{X}_{\mathrm{Res}}(Y) := -\mathrm{i}Op^{\mathrm{BW}}\big(\mathtt{D}(Y;\xi)\big)[Y] + \widetilde{\mathbb{R}}^{\mathrm{res}}(Y)[Y], \tag{6.21}$$

coincides with the Hamiltonian vector field $X_{H_{ZD}^{(4)}}$*:*

$$\mathcal{X}_{\mathrm{Res}} = X_{\Pi_{\ker}(H_{\mathbb{C}}^{(4)} + \frac{1}{2}\{F_{\mathbb{C}}^{(3)}, H_{\mathbb{C}}^{(3)}\})} = X_{H_{ZD}^{(4)}} \tag{6.22}$$

where the Hamiltonians $H_{\mathbb{C}}^{(l)}$*,* $l = 3, 4$*, are defined in* (6.10) *and* (6.11)*,* $F_{\mathbb{C}}^{(3)}$ *is the unique solution of* (6.13)*, and* $\Pi_{\ker}$ *is defined in* (6.16)*.*

The rest of the section is devoted to the proof of Proposition 6.2, which is based on a uniqueness argument for the Poincaré-Birkhoff normal form up to quartic remainders. The idea is the following. We first expand the water waves Hamiltonian vector field in (1.3), (1.5) in degrees of homogeneity

$$\begin{gathered} X_H = X_1 + X_2 + X_3 + X_{\geq 4} \\ \text{where } X_1 := X_{H^{(2)}},\ X_2 := X_{H^{(3)}},\ X_3 := X_{H^{(4)}}, \end{gathered} \tag{6.23}$$

and $X_{\geq 4}$ collects the higher-order terms and $H^{(p)} := H_{\mathbb{C}}^{(p)} \circ \Lambda$, $p = 2, 3, 4$, see (6.8). Then, we express the transformed system (5.10), obtained conjugating (1.3) via the good-unknown transformation $\mathcal{G}$ in (3.1) and $\mathbf{F}_T^1$ in Proposition 5.2, by a Lie commutator expansion up to terms of homogeneity at least 4. See Lemma A.10. Then, after some algebraic manipulation, we obtain (6.33). Since the adjoint operator $[\,\cdot, X_{H^{(2)}}^{\mathbb{C}}]$ is injective and surjective, we then obtain the identity (6.35), and can eventually deduce (6.22).

Notation. We use the Lie expansion (A.33) induced by a time-dependent vector field S, which contains quadratic and cubic terms. Given a homogeneous vector field X, we denote by $\Phi_S^{\star}X$ the induced (formal) push-forward

$$\Phi_S^{\star}X = X + [\![S, X]\!]_{|\theta=0} + \frac{1}{2}[\![S, [\![S, X]\!]]\!]_{|\theta=0} + \frac{1}{2}[\![\partial_\theta S_{|\theta=0}, X]\!] + \cdots \tag{6.24}$$

where $[\![\cdot, \cdot]\!]$ is the nonlinear commutator defined in (A.32).

Step 1. THE GOOD UNKNOWN CHANGE OF VARIABLE $\mathcal{G}$ IN (3.1)] We first provide the Lie expansion up to degree 4 of the vector field in (3.2)–(3.3), which is obtained by transforming the water waves vector field $X_1 + X_2 + X_3$ in (6.23) under the nonlinear map $\mathcal{G}$ in (3.1).

We first note that $\mathcal{G}(\eta, \psi) = (\Phi^\theta(\eta, \psi))_{\theta=1}$ where

$$\Phi^\theta(\eta, \psi) := (\eta, \psi - \theta Op^{\mathrm{BW}}(B(\eta, \psi))\eta), \ \theta \in [0, 1].$$

Since $B(\eta, \psi)$ is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}$ we have, using the remarks under Definition 2.7, that the map $\Phi^\theta(\eta, \psi)$ has the form (A.27) in which U denotes the real variables (η, ψ), plus a map in $\mathcal{M}_{K,0,3} \otimes \mathcal{M}_2(\mathbb{C})$. By Lemma A.9 we regard the inverse of the map $\mathcal{G}_{\le 3}$, obtained approximating $\mathcal{G}$ up to quartic remainders as the (formal) time one flow of a nonautonomous vector field of the form

$$S(\theta) := \mathtt{S}_2 + \theta \mathtt{S}_3 \quad \text{where } \mathtt{S}_2 := S_1(\eta, \psi)\begin{bmatrix}\eta \\ \psi\end{bmatrix}, \ \mathtt{S}_3 := S_2(\eta, \psi)\begin{bmatrix}\eta \\ \psi\end{bmatrix}, \tag{6.25}$$

where $S_1(\eta, \psi) \in \widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$ and $S_2(\eta, \psi) \in \widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$. By (6.23)–(6.25), we get

$$\Phi^\star_S(X_1 + X_2 + X_3) = X_1 + X_{2,1} + X_{3,1} + \cdots \tag{6.26}$$

$$X_{2,1} := X_2 + [\![\mathtt{S}_2, X_1]\!], \tag{6.27}$$

$$X_{3,1} := X_3 + [\![\mathtt{S}_2, X_2]\!] + \tfrac{1}{2}[\![\mathtt{S}_2, [\![\mathtt{S}_2, X_1]\!]]\!] + \tfrac{1}{2}[\![\mathtt{S}_3, X_1]\!]. \tag{6.28}$$

COMPLEX COORDINATES Λ IN (6.2). In the complex coordinates (6.2), the vector field (6.26) reads, recalling the notation (6.3),

$$\begin{aligned}\Lambda^\star\Phi^\star_S(X_1 + X_2 + X_3) &= \Lambda^\star X_1 + \Lambda^\star X_{2,1} + \Lambda^\star X_{3,1} + \cdots \\ &= X_1^{\mathbb{C}} + X_{2,1}^{\mathbb{C}} + X_{3,1}^{\mathbb{C}} + \cdots\end{aligned} \tag{6.29}$$

where $X_1^{\mathbb{C}}$ is the linear Hamiltonian vector field

$$X_1^{\mathbb{C}} = X^{\mathbb{C}}_{H^{(2)}} = -\mathrm{i}\sum_{j,\sigma} \sigma\omega_j u_j^\sigma \partial_{u_j^\sigma}.$$

Step 2. THE TRANSFORMATION $\mathbf{F}^1_T$ IN PROPOSITION 5.2. We consider the nonlinear map $(\mathbf{F}^1_T)_{\le 3}$ which retains only the terms of the map $\mathbf{F}^1_T := (\mathbf{F}^\theta_T)_{|\theta=1}$ up to quartic remainders. The approximate inverse of the map $(\mathbf{F}^1_T)_{\le 3}$ provided by Lemma A.8, can be regarded, by Lemma A.9, as the (formal) approximate time-one flow of a nonautonomous vector field $T(\theta) := \mathtt{T}_2 + \theta \mathtt{T}_3$ where $\mathtt{T}_2(U) := T_1(U)[U]$, $\mathtt{T}_3(U) := T_2(U)[U]$, for some $T_1(U)$ in $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$ and $T_2(U) \in \widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$. We transform the system obtained retaining only the components $X_1^{\mathbb{C}} + X_{2,1}^{\mathbb{C}} + X_{3,1}^{\mathbb{C}}$ in (6.29). By (6.24) we get

$$\Phi^\star_T\Lambda^\star\Phi^\star_S(X_1 + X_2 + X_3) = X_1^{\mathbb{C}} + X_{2,2}^{\mathbb{C}} + X_{3,2}^{\mathbb{C}} + \cdots \tag{6.30}$$

$$X^{\mathbb{C}}_{2,2} := X^{\mathbb{C}}_{2,1} + [\![\mathtt{T}_2, X^{\mathbb{C}}_1]\!],$$
$$X^{\mathbb{C}}_{3,2} := X^{\mathbb{C}}_{3,1} + [\![\mathtt{T}_2, X^{\mathbb{C}}_{2,1}]\!] + \tfrac{1}{2}[\![\mathtt{T}_2, [\![\mathtt{T}_2, X^{\mathbb{C}}_1]\!]]\!] + \tfrac{1}{2}[\![\mathtt{T}_3, X^{\mathbb{C}}_1]\!]$$

and, recalling the expressions of $X_{2,1}$, $X_{3,1}$ in (6.27), the quadratic and the cubic components of the vector field (6.30) are given by

$$X^{\mathbb{C}}_{2,2} = X^{\mathbb{C}}_2 + [\![\mathtt{S}^{\mathbb{C}}_2 + \mathtt{T}_2, X^{\mathbb{C}}_1]\!] \tag{6.31}$$

$$\begin{aligned} X^{\mathbb{C}}_{3,2} = X^{\mathbb{C}}_3 &+ [\![\mathtt{S}^{\mathbb{C}}_2 + \mathtt{T}_2, X^{\mathbb{C}}_2]\!] + \tfrac{1}{2}[\![\mathtt{S}^{\mathbb{C}}_2 + \mathtt{T}_2, [\![\mathtt{S}^{\mathbb{C}}_2 + \mathtt{T}_2, X^{\mathbb{C}}_1]\!]]\!] \\ &+ \tfrac{1}{2}[\![[\![\mathtt{S}^{\mathbb{C}}_2 + \mathtt{T}_2, \mathtt{S}^{\mathbb{C}}_2]\!] + \mathtt{S}^{\mathbb{C}}_3 + \mathtt{T}_3, X^{\mathbb{C}}_1]\!], \end{aligned} \tag{6.32}$$

where, to obtain the (6.32), we also used the Jacobi identity.

Step 3. IDENTIFICATION OF QUADRATIC AND CUBIC VECTOR FIELDS. The vector field $\Phi^{\star}_T \Lambda^{\star} \Phi^{\star}_S (X_1 + X_2 + X_3)$ in (6.30) is the vector field in the right-hand side of (6.1), up to quartic remainders. Thus, recalling the expression of the quadratic, resp. cubic, vector field in (6.31), resp. (6.32), the expansion (6.23), formula (6.6), and the definition of $\mathcal{X}_{\rm Res}$ in (6.21), we have the identification order by order:

$$X^{\mathbb{C}}_1(Y) = -\mathrm{i}\Omega Y, \quad X^{\mathbb{C}}_{H^{(3)}} + [\![\mathtt{S}^{\mathbb{C}}_2 + \mathtt{T}_2, X^{\mathbb{C}}_{H^{(2)}}]\!] = 0, \quad X^{\mathbb{C}}_{3,2} = \mathcal{X}_{\rm Res}. \tag{6.33}$$

QUADRATIC VECTOR FIELDS. Since $F^{(3)}_{\mathbb{C}}$ solves (6.13), by (6.7), we have

$$X^{\mathbb{C}}_{H^{(3)}} + [\![X_{F^{(3)}_{\mathbb{C}}}, X_{H^{(2)}_{\mathbb{C}}}]\!] = 0. \tag{6.34}$$

Subtracting the second identity in (6.33) and (6.34), and since $X_{H^{(2)}_{\mathbb{C}}} = X^{\mathbb{C}}_{H^{(2)}}$, we deduce

$$[\![\mathtt{S}^{\mathbb{C}}_2 + \mathtt{T}_2 - X_{F^{(3)}_{\mathbb{C}}}, X^{\mathbb{C}}_{H^{(2)}}]\!] = 0.$$

Since the adjoint operator $\mathrm{Ad}_{X^{\mathbb{C}}_{H^{(2)}}} := [\cdot, X^{\mathbb{C}}_{H^{(2)}}]$ acting on quadratic monomial vector fields $u^{\sigma_1}_{j_1} u^{\sigma_2}_{j_2} \partial_{u^{\sigma}_j}$ satisfying the momentum conservation property $\sigma j = \sigma_1 j_1 + \sigma_2 j_2$ is injective and surjective (indeed we have that $[\![u^{\sigma_1}_{j_1} u^{\sigma_2}_{j_2} \partial_{u^{\sigma}_j}, X^{\mathbb{C}}_{H^{(2)}}]\!] = -\mathrm{i}(\sigma\omega(j) - \sigma_1\omega(j_1) - \sigma_2\omega(j_2)) u^{\sigma_1}_{j_1} u^{\sigma_2}_{j_2} \partial_{u^{\sigma}_j}$ and the system (6.14) has no solutions), we obtain

$$\mathtt{S}^{\mathbb{C}}_2 + \mathtt{T}_2 = X_{F^{(3)}_{\mathbb{C}}}. \tag{6.35}$$

CUBIC VECTOR FIELDS. The vector field $\mathcal{X}_{\rm Res} r$ defined in (6.21) is in Poincaré-Birkhoff normal form, since the symbol $\mathtt{D}(Y;\xi)$ is *integrable* (Definition 4.1) and

$\widetilde{\mathtt{R}}^{\mathrm{res}}(U)$ is *Birkhoff resonant* (Definition 5.1). Therefore, defining the linear operator $\Pi_{\ker}$ acting on a cubic monomial vector field $u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}u_{j_3}^{\sigma_3}\partial_{u_j^\sigma}$ as

$$\Pi_{\ker}\Big(u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}u_{j_3}^{\sigma_3}\partial_{u_j^\sigma}\Big) := \tag{6.36}$$
$$\begin{cases} u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}u_{j_3}^{\sigma_3}\partial_{u_j^\sigma} & \text{if } -\sigma\omega(j)+\sigma_1\omega(j_1)+\sigma_2\omega(j_2)+\sigma_3\omega(j_3)=0,\\ 0 & \text{otherwise,}\end{cases}$$

we have

$$\Pi_{\ker}(\mathcal{X}_{\mathrm{Res}}) = \mathcal{X}_{\mathrm{Res}}. \tag{6.37}$$

From the expression for $[\![u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}u_{j_3}^{\sigma_3}\partial_{u_j^\sigma}, X^{\mathbb{C}}_{H^{(2)}}]\!]$ we deduce that, for any cubic vector field G_3,

$$\Pi_{\ker}[\![G_3, X^{\mathbb{C}}_{H^{(2)}}]\!] = 0. \tag{6.38}$$

We can then calculate

$$\begin{aligned}
\mathcal{X}_{\mathrm{Res}} &\overset{(6.37)}{=} \Pi_{\ker}(\mathcal{X}_{\mathrm{Res}})\\
&\overset{(6.33),(6.32),(6.38)}{=} \Pi_{\ker}\Big(X^{\mathbb{C}}_{H^{(4)}} + [\![\mathtt{S}_2^{\mathbb{C}}+\mathtt{T}_2, X_2^{\mathbb{C}}]\!] + \frac{1}{2}[\![\mathtt{S}_2^{\mathbb{C}}+\mathtt{T}_2, [\![\mathtt{S}_2^{\mathbb{C}}+\mathtt{T}_2, X_1^{\mathbb{C}}]\!]]\!]\Big)\\
&\overset{(6.35),(6.23)}{=} \Pi_{\ker}\Big(X^{\mathbb{C}}_{H^{(4)}} + [\![X_{F^{(3)}_{\mathbb{C}}}, X^{\mathbb{C}}_{H^{(3)}}]\!] + \frac{1}{2}[\![X_{F^{(3)}_{\mathbb{C}}}, [\![X_{F^{(3)}_{\mathbb{C}}}, X_{H^{(2)}_{\mathbb{C}}}]\!]]\!]\Big)\\
&\overset{(6.6),(6.7),(6.13)}{=} \Pi_{\ker}\big(X_{H^{(4)}_{\mathbb{C}}+\frac12\{F^{(3)}_{\mathbb{C}},H^{(3)}_{\mathbb{C}}\}}\big) \overset{(6.36),(6.16)}{=} X_{\Pi_{\ker}(H^{(4)}_{\mathbb{C}}+\frac12\{F^{(3)}_{\mathbb{C}},H^{(3)}_{\mathbb{C}}\})},
\end{aligned}$$

which is (6.22); the second identity follows by the definition of $H^{(4)}_{ZD}$ in (6.17).

6.3 Energy estimate and proof of Theorem 1.1

We first prove the following lemma.

LEMMA 6.3. *Let $K \in \mathbb{N}^*$. There is $s_0 > 0$ such that, for any $s \geq s_0$, for all $0 < r \leq r_0(s)$ small enough, if U belongs to $B^K_s(I;r)$ and solves* (3.7)*, then there is a constant $C_{s,K} > 0$ such that*

$$\big\|\partial_t^k U(t,\cdot)\big\|_{\dot H^{s-k}} \leq C_{s,K}\|U(t,\cdot)\|_{\dot H^s} \quad \forall\, 0 \leq k \leq K. \tag{6.39}$$

In particular, we deduce that the norm $\|U(t,\cdot)\|_{K,s}$ defined in (2.1) *is equivalent to the norm $\|U(t,\cdot)\|_{\dot H^s}$ for $U(t,\cdot)$ a solution of* (3.7)*.*

PROOF. For $k = 0$ the estimate (6.39) is trivial. We are going to estimate $\partial_t^k U$ by (3.7). Since the matrix of symbols $\mathrm{i}A_1(U;x)\xi + \mathrm{i}A_{1/2}(U;x)|\xi|^{1/2} + A_0(U;x,\xi) + A_{-1/2}(U;x,\xi)$ in (3.7) belongs to $\Sigma\Gamma^1_{K,1,0}\otimes\mathcal{M}_2(\mathbb{C})$ and the smoothing operator $R(U)$ is in $\Sigma\mathcal{R}^{-\rho}_{K,1,1}\otimes\mathcal{M}_2(\mathbb{C})$, applying Proposition 2.6(ii) (with $K' = 1, k = 0$), the estimate (2.20) for $R(U)$ (with $K' = 1, k = 0, N = 1$), and recalling (2.1), we deduce, for $s \geq s_0$ large enough,

$$\begin{aligned} \|\partial_t U(t,\cdot)\|_{\dot{H}^{s-1}} \lesssim_s \|U(t,\cdot)\|_{\dot{H}^s}\big(1 + \|U(t,\cdot)\|_{\dot{H}^{s_0}} + \|\partial_t U(t,\cdot)\|_{\dot{H}^{s_0-1}}\big) \\ + \|\partial_t U(t,\cdot)\|_{\dot{H}^{s-1}}\|U(t,\cdot)\|_{\dot{H}^{s_0}}. \end{aligned} \tag{6.40}$$

Evaluating (6.40) at $s = s_0$ and since $\|U(t,\cdot)\|_{\dot{H}^{s_0}}$ is small, we get

$$\|\partial_t U(t,\cdot)\|_{\dot{H}^{s_0-1}} \lesssim_s \|U(t,\cdot)\|_{\dot{H}^{s_0}}.$$

This and (6.40) imply (6.39) for $k = 1$, for any $s \geq s_0$. Differentiating in t the system (3.7) and arguing by induction on k, one can similarly obtain (6.39) for any $k \geq 2$. □

We now prove the following energy estimate.

LEMMA 6.4 (*Energy estimate*). *Under the same assumptions as Proposition* 5.2 *the vector field* $\mathcal{X}_{\geq 4}(U,Y) = [\mathcal{X}^+_{\geq 4}(U,Y), \overline{\mathcal{X}^+_{\geq 4}(U,Y)}]$ *in* (5.11) *satisfies, for any* $t \in I$, *the energy estimate*

$$\mathrm{Re}\int_{\mathbb{T}} |D|^s \mathcal{X}^+_{\geq 4}(U,Y)\cdot \overline{|D|^s y}\, dx \lesssim_s \|y\|^5_{\dot{H}^s}. \tag{6.41}$$

PROOF. By (5.11) and (4.3), we have that

$$\mathcal{X}^+_{\geq 4}(U,Y) = Op^{\mathrm{BW}}(H_{\geq 3})[y] + \mathfrak{R}^+_{\geq 3}(U)[Y],$$

where $H_{\geq 3}$ is an admissible symbol as in (4.2) that we write

$$\begin{aligned} H_{\geq 3} &= h^+_{\geq 3}(U;x,\xi) + \gamma_{\geq 3}(U;x,\xi), \\ h^+_{\geq 3}(U;x,\xi) &:= \mathrm{i}\alpha_{\geq 3}(U;x)\xi + \mathrm{i}\beta_{\geq 3}(U;x)|\xi|^{\frac{1}{2}}, \end{aligned} \tag{6.42}$$

and $\mathfrak{R}^+_{\geq 3}(U)$ denotes the first row of $\mathfrak{R}_{\geq 3}$. Then the left-hand side of (6.41) is equal to

$$\frac{1}{2}\big(|D|^s y, |D|^s Op^{\mathrm{BW}}(h^+_{\geq 3})[y]\big)_{L^2} \tag{6.43}$$

$$+ \frac{1}{2}\big(|D|^s Op^{\mathrm{BW}}(h^+_{\geq 3})[y], |D|^s y\big)_{L^2}$$

$$+ \mathrm{Re}\int_{\mathbb{T}} |D|^s Op^{\mathrm{BW}}(\gamma_{\geq 3})[y]\cdot\overline{|D|^s y}\, dx \tag{6.44}$$

$$+ \mathrm{Re}\int_{\mathbb{T}} |D|^s \mathfrak{R}^+_{\geq 3}(U)[Y]\cdot\overline{|D|^s y}\, dx.$$

Since $\gamma_{\geq 3} \in \Gamma^0_{K,K',3}$ and $\mathfrak{R}^+_{\geq 3}(U)$ is a 1×2 matrix of smoothing operators in $\mathcal{R}^0_{K,K',3}$, the Cauchy-Schwarz inequality, Proposition 2.6, and (2.20) imply that

$$|(6.44)| \lesssim_s \|y(t,\cdot)\|^2_{\dot{H}^s}\|U(t,\cdot)\|^3_{K,s}. \tag{6.45}$$

Since the symbol $h^+_{\geq 3}$ has positive order we write the quantity in (6.43) as

$$\begin{aligned}(6.46)\qquad &\tfrac{1}{2}\big(|D|^s y, |D|^s(\mathcal{H}_{\geq 3} + \mathcal{H}^*_{\geq 3})y\big)_{L^2} + \tfrac{1}{2}\big(|D|^s y, [\mathcal{H}^*_{\geq 3}, |D|^s]y\big)_{L^2}\\ &+ \tfrac{1}{2}\big([|D|^s, \mathcal{H}_{\geq 3}]y, |D|^s y\big)_{L^2},\end{aligned}$$

where $\mathcal{H}_{\geq 3} := Op^{\mathrm{BW}}(h^+_{\geq 3}(U; x, \xi))$ and $\mathcal{H}^*_{\geq 3} = Op^{\mathrm{BW}}(\overline{h^+_{\geq 3}(U; x, \xi)})$ is its adjoint with respect to the L^2-scalar product. Recalling (6.42) and that the functions $\alpha_{\geq 3}(U; x)$, $\beta_{\geq 3}(U; x)$ are real, we have

$$(6.47)\qquad \mathcal{H}_{\geq 3} + \mathcal{H}^*_{\geq 3} = Op^{\mathrm{BW}}\big(h^+_{\geq 3} + \overline{h^+_{\geq 3}}\big) = 0.$$

Furthermore, by Proposition 2.9 and the remark after the proof of proposition 3.12 in [6], the commutators $[\mathcal{H}^*_{\geq 3}, |D|^s]$, $[|D|^s, \mathcal{H}_{\geq 3}]$ are paradifferential operators with symbol in $\Gamma^s_{K,K',3}$, up to a bounded operator in $\mathcal{L}(\dot H^s, \dot H^0)$ with operator norm bounded by $\|U\|^3_{K,s_0}$. Then, applying Proposition 2.6 we get

$$\begin{aligned}(6.48)\qquad &\big|\big(|D|^s y, \big[\mathcal{H}^*_{\geq 3}, |D|^s\big]y\big)_{L^2}\big| + \big|\big(\big[|D|^s, \mathcal{H}_{\geq 3}\big]y, |D|^s y\big)_{L^2}\big|\\ &\lesssim_s \|y(t,\cdot)\|^2_{\dot H^s}\|U(t,\cdot)\|^3_{K,s}.\end{aligned}$$

In conclusion, by (6.45)–(6.48), and using Lemma 6.3 and by (5.13), we deduce

$$\mathrm{Re}\int_{\mathbb{T}} |D|^s \mathcal{X}^+_{\geq 4}(U, Y)\cdot \overline{|D|^s y}\, dx \lesssim_s \|y(t,\cdot)\|^2_{\dot H^s}\|U(t,\cdot)\|^3_{\dot H^s} \lesssim_s \|y(t,\cdot)\|^5_{\dot H^s}$$

proving the estimate (6.41). □

We can now prove Theorem 1.1.

Proof of Theorem 1.1. By (1.14), the function $U = [\begin{smallmatrix} u \\ \bar u\end{smallmatrix}]$, where u is the variable defined in (1.12) and ω in (1.8), belongs to the ball $B^K_N(I; r)$ (recall (2.2)) with $r = \bar\varepsilon \ll 1$ and $I = [-T, T]$. By Proposition 3.3 the function U solves system (3.7). Then we apply the Poincaré-Birkhoff proposition 5.2 with $s \rightsquigarrow N \gg K \geq 2\rho + 2 \geq 2\rho_0 + 2$. The map $\mathbf{F}^1_T(U) = \mathfrak{C}^1(U)[U]$ in (5.9) transforms the water waves system (3.7) into (5.10), which, thanks to Proposition 6.2, is expressed in terms of the Dyachenko-Zakharov Hamiltonian H_{ZD} in (1.17), as $\partial_t Y = X_{H_{ZD}}(Y) + \mathcal{X}_{\geq 4}(U, Y)$. Renaming $y \rightsquigarrow z$ and recalling (6.4), the first component of the above system is the equation (1.16), denoting $\mathfrak{B}(u)u$ the first component of $\mathfrak{C}^1(U)[U]$. The bound (1.15) follows by (5.12) with $s \rightsquigarrow N$ and $k = 0$, and Lemma 6.3. The energy estimate (1.19) is proved in Lemma 6.4. □

6.4 Proof of Theorem 1.2

The next bootstrap Proposition 6.5 is the main ingredient for the proof of the long time existence Theorem 1.2. Proposition 6.5 is a consequence of Theorem 1.1 and the integrability of the fourth-order Hamiltonian $H^{(4)}_{ZD}$ in (1.18). By time reversibility we may, without loss of generality, look only at positive times $t > 0$.

PROPOSITION 6.5 (*Main bootstrap*). *Fix the constants* $\overline{\varepsilon}, K, N$ *as in Theorem* 1.1 *and let the function* $u \in C^0([0,T]; H^N)$ *be defined as in* (1.12), *with* ω *in* (1.8) *and* (η,ψ) *the solution of* (1.3) *satisfying* (1.9), (1.10). *The function* u *satisfies* (1.13). *Then there exists* $c_0 > 0$ *such that, for any* $0 < \varepsilon_1 \le \overline{\varepsilon}$, *if*

$$\|u(0)\|_{H^N} \le c_0\varepsilon_1, \quad \sup_{t\in[0,T]} \sum_{k=0}^{K} \big\|\partial_t^k u(t)\big\|_{H^{N-k}} \le \varepsilon_1, \quad T \le c_0\varepsilon_1^{-3}, \tag{6.49}$$

then we have the improved bound

$$\sup_{t\in[0,T]} \sum_{k=0}^{K} \big\|\partial_t^k u(t)\big\|_{H^{N-k}} \le \frac{\varepsilon_1}{2}. \tag{6.50}$$

PROOF. In view of (6.49) the smallness condition (1.14) holds and we can apply Theorem 1.1 obtaining the new variable $z = \mathfrak{B}(u)u$ satisfying the equation (1.16)–(1.19). The integrability of $H^{(4)}_{ZD}$ in Theorem 1.4 gives

$$\mathrm{Re}\int_{\mathbb{T}} |D|^N \big(\mathrm{i}\partial_{\overline{z}} H^{(4)}_{ZD}\big)\cdot \overline{|D|^N z}\, dx = 0.$$

From this, (1.16), and (1.19) we obtain the energy estimate

$$\frac{d}{dt}\|z(t)\|^2_{\dot H^N} \lesssim_N \|z(t)\|^5_{\dot H^N}. \tag{6.51}$$

Using (1.15) and (6.49) we deduce that, for all $0 \le t \le T$,

$$\begin{aligned}\|u(t)\|^2_{\dot H^N} \lesssim_N \|z(t)\|^2_{\dot H^N} &\overset{(6.51)}{\lesssim_N} \|z(0)\|^2_{\dot H^N} + \int_0^t \|z(\tau)\|^5_{\dot H^N}\, d\tau \\ &\le C\|u(0)\|^2_{\dot H^N} + C\int_0^t \|u(\tau)\|^5_{\dot H^N}\, d\tau\end{aligned}$$

for some $C = C(N) > 0$. Then, by the a priori assumption (6.49) we get, for all $0 \le t \le T \le c_0\varepsilon_1^{-3}$,

$$\|u(t)\|^2_{\dot H^N} \le Cc_0^2\varepsilon_1^2 + CT\varepsilon_1^5 \le \varepsilon_1^2\big(Cc_0^2 + Cc_0\big). \tag{6.52}$$

The desired conclusion (6.50) on the norms $C_t^k H_x^{N-k}$ follows by Lemma 6.3, (6.52), and recalling that $\int_{\mathbb{T}} u(t,x)dx = 0$, choosing c_0 small enough depending on N. □

We now prove the long time existence Theorem 1.2, by Theorem 1.1 and Proposition 6.5.

Step 1. LOCAL EXISTENCE AND PRELIMINARY ESTIMATES. Let $s > 3/2$. By the assumption (1.20), Theorem 1.3 guarantees the existence of a time $T_{\mathrm{loc}} > 0$ and a unique classical solution $(\eta,\psi)\in C^0([0,T_{\mathrm{loc}}];H^{s+(1/2)}\times H^{s+(1/2)})$ of (1.3), with initial data as in (1.20) such that

$$\sup_{t\in[0,T_{\mathrm{loc}}]}\|(\eta,\psi,V,B)(t)\|_{X^s}\le C\varepsilon,\quad \int_{\mathbb{T}}\eta(t,x)dx=0. \tag{6.53}$$

We now show that for any $K>0$, if $s\ge K+\sigma_0$ for some σ_0 large enough, and ε is small enough, then the time derivatives $(\partial_t^k\eta,\partial_t^k\psi)$, $k=0,\dots,K$, satisfy, for all $t\in[0,T_{\mathrm{loc}}]$,

$$\big\|\partial_t^k\eta\big\|_{H^{s+\frac12-k}}+\big\|\partial_t^k\psi\big\|_{H^{s+\frac12-k}}\lesssim_s\|\eta\|_{H^{s+\frac12}}+\|\psi\|_{H^{s+\frac12}}\lesssim_s\varepsilon. \tag{6.54}$$

One argues by induction on k. For $k=0$ the second estimate in (6.54) is (6.53). Assume that (6.54) holds for any $0\le j\le k-1\le K-1$, $k\ge1$. By differentiating in t the water waves system (1.3) we get, for any $k=1,\dots,K$,

$$\partial_t^k\eta=\partial_t^{k-1}(G(\eta)\psi),\quad \partial_t^k\psi=\partial_t^{k-1}\big(\mathcal{F}(\eta,\eta_x,\psi_x,G(\eta)\psi)\big), \tag{6.55}$$

where $\mathcal{F}$ is an analytic function vanishing at the origin. Then, using that $G(\eta)\psi$ is expressed from the side of (3.2), Proposition 2.6, (2.20), and the inductive hypothesis, we get

$$\begin{aligned}\big\|\partial_t^{k-1}(G(\eta)\psi)\big\|_{H^{s+\frac12-k}}&\lesssim_s\sum_{k'\le k-1}\big\|\partial_t^{k'}\psi\big\|_{H^{s+\frac12-k+1}}+\big\|\partial_t^{k'}\eta\big\|_{H^{s+\frac12-k+1}}\\&\lesssim_s\|\eta\|_{H^{s+\frac12}}+\|\psi\|_{H^{s+\frac12}}.\end{aligned}$$

This implies, in view of the first equation in (6.55), that $\partial_t^k\eta$ is bounded as in (6.54). To estimate $\partial_t^k\psi$ we use the second equation in (6.55), the inductive estimates for $(\partial_t^j\eta,\partial_t^j\psi)$, $0\le j\le k-1$, and the previous bound on $\|\partial_t^{k-1}(G(\eta)\psi)\|_{H^{s+1/2-k}}$.

Step 2. A PRIORI ESTIMATE FOR THE BASIC DIAGONAL COMPLEX VARIABLE. We now look at the complex variable u defined in (1.12) and (1.8). Since the function B is in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}$ (Proposition 3.1), we deduce, applying Proposition 2.6 for $s\ge s_0$ large enough, that $\omega\in C^0([0,T_{\mathrm{loc}}];\dot H^{s+1/2})$, and so $u\in C^0([0,T_{\mathrm{loc}}];H^N)$ with $N=s+1/4$. Moreover, using (1.20), (6.53)–(6.54), we estimate, for $k=0,\dots,K$, $N\gg K$,

$$\big\|\partial_t^k u\big\|_{H^{N-k}}\overset{(1.12),(1.8),\text{Prop.2.6}}{\lesssim_N}\big\|\partial_t^k\eta\big\|_{H^{N+\frac14-k}}+\big\|\partial_t^k\psi\big\|_{H^{N+\frac14-k}}\overset{(6.54)}{\lesssim_N}\varepsilon,$$

for any $t\in[0,T_{\mathrm{loc}}]$. In conclusion, there is $C_1=C_1(N)>0$ such that

$$\|u(0)\|_{H^N}\le2\varepsilon,\quad \sup_{t\in[0,T_{\mathrm{loc}}]}\sum_{k=0}^K\|\partial_t^k u(t)\|_{H^{N-k}}\le C_1\varepsilon,\quad \int_{\mathbb{T}}u(t,x)dx=0. \tag{6.56}$$

Step 3. BOOTSTRAP ARGUMENT AND CONTINUATION CRITERION. With $\overline{\varepsilon}$, K,N given by Theorem 1.1, and c_0 by Proposition 6.5, we choose ε_0 in (1.20) small enough so that, for $0<\varepsilon\le\varepsilon_0$, we have $2\varepsilon\le c_0\overline{\varepsilon}$, $C_1\varepsilon\le\overline{\varepsilon}$ where C_1 is the constant in (6.56). Moreover, we take $s \geq s_0$ large enough in such a way that (6.56) hold with N given by Theorem 1.1. Hence the first two assumptions in (6.49) hold with $\varepsilon_1 = \varepsilon \max\{2c_0^{-1}, C_1\}$ on the time interval $[0, T_{\mathrm{loc}}]$. Then Proposition 6.5 and a standard bootstrap argument guarantees that $u(t)$ can be extended up to a time $T_\varepsilon := c_0 \varepsilon_1^{-3}$, consistently with (1.21), and that

$$\sup_{[0,T_\varepsilon]} \|u(t)\|_{H^N} \leq \varepsilon_1, \quad \int_{\mathbb{T}} u(t,x)dx = 0. \tag{6.57}$$

Finally, we prove that the solution of (1.3) satisfies (1.22) and that $(\eta, \psi, V, B)(t)$ takes values in X^s for all $t \in [0, T_\varepsilon]$. Expressing (η, ω) in terms of $u, \overline{u}$ as in (3.21), we deduce by (6.57) that

$$\sup_{[0,T_\varepsilon]} \big(\|\eta(t)\|_{H^s} + \|\omega(t)\|_{H^{s+1/2}} \big) \lesssim_s \varepsilon. \tag{6.58}$$

Then, by (1.8), (6.58), and Proposition 2.6, using (3.2) for $G(\eta)\psi$, we estimate

$$\sup_{[0,T_\varepsilon]} \|\psi(t)\|_{H^s} + \sup_{[0,T_\varepsilon]} \|(V,B)(t)\|_{H^{s-1}\times H^{s-1}} \lesssim_s \varepsilon.$$

The estimates above imply (1.22) and, in particular, that

$$\sup_{[0,T_\varepsilon]} \|(\eta,\psi,V,B)(t)\|_{X^{s-1}} \lesssim_s \varepsilon,$$

thus guaranteeing (1.23), for $s - 1 \geq 5$, on the time interval $[0, T_\varepsilon]$. The continuation criterion in Theorem 1.3 implies that the solution (η, ψ, V, B) is in $C^0([0,T], X^s)$ for $T \geq T_\varepsilon$.

Appendix: Flows and Conjugations

In this appendix we study the conjugation rules of a vector field under flow maps.

A.1 Conjugation rules

We first give this simple lemma that we use in sections 3.2 and 5.

LEMMA A.1. *For* $U = \left[\begin{smallmatrix} u \\ \overline{u} \end{smallmatrix}\right]$ *consider a system* $\partial_t U = X(U)U$ *with* $X(U)$ *in* $\Sigma\mathcal{M}_{K,K',0} \otimes \mathcal{M}_2(\mathbb{C})$ *and let* $\Phi^\theta(U)$ *be the flow of*

$$\partial_\theta \Phi^\theta(U) = \mathrm{i}\mathbf{A}(U)\Phi^\theta(U), \quad \Phi^0(U) = \mathrm{Id}, \tag{A.1}$$

where $\mathbf{A} := \mathbf{A}(U)$ *is in* $\Sigma\mathcal{R}^0_{K,K',1} \otimes \mathcal{M}_2(\mathbb{C})$. *Under the change of variable* $V := (\Phi^\theta(U))_{\theta=1}U$, *the new system becomes*

$$\partial_t V = X^+(U)V, \quad X^+(U) := (\partial_t \Phi^1(U))(\Phi^1(U))^{-1} + \Phi^1(U)X(U)(\Phi^1(U))^{-1}. \tag{A.2}$$

The operator $X^+(U)$ *is in* $\Sigma\mathcal{M}_{K,K'+1,0}\otimes\mathcal{M}_2(\mathbb{C})$ *and, setting* $\mathrm{Ad}_{\mathrm{i}\mathbf{A}}[X]:=[\mathrm{i}\mathbf{A},X]$*, it admits the Lie expansion*

$$\Phi^1(U)X(U)(\Phi^1(U))^{-1}=X+\sum_{q=1}^{L}\frac{1}{q!}\mathrm{Ad}^q_{\mathrm{i}\mathbf{A}}[X] \\ +\frac{1}{L!}\int_0^1(1-\theta)^L\Phi^\theta(U)\mathrm{Ad}^{L+1}_{\mathrm{i}\mathbf{A}}[X](\Phi^\theta(U))^{-1}\,d\theta, \tag{A.3}$$

$$(\partial_t\Phi^1(U))(\Phi^1(U))^{-1}=\sum_{q=1}^{L}\frac{1}{q!}\mathrm{Ad}^{q-1}_{\mathrm{i}\mathbf{A}}[\mathrm{i}\partial_t\mathbf{A}] \\ +\frac{1}{L!}\int_0^1(1-\theta)^L\Phi^\theta(U)\mathrm{Ad}^{L}_{\mathrm{i}\mathbf{A}}[\mathrm{i}\partial_t\mathbf{A}](\Phi^\theta(U))^{-1}\,d\theta. \tag{A.4}$$

PROOF. The expression (A.2) follows by an explicit computation. In order to prove (A.3) note that the vector field $P(\theta):=\Phi^\theta(U)X(U)(\Phi^\theta(U))^{-1}$ satisfies the Heisenberg equation $\partial_\theta P(\theta)=[\mathrm{i}\mathbf{A},P(\theta)]$ with $P(0)=X(U)$. We also have $\partial_\theta P(\theta)=\Phi^\theta(U)\mathrm{Ad}_{\mathrm{i}\mathbf{A}}[X]\Phi^\theta(U)^{-1}$. Then (A.3) follows by a Taylor expansion. To prove (A.4) we reason as follows. We have that

$$\begin{aligned}\Phi^1(U)\circ\partial_t\circ(\Phi^1(U))^{-1}&=\partial_t+\Phi^1(U)\big[\partial_t(\Phi^1(U))^{-1}\big]\\&=\partial_t-\big(\partial_t\Phi^1(U)\big)(\Phi^1(U))^{-1}.\end{aligned} \tag{A.5}$$

Using the expansion (A.3) with $X\rightsquigarrow\partial_t$ and (A.5) we get (A.4). By Taylor-expanding $\Phi^1(U)$ using (A.1), we derive that $\Phi^1(U)-\mathrm{Id}$ is in $\Sigma\mathcal{M}_{K,K',1}\otimes\mathcal{M}_2(\mathbb{C})$. The translation invariance property (2.19) of the homogeneous components of $\Phi^1(U)$ follows since the generator $\mathbf{A}(U)$ satisfies (2.19). Then, the operator $X^+(U)$ in (A.2) belongs to $\Sigma\mathcal{M}_{K,K'+1,0}\otimes\mathcal{M}_2(\mathbb{C})$ by Proposition 2.10 and the remarks after Definition 2.7. Let us justify the translation invariance property of the homogeneous components of $X^+(U)$. Denoting by $\Phi^1_{\le 2}(U)$ the sum of its homogeneous components of degree less than or equal to 2 we get, for any $\vartheta\in\mathbb{R}$, $\tau_\vartheta\Phi^1_{\le2}(U)=\Phi^1_{\le2}(\tau_\vartheta U)\tau_\vartheta$, and so $\tau_\vartheta d_U\Phi^1_{\le2}(U)[\widehat{H}]=d_U\Phi^1_{\le2}(\tau_\vartheta U)[\tau_\vartheta\widehat{H}]\tau_\vartheta$. Then we deduce $\tau_\vartheta(\partial_t\Phi^1_{\le2}(U))=(\partial_t\Phi^1_{\le2}(\tau_\vartheta U))\tau_\vartheta$ using the translation invariance of $X(U)U$. By composition we deduce that the homogeneous components of $X^+(U)$ in (A.2) satisfy (2.19). □

In the next subsection we analyze how paradifferential operators change under the flow maps generated by paradifferential operators.

A.2 Conjugation of paradifferential operators via flows

We consider the flow equation

$$\partial_\theta\Phi^\theta=\mathrm{i}Op^{\mathrm{BW}}(f(\theta,U;x,\xi))\Phi^\theta,\quad \Phi^0=\mathrm{Id}, \tag{A.6}$$

where f is a symbol assuming one of the following forms:

$$f(\theta,U;x,\xi):=b(\theta,U;x)\xi:=\frac{\beta(U;x)}{1+\theta\beta_x(U;x)}\xi,\quad \beta(U;x)\in\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',1},\tag{A.7}$$

$$f(\theta,U;x,\xi):=f(U;x,\xi):=\beta(U;x)|\xi|^{\frac{1}{2}},\quad \beta(U;x)\in\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',1},\tag{A.8}$$

$$f(\theta,U;x,\xi):=f(U;x,\xi)\in\Sigma\Gamma^{m}_{K,K',1},\quad m\le 0.\tag{A.9}$$

Note that (A.6) with f as in (A.7) is a paradifferential transport equation. This is used in Section 4.1 and Section 4.2. Flows with f as in (A.8) are used in Section 4.2 and with f as in (A.9) in Section 4.2 and Section 4.3.

LEMMA A.2 (Linear flows generated by a paradifferential operator). *Assume that f has the form* (A.7) *or* (A.8) *or* (A.9). *Then, there is $s_0>0$ and $r>0$ such that, for any $U\in C^K_{*\mathbb{R}}(I;\dot H^s)\cap B^K_{s_0}(I;r)$, for any $s>0$, the equation* (A.6) *has a unique solution $\Phi^\theta(U)$ satisfying the following*:

(i) *the linear map $\Phi^\theta(U)$ is invertible and, for some $C_s>0$, $\forall\, 0\le k\le K-K'$,*

$$\begin{aligned}&\|\partial_t^k\Phi^\theta(U)[v]\|_{\dot H^{s-k}}+\|\partial_t^k(\Phi^\theta(U))^{-1}[v]\|_{\dot H^{s-k}}\\&\quad\le\|v\|_{k,s}\big(1+C_s\|U\|_{K,s_0}\big),\end{aligned}\tag{A.10}$$

$$C_s^{-1}\|v\|_{\dot H^s}\le\|\Phi^\theta(U)[v]\|_{\dot H^s}\le C_s\|v\|_{\dot H^s},\tag{A.11}$$

for any $v\in C^{K-K'}_*(I;\dot H^s)$ and uniformly in $\theta\in[0,1]$.

(ii) *The map $\Phi^\theta(U)$ admits an expansion in multilinear maps as $\Phi^\theta(U)-\mathrm{Id}\in\Sigma\mathcal{M}_{K,K',1}$, $\theta\in[0,1]$. More precisely, there are $M_1(U)$ in $\widetilde{\mathcal{M}}_1$ and $M_2^{(1)}(U)$ and $M_2^{(2)}(U)$ in $\widetilde{\mathcal{M}}_2$ (independent of θ) such that*

$$\begin{aligned}\Phi^\theta(U)[U]=U+\theta\big(M_1(U)[U]+M_2^{(1)}(U)[U]\big)+\theta^2M_2^{(2)}(U)[U]\\+M_{\ge3}(\theta;U)[U]\end{aligned}\tag{A.12}$$

where $M_{\ge3}(\theta;U)$ is in $\mathcal{M}^m_{K,K',3}$ with estimates uniform in $\theta\in[0,1]$.

The same result holds for a matrix-valued system $\partial_\theta\Phi^\theta(U)=\mathbf{B}(U)\Phi^\theta(U)$, $\Phi^0(U)=\mathrm{Id}$, where $\mathbf{B}(U)=Op^{\mathrm{BW}}(B(U;x,\xi))$ and $B(U;x,\xi)$ is a matrix of symbols in $\Sigma\Gamma^0_{K,K',1}\otimes\mathcal{M}_2(\mathbb{C})$.

PROOF. See lemma 3.22 in [6]. The property (2.19) of the flow map $\Phi^\theta(U)$ defined by (A.6) follows by the fact that the homogeneous components of the symbol $f(\theta,U;x,\xi)$ satisfy (2.7). □

The proof of the next lemma follows by standard theory of Banach space ODEs.

LEMMA A.3 (Linear flows generated by a smoothing operator). *Assume that $\mathbf{A}(U)$ in* (A.1) *is a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}\otimes\mathcal{M}_2(\mathbb{C})$ for some $\rho\ge0$. Then,* *there is* $s_0 > 0$*,* $r > 0$ *such that, for any* $U \in B_s^K(I;r)$*, for any* $s > s_0$*, the equation* (A.1) *has a unique solution* $\Phi^\theta(U)$ *satisfying, for some* $C_s > 0$*,*

$$\begin{aligned}\|\partial_t^k(\Phi^\theta(U))^{\pm 1}[v]\|_{\dot H^{s+\rho-k}} \\ \le \|v\|_{k,s}\big(1 + C_s\|U\|_{K,s_0}\big) + C_s\|v\|_{k,s_0}\|U\|_{K,s},\end{aligned} \tag{A.13}$$

for any $v \in C_*^{K-K'}(I;\dot H^s)$*,* $0 \le k \le K-K'$*, and uniformly in* $\theta \in [0,1]$*. Moreover,* $\Phi^\theta(U)$ *satisfies a bound like* (A.11) *and* (ii) *of Lemma* A.2*.*

We now provide the conjugation rules of a paradifferential operator under the flow $\Phi^\theta(U)$ in (A.6). We first give the result in the case when f has the form (A.7); i.e., (A.6) is a transport equation.

LEMMA A.4 (Conjugation of a paradifferential operator under transport flow). *Let* $\Phi^\theta(U)$ *be the flow of* (A.6) *given by Lemma* A.2 *with* $f(\theta,U;x,\xi)$ *as in* (A.7) *and* $U \in C_{*\mathbb{R}}^K(I;\dot H^{s_0}) \cap B_{s_0}^K(I;r)$*. Consider the diffeomorphism of* $\mathbb{T}$ *given by* $\psi_U : x \mapsto x + \beta(U;x)$*. Let* $a(U;x,\xi)$ *be a symbol in* $\Sigma\Gamma_{K,K',q}^m$ *for some* $q \in \mathbb{N}$*,* $q \le 2$*,* $K' \le K$*,* $r > 0$*, and* $m \in \mathbb{R}$*. If* s_0 *is large enough and* r *small enough, then there is a symbol* $a_\Phi(U;x,\xi)$ *in* $\Sigma\Gamma_{K,K',q}^m$ *such that*

$$\Phi^1(U)Op^{\mathrm{BW}}(a(U;x,\xi))(\Phi^1(U))^{-1} = Op^{\mathrm{BW}}(a_\Phi(U;x,\xi)) + R(U) \tag{A.14}$$

where $R(U)$ *is a smoothing remainder in* $\Sigma\mathcal{R}_{K,K',q+1}^{-\rho+m}$*. Moreover,* a_Φ *admits an expansion as*

$$a_\Phi(U;x,\xi) = a_\Phi^{(0)}(U;x,\xi) + a_\Phi^{(1)}(U;x,\xi) \tag{A.15}$$

where

$$a_\Phi^{(0)}(U;x,\xi) = a\big(U;\psi_U(t,x),\xi\partial_y(\psi_U^{-1}(t,y))_{|y=\psi_U(t,x)}\big) \in \Sigma\Gamma_{K,K',q}^m \tag{A.16}$$

and $a_\Phi^{(1)}(U;x,\xi)$ *is a symbol in* $\Sigma\Gamma_{K,K',q+1}^{m-2}$*. In addition, if* $a(U;x,\xi) = g(U;x)\xi$*, then* $a_\Phi^{(1)} = 0$*.*

Furthermore, the symbol $a_\Phi^{(0)}$ *in* (A.16) *admits an expansion in degrees of homogeneity as*

$$a_\Phi^{(0)} = a + \{\beta\xi, a\} + \frac{1}{2}\big(\{\beta\xi,\{\beta\xi,a\}\} + \{-\beta\beta_x\xi, a\}\big) \tag{A.17}$$

up to a symbol in $\Gamma_{K,K',3}^m$ *which is real-valued as* $a_\Phi^{(0)}$ *if* $a(U;x,\xi)$ *is real-valued.*

PROOF. Formulas (A.14)–(A.16) are proved in [6, theorem 3.27] (with homogeneity degree $N = 3$), and it is shown that the symbol

$$a_\Phi^{(0)}(U;x,\xi) = a_0(\theta,U;x,\xi)_{|\theta=1}$$

and $a_0(\theta)$ solves the transport equation

$$\frac{d}{d\theta}a_0(\theta) = \{b(\theta,U;x)\xi, a_0(\theta)\}, \quad a_0(0) = a. \tag{A.18}$$

The claim that, if $a(U;x,\xi) = g(U;x)\xi$, then $a_{\Phi}^{(1)} = 0$ follows because in formula (3.5.37) of [6], the symbol $r_{-\rho,3} = 0$. Finally, we deduce (A.17) by a Taylor expansion in θ using (A.18) (note that b and β have degree of homogeneity 1 in u). Since the homogeneous components of $\beta(U;x)$ satisfy the invariance condition (2.7), the flow $\Phi^1(U)$ satisfies (2.19), and so the left-hand side in (A.14). The proof shows that the symbol a_Φ in (A.15) satisfies (2.7), and therefore the remainder $R(U)$ in (A.14) satisfies (2.19) by difference. □

LEMMA A.5 (*Conjugation of* ∂_t *under transport flow*). *Let* $\Phi^\theta(U)$ *be the flow of* (A.6) *given by Lemma* A.2 *with* $f(\theta,U;x,\xi)$ *as in* (A.7). *Then*

$$\big(\partial_t \Phi^1(U)\big)\big(\Phi^1(U)\big)^{-1} = \mathrm{i}Op^{\mathrm{BW}}(g(U;x)\xi) + R(U) \tag{A.19}$$

where $g(U;x)$ *is a function in* $\Sigma\mathcal{F}^{\mathbb{R}}_{K,K'+1,1}$ *and* $R(U)$ *is a smoothing operator in* $\Sigma\mathcal{R}^{-\rho}_{K,K'+1,1}$. *In addition, the function* $g(U;x)$ *admits the expansion in degrees of homogeneity*

$$g(U;x) = \beta_t - \beta_x\beta_t + g_{\ge 3}(U;x), \quad g_{\ge 3}(U;x) \in \mathcal{F}^{\mathbb{R}}_{K,K'+1,3}. \tag{A.20}$$

PROOF. By the proof of proposition 3.28 of [6] (see formulæ (A.5) and (3.5.55) in [6]) the operator $P(\theta) := (\partial_t\Phi^\theta(U))(\Phi^\theta(U))^{-1}$ solves

$$\frac{d}{d\theta}P(\theta) = \big[\mathrm{i}Op^{\mathrm{BW}}(b(\theta,U;x)\xi), P(\theta)\big] + \mathrm{i}Op^{\mathrm{BW}}(\partial_t b(\theta,U;x)\xi), \ P(0) = 0. \tag{A.21}$$

We claim that the solution of (A.21) is, up to smoothing remainders, $P(\theta) = Op^{\mathrm{BW}}(p_0(\theta,x,\xi))$, where the symbol $p_0(\theta,x,\xi)$ solves the forced transport equation

$$\begin{gathered}\frac{d}{d\theta}p_0(\theta,x,\xi) = \{b(\theta,U;x)\xi, p_0(\theta,x,\xi)\} + \mathrm{i}\partial_t b(\theta,U;x)\xi,\\ p_0(0) = 0,\end{gathered} \tag{A.22}$$

Indeed, the solution of (A.22) is

$$p_0(\theta,x,\xi) = \mathrm{i}\int_0^\theta \partial_t f(s,U;\phi^{\theta,s}(x,\xi))ds \tag{A.23}$$

where $f(s,U;x,\xi) := b(s,U;x)\xi$ and $\phi^{\theta,s}(x,\xi)$ is the solution of the characteristic Hamiltonian system

$$\begin{cases}\frac{d}{ds}x(s) = -b(s,x(s)),\\ \frac{d}{ds}\xi(s) = b_x(s,x(s))\xi(s),\end{cases}$$

with initial condition $\phi^{\theta,\theta} = \mathrm{Id}$. Note that $\phi^{\theta,s}(x,\xi) = \phi^{0,s}\phi^{\theta,0}$ where

$$\phi^{\theta,0}(x,\xi) = (x+\theta\beta(U;x), \xi(1+\partial_y\gamma(U;\theta,y)_{|y=x+\theta\beta(U;x)})$$

where $y+\gamma(U;\theta,y)$ is the inverse diffeomorphism of $x+\theta\beta(U;x)$ (see lemma 3.21 in [6]). Then $\phi^{\theta,s}(x,\xi)$ is linear in ξ; hence also $p_0(\theta,x,\xi)$ in (A.23) is linear in ξ. Since both $b(\theta,U;x)\xi$ and $p_0(\theta,x,\xi)$ are linear in ξ we deduce that the commutator $\left[\mathrm{i}Op^{\mathrm{BW}}(b(\theta, U; x)\xi), Op^{\mathrm{BW}}(p_0(\theta, x, \xi))\right]$ is given, up to smoothing operators, by $Op^{\mathrm{BW}}(\{b(\theta, U; x)\xi, p_0(\theta, x, \xi)\})$. Moreover, by lemma 3.23 in [6], $f(s, U; \phi^{\theta,s}(x, \xi))$ is in $\Sigma\Gamma^1_{K,K',1}$ with estimates uniform in $|\theta|, |s| \leq 1$. Then (A.19) follows by setting $\mathrm{i}g(U; x)\xi := p_0(1, x, \xi)$. Finally, we deduce (A.20) by a Taylor expansion in θ of the symbol $p_0(\theta)$, using (A.22). The function $\beta_t - \beta_x\beta_t$ satisfies the translation invariance property (2.7) as β. As in Lemma A.1 the operator $(\partial_t\Phi^1(U))(\Phi^1(U))^{-1}$ in (A.19) is translation invariant, and $R(U)$ satisfies the property (2.19) by difference. □

We now provide the conjugation of a paradifferential operator under the flow $\Phi^\theta(U)$ in (A.6) if f has the form (A.8) or (A.9).

LEMMA A.6 (*Conjugation of a paradifferential operator*). *Let $\Phi^\theta(U)$ be the flow of* (A.6) *given by Lemma* A.2 *with symbol $f(U; x, \xi)$ in $\Sigma\Gamma^m_{K,K',1}$ with $m \leq 1/2$, of the form* (A.8) *or* (A.9). *Let $a(U; x, \xi)$ be a symbol in $\Sigma\Gamma^{m'}_{K,K',q}$ for some $q \in \mathbb{N}$, $q \leq 2$, $K' \leq K$, $r > 0$, and $m' \in \mathbb{R}$. Then*

$$\begin{aligned}
&\Phi^1(U)Op^{\mathrm{BW}}(a(U; x, \xi))(\Phi^1(U))^{-1} \\
&\quad = Op^{\mathrm{BW}}\left(a + \{f, a\} + \frac{1}{2}\{f, \{f, a\}\} + r_1 + r_2 + r_3\right) + R(U)
\end{aligned} \tag{A.24}$$

where $r_1 \in \Sigma\Gamma^{m+m'-3}_{K,K',q+1}$, $r_2 \in \Sigma\Gamma^{2m+m'-4}_{K,K',q+2}$, $r_3 \in \Gamma^{3m+m'-3}_{K,K',3}$, and $R(U) \in \Sigma\mathcal{R}^{-\rho}_{K,K',q+1}$. In addition, if $a(U; x, \xi)$ is real, then also the symbols r_i, $i = 1, 2, 3$, are real-valued as well.

PROOF. The result follows by a Lie expansion. Using (A.3) we have, for $L \geq 3$,

$$\begin{aligned}
&\Phi^1(U)Op^{\mathrm{BW}}(a)(\Phi^1(U))^{-1} \\
&\quad = Op^{\mathrm{BW}}(a) + \left[Op^{\mathrm{BW}}(\mathrm{i}f), Op^{\mathrm{BW}}(a)\right] + \frac{1}{2}\mathrm{Ad}^2_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(a)] \\
&\qquad + \sum_{k=3}^{L} \frac{1}{k!}\mathrm{Ad}^k_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(a)] \\
&\qquad + \frac{1}{L!}\int_0^1 (1-\theta)^L \Phi^\theta(U)\big(\mathrm{Ad}^{L+1}_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(a)]\big)(\Phi^\theta(U))^{-1}\, d\theta.
\end{aligned} \tag{A.25}$$

By Propositions 2.9, 2.10 replacing the smoothing index ρ by some $\widetilde{\rho}$ chosen below large enough, we get

$$\begin{aligned}
\mathrm{Ad}_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(a)] &= \left[Op^{\mathrm{BW}}(\mathrm{i}f), Op^{\mathrm{BW}}(a)\right] \\
&= Op^{\mathrm{BW}}\big(\{f, a\} + r_1\big),\ r_1 \in \Sigma\Gamma^{m+m'-3}_{K,K',q+1},
\end{aligned}$$

up to a smoothing operator in $\Sigma\mathcal{R}^{-\widetilde{\rho}+m+m'}_{K,K',q+1}$. Moreover,

$$\mathrm{Ad}^2_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(a)] = Op^{\mathrm{BW}}(\{f, \{f, a\}\} + r_2), \quad r_2 \in \Sigma\Gamma^{2m+m'-4}_{K,K',q+2},$$

up to a smoothing operator in $\Sigma\mathcal{R}^{-\widetilde{\rho}+2m+m'}_{K,K',q+2}$. By induction, for $k \geq 3$ we have

$$\mathrm{Ad}^k_{Op^{\mathrm{BW}}(\mathrm{i}f)}[Op^{\mathrm{BW}}(a)] = Op^{\mathrm{BW}}(b_k), \quad b_k \in \Sigma\Gamma^{k(m-1)+m'}_{K,K',q+k},$$

up to a smoothing operator in $\Sigma\mathcal{R}^{-\widetilde{\rho}+m'+km}_{K,K',q+k}$. We choose L in such a way that $(L+1)(1-m)-m' \geq \rho$ and $L+1 \geq 3$ so that the operator $Op^{\mathrm{BW}}(b_{L+1})$ belongs to $\mathcal{R}^{-\rho}_{K,K',3}$. The integral Taylor remainder in (A.25) belongs to $\mathcal{R}^{-\rho}_{K,K',3}$ as well; see lemma 5.6 in [6]. Then we choose $\widetilde{\rho}$ large enough so that $\widetilde{\rho}-m'-(L+1)m \geq \rho$ and the remainders are ρ-smoothing. By the third remark under Definition 2.8 we deduce that if $a(U;x,\xi)$ is real, then the symbol of $\left[Op^{\mathrm{BW}}(\mathrm{i}f), Op^{\mathrm{BW}}(a)\right]$ is real, and so r_1, r_2, r_3 are real-valued as well. □

LEMMA A.7 (*Conjugation of* ∂_t). *Let* $\Phi^\theta(U)$ *be the flow of* (A.6) *with symbol* $f(U;x,\xi)$ *in* $\Sigma\Gamma^m_{K,K',1}$ *with* $m \leq 1/2$, *of the form* (A.8) *or* (A.9). *Then*

$$\begin{aligned}(\text{A.26})\qquad &\big(\partial_t \Phi^1(U)\big)\big(\Phi^1(U)\big)^{-1} \\ &= \mathrm{i}Op^{\mathrm{BW}}\Big(\partial_t f + \frac{1}{2}\{f, \partial_t f\}\Big) + Op^{\mathrm{BW}}(r_1 + r_2) + R(U)\end{aligned}$$

where $r_1 \in \Sigma\Gamma^{2m-3}_{K,K'+1,2}$, $r_2 \in \Gamma^{3m-2}_{K,K'+1,3}$ *and* $R(U) \in \Sigma\mathcal{R}^{-\rho}_{K,K'+1,2}$.

PROOF. The result follows by using the Lie expansion (A.4) and arguing as in Lemma A.6. □

A.3 Lie expansions of vector fields up to quartic degree

In this subsection the variable U may denote both the couple of complex variables $(u, \bar{u})$ or the real variables (η, ψ).

LEMMA A.8 (*Inverse of* $\mathbf{F}^\theta_{\leq 3}(U)$ *up to* $O(u^4)$). *Consider a map* $\theta \mapsto \mathbf{F}^\theta_{\leq 3}(U)$, $\theta \in [0,1]$, *of the form*

$$(\text{A.27})\qquad \mathbf{F}^\theta_{\leq 3}(U) = U + \theta\big(M_1(U)[U] + M_2^{(1)}(U)[U]\big) + \theta^2 M_2^{(2)}(U)[U]$$

where $M_1(U)$ *is in* $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$ *and the maps* $M_2^{(1)}(U), M_2^{(2)}(U)$ *are in* $\widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$. *Then there is a family of maps* $\mathbf{G}^\theta_{\leq 3}(V)$ *of the form*

$$(\text{A.28})\qquad \mathbf{G}^\theta_{\leq 3}(V) = V - \theta\big(M_1(V)[V] + M_2^{(1)}(V)[V]\big) + \theta^2 \breve{M}_2^{(2)}(V)[V]$$

where $\breve{M}_2^{(2)}(V)$ *is in* $\widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$ *such that*

$$(\text{A.29})\qquad \begin{aligned}\mathbf{G}^\theta_{\leq 3} \circ \mathbf{F}^\theta_{\leq 3}(U) &= U + M_{\geq 3}(\theta;U)[U], \\ \mathbf{F}^\theta_{\leq 3} \circ \mathbf{G}^\theta_{\leq 3}(V) &= V + M_{\geq 3}(\theta;U)[U],\end{aligned}$$

where $M_{\geq 3}(\theta;U)$ *is a polynomial in* θ *and finitely many monomials* $M_p(U)[U]$ *for maps* $M_p(U) \in \widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$, $p \geq 3$.

PROOF. Set $V = \mathbf{F}^{\theta}_{\leq 3}(U)$ and substitute iteratively twice the expansion (A.27) to express U as a function of V, up to terms of higher homogeneity (using the last two remarks under Definition 2.7). □

We regard the map $\theta \mapsto \mathbf{G}^{\theta}_{\leq 3}(V)$ in (A.28) as the formal flow of a non-autonomous vector field $S(\theta; U)$ up a remainder of degree of homogeneity 4; see (A.30).

LEMMA A.9. *Consider a map* $\mathbf{F}^{\theta}_{\leq 3}(U)$ *as in* (A.27) *and let* $\mathbf{G}^{\theta}_{\leq 3}(V)$ *be its approximate inverse as in* (A.28) *up to quartic remainders. Then*

$$\partial_\theta \mathbf{G}^{\theta}_{\leq 3}(V) = S(\theta; \mathbf{G}^{\theta}_{\leq 3}(V)) + M_{\geq 3}(\theta; U)[U], \quad \mathbf{G}^{0}_{\leq 3}(V) = V, \tag{A.30}$$

where $S(\theta; U)$ *is a vector field of the form*

$$S(\theta; U) = S_1(U)[U] + \theta S_2(U)[U] \tag{A.31}$$

where $S_1(U)$ *is a map in* $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$ *and* $S_2(U)$ *in* $\widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$*, and* $M_{\geq 3}(\theta; U)$ *is a polynomial in* θ *and finitely many monomials* $M_p(U)[U]$ *for maps* $M_p(U) \in \widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$*,* $p \geq 3$*.*

PROOF. It follows by explicit computation differentiating (A.28), using the expansions (A.27), (A.28), and the last two remarks under Definition 2.7. □

Given polynomials vector fields $X(U)$ and $Y(U)$ we define the nonlinear commutator

$$[\![X, Y]\!](U) := d_U Y(U)[X(U)] - d_U X(U)[Y(U)]. \tag{A.32}$$

Under the same notation of Lemmata A.8, A.9, we have the following result.

LEMMA A.10 (Lie expansion). *Consider a vector field* X *of the form* $X(U) = M(U)U$ *for some map* $M(U) = M_0 + M_1(U) + M_2(U)$ *where* M_0 *is in* $\widetilde{\mathcal{M}}_0 \otimes \mathcal{M}_2(\mathbb{C})$*,* $M_1(U)$ *is in* $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$*, and* $M_2(U)$ *in* $\widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$*. Consider a transformation* $\mathbf{F}^{\theta}_{\leq 3}(U)$ *as in* (A.27) *and let* $S(\theta; U)$ *be the vector field of the form* (A.31) *such that* (A.30) *holds true. Then, if* U *solves* $\partial_t U = X(U)$*, the function* $V := \mathbf{F}^{1}_{\leq 3}(U)$ *solves*

$$\partial_t V = X(V) + [\![S, X]\!]_{|\theta=0}(V) + \frac{1}{2}[\![S, [\![S, X]\!]]\!]_{|\theta=0}(V) + \frac{1}{2}[\![\partial_\theta S_{|\theta=0}, X]\!](V) + \cdots \tag{A.33}$$

up to terms of degree of homogeneity greater than or equal to 4*.*

PROOF. In order to find the quadratic and cubic components of the transformed system, it is sufficient to write $V := \mathbf{F}^{\theta}_{\leq 3}(U)$, $\theta \in [0, 1]$, and the first identity in (A.29) as $U = \mathbf{G}^{\theta}_{\leq 3}(V) + M_{\geq 3}(\theta; U)[U]$. Then, differentiating with ∂_t the first identity in (A.29) we obtain, up to a quartic term,

$$X\big(\mathbf{G}^{\theta}_{\leq 3}(V)\big) = d\mathbf{G}^{\theta}_{\leq 3}(V)[V_t] = (\mathrm{Id} - M(\theta; V))[V_t] \tag{A.34}$$

where $M(\theta; V) = \theta(\check{M}_1(V) + \check{M}_2^{(1)}(V)) + \theta^2 \check{M}_2(V)$ for suitable maps $\check{M}_1(V)$ in $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$ and $\check{M}_2(V)$, $\check{M}_2^{(1)}(V)$ in $\widetilde{\mathcal{M}}_2 \otimes \mathcal{M}_2(\mathbb{C})$; recall (A.28). Applying in (A.34) the “pseudo-inverse"

$$\big(d\mathbf{G}^{\theta}_{\leq 3}(V)\big)^{-1} := \mathrm{Id} + M(\theta; V) + M^2(\theta; V),$$

and since we have $\partial_t V = \partial_t U$ plus a quadratic term in U, we deduce that, up to a quartic term,

$$\big(d\mathbf{G}^{\theta}_{\leq 3}(V)\big)^{-1} X(\mathbf{G}^{\theta}_{\leq 3}(V)) = V_t.$$

The left-hand side of this formula can be expanded in Taylor at $\theta = 0$ up to degree 2, obtaining, using (A.30), the usual Lie formula

$$X(V) + \theta [\![S, X]\!]_{|\theta=0}(V) + \frac{\theta^2}{2}\big([\![S, [\![S, X]\!]]\!]_{|\theta=0}(V) + [\![(\partial_\theta S(\theta))_{|\theta=0}, X]\!](V)\big) \tag{A.35}$$

up to terms of degree 4. Evaluating (A.35) at $\theta = 1$ we get (A.33). □

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