On the genus of projective curves not contained in hypersurfaces of given degree

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Abstract

Fix integers $r \geq 4$ and $i \geq 2$ (for $r = 4$ assume $i \geq 3$). Assume that the rational number s defined by the equation $\binom{i+1}{2} s + (i+1) = \binom{r+i}{i}$ is an integer. Fix an integer $d \geq s$. Divide $d - 1 = ms + \epsilon$, $0 \leq \epsilon \leq s - 1$, and set $G(r;d,i) := \binom{m}{2} s + m\epsilon$. As a number, $G(r; d, i)$ is nothing but the Castelnuovo's bound $G(s + 1;d)$ for a curve of degree d in $\mathbb{P}^{s+1}$. In the present paper we prove that $G(r; d, i)$ is also an upper bound for the genus of a reduced and irreducible complex projective curve in $\mathbb{P}^r$, of degree $d \gg \max\{r, i\}$, not contained in hypersurfaces of degree $\leq i$. We prove that the bound $G(r; d, i)$ is sharp if and only if there exists an integral surface $S \subset \mathbb{P}^r$ of degree s, not contained in hypersurfaces of degree $\leq i$. Such a surface, if existing, is necessarily the isomorphic projection of a rational normal scroll surface of degree s in $\mathbb{P}^{s+1}$. The existence of such a surface S is known for $r \geq 5$, and $2 \leq i \leq 3$. It follows that, when $r \geq 5$, and $i = 2$ or $i = 3$, the bound $G(r; d, i)$ is sharp, and the extremal curves are isomorphic projection in $\mathbb{P}^r$ of Castelnuovo's curves of degree d in $\mathbb{P}^{s+1}$. We do not know whether the bound $G(r; d, i)$ is sharp for $i > 3$.

Keywords Projective curve · Castelnuovo-Halphen theory · Quadric and cubic hypersurfaces · Projection of a rational normal scroll surface · Maximal rank

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1 Introduction

A classical problem in the theory of projective curves is the classification of all their possible genera in terms of the degree and the dimension of the space where they are embedded [2, 10, 13]. Fixed integers r, d, s, Castelnuovo-Halphen's theory states a sharp upper bound

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for the genus of a complex, non-degenerate, reduced and irreducible curve of degree d in $\mathbb{P}^r$, under the condition of being not contained in a surface of degree $< s$ [2, 10, 11].

This theory leads naturally to the consideration of a more general problem, namely the determination of a sharp upper bound for the genus of curves verifying *flag conditions*. Fixed $l + 1$ integers $r, d, s_2^{(i_2)}, s_3^{(i_3)}, \ldots, s_l^{(i_l)}$, with $2 \leq l \leq r - 1$, and $2 \leq i_2 < i_3 < \cdots < i_l \leq r - 1$, we say that a curve verifies the flag condition $\left(r; d, s_2^{(i_2)}, s_3^{(i_3)}, \ldots, s_l^{(i_l)} \right)$ if it is contained in $\mathbb{P}^r$, it has degree d, and it is not contained in a subvariety of $\mathbb{P}^r$ of dimension i_j and degree $< s_j^{(i_j)}$ for all $j = 2, \ldots, l$. This problem can be posed not only for curves, but also for the study of the genus of varieties of dimension > 1. We refer to [4, Introduction] and [6, Introduction] for a general discussion on this topic. In the case of curves, we refer to [3, 4, 7, 8] for some results, mostly asymptotic, when $i_j = j$ for all $j = 2, \ldots, l$. In the same range, for varieties of dimension > 1, we refer to [5, 6, 12, 14].

As far as we know, in the general case, i.e. without the restriction $i_j = j$ on the flag conditions, there are no results, except some remarks in the case $l = 2$ and $i_2 = r - 1$ [4, p. 725, Remark 2.9], [6, p. 1130, Remark 2.6], [14, Section 5, p.179]. In this case, simplifying the notation, we may state the problem as follows (compare with [2, Introduction, lines 18-23 from above]).

Fixed integers r, d, i, one may ask for the maximal genus of a curve of degree d in $\mathbb{P}^r$, not contained in a hypersurface of degree $\leq i$. Apart from some rough estimates [4, p. 726, (2.10)], this problem has remained completely open until our recent paper [9]. In this paper we determined sharp results for $4 \leq r \leq 5$, $i = 2$ and $d \gg 0$ [9, Theorem 1.1 and Theorem 1.2], and asymptotic sharp results for certain values of $r \geq 5$, and for $i = 2$ and $i = 3$ [9, Proposition 1.3 and Proposition 1.4]. In these cases, we obtained also some initial information on the extremal curves, e.g. they should lie on surfaces of a certain degree s, depending on r and i (see definition (1) below), with sectional genus 0.

The aim of this note is to improve [9, Proposition 1.3 and Proposition 1.4], to say something more specific on the extremal curves, to simplify the proof of [9, Theorem 1.2], and to add some information in the case $i > 3$. For a precise description of the results see the statement of Theorem 1.1 below. As well as the quoted results in [9], its proof relies on Castelnuovo-Halphen's theory, in particular on [2, Main Theorem]. But also, and this is a little novelty, on Lemma 2.1 below, which, although elementary, seems to have escaped explicit notice. We think it may play a role in any subsequent developments.

From Theorem 1.1, taking into account [1, p. 478, Theorem 2], first we deduce sharp results (not only asymptotic sharp) for certain values of $r \geq 5$, and for $i = 2$ and $i = 3$ (see Corollary 1.3 below, and compare with [9, Proposition 1.3 and Proposition 1.4]). In these cases, we are able to identify the extremal curves. In fact, they lie on isomorphic projections of smooth rational normal scrolls surfaces, and correspond to the Castelnuovo's curves lying on such surfaces. Comparing with [9], this is completely new (although this phenomenon, in a certain sense, already appears for $r = 4$ and $i = 2$ in [9, Theorem 1.1]). Moreover, in the case $r = 5$, the proof of Theorem 1.1 simplifies the proof of [9, Theorem 1.2], because, relying on Lemma 2.1, it does not require the analysis of the arithmetic genus of a surface in $\mathbb{P}^5$. Theorem 1.1 states an upper bound also for $i > 3$. Unfortunately, in this case we cannot apply [1, p. 478, Theorem 2], and so we do not know whether the bound is sharp (compare with Remark 1.2, (*v*), below), as we expect at least in some cases.

For more specific comments on the claim of Theorem 1.1, we refer to Remark 1.2 and Remark 2.2 below.

Theorem 1.1 *Fix integers $r \geq 4$ and $i \geq 2$ (for $r = 4$ assume $i \geq 3$). Assume that the rational number s defined by the equation*

$$\binom{i+1}{2} s + (i+1) = \binom{r+i}{i} \tag{1}$$

is an integer. Let $C \subset \mathbb{P}^r$ be a reduced and irreducible complex curve, of degree d and arithmetic genus $p_a(C)$, not contained in a hypersurface of degree $\leq i$. Assume $d \gg \max\{r, i\}$ (for a precise inequality see Remark 1.2, (vi), below), and divide $d - 1 = ms + \epsilon$, $0 \leq \epsilon \leq s - 1$. Then

$$p_a(C) \leq \binom{m}{2} s + m\epsilon =: G(r;d, i). \tag{2}$$

Moreover, this bound $G(r; d, i)$ is sharp if and only if there exists an integral surface in $\mathbb{P}^r$ of degree s, not contained in hypersurfaces of degree $\leq i$. In this case, every extremal curve D is not a.C.M., and it is contained in a flag $S \subset T \subset \mathbb{P}^r$, where S is a surface of degree s, uniquely determined by D, not contained in hypersurfaces of degree i, and T is a hypersurface of $\mathbb{P}^r$ of degree $i + 1$. Furthermore, S is the isomorphic projection in $\mathbb{P}^r$ of a rational normal scroll surface S' of degree s in $\mathbb{P}^{s+1}$, and, via this isomorphism, D corresponds to a Castelnuovo's curve of S' of degree d.

Remark 1.2

(i) For instance, if $i = 2$, the number s defined by the equation (1) is an integer if and only if r is not divisible by 3 [1, p. 484, line 12 from below]. If $i = 3$, the number s is an integer if and only if $r = 1, 2, 9, 10, 11, 18, 19, 27, 29$ modulo 36. When $r = 4$, the number s is an integer if and only if $i = 1, 2, 5, 10$ modulo 12. When $r = 5$, the number s is an integer if and only if $i = 1, 2, 7, 11, 13, 17, 22, 23, 26, 31, 37, 38, 41, 43, 46, 47, 53, 58$ modulo 60.

(ii) For the case $r = 4$ and $i = 2$, we refer to [9, Theorem 1.1].

(iii) The number $G(r; d, i)$ defined in (2) is nothing but the Castelnuovo's bound $G(s + 1; d)$ for the genus of a non-degenerate integral curve of degree d in $\mathbb{P}^{s+1}$ [10, p. 87, Theorem (3.7)]. Therefore, the number $G(r; d, i)$ is both an upper bound for the genus of a degree d non-degenerate curve in $\mathbb{P}^{s+1}$, and for the genus of a degree d curve in $\mathbb{P}^r$ not contained in hypersurfaces of degree $\leq i$ (recall that s is defined by (1)).

(iv) Recall that $h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(i)) = \binom{r+i}{i}$, and that, if $S' \subseteq \mathbb{P}^{s+1}$ is a rational normal scroll surface of degree s, then $h^0(S', \mathcal{O}_{S'}(i)) = \binom{i+1}{2} s + (i+1)$. Moreover, notice that for an integral surface in $\mathbb{P}^r$ ($r \geq 4$) of degree σ, not contained in a hypersurface of degree $\leq i$, one has $\binom{i+1}{2} \sigma + (i+1) \geq \binom{r+i}{i}$ [9, inequality (11)]. When the number s defined by (1) is not an integer, then one may define s differently as the minimal integer such that $\binom{i+1}{2} s + (i+1) \geq \binom{r+i}{i}$ (compare with [9, inequal-

ity (11)]). For instance, when $i = 2$, we have $3s + 3 = \begin{pmatrix} r + 2 \\ 2 \end{pmatrix}$ if r is not divisible by 3, and $3s + 1 = \begin{pmatrix} r + 2 \\ 2 \end{pmatrix}$ otherwise. More generally, divide

$$\begin{pmatrix} r + i \\ i \end{pmatrix} - (i + 1) = \alpha \begin{pmatrix} i + 1 \\ 2 \end{pmatrix} + \beta, \quad 0 \leq \beta \leq \begin{pmatrix} i + 1 \\ 2 \end{pmatrix} - 1.$$

Then one has $s = \alpha$ if $\beta = 0$, and $s = \alpha + 1$ if $\beta > 0$. We hope to give some information in the case $\beta > 0$ in a forthcoming paper.

(v) Let S be an integral surface in $\mathbb{P}^r$, not contained in hypersurfaces of degree $\leq i$, of degree s given by the equation (1). In view of Lemma 2.1 (see below), S is the isomorphic projection in $\mathbb{P}^r$ of a rational normal scroll surface $S' \subset \mathbb{P}^{s+1}$ of degree s. Therefore, since $h^0(\mathbb{P}^r, \mathcal{I}_S(i)) = 0$, the restriction map $h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(i)) \to h^0(S, \mathcal{O}_S(i))$ is bijective (compare with (iv) above). It follows that S is of maximal rank [1, p. 482, line 14 from below]. We deduce that *the existence of an integral surface in $\mathbb{P}^r$,not contained in hypersurfaces of degree $\leq i$,of degree s given by the equation (1), is equivalent to the existence of a rational normal scroll surface of degree s in $\mathbb{P}^{s+1}$ which can be projected isomorphically into $\mathbb{P}^r$ as a surface of maximal rank*. By [1, p. 478, Theorem 2] one knows that, when $r \geq 5$, and for $2 \leq i \leq 3$, a general projection in $\mathbb{P}^r$ of a smooth rational normal scroll surface of degree s is of maximal rank (our numbers r and s play the role of k and d in the claim of the quoted paper [1, p. 478, Theorem 2]). Therefore, at least for $r \geq 5$, and $i = 2$ or $i = 3$, the bound (2) is sharp. We do not know whether there are other values of i for which this is true.

(vi) Taking into account the proof of Theorem 1.1 (see below), one may explicit the assumption $d \gg \max\{r, i\}$. In fact, an elementary computation, that we omit, proves that it suffices to assume $d > 179$ in the case $i = 2$ and $r = 5$, and

$$d > \max \left\{ \frac{2s}{r - 2} \prod_{j=1}^{r-2} [(r - 1)! s]^{\frac{1}{r-1-j}}, \; \frac{4s}{r - 2}(s + 1)^3 \right\}$$

otherwise (compare with [9, Section 2, (v), (vi), (vii)]). Observe that the number s defined in (1) depends on r and i. Also, notice that, for $r \geq 4$ and $i \geq 2$, one has $\frac{4s}{r-2}(s + 1)^3 \geq is$.

In view of previous Remark 1.2, (v), we have:

Corollary 1.3 *When $r \geq 5$, and $i = 2$ or $i = 3$, the bound (2) is sharp.*

2 The proof of Theorem 1.1

Lemma 2.1 *Fix integers $r \geq 4$ and $i \geq 2$ (for $r = 4$ assume $i \geq 3$). Assume that the rational number s defined by the equation (1) is an integer. Let $S \subset \mathbb{P}^r$ be an integral surface of degree s, not contained in a hypersurface of $\mathbb{P}^r$ of degree $\leq i$. Then S is the isomorphic projection in $\mathbb{P}^r$ of a rational normal scroll surface $S' \subset \mathbb{P}^{s+1}$ of degree s.*

Proof It suffices to prove that $h^0(S, \mathcal{O}_S(1)) = s + 2$ (notice that $s \geq 5$).

Let $\Sigma \subset \mathbb{P}^{r-1}$ be a general hyperplane section of S. For every $j \geq 1$ one has $h^0(\Sigma, \mathcal{O}_\Sigma(j)) \leq 1 + js$ [9, inequality (10)]. Hence, from the natural exact sequence $0 \to \mathcal{O}_S(j-2) \to \mathcal{O}_S(j-1) \to \mathcal{O}_\Sigma(j-1) \to 0$, we obtain $h^0(S, \mathcal{O}_S(1)) \leq s + 2$, and:

$$h^0(S, \mathcal{O}_S(i-1)) \leq h^0(S, \mathcal{O}_S(1)) + \sum_{j=2}^{i-1} [1 + js] \leq i + \binom{i}{2} s.$$

We deduce that if $h^0(S, \mathcal{O}_S(i-1)) = i + \binom{i}{2} s$, then $h^0(S, \mathcal{O}_S(1)) = s + 2$. Therefore, it suffices to prove that $h^0(S, \mathcal{O}_S(i-1)) = i + \binom{i}{2} s$. Since $h^0(\mathbb{P}^r, \mathcal{I}_S(i-1)) = 0$, from the natural exact sequence $0 \to \mathcal{I}_S(i-1) \to \mathcal{O}_{\mathbb{P}^r}(i-1) \to \mathcal{O}_S(i-1) \to 0$, we get:

$$h^1(\mathbb{P}^r, \mathcal{I}_S(i-1)) = h^0(S, \mathcal{O}_S(i-1)) - \binom{r+i-1}{i-1} \leq i + \binom{i}{2} s - \binom{r+i-1}{i-1}.$$

Hence, it suffices to prove that

$$h^1(\mathbb{P}^r, \mathcal{I}_S(i-1)) = i + \binom{i}{2} s - \binom{r+i-1}{i-1}. \tag{3}$$

To this purpose notice that, since $h^0(\mathbb{P}^r, \mathcal{I}_S(i)) = 0$, from the natural exact sequence $0 \to \mathcal{I}_S(i-1) \to \mathcal{I}_S(i) \to \mathcal{I}_{\Sigma, \mathbb{P}^{r-1}}(i) \to 0$, we have:

$$h^0(\mathbb{P}^{r-1}, \mathcal{I}_{\Sigma, \mathbb{P}^{r-1}}(i)) \leq h^1(\mathbb{P}^r, \mathcal{I}_S(i-1)) \leq i + \binom{i}{2} s - \binom{r+i-1}{i-1}. \tag{4}$$

Now, from the natural exact sequence $0 \to \mathcal{I}_{\Sigma, \mathbb{P}^{r-1}}(i) \to \mathcal{O}_{\mathbb{P}^{r-1}}(i) \to \mathcal{O}_\Sigma(i) \to 0$, we get:

$$h^1(\mathbb{P}^{r-1}, \mathcal{I}_{\Sigma, \mathbb{P}^{r-1}}(i)) = h^0(\mathbb{P}^{r-1}, \mathcal{I}_{\Sigma, \mathbb{P}^{r-1}}(i)) - \binom{r+i-1}{r-1} + h^0(\Sigma, \mathcal{O}_\Sigma(i))$$

$$\leq i + \binom{i}{2} s - \binom{r+i-1}{i-1} - \binom{r+i-1}{r-1} + (1 + is),$$

and this number is equal to 0 in view of the equation (1). It follows that:

$$h^0(\mathbb{P}^{r-1}, \mathcal{I}_{\Sigma, \mathbb{P}^{r-1}}(i)) = i + \binom{i}{2} s - \binom{r+i-1}{i-1}.$$

And so from (4) we obtain (3). □

Remark 2.2 Notice that the surface S in Lemma 2.1 is necessarily smooth. In fact, if $S' \subset \mathbb{P}^{s+1}$ is a singular rational normal scroll, then it is a cone, and so it can not be projected isomorphically into a space of lower dimension. It follows that *if $S' \subset \mathbb{P}^{s+1}$ is a singular rational normal scroll surface of degree s given by the equation* (1), *and S is the projection of S' into $\mathbb{P}^r$ from a $\mathbb{P}^{s-r}$ disjoint from S' and $\mathbb{P}^r$, then S is contained in some hypersurface of degree i*. This is completely in contrast with the smooth case [1, Theorem 2]. Another consequence of previous remark is that, *when $i \geq 3$, every integral surface of degree s given by the equation* (1) *in $\mathbb{P}^4$, is contained in some hypersurface of degree i*. It follows that, in the case $r = 4$ and $i \geq 3$, the bound (2) is not sharp. Hence, when studying curves in $\mathbb{P}^4$ in our setting, one only has to consider curves contained in surfaces of degree $\geq s + 1$. But we will not dwell further with the case $r = 4$ in this paper.

Proof of Theorem 1.1 First we notice that C cannot be contained in a surface of degree $< s$, because every such a surface is contained in a hypersurface of degree i [9, inequality (11)]. On the other hand, if C is not contained in a surface of degree $< s + 1$, since $d \gg \max\{r, i\}$, by [2, Main Theorem] we have [9, Section 2, (*v*), (*vi*), (*vii*)]:

$$p_a(C) \leq \frac{d^2}{2(s+1)} + O(d) < \frac{d^2}{2s} + O(d) = \binom{m}{2}s + m\epsilon.$$

Therefore, in order to prove Theorem 1.1, we may assume C is contained in a surface S of degree s, not contained in a hypersurface of degree $\leq i$. By the previous Lemma 2.1, we know that S is an isomorphic projection of a rational normal scroll surface $S' \subset \mathbb{P}^{s+1}$. Hence, C is isomorphic to a curve $C' \subseteq S'$ of degree d, in particular $p_a(C) = p_a(C')$. Since $d \gg \max\{r, i\}$, by Bezout's theorem C' is non-degenerate. Therefore, the bound (2) for $p_a(C)$ follows from Castelnuovo's bound for $p_a(C')$ [10, loc. cit.]. Observe that previous argument shows also that if the bound (2) is sharp, then every extremal curve $D \subset \mathbb{P}^r$ is contained in a surface $S \subset \mathbb{P}^r$ of of degree s, not contained in a hypersurface of degree $\leq i$. Conversely, if such a surface S exists, with notations as before, let $D' \subset S'$ be a Castelnuovo's curve of degree d. Let $D \subset S$ be the projection of D' in S. Then D is an extremal curve, and so the bound (2) is sharp. In fact, $p_a(D) = p_a(D')$. Moreover, since $d > is$, D cannot be contained in a hypersurface of degree i by Bezout's theorem (compare with Remark 1.2, (*vi*)). Now, let $D \subset \mathbb{P}^r$ be an extremal curve. We just proved that D is contained in a surface S of degree s, a fortiori not contained in a hypersurface of degree $\leq i$. The surface S is an isomorphic projection of a rational normal scroll surface $S' \subset \mathbb{P}^{s+1}$, and, as before, via this isomorphism, D corresponds to a Castelnuovo's curve of S'. Since $d \gg \max\{r, i\}$, again by Bezout's theorem, the surface S of degree s containing D is unique. By [9, inequality (11)] and the definition of s, it follows that S is contained in a hypersurface T of degree $i + 1$. Moreover, D cannot be a.C.M., since it is an isomorphic projection. □

Declarations

References

1. Ballico, E., Ellia, P.: On projections of ruled and veronese surfaces. J. Algebra **121**, 477–487 (1989)
2. Chiantini, L., Ciliberto, C., Di Gennaro, V.: The genus of projective curves. Duke Math. J. **70**(2), 229–245 (1993)
3. Chiantini, L., Ciliberto, C., Di Gennaro, V.: The genus of curves in $\mathbf{P}^4$ verifying certain flag conditions. Manuscr. Math. **88**(1), 119–134 (1995)
4. Chiantini, L., Ciliberto, C., Di Gennaro, V.: On the genus of projective curves verifying certain flag conditions. Bollettino U.M.I. **10–B**(7), 701–732 (1996)
5. Di Gennaro, V.: Generalized Castelnuovo varieties. Manuscr. Math. **81**(3–4), 311–328 (1993)
6. Di Gennaro, V.: A bound on the geometric genus of projective varieties verifying certain flag conditions. Trans. Am. Math. Soc. **349**(3), 1121–1151 (1997)
7. Di Gennaro, V.: Hierarchical structure of the family of curves with maximal genus verifying flag conditions. Proc. Am. Math. Soc. **136**(3), 791–799 (2008)
8. Di Gennaro, V.: A remark on the genus of curves in $\mathbb{P}^4$. Rend. Circ. Mat. Palermo **69**(3), 1079–1091 (2020)
9. Di Gennaro, V.: The genus of curves in $\mathbb{P}^4$ and $\mathbb{P}^5$ not contained in quadrics. Rendiconti del Circolo Matematico di Palermo, Series 2 (2022), Online first articles, 29 July 2022. https://doi.org/10.1007/s12215-022-00788-x
10. Eisenbud, D. , Harris, J.: Curves in projective space. Sém. Math. Sup. 85 Les Presses de l'Université de Montréal, (1982)
11. Gruson, L, Peskine, C.: Genre des courbes dans l'espace projectif. In: Algebraic Geometry: Proceedings, Lecture Notes in Math., pp. 31-59, Norway, (1978)
12. Harris, J.: A bound on the geometric genus of projective varieties. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **8**, 35–68 (1981)
13. Hartshorne, R.: Algebraic Geometry GTM. Springer, Berlin (1983)
14. Nagel, U., Vogel, W.: Bounds for Castelnuovo's regularity and the genus of projective varieties. Banach Center Publications, Warsaw (1990)