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by

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RANKING BY RELATIONAL POWER BASED ON DIGRAPHS

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ABSTRACT. In this paper we examine the ranking for the case of many judges and many objects. We use a directed graph to determine the ranking of the objects. A measure is the function whose domain is the collection of all directed graphs and range is the set of real vectors of as many components as the number of nodes, and the components are called relational power. We proposed two measures and showed the validity of the measures from two aspects: axiomatization and the Shapley value. We also showed the character of measures by some numerical examples.

1. INTRODUCTION

We need the ranking in various situations such as competitive sport and contest. The ranked objects are players or participants in the contest or the nominated papers. There are various methods to rank each objects such as based on their points or evaluations from judges. In this paper we consider the case of many judges and many objects, and every judge can not order all objects, i.e., they order some part of the objects. From such orders of all judges we determine the ranking of all objects.

First we will make the following four assumptions. The first one is that

· each judge orders some objects allocated to him.

It is natural to assume that the order by each judge has transitivity, i.e., when a judge ranks A higher than B ($A \succ B$) and B higher than C ($B \succ C$), then he also ranks A higher than C ($A \succ C$). Then we assume that

· each judge's order has transitivity.

However it is not rational to conclude $A \succ C$ when someone orders $A \succ B$ and another orders $B \succ C$. Then we make following assumption that

· the orders of different judges do not imply the transitivity.

Since it is difficult to ask their ability in real situations, we assume

· there is no difference among the importance of judge's order.

The Borda rule [5] is a method for the problem of aggregating the orders of many judges. Borda rule is based on the Borda point where $n-1$, $n-2$, $\ldots$, $n-n=0$ points are assigned, respectively, to a judge's first, second, $\ldots$, nth ranked object. The object are ranked according to the sum of the Borda points. However Borda rule has the assumption that every judge ranks all objects, then the Borda rule is not appropriate to our case.

In this paper we propose two measures based on the idea of van den Brink and Gilles [2]. They use the directed graph to determine the ranking, and show the adequacy of their measures by axioms and the Shapley value. We also show the adequacy of our proposed measures. Furthermore we show the character of each measure by some numerical examples.

Now we describe our problem. We shall label the objects $1, 2, \ldots, n$, then $N = \{\, 1, 2, \ldots n \,\}$ is the set of all objects. We define the directed graph with node set N in the following way. When judge r compares the object i and j, and if he orders $i \succ j$, then we make an arc

$$i \to j.$$

The directed graph constructed this way is denoted by (N, D_r). We make such directed digraphs for all judges. Then we overlap all arcs to make a directed graph (N, D),

$$D = \bigcup_r D_r,$$

where the union is taken over all judges. For simplicity we assume that every pair of objects is ordered by at most one judge.

2. Notation and Axioms

2.1. **Notation.** We consider the directed graph or digraph being a pair (N, D), where $N = \{\, 1, 2, \ldots, n \,\}$ is a finite set of nodes and $D \subseteq N \times N$ is a binary relation on N representing the set of arcs. We assume that the binary relation D is irreflexive, i.e., $(i, i) \notin D$ for any $i \in N$. If $(i, j) \in D$, we say "node i wins node j" or "node j loses node i". Since we take the set of nodes N to be fixed, we can represent the digraph (N, D) by its binary relation D alone. We denote by $\mathcal{D}$ the collection of all digraphs on N. Let $D \in \mathcal{D}$. For all $i \in N$ the nodes in $S_D(i) := \{\, j \in N \mid (i, j) \in D \,\}$ are called the *successors* of i in D, and the nodes in $P_D(i) := \{\, j \in N \mid (j, i) \in D \,\}$ are called the *predecessors* of i in D. Further we define $s_D(i) = \sharp S_D(i)$, and $p_D(i) = \sharp P_D(i)$, where $\sharp A$ means the number of elements in the set A.

In this paper we use the function $f : \mathcal{D} \to \mathbb{R}^N$, where its domain is the collection $\mathcal{D}$ of all digraphs and its range is the set of n-dimensional real vectors whose components correspond to N. We refer to the ith component of this vector as node i's *relational power*. Moreover we call such function the *measure*.

2.2. **α plus measure and β plus measure.** The α plus measure introduced in van den Brink and Gilles [2] is the function $\alpha^+ : \mathcal{D} \to \mathbb{R}^N$ given by

$$(2.1) \qquad\qquad \alpha_i^+(D) := s_D(i) \quad (\forall i \in N, \forall D \in \mathcal{D}).$$

They introduce the four axioms that the measure should satisfy.

Axiom 2.1 (Normalization)**.** The sum of all relational powers is equal to the number of arcs, i.e.,

$$\sum_{i \in N} f_i(D) = \sharp D \quad (\forall D \in \mathcal{D}).$$

Axiom 2.2 (Dummy node property)**.** The relational power of the node which does not win any nodes is zero, i.e.,

$$S_D(i) = \emptyset \Rightarrow f_i(D) = 0 \quad (\forall i \in N, \forall D \in \mathcal{D}).$$

Axiom 2.3 (Monotonicity)**.**

$$S_D(i) \supseteq S_D(j) \Rightarrow f_i(D) \geq f_j(D) \quad (\forall D \in \mathcal{D} \,, \, \forall i, j \in N).$$

In order to give the fourth axiom we introduce a partition of D. A partition of D is a collection $\mathcal{S} = \{\, D_1, \ldots, D_m \,\}$ that satisfies
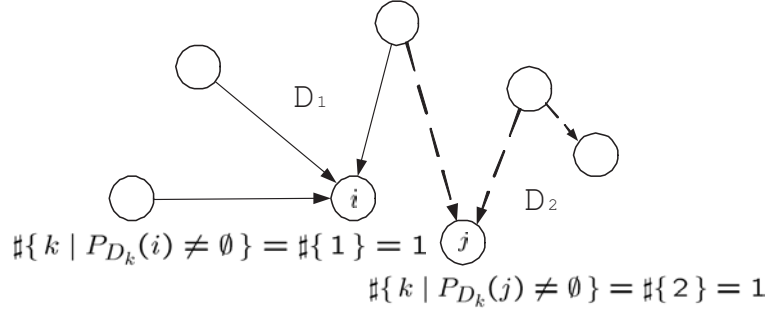
· $\bigcup_{k=1}^m D_k = D$;

$$\sharp\{\,k\mid P_{D_k}(i)\neq\emptyset\,\}=\sharp\{\,1\,\}=1$$

$$\sharp\{\,k\mid P_{D_k}(j)\neq\emptyset\,\}=\sharp\{\,2\,\}=1$$

FIG. 2.1. Independent partition

$$\cdot\; D_k\cap D_l=\emptyset \quad (\forall k\,,\,\forall l\,;\,1\le k<l\le m).$$

We define the independence of partition $\mathcal{S}$.

Definition 2.4. Partition $\mathcal{S}$ of D is said to be independent if and only if

$$\sharp\{\,k\mid P_{D_k}(i)\neq\emptyset\,\}\le 1 \quad (\forall i\in N).$$

Figure 2.1 shows an example of independent partition.

Axiom 2.5 (Additivity over independent partitions). The sum of the node relational powers which are measured on an independent partition (Definition 2.4) is equal to the node relational power on digraph D, i.e.,

$$f_i(D)=\sum_{D_k\in\mathcal{S}} f_i(D_k) \quad (\forall i\in N\,,\,\forall\text{independent partition }\mathcal{S}=\{\,D_1,\cdots,D_m\,\}\text{ of }D).$$

Then they show the following theorem that the α plus measure is characterized by the four axioms introduced above.

Theorem 2.6 (Theorem 3.3 [2]). *A function $f:\mathcal{D}\to\mathbb{R}^N$ is equal to the α plus measure on N if and only if it satisfies Axiom 2.1, 2.2, 2.3 and 2.5.*

They also introduce the β plus measure, which is given by

$$(2.2) \qquad \beta_i^+(D):=\sum_{j\in S_D(i)}\frac{1}{p_D(i)} \quad (\forall i\in N,\forall D\in\mathcal{D}).$$

The β plus measure can be axiomatized by replacing Axiom 2.1 in Theorem 2.6 with the following axiom.

Axiom 2.7 (Normalization). The sum of all relational powers is equal to the number of the node which loses some nodes, i.e.,

$$\sum_{i\in N} f_i(D)=\sharp\{\,j\in N\mid P_D(j)\neq\emptyset\,\} \quad (\forall D\in\mathcal{D}).$$

They show the following theorem.

Theorem 2.8 (Theorem 2.7 [2]). *A function $f:\mathcal{D}\to\mathbb{R}^N$ is equal to the β plus measure on N if and only if it satisfies Axiom 2.2, 2.3, 2.5 and 2.7.*

2.3. **The Shapley value.** Van den Brink and Gilles [2] also show a relationship among the α plus measure, the β plus measure and the Shapley value [9]. In fact the axioms introduced above is similar to the axioms of the Shapley value: efficiency, dummy, symmetry, and additivity.

Following Owen [4], we will introduce several terms. The Shapley value is one of the index to measure the player's power for cooperative games. In cooperative games we use the coalition formation. We shall label the players $1, 2, \ldots, n$. Then $N = \{ 1, 2, \ldots, n \}$ is the set of all players in the game. Each the non-empty subset $E \subseteq N$ is called a *coalitions*. We assume that each coalition has certain strategies and also that it would know how best to use these strategies, so as to maximize the amount of utility received all of its members. So we use a characteristic function $v : 2^N \to \mathbb{R}$ which tells us the maximum that each coalition can obtain. And a pair of (N, v) is called a characteristic function game.

Given a characteristic function game (N, v), the Shapley value of player i is defined

$$(2.3) \qquad \varphi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} [v(V(i, \pi) \cup \{ i \}) - v(V(i, \pi))],$$

where π is a permutation of N and $\Pi(N)$ is the collection of all permutations of N, $V(i, \pi)$ is the set of players that precede player i in permutation π, i.e., $V(i, \pi) = \{ j \in N \mid \pi(j) < \pi(i) \}$. A permutation π means the order that all players enter the coalition. And $v(V(i, \pi) \cup \{ i \}) - v(V(i, \pi))$ is the increment of the coalition value that player i enters the coalition $V(i, \pi)$, that is to say it is the contribution to coalition $V(i, \pi) \cup \{ i \}$. Therefore, the Shapley value of player i is the expected value of player i's contribution on all permutations.

Next we regard the players as the nodes on digraph D and describe the relation among the Shapley value, the α plus measure and the β plus measure. For every $D \in \mathcal{D}$ and $E \subseteq N$, we define the set of successors of E in D as $S_D(E) := \bigcup_{i \in E} S_D(i)$ and $s_D(E) = \sharp S_D(E)$. Similarly we define the set of predecessors of E in D as $P_D(E) := \bigcup_{i \in E} P_D(i)$ and $p_D(E) = \sharp P_D(E)$. Van den Brink and Gilles [2] show following Theorem 2.9 and we show Theorem 2.10, however the step of the proof is similar to Theorem 2.9 so we omit the proof.

Theorem 2.9 (Theorem 4.2 [2])**.** *For every $E \subseteq N$ let*

$$v_D(E) = s_D(E),$$

*then the Shapley value $\varphi(v_D)$ on characteristic function game (N, v_D) is equal to the β plus measure.*

Theorem 2.10. *For every $E \subseteq N$ let*

$$v_D(E) = \sharp\{ (i, j) \mid i \in E, (i, j) \in D \},$$

then the Shapley value $\varphi(v_D)$ on characteristic function game (N, v_D) is equal to the α plus measure.

3. New measures

In the discussion of the α plus measure and the β plus measure, we focus on only the nodes that node i wins, that is we does not fully see the information around node i. Now we propose new measures, the γ plus measure and the δ plus measure.
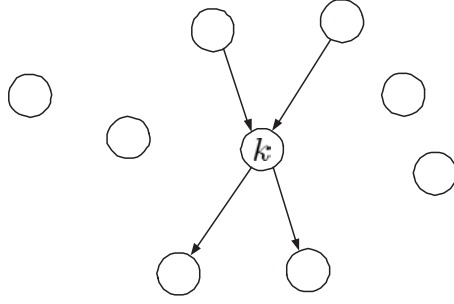
FIG. 3.1. Subdigraph D_k

3.1. γ plus measure and δ plus measure. We introduce the γ plus measure $\gamma^+ : \mathcal{D} \to \mathbb{R}^N$ given by

$$\gamma_i^+(D) := \sum_{j \in S_D(i)} \frac{s_D(j) + 1}{p_D(j)} \quad (\forall i \in N, \forall D \in \mathcal{D}).$$

The γ plus measure focuses on the nodes that node i wins, and considers how many times those nodes win or lose.

Axiom 3.1 (Normalization)**.** The sum of all relational powers is equal to the sum of two terms: the first term is the sum of the numbers of successors where the sum is taken over all nodes which lose some nodes, and the second term is the sum of the β plus measures over all nodes, i.e.,

$$\sum_{i \in N} f_i(D) = \sum_{\substack{j \in N \\ P_D(j) \neq \emptyset}} s_D(j) + \sum_{i \in N} \beta_i^+(D) \quad (\forall D \in \mathcal{D}).$$

Axiom 3.2 (Extended dummy node property)**.** If no successors of node i win a node, then node i's relational power is given by β plus measure, i.e.,

$$S_D(S_D(i)) = \emptyset \Rightarrow f_i(D) = \beta_i^+(D) \quad (\forall i \in N , \forall D \in \mathcal{D}).$$

Axiom 3.3 (Monotonicity)**.**

$$S_D(i) \supseteq S_D(j) \Rightarrow f_i(D) \geq f_j(D) \quad (\forall i, j \in N , \forall D \in \mathcal{D}).$$

In order to give the fourth axiom we define a subdigraph D_k for each $k \in N$. Figure 3.1 show this subdigraph D_k.

Definition 3.4. For every $k \in N$, a subdigraph D_k is defined by

$$D_k = \{ (i, k) \mid (i, k) \in D \} \cup \{ (k, j) \mid (k, j) \in D \}.$$

Axiom 3.5 (Additivity over subdigraphs)**.** Node i's relational power on the digraph D is equal to the subtraction of the following two terms: the first term is the sum of node i's relational power on subdigra phs D_k, and the second term is node i's value of the α plus measure.

$$(3.1) \qquad f_i(D) = \sum_{k \in N} f_i(D_k) - \alpha_i^+(D) \quad (\forall i \in N , \forall D \in \mathcal{D}).$$

The following theorem shows that the γ plus measure is characterized by the four axioms introduced above.

Theorem 3.6. *A function $f : \mathcal{D} \to \mathbb{R}^N$ is equal to the γ plus measure on N if and only if it satisfies Axiom 3.1, 3.2, 3.3 and 3.5.*

Proof. It can easily be seen that the γ plus measure satisfies Axiom 3.1, 3.2 and 3.3. Now we will show that γ plus measure satisfies Axiom 3.5. Let $i \in N$ be fixed. For $k \in N$, there are four possible cases: $k \in S_D(i)$, $k \in P_D(i)$, $k = i$ and the rest. Then the right hand side of (3.1) is rewritten as

$$\sum_{k \in N} \gamma_i^+(D_k) - \alpha_i^+(D) = \sum_{k \in S_D(i)} \gamma_i^+(D_k) + \sum_{k \in P_D(i)} \gamma_i^+(D_k) + \gamma_i^+(D_i)$$

$$(3.2) \hspace{2cm} + \sum_{k \in N \setminus (S_D(i) \cup P_D(i) \cup \{\, i \,\})} \gamma_i^+(D_k) - \alpha_i^+(D).$$

Next we calculate the value of $\gamma_i^+(D_k)$ in each case.

- case 1: $k \in S_D(i)$
  It holds that $S_{D_k}(i) = \{\, k \,\}$ and $j = k$ if $j \in S_{D_k}(i)$, then

$$(3.3) \hspace{1cm} \gamma_i^+(D_k) = \sum_{j \in S_{D_k}(i)} \frac{s_{D_k}(j) + 1}{p_{D_k}(j)} = \frac{s_{D_k}(k) + 1}{p_{D_k}(k)} = \frac{s_D(k) + 1}{p_D(k)} \hspace{0.5cm} (\forall k \in S_D(i)).$$

- case 2: $k \in P_D(i)$
  It holds $S_{D_k}(i) = \emptyset$, then

$$(3.4) \hspace{2cm} \gamma_i^+(D_k) = 0 \hspace{0.5cm} (\forall k \in P_D(i)).$$

- case 3: $k = i$
  It holds that $S_{D_i}(j) = \emptyset$ and $P_{D_i}(j) = \{\, i \,\}$ for every $j \in S_{D_i}(i)$, then

$$(3.5) \hspace{1cm} \gamma_i^+(D_i) = \sum_{j \in S_{D_i}(i)} \frac{s_{D_i}(j) + 1}{p_{D_i}(j)} = \sum_{j \in S_{D_i}(i)} 1 = s_D(i) \hspace{0.5cm} (k = i).$$

- case 4: $k \in N \setminus (S_D(i) \cup P_D(i) \cup \{\, i \,\})$
  It holds $S_{D_k}(i) = \emptyset$, then

$$(3.6) \hspace{1cm} \gamma_i^+(D_k) = 0 \hspace{0.5cm} (\forall k \in N \setminus (S_D(i) \cup P_D(i) \cup \{\, i \,\})).$$

Since $\alpha_i^+(D) = s_D(i)$, according to (3.3), (3.5) and (3.6), we obtain that (3.2) is

$$\sum_{k \in N} \gamma_i^+(D_k) - \alpha_i^+(D) = \sum_{k \in S_D(i)} \frac{s_D(k) + 1}{p_D(k)} + 0 + s_D(i) + 0 - s_D(i)$$

$$(3.7) \hspace{2cm} = \sum_{k \in S_D(i)} \frac{s_D(k) + 1}{p_D(k)} = \gamma_i^+(D).$$

Next we suppose that a function $f : \mathcal{D} \to \mathbb{R}^N$ satisfies the four axioms and show that f is equal to the γ plus measure. Let $D \in \mathcal{D}$, and we consider the value of the function f on subdigraph D_k. Since for every $i \in P_D(k)$, i.e., $k \in S_D(i)$ it follows from the above discussion and Axiom 3.3 that there is a constant $c \in \mathbb{R}$ such that

$$(3.8) \hspace{2cm} f_i(D_k) = c \hspace{0.5cm} (\forall i \in P_D(k)).$$

For every $i \in S_D(k)$ it also follows from the above discussion and Axiom 3.2 that

$$(3.9) \hspace{2cm} f_i(D_k) = 0 \hspace{0.5cm} (\forall i \in S_D(k)).$$

For $i = k$ it also follows from the above discussion and Axiom 3.2 that

$$(3.10) \qquad f_k(D_k) = \beta_k^+(D_k) = \sum_{j \in S_{D_k}(k)} \frac{1}{p_{D_k}(j)} = \sum_{j \in S_{D_k}(k)} \frac{1}{1} = s_D(k).$$

For $i \in N \setminus (P_D(k) \cup S_D(k) \cup \{k\})$ we have from the above discussion and Axiom 3.2 that

$$(3.11) \qquad f_i(D_k) = 0 \quad (\forall i \in N \setminus (P_D(k) \cup S_D(k) \cup \{k\})\ ).$$

Applying Axiom 3.1 to the subdigraph D_k, then we obtain that

$$\begin{aligned}
\sum_{i \in N} f_i(D_k) &= \sum_{\substack{k \in N \\ P_{D_k}(j) \neq \emptyset}} s_{D_k}(j) + \sum_{i \in N} \beta_i^+(D_k) \\
&= s_D(k) + (s_D(k) + 1) \\
(3.12) \qquad &= 2s_D(k) + 1.
\end{aligned}$$

According to (3.8), (3.9), (3.10) and (3.11)

$$\begin{aligned}
\sum_{i \in N} f_i(D_k) &= \sum_{i \in P_D(k)} f_i(D_k) + \sum_{i \in S_D(k)} f_i(D_k) \\
&\qquad + f_k(D_k) + \sum_{i \in N \setminus (P_D(k) \cup S_D(k) \cup \{k\})} f_i(D_k) \\
&= p_D(k) \times c + 0 + s_D(k) + 0 \\
(3.13) \qquad &= p_D(k) \times c + s_D(k).
\end{aligned}$$

Then from (3.12) and (3.13) we are that $c = \dfrac{s_D(k) + 1}{p_D(k)}$.

According to Axiom 3.5, we obtain that for every i

$$\begin{aligned}
f_i(D) &= \sum_{k \in N} f_i(D_k) - \alpha_i^+(D) \\
&= \left(\sum_{k \in S_D(i)} f_i(D_k) + \sum_{k \in P_D(i)} f_i(D_k) \right. \\
&\qquad \left. + f_i(D_i) + \sum_{k \in N \setminus (S_D(i) \cup P_D(i) \cup \{i\})} f_i(D_k) \right) - s_D(i) \\
&= \sum_{k \in S_D(i)} \frac{s_D(k) + 1}{p_D(k)} + 0 + s_D(i) + 0 - s_D(i) \\
(3.14) \qquad &= \sum_{k \in S_D(i)} \frac{s_D(k) + 1}{p_D(k)}.
\end{aligned}$$

Therefore we conclude that

$$(3.15) \qquad f_i(D) = \gamma_i^+(D) \quad (\forall i \in N).$$

$\square$

We have provided the axiomatization of the γ plus measure but it does not seem natural. Since the γ plus measure is an extension of the α plus measure and the β plus measure, it is suitable that Axiom 3.5 contains α^+ and Axiom 3.3 contains β^+, however we cannot explain why α^+ and β^+ appear there.

The following measure of the relational power was introduced by Borm *et al.* [1] but they did not axiomatize it. We will call it the δ plus measure.

$$\delta_i^+(D) := \sum_{j \in S_D(i) \cup \{i\}} \frac{1}{p_D(j) + 1} \quad (\forall i \in N, \forall D \in \mathcal{D}).$$

The definition of the δ plus measure is similar to the β plus measure, however we consider node i itself as well as the nodes that node i wins.

Axiom 3.7 (Normalization)**.** The sum of all relational powers is equal to n, the number of elements of set N, i.e.,

$$\sum_{i \in N} f_i(D) = n \quad (\forall D \in \mathcal{D}).$$

Axiom 3.8 (Dummy node property)**.** The relational power of the node which does not win any nodes is one divided by the number of node i's predecessors incremented by one, i.e.,

$$S_D(i) = \emptyset \Rightarrow f_i(D) = \frac{1}{p_D(i) + 1} \quad (\forall i \in N, \forall D \in \mathcal{D}).$$

Axiom 3.9 (Monotonicity)**.**

$$S_D(i) \supseteq S_D(j) \Rightarrow f_i(D) \geq f_j(D) \quad (\forall i, j \in N, \forall D \in \mathcal{D}).$$

In order to give the fourth axiom we define another partition $\mathcal{T} = \{\, D_k' \mid k \in N \,\}$,

Definition 3.10. For every $k \in N$,

$$D_k' = \{\, (i, k) \mid i \in P_D(k) \,\}.$$

Axiom 3.11 (Additivity over partition $\mathcal{T}$)**.**

$$(3.16) \qquad f_i(D) - 1 = \sum_{k \in N} \big(f_i(D_k') - 1 \big) \quad (\forall i \in N, \forall D \in \mathcal{D}).$$

We obtain the following theorem that the δ plus measure is characterized by the four axioms introduced above.

Theorem 3.12. *A function* $f : \mathcal{D} \to \mathbb{R}^N$ *is equal to the δ plus measure on N if and only if it satisfies Axiom 3.7, 3.8, 3.9 and 3.11.*

Proof. It can easily be seen that the δ plus measure satisfies Axiom 3.7, Axiom 3.8 and Axiom 3.9. Now we will show that it satisfies Axiom 3.11. Let $i \in N$ be fixed. For each $k \in N$, consider the possible three cases: $k \in S_D(i)$, $k = i$ and the rest. Then the right hand side of (3.16) is rewritten as

$$(3.17) \sum_{k \in N} (\delta_i^+(D_k') - 1) = \sum_{k \in S_D(i)} \delta_i^+(D_k') + \delta_i^+(D_i') + \sum_{k \in N \setminus (S_D(i) \cup \{i\})} \delta_i^+(D_k') - n.$$

Next we calculate the value of $\delta_i^+(D_k')$ in each cases.

- case 1: $k \in S_D(i)$

  It holds that $S_{D_k'}(i) = \{k\}$ and $P_{D_k'}(i) = \emptyset$, then

$$(3.18) \quad \delta_i^+(D_k') = \sum_{j \in S_{D_k'}(i) \cup \{i\}} \frac{1}{p_{D_k'}(j) + 1} = \frac{1}{p_D(k) + 1} + 1 \quad (\forall k \in S_D(i)).$$

- case 2: $k = i$

    It holds $S_{D'_i}(i) = \emptyset$, then

$$
(3.19) \qquad \delta_i^+(D'_i) = \sum_{j \in S_{D'_i}(i) \cup \{\,i\,\}} \frac{1}{p_{D'_i}(j) + 1} = \frac{1}{p_D(i) + 1} \quad (k = i).
$$

- case 3: $k \in N \setminus (S_D(i) \cup \{\,i\,\})$

    It holds that $S_{D'_k}(i) = \emptyset$ and $P_{D'_k}(i) = \emptyset$, then

$$
(3.20) \qquad \delta_i^+(D'_k) = \sum_{j \in S_{D'_k}(i) \cup \{\,i\,\}} \frac{1}{p_{D'_k}(j) + 1} = 1 \quad (\forall k \in N \setminus (S_D(i) \cup \{\,i\,\})).
$$

According to (3.18), (3.19) and (3.20), we obtain that (3.17) is

$$
\begin{aligned}
\sum_{k \in N} (\delta_i^+(D'_k) - 1) &= \sum_{k \in S_D(i)} \delta_i^+(D'_k) + \delta_i^+(D'_i) + \sum_{k \in N \setminus (S_D(i) \cup \{\,i\,\})} \delta_i^+(D'_k) - n \\
&= \sum_{k \in S_D(i)} \left(\frac{1}{p_D(k) + 1} + 1 \right) + \frac{1}{p_D(i) + 1} + 1 \times (n - (s_D(i) + 1)) - n \\
&= \sum_{k \in S_D(i)} \frac{1}{p_D(k) + 1} + s_D(i) + \frac{1}{p_D(i) + 1} - s_D(i) - 1 \\
&= \sum_{k \in S_D(i) \cup \{i\}} \frac{1}{p_D(k) + 1} - 1 \\
&= \delta_i^+(D) - 1.
\end{aligned}
$$

Next we suppose that a function $f : \mathcal{D} \to \mathbb{R}^N$ satisfies the four axioms then show that f is equal to the δ plus measure. Let $\mathcal{D} \in D$, and we consider the value of the function f on $D_k = \{\, (i,k) \in N \times N \mid i \in P_D(k) \,\}$. Since for every $i \in P_D(k)$ i.e., $k \in S_D(i)$ it follows from the above discussion and Axiom 3.9 that there is a constant $c \in \mathbb{R}$ such that

$$
(3.21) \qquad f_i(D_k) = c \quad (\forall i \in P_D(k)).
$$

For $i = k$ it follows from the above discussion and Axiom 3.8 that

$$
(3.22) \qquad f_k(D_k) = \frac{1}{p_D(k) + 1}.
$$

For $i \in N \setminus (P_D(j) \cup \{\,j\,\})$ we have from the above discussion and Axiom 3.8 that

$$
(3.23) \qquad f_i(D_k) = 1 \quad (\forall i \in N \setminus (P_D(k) \cup \{\,k\,\})).
$$

We apply Axiom 3.7 to D_k, and we obtain that

$$
(3.24) \qquad \sum_{i \in N} f_i(D_k) = n,
$$

while according to (3.21), (3.22) and (3.23),

$$
\begin{aligned}
\sum_{i \in N} f_i(D_k) &= \sum_{i \in P_D(k)} f_i(D_k) + f_i(D_i) + \sum_{i \in N \setminus (P_D(k) \cup \{k\})} f_i(D_k) \\
&= p_D(k) \times c + \frac{1}{p_D(k) + 1} + n - (p_D(k) + 1).
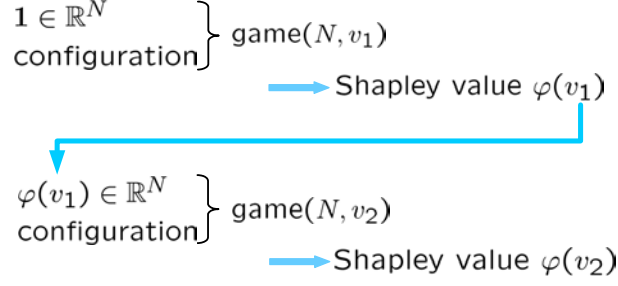\end{aligned}
$$

$$\begin{aligned}
&\left.\begin{array}{l} \mathbf{1} \in \mathbb{R}^N \\ \text{configuration} \end{array}\right\} \text{game}(N, v_1) \\
&\qquad\qquad\qquad \longrightarrow \text{Shapley value } \varphi(v_1) \\[2ex]
&\left.\begin{array}{l} \varphi(v_1) \in \mathbb{R}^N \\ \text{configuration} \end{array}\right\} \text{game}(N, v_2) \\
&\qquad\qquad\qquad \longrightarrow \text{Shapley value } \varphi(v_2)
\end{aligned}$$

FIG. 3.2. Idea of the repeated game

Then from (3.24) we see $c = 1 + \dfrac{1}{p_D(j) + 1}$. Since digraph D_k is a partition $\mathcal{T}$ it follows from Axiom 3.11 that for every $i \in N$

$$\begin{aligned}
f_i(D) &= \sum_{k \in N} (f_i(D_k) - 1) + 1 \\
&= \sum_{k \in N} f_i(D_k) - n + 1 \\
&= \sum_{k \in S_D(i)} f_i(D_k) + f_i(D_i) \sum_{k \in N \setminus (S_D(i) \cup \{i\})} f_i(D_k) - (n-1) \\
&= \sum_{k \in S_D(i)} \left(1 + \frac{1}{p_D(k) + 1} \right) + \frac{1}{p_D(i) + 1} + n - (s_D(i) + 1) - (n-1) \\
&= s_D(i) + \sum_{k \in S_D(i)} \frac{1}{p_D(k) + 1} + \frac{1}{p_D(i) + 1} + n - s_D(i) - 1 - n + 1 \\
&= \sum_{k \in S_D(i) \cup \{i\}} \frac{1}{p_D(k) + 1}.
\end{aligned}$$

Therefore we conclude that

$$f_i(D) = \delta_i^+(D) \quad (\forall i \in N).$$

$\square$

3.2. **Repeated game.** We will also show a relationship between the γ plus measure and the Shapley value. We propose to repeat the game, and consider the Shapley value, namely we make the second game based on the Shapley value that calculated in the first game, and then calculate the Shapley value of the second game. We call such game the *repeated game*. Here it is important to devise the configuration of the game. This idea is illustrated in Figure 3.2.

In order to give the following lemmas, we introduce the notation. For every $D \in \mathcal{D}$ and $E \subseteq N$, we define the set of successors and itself as $\bar{S}_D(E) := \bigcup_{i \in E}(S_D(i) \cup \{i\})$ and $\bar{s_D}(E) = \sharp \bar{S}_D(E)$. Also, we define the function $c(\cdot)$ that counts the number of elements in the coalition E, i.e., $c(E) = |E|$.

Lemma 3.13. *For every $E \subseteq N$, we define a characteristic function v_D as*

$$v_D(E) = \sum_{i \in E} \sum_{j \in \bar{S}_{D(i)}} 1,$$

and consider the game (N, v_D), *then node* i's *Shapley value* $\varphi_i(v_D)$ *is* $s_D(i) + 1$.

Proof. First we rewrite $v_D(E)$, then

$$
\begin{aligned}
v_D(E) &= \sum_{i \in E} \sum_{j \in \bar{S}_D(i)} 1 \\
&= \sum_{i \in E} \sum_{j \in N} [\, j \in \bar{S}_D(i) \,] \\
&= \sum_{i \in E} \sum_{j \in N} [\, j \in (S_D(i) \cup \{i\}) \,] \\
&= \sum_{i \in N} \sum_{j \in N} [\, i \in E \,][\, j \in (S_D(i) \cup \{i\}) \,] \\
&= \sum_{i \in N} \sum_{j \in N} [\, i \in E, (i,j) \in D \,] + \sum_{i \in N} [\, i \in E \,] \\
&= \sum_{i \in N} \sum_{j \in N} [\, i \in E, (i,j) \in D \,] + c(E),
\end{aligned}
$$
(3.25)

where $[\,]$ is the indicator function which is one if the statement in the brackets $[\,]$ is true and zero otherwise, defined by Graham *et al.* [3]. For every $j \in N$, we consider the digraph $D_j = \{\, (i,j) \in N \times N \mid i \in P_D(j) \,\}$. Also by the structure of digraph D_j it holds that $(i,k) \in D_j$ when only $k = j$, and $(i,k) \in D_j$ means $(i,j) \in D$. Then

$$
\begin{aligned}
\sum_{j \in N} v_{D_j}(E) &= \sum_{j \in N} \left(\sum_{i \in N} \sum_{k \in N} [\, i \in E, (i,k) \in D_j \,] + c(E) \right) \\
&= \sum_{j \in N} \left(\sum_{i \in N} [\, i \in E, (i,j) \in D \,] + c(E) \right) \\
&= \sum_{i \in N} \sum_{j \in N} [\, i \in E, (i,j) \in D \,] + nc(E) \\
&= \sum_{i \in N} \sum_{j \in N} [\, i \in E, (i,j) \in D \,] + c(E) + (n-1)c(E).
\end{aligned}
$$
(3.26)

According to (3.25) and (3.26), we see

(3.27)
$$
v_D(E) = \sum_{j \in N} v_{D_j}(E) - (n-1)c(E).
$$

From the the additivity property of the Shapley value, we obtain

$$
\begin{aligned}
\varphi_i(v_D) &= \varphi_i \left(\sum_{j \in N} v_{D_j} - (n-1)c \right) \\
&= \sum_{j \in N} \varphi_i(v_{D_j}) - (n-1)\varphi_i(c).
\end{aligned}
$$
(3.28)

Let us consider digraph D_j, coalition $E \subseteq N$ and node $i \in N \setminus E$. Then

(3.29)
$$
v_{D_j}(E \cup \{i\}) - v_{D_j}(E) =
\begin{cases}
2 & \text{if } i \in P_D(j) \\
1 & \text{if } i = j \\
1 & \text{otherwise.}
\end{cases}
$$

We consider all the permutations of N. Let π be a permutation of N. In the case where $i \in P_D(j)$, node i's contribution value to coalition E is always two whether there may be node j's predecessors before node i in π. In the other case, we can see in the similar way that the contribution value is always one. Therefore, from the definition of the Shapley value (2.3) and (3.29), the Shapley value of node i in game (N, v_{D_j}) is given by

$$(3.30) \qquad \varphi_i(v_{D_j}) = \begin{cases} 2 & \text{if } i \in P_D(j) \\ 1 & \text{if } i = j \\ 1 & \text{othewise.} \end{cases}$$

Note that $\varphi_i(c) = 1$ since $c(E \cup \{i\}) - c(E) = 1$ in all cases. According to (3.28) and (3.30), we obtain that

$$\begin{aligned}
\varphi_i(v_D) &= \sum_{j \in N} \varphi_i(v_{D_j}) - (n-1)\varphi_i(c) \\
&= \sum_{j \in S_D(i)} \varphi_i(v_{D_j}) + \varphi_i(v_{D_i}) + \sum_{j \in N \setminus (S_D(i) \cup \{i\})} \varphi_i(v_{D_j}) - (n-1)\varphi_i(c) \\
&= \sum_{j \in S_D(i)} 2 + 1 + \sum_{j \in N \setminus (S_D(i) \cup \{i\})} 1 - (n-1)1 \\
&= 2s_D(i) + 1 + n - s_D(i) - 1 - n + 1 \\
&= s_D(i) + 1.
\end{aligned}$$

$\square$

Lemma 3.14. *Given $\theta \in \mathbb{R}^N$, we define a characteristic function v_D as*

$$v_D(E) = \sum_{j \in S_{D(E)}} \theta_j,$$

and consider the game (N, v_D). Then node i's the Shapley value $\varphi_i^S(v_D)$ is $\displaystyle \sum_{j \in S_D(i)} \frac{\theta_j}{p_D(j)}$.

Proof. First we rewrite $v_D(E)$, then

$$(3.31) \qquad v_D(E) = \sum_{j \in S_D(E)} \theta_j = \sum_{j \in N} \theta_j [\, \exists i \in E, (i,j) \in D \,].$$

For every $j \in N$, we consider the digraph $D_j = \{\, (i,j) \in N \times N \mid i \in P_D(j) \,\}$. Also by the structure of digraph D_j it holds that $(i,k) \in D_j$ only if $k = j$, and $(i,k) \in D_j$ means $(i,j) \in D$. Then

$$\begin{aligned}
\sum_{j \in N} v_{D_j}(E) &= \sum_{j \in N} \sum_{k \in S_{D_j}(E)} \theta_k \\
&= \sum_{j \in N} \sum_{k \in N} \theta_k [\, \exists i \in E, (i,k) \in D_j \,] \\
(3.32) \qquad &= \sum_{j \in N} \theta_j [\, \exists i \in E, (i,j) \in D \,].
\end{aligned}$$

According to (3.31) and (3.32), we see

$$(3.33) \qquad v_D(E) = \sum_{j \in N} v_{D_j}(E)$$

Let us consider digraph D_j, coalition $E \subseteq N$ and node $i \in N \setminus E$. Then

$$(3.34) \qquad v_{D_j}(E \cup \{i\}) - v_{D_j}(E) = \begin{cases} \theta_j & \text{if } i \in P_D(j) \text{ and } E \cap P_D(j) = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

We consider all the permutations of N. Let elements of $P_D(j)$ be $\{i_1, i_2, \ldots i_l\}$, where $l = p_D(j)$. We let the group $G1$ consist of the permutations where i_1 is in front of $P_D(j)$. Similarly we let the group $G2$ consist of the permutations where i_2 is in front of $P_D(j)$. Then we have $p_D(j)$ groups. The one of these groups corresponds to the case of $i \in P_D(j)$ and $E \cap P_D(j) = \emptyset$. Therefore, from the definition of the Shapley value (2.3) and (3.34), the Shapley value of node i in game (N, v_{D_j}) is given by

$$(3.35) \qquad \varphi_i(v_{D_j}) = \begin{cases} \dfrac{\theta_j}{p_D(j)} & (\forall i \in P_D(j)) \\ 0 & (\forall i \in N \setminus P_D(j)). \end{cases}$$

From the additivity property of the Shapley value, (3.33) and (3.35) it holds that for all $i \in N$,

$$\begin{aligned}
\varphi_i(v_D) &= \varphi_i \left(\sum_{j \in N} v_{D_j} \right) \\
&= \sum_{j \in N} \varphi_i(v_{D_j}) \\
&= \sum_{j \in S_D(i)} \varphi_i(v_{D_j}) + \sum_{j \in N \setminus S_D(i)} \varphi_i(v_{D_j}) \\
&= \sum_{j \in S_D(i)} \frac{\theta_j}{p_D(j)}.
\end{aligned}$$

$\square$

Theorem 3.15. *Let $v_D^1(E)$ be defined as*

$$v_D^1(E) = \sum_{i \in E} \sum_{j \in \bar{S}_{D(i)}} 1$$

for every $E \subseteq N$, and let $\varphi(v_D^1)$ be the Shapley value of the game (N, v_D^1). Let $v_D^2(E)$ be defined as

$$v_D^2(E) = \sum_{j \in S_{D(E)}} \varphi_j(v_D^1).$$

Then the Shapley value of the game (N, v_D^2) is equal to the γ plus measure.

Proof. Straightforward from Lemma 3.13 and 3.14. $\square$

3.3. **List of the measures.** We also propose other measures, one is the minus measure which is defined in the way opposite to the plus measure, i.e., replacing $S_D(i)$ with $P_D(i)$ and $P_D(i)$ with $S_D(i)$. Note that the more loses, the more high relational power obtains in the minus measure. Another measure is the plus minus measure that is a combination of the plus and minus measures. We show the definition and the characteristic function v_D of these measures in Table 3.1. The γ plus and minus measure are defined by the twice repeated game, then we do not show the characteristic function of the γ plus minus measure.

TABLE 3.1. List of measures

name	symbol	definition	characteristic function $v_D(E)$
α plus [2]	$\alpha_i^+(D)$	$s_D(i)$	$\sharp\{(i,j) \mid (i,j) \in D, i \in E\}$
α minus	$\alpha_i^-(D)$	$p_D(i)$	$\sharp\{(j,i) \mid (j,i) \in D, i \in E\}$
α plus minus	$\alpha_i^\pm(D)$	$\alpha_i^+(D) - \alpha_i^-(D)$	$\sharp\{(i,j) \mid (i,j) \in D, i \in E\}$ $-\sharp\{(j,i) \mid (j,i) \in D, i \in E\}$
β plus [2]	$\beta_i^+(D)$	$\displaystyle\sum_{j \in S_D(i)} \frac{1}{p_D(j)}$	$\sharp\{k \mid i \in E, (i,k) \in D\}$ [2]
β minus	$\beta_i^-(D)$	$\displaystyle\sum_{j \in P_D(j)} \frac{1}{s_D(j)}$	$\sharp\{k \mid j \in E, (k,j) \in D\}$
β plus minus	$\beta_i^\pm(D)$	$\beta_i^+(D) - \beta_i^-(D)$	$\sharp\{k \mid i \in E, (i,k) \in D\}$ $-\sharp\{k \mid j \in E, (k,j) \in D\}$
γ plus	$\gamma_i^+(D)$	$\displaystyle\sum_{j \in S_D(i)} \frac{s_D(j)+1}{p_D(j)}$	first : $\displaystyle\sum_{i \in E} \sum_{j \in \bar{S}_D(i)} 1$ second : $\displaystyle\sum_{j \in S_D(E)} \varphi_j(v_D^1)$
γ minus	$\gamma_i^-(D)$	$\displaystyle\sum_{j \in P_D(i)} \frac{p_D(j)+1}{s_D(j)}$	first : $\displaystyle\sum_{i \in E} \sum_{j \in \bar{P}_D(i)} 1$ second : $\displaystyle\sum_{j \in P_D(E)} \varphi_j(v_D^1)$
γ plus minus	$\gamma_i^\pm(D)$	$\gamma_i^+(D) - \gamma_i^-(D)$	
δ plus [1]	$\delta_i^+(D)$	$\displaystyle\sum_{j \in S_D(i) \cup \{i\}} \frac{1}{p_D(j)+1}$	$\sharp\{j \in N \mid P_D(j) \cup \{j\} \subset E\}$ [1]
δ minus	$\delta_i^-(D)$	$\displaystyle\sum_{j \in P_D(i) \cup \{i\}} \frac{1}{s_D(j)+1}$	$\sharp\{j \in N \mid S_D(j) \cup \{j\} \subset E\}$
δ plus minus	$\delta_i^\pm(D)$	$\delta_i^+(D) - \delta_i^-(D)$	$\sharp\{j \in N \mid P_D(j) \cup \{j\} \subset E\}$ $-\sharp\{j \in N \mid S_D(j) \cup \{j\} \subset E\}$

4. NUMERICAL EXAMPLE

In this section we compute the relational power on various digraphs, and rank the nodes based on their relational power. Here we compare the ranking based on binary ANP when the digraph is complete. We also compute the relational power on incomplete digraphs.

4.1. **Methods.** First we briefly introduce the ANP (Analytic Network Process) by Saaty [**?**]. In ANP the paired relation of objects i and j is given as follows.

$$\text{If object } i \text{ wins object } j \text{ then } \begin{array}{l} a_{ij} := \theta \\ a_{ji} := 1/\theta, \end{array}$$

where θ is a parameter greater than one. Then we make the $n \times n$ matrix, called a pairwise comparison matrix,

$$A = [a_{ij}],$$

where the diagonal component a_{ii} is one. Next we compute an eigen vector of A, and we regard the ith component of this eigen vector as object i's value. The off-diagonal elements of A are either θ or $1/\theta$. Then this is called a binary problem of ANP.

Generally the pairwise relation of the binary problem can be shown by a digraph, i.e., we make an arc from node i to node j if object i wins object j. When for any pair of nodes one wins the other, and the digraph has an arc between any pair of nodes, we say that the situation has the complete information.

We say that the digraph D has consistency when for any triplet of nodes i, j and k,

(4.1) $$(i,j) \in D \ \text{and} \ (j,k) \in D \ \text{implies} \ (i,k) \in D.$$

A cycle exists in digraph D when (4.1) does not hold. Therefore we see that the consistency of the digraph is low if there are many cycles in the digraph. In the following discussion we will focus on the number of cycles in the digraph. Let $B = [b_{ij}]$ be an $n \times n$ matrix defined as

$$\text{if object } i \text{ wins object } j \text{ then } \begin{array}{l} b_{ij} := 1 \\ b_{ji} := 0, \end{array}$$

where diagonal components of B are zero, which corresponds to the fact that the digraph is irreflexive. An example of matrices A and B is given below.

$$A = \begin{bmatrix} 1 & \theta & \theta & \theta \\ 1/\theta & 1 & 1/\theta & \theta \\ 1/\theta & \theta & 1 & 1/\theta \\ 1/\theta & 1/\theta & \theta & 1 \end{bmatrix} \longleftrightarrow B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We know that the parameter θ effects the eigen vector. In the numerical example that we will show in the next section the value of θ is fixed to two. We fix n and produce every possible matrix A and B. Then we compute the eigen vector as well as the node relational power, and we determine the ranking. We calculate the relational power of each measure listed on Table 3.1. We multiply minus measure by -1. We will show some of the results for $n = 4$ and 5.

4.2. **Result and discussion.** First we introduce the term for the following discussion. We say that node i and node j are equivalent when there are paths from i to j and j to i, and write $i \approx j$. The relation $\approx$ is an equivalence relation. An equivalence class given by the equivalence relation is said to be a strongly-connected component in digraph D. There is a partial order among these strongly-connected components. In this paper, we regard this partial order as the order among strongly-connected components. Note that this partial order is a total order if digraph D is a complete digraph.

First we discuss the character of the ranking when the digraph is complete. Each strongly-connected component consists of one node when there is no cycle in the digraph. Then the ranking of strongly-connected components is identical to the total order that is consistent with the direction of arcs. We observed that each ranking of measure is equal to such total order, while each node relational power varies. See Table 4.1. When there is one cycle, the digraph is divided into several strongly-connected components, the one consists of three nodes which form a cycle and others consist of a single node. Then we find that all rankings given by measures are equal to the total order, and the nodes on a cycle are given the same rank. We obtain different rankings when there are more than one cycle. The more the number of cycles is, the more we obtain the different rankings. Also the ranking of the γ plus measure and the γ minus measure are totally different from other measures. This is because the power of the node which loses node i effects node

TABLE 4.1. Complete digraph with no cycle

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

0	relational power					ranking				
measure	A	B	C	D	E	A	B	C	D	E
ANP	0.323	0.245	0.185	0.141	0.107	1	2	3	4	5
α^+	4.000	3.000	2.000	1.000	0.000	1	2	3	4	5
α^-	0.000	-1.000	-2.000	-3.000	-4.000	1	2	3	4	5
$\alpha^\pm$	4.000	2.000	0.000	-2.000	-4.000	1	2	3	4	5
β^+	2.083	1.083	0.583	0.250	0.000	1	2	3	4	5
β^-	0.000	-0.250	-0.583	-1.083	-2.083	1	2	3	4	5
$\beta^\pm$	2.083	0.833	0.000	-0.833	-2.083	1	2	3	4	5
γ^+	6.417	2.417	0.917	0.250	0.000	1	2	3	4	5
γ^-	0.000	-0.250	-0.917	-2.417	-6.417	1	2	3	4	5
$\gamma^\pm$	6.417	2.167	0.000	-2.167	-6.417	1	2	3	4	5
δ^+	2.283	1.283	0.783	0.450	0.200	1	2	3	4	5
δ^-	-0.200	-0.450	-0.783	-1.283	-2.283	1	2	3	4	5
$\delta^\pm$	2.083	0.833	0.000	-0.833	-2.083	1	2	3	4	5

TABLE 4.2. Complete digraph with three cycles

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$
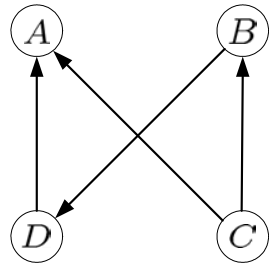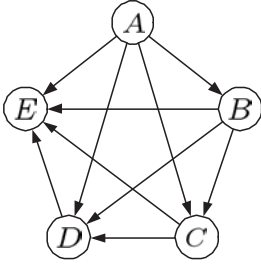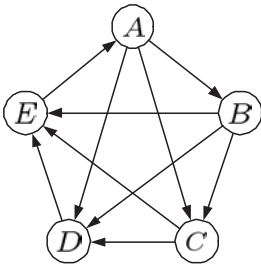

3	relational power					ranking				
measure	A	B	C	D	E	A	B	C	D	E
ANP	0.259	0.240	0.186	0.145	0.170	1	2	3	5	4
α^+	3.000	3.000	2.000	1.000	1.000	1	1	3	4	4
α^-	-1.000	-1.000	-2.000	-3.000	-3.000	1	1	3	4	4
$\alpha^\pm$	2.000	2.000	0.000	-2.000	-2.000	1	1	3	4	4
β^+	1.833	1.167	0.667	0.333	1.000	1	2	4	5	3
β^-	-1.000	-0.333	-0.667	-1.167	-1.833	3	1	2	4	5
$\beta^\pm$	0.833	0.833	0.000	-0.833	-0.833	1	1	3	4	4
γ^+	6.167	2.833	1.333	0.667	4.000	1	3	4	5	2
γ^-	-4.000	-0.667	-1.333	-2.833	-6.167	4	1	2	3	5
$\gamma^\pm$	2.167	2.167	0.000	-2.167	-2.167	1	1	3	4	4
δ^+	1.583	1.333	0.833	0.500	0.750	1	2	3	5	4
δ^-	-0.750	-0.500	-0.833	-1.333	-1.583	2	1	3	4	5
$\delta^\pm$	0.833	0.833	0.000	-0.833	-0.833	1	1	3	4	4

i's relational power in γ plus measure similarly the power of the node which wins node i effects node i's relational power in γ minus measure, while we only use the information of nodes links directly to node i in other measures. See Table 4.2.
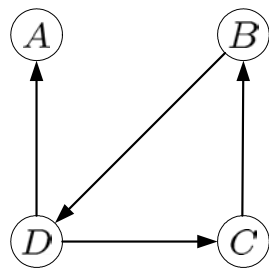
We also consider the β plus measure and δ plus measure. They have similar definitions, however they give different ranking for the digraph with many cycles. Interestingly the ranking of the δ plus measure is almost equal to that of ANP. See Table 4.2.

TABLE 4.3. Identical comparison number

	ranking			
measure	A	B	C	D
β^+	4	2	1	3
β^-	4	2	1	3
$\beta^\pm$	4	2	1	3
γ^+	4	2	1	3
γ^-	4	2	1	3
$\gamma^\pm$	4	2	1	3
δ^+	4	2	1	3
δ^-	4	2	1	3
$\delta^\pm$	4	2	1	3

TABLE 4.4. Different comparison number

	ranking			
measure	A	B	C	D
β^+	4	2	2	1
β^-	1	3	1	3
$\beta^\pm$	4	3	2	1
γ^+	4	1	3	1
γ^-	1	3	1	3
$\gamma^\pm$	4	1	1	1
δ^+	4	2	2	1
δ^-	4	3	1	1
$\delta^\pm$	4	3	2	1

We also generate an incomplete digraphs, by making matrices of zero and one. In our numerical example, we fix $n = 4$ because there are too many possible matrices[1]. We do not consider the α^+, α^- and $\alpha^\pm$ measures because these measures only count the number of objects linked directly to each node then they are not useful in the incomplete digraph. Here we focus on how many times each object is compared with and we call this number *comparison number*. For example the digraph in Table 4.4 the object A's comparison number is one, similarly that of B and C is two, and that of D is three.

We find that if all comparison numbers are identical then we obtain the same ranking for each measure, which is shown in Table 4.3. Conversely, if all comparison numbers are not identical then we obtain different rankings, as shown in Table 4.4. Especially the ranking of β minus measure and γ minus measure is different from that of others. The object A is ranked almost at the bottom, however these measures rank this at the top. Similar situation are observed for the β plus measure and the γ plus measure in other digraphs. Therefore when we use these measures in complete digraph, we should look out for the structure of the digraph.

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[1]When the dimension is $n = 4$, there are 41 matrices.

5. Conclusion and Remarks

In this paper, we consider the case of many judges and many objects, and every judge compares some of the objects. We proposed two measures, the γ plus measure and the δ plus measure based on van den Brink and Gilles [2]. Also we axiomatized these two measures. Especially the γ plus measure is an extension of the α plus measure and the β plus measure, and its axioms share some character with those measures. We showed that the Shapley value of the second game is equal to the γ plus measure. Then we showed the validity of these measures from two aspects.

In the numerical example we computed the relational power on complete digraphs and incomplete digraphs and made the ranking. In the case of complete digraphs we observed that the number of cycles in the digraph effects the ranking, and the δ plus measure and ANP have almost the same ranking. In the case of incomplete digraphs we observed that the comparison number effects the ranking. If we can decide the combination of judges and objects then it is important to set each object's comparison number be equal.

We rewrite each measure in terms of the following matrix,

$$(5.1) \qquad A = [a_{ij}] = \left\{ \begin{array}{ll} 1 & \text{if } (i,j) \in D \\ 0 & \text{otherwise,} \end{array} \right.$$

$$(5.2) \qquad B = [b_{ij}] = \left\{ \begin{array}{ll} \dfrac{1}{p_D(j)} & \text{if } (i,j) \in D \\ 0 & \text{otherwise,} \end{array} \right.$$

$$(5.3) \qquad D = [d_{ij}] = \left\{ \begin{array}{ll} \dfrac{1}{p_D(j)+1} & \text{if } (i,j) \in D \text{ or } i = j \\ 0 & \text{otherwise.} \end{array} \right.$$

Letting $\mathbf{e}$ be the vector of ones. We obviously see that $A\mathbf{e}$, $B\mathbf{e}$ and $D\mathbf{e}$ are equal to the α plus measure, the β plus measure and the γ plus measure, respectively. Then we can easily write the Shapley value of the second and more repeated game by these matrices, for example, if we repeat twice the game that makes the α plus measure then the Shapley value is given as $A^2\mathbf{e}$. Therefore we see that the γ plus measure is given as $(A + E)B\mathbf{e}$, where E is the matrix of ones. When the game is repeated infinitely many times, the Shapley value is given as $A^\infty\mathbf{e}$, $B^\infty\mathbf{e}$ and $D^\infty\mathbf{e}$, respectively.

We can regard matrix D as the transposed transition probability matrix since for all $i, j \in N$, $d_{ij} \geq 0$ and $\sum_{i=1}^N d_{ij} = 1$. The vector $\omega \in \mathbb{R}^N$ satisfying $\omega = D\omega$ is an eigen vector of D or a stationary distribution. On the other hand, ANP provides the principal eigen vector of the super matrix as a solution. Therefore the solution of ANP is considered as the Shapley value of the infinitely repeated game.

When the overall judgement of objects in ANP is considered Sekitani [7], [8] make a new super matrix whose $(1,1)$ element is one. Since the solution of ANP is the Shapley value of the infinitely repeated game, it can be viewed to use the information of the whole structure of the digraph. While our proposed measures is at most two times game, therefore we use the information of the node around node i when we compute node i's relational power.

Acknowledgement

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