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# The p-Adic Numbers and Conic Sections

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# The p-Adic Numbers and Conic Sections

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ABSTRACT

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This thesis introduces the p-adic metric on the rational numbers. We then present the basic properties of this metric. Using this metric, we explore conic sections, viewed as equidistant sets. Lastly, we move on to sequences and series, and from there, we define p-adic expansions and the analytic completion of $\mathbb{Q}$ with respect to the p-adic metric, which leads to exploring some arithmetic properties of $\mathbb{Q}_p$.

## ACKNOWLEDGMENTS

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TABLE OF CONTENTS

LIST OF FIGURES

# 1 INTRODUCTION

The set of real numbers is the ordered complete field. It contains rational numbers and irrational numbers. A rational number is defined as the quotient of two integer numbers, and the set of rational numbers is $\mathbb{Q} = \left\{ \frac{x}{y} : x, y \in \mathbb{Z} \right\}$. An irrational number such as $\pi$ can be represented as the limit of a sequence of rational numbers. That is $\pi = \lim_{n \to \infty} a_n$, where $\{a_n\} = \left\{ 3, \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000} .... \right\}$. The subsequent terms of this sequence are getting closer and closer to each other. So, $\{a_n\}$ is a Cauchy Sequence, and any irrational number is defined as the limit of a Cauchy Sequence. Now, one can define the set of real numbers as the rational numbers plus all of the limits of Cauchy Sequences of rational numbers. So, the set of all real numbers is the analytic completion of the rational numbers. Note that the decreasing distance between the terms of the Cauchy Sequence above is given using the usual absolute value. However, this is not the only way to define a metric on the rational numbers. There is a different type of distance called a p-adic metric [8]. In this thesis, we consider this p-adic metric and consequences for conic sections viewed as equidistant sets. In addition, we present the analytic completion of the rationals with respect to this metric.

## 2 P-ADIC METRIC

In this section, we will introduce the p-adic distance, which will be used later in some conic sections' equations. First, we begin by defining the usual absolute value or modulus for an arbitrary set $\mathbb{X}$.

**Definition**. Let $\mathbb{K}$ be a field. An *absolute value* is a function $|\cdot| : \mathbb{K} \longrightarrow \mathbb{R}_{\geq 0}$. For all absolute values, the following properties hold:

1. $|x| \geq 0$, for all $x \in \mathbb{K}$.

2. $|x| = 0$, if and only if $x = 0$.

3. $|xy| = |x||y|$, for all $x, y \in \mathbb{K}$.

4. $|x+y| \leq |x| + |y|$, for all $x, y \in \mathbb{K}$. (triangle inequality)

Additionally, if $|x+y| \leq max\Big(|x|, |y|\Big)$, for all $x, y \in \mathbb{K}$, then $|\cdot|$ is *non-archimedean*.

Note that it is worth mentioning two absolute values. The first is the usual absolute value, which is defined as:

$$|x|_\infty = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

The second one is the trivial absolute value, and it's defined as:

$$|x|_t = \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

From the absolute value, we define the following distance function:

**Definition**. Let $\mathbb{X}$ be a set and $d$ be a function mapping $\mathbb{X} \times \mathbb{X}$ into $\mathbb{R}$. The pair $(\mathbb{X}, d)$ is called a metric space, where $d$ is the metric such that for all $x, y, z \in \mathbb{X}$, we have:

1. $d(x, y) \geq 0$.

2. $d(x, y) = 0$ if and only if $x = y$.

3. $d(x, y) = d(y, x)$. (Symmetry)

4. $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle Inequality)

**Examples** :

1. $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$. (Euclidean distance)

2. $(\mathbb{R}^2, d)$, where $d((x_1, x_2), (y_1, y_2)) = |y_1 - x_1| + |y_2 - x_2|$. (Taxicab distance)

3. $(\mathbb{R}^2, d)$, where $d((x_1, x_2), (y_1, y_2)) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$. (Euclidean distance)

**Definition**. A metric $d$ is *ultrametric* or *non-archimedean*, if the *ultrametric inequality* is satisfied. That is for any x, y, and z in $\mathbb{X}$, $d(x, z) \leq max\left\{d(x, y), d(y, z)\right\}$, which is a *stronger* property than triangle inequality.

**Example**. Let $\mathbb{X}$ be any set, and for any $x, y \in \mathbb{X}$, we have:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y. \end{cases}$$

So, $d$ is ultrametric, and it's called the *trivial metric*, as defined in [11].

For the rest of this paper, we're interested in a particular example of a non-archimedean metric, namely, the p-adic distance. First, we need to define the p-adic valuation.

## 2.1 p-Adic Valuation

**Definition**. [10] Let $a$ and $b$ be integers, where $a \neq 0$. If $a^k | b$, but $a^{k+1} \nmid b$ for some $k \in \mathbb{Z}_{>0}$, then we say that $a^k$ *exactly divides* $b$, and we write $a^k || b$.

**Definition**. Let $a \in \mathbb{Q}$, with $a \neq 0$. Then, $a$ can be written as $a = p^n \left(\frac{x}{y}\right)$, where $p$ is a prime number, $n \in \mathbb{Z}$, $x, y \in \mathbb{Z}^*$, $\gcd(x, y) = 1$, and $p \nmid xy$. We call n the *p-adic valuation* of $a$, and we write $\nu_p(a) = n$ or $ord_p(a) = n$.

The p-adic valuation is the function as: $\nu_p : \mathbb{Q} \longrightarrow \mathbb{Z} \cup \{\infty\}$, and $\nu_p(0) = \infty$.

**Example**. We compute $\nu_p(45)$ for different prime numbers, and we get:

$\nu_3(45) = \nu_3(5^1 \cdot 3^2) = 2,$

$\nu_5(45) = \nu_5(5^1 \cdot 3^2) = 1,$

$\nu_7(45) = \nu_7(5^1 \cdot 3^2 \cdot 7^0) = 0.$

**Proposition 2.1:** For all $x, y \in \mathbb{Q}$, we have:

1. $\nu_p(xy) = \nu_p(x) + \nu_p(y)$.

2. $\nu_p\left(\frac{x}{y}\right) = \nu_p(x) - \nu_p(y)$.

3. $\nu_p(x - y) \geq \min\Big(\nu_p(x), \nu_p(y)\Big)$, with equality when $\nu_p(x) \neq \nu_p(y)$.

*Proof.* Let $x, y, x', y' \in \mathbb{Q}$ and $p$ a prime number. Write $x = p^n x'$ and $y = p^m y'$, with $\nu_p(x') = 0$ and $\nu_p(y') = 0$. Then, 1.

$$\begin{aligned}\nu_p(xy) &= \nu_p\Big(p^{n+m}(x'y')\Big)\\ &= n+m\\ &= \nu_p(x)+\nu_p(y).\end{aligned}$$

2.

$$\begin{aligned}\nu_p\left(\frac{x}{y}\right) &= \nu_p\Big(p^{n-m}(x'y')\Big)\\ &= n-m\\ &= \nu_p(x)-\nu_p(y).\end{aligned}$$

3. Consider two cases:

<u>**Case 1**. $n = m$</u>:

$$\begin{aligned}\nu_p(x-y) &= \nu_p\Big(p^{n}(x'-y')\Big)\\ &= \nu_p(p^n)+\nu_p(x'-y')\\ &= n+\nu_p(x'-y').\end{aligned}$$

We can rewrite $x'$ and $y'$ as: $x' = \frac{a}{b}$ and $y' = \frac{c}{d}$, where $a, b, c, d, \in \mathbb{Z}^*$, $gcd(a,b) = 1$, and $gcd(c,d) = 1$. Then, $\nu_p(x'-y') = \nu_p\left(\frac{a}{b}-\frac{c}{d}\right) = \nu_p\left(\frac{ad-bc}{bd}\right)$

Note that $\nu_p(x') = 0$ implies that $p \nmid b$, and $\nu_p(y') = 0$ implies that $p \nmid d$. So, $p \nmid bd$.

Hence, $\nu_p(x'-y') = k \geq 0$.

That is $\nu_p(x-y) = n + \nu_p(x'-y') = n+k \geq \min\Big(\nu_p(x), \nu_p(y)\Big)$.

**<u>Case 2</u>**<u>. $n \neq m$</u> :

Without loss of generality, suppose that $n > m$. Then,

$$\begin{aligned}\nu_p(x-y) &= \nu_p\Big(p^n x' - p^m y'\Big)\\ &= \nu_p\Big(p^m\Big(p^{n-m}x' - y'\Big)\Big)\\ &= \nu_p(p^m) + \nu_p\Big(p^{n-m}x' - y'\Big)\end{aligned}$$

Because $p$ divides $p^{n-m}x'$, but it does not divide $y'$, $p$ does not divide $p^{n-m}x' - y'$. That is $\nu_p(p^{n-m}x' - y') = 0$. Then, $\nu_p(x-y) = m + 0 = m = \nu_p(y) = min\Big(\nu_p(x), \nu_p(y)\Big)$.

□

## 2.2 One-Dimensional p-Adic Absolute Value

**Definition** The p-adic absolute value of $a \in \mathbb{Q}$ is a map $|\cdot|_p : \mathbb{Q} \longrightarrow \mathbb{R}_{\geq 0}$, that is defined by:

$$|a|_p = \begin{cases} \frac{1}{p^{\nu_p(a)}}, & \text{if} \quad a \neq 0\\ 0, & \text{if} \quad a = 0\end{cases}$$

**Example**. We compute $|45|_p$ for different prime numbers, and we get:

$|45|_3 = \left|3^2 \cdot 5^1\right|_3 = \frac{1}{3^2} = \frac{1}{9}$,

$|45|_5 = \left|3^2 \cdot 5^1\right|_5 = \frac{1}{5}$,

$|45|_7 = \left|3^2 \cdot 5^1 \cdot 7^0\right|_7 = \frac{1}{7^0} = 1$.

Now, let's compare the distances between the same pair of points, using the usual absolute value $|\cdot|$ $\Big($also called the absolute value at infinity $|\cdot|_\infty\Big)$ and the p-adic absolute value $|\cdot|_p$, where p is a prime number.

Let d: $\mathbb{Q}\times\mathbb{Q} \longrightarrow \mathbb{R}_{\geq 0}$ be defined by: $d(x,y) = |x-y|$, for all $x, y \in \mathbb{Q}$.

Consider $x = \frac{7}{8}$, $y = \frac{3}{32}$, and $z = \frac{7}{16}$. Then,

$$d(x,y) = \left|\frac{7}{8} - \frac{3}{32}\right| = \frac{25}{32},$$

$$d(x,z) = \left|\frac{7}{8} - \frac{7}{16}\right| = \frac{7}{16},$$

$$d(z,y) = \left|\frac{7}{16} - \frac{3}{32}\right| = \frac{11}{32}.$$

This function satisfies the four conditions in the metric space definition above. Thus, $(\mathbb{Q}, d)$ is a metric space. However, $d$ is not ultrametric, since $d(x,y) \nleq max\Big(d(x,z), d(z,y)\Big)$.

**Theorem**. The p-adic distance $d_p(x,y)$ is ultrametric.

*Proof.* Let $x, y \in \mathbb{Q}$ such that $|x|_p = p^{-\nu_p(x)}$ and $|y|_p = p^{-\nu_p(y)}$. From Proposition 2.1/(3), we have that $\nu_p(x-y) \geq min\Big(\nu_p(x), \nu_p(y)\Big)$. So, $p^{-\nu_p(x-y)} \leq p^{-min\big(\nu_p(x),\nu_p(y)\big)}$. It implies that $|x-y|_p \leq max\Big(|x|_p, |y|_p\Big)$, as desired. □

**Example**. We compute the 2-adic distances $d_2(x,y)$, $d_2(x,z)$, $d_2(z,y)$, where $x = \frac{7}{8}$, $y = \frac{3}{32}$, and $z = \frac{7}{16}$. So, we get:

$$d_2(x,y) = \left|\frac{7}{8} - \frac{3}{32}\right|_2 = \left|25\cdot 2^{-5}\right|_2 = 2^5,$$

$$d_2(x,z) = \left|\frac{7}{8} - \frac{7}{16}\right|_2 = \left|7 \cdot 2^{-4}\right|_2 = 2^4,$$

$$d_2(z,y) = \left|\frac{7}{16} - \frac{3}{32}\right|_2 = \left|11 \cdot 2^{-5}\right|_2 = 2^5.$$

So, the ultrametric property is satisfied, since $d_2(x,y) \leq max\Big(d_2(x,z), d_2(z,y)\Big)$.

Note that $d_2(x,y) = d_2(z,y)$. This equality can be explained by the following theorem.

**Theorem**. Using the p-adic absolute value on the field of rationals $\mathbb{Q}$, all triangles are isosceles, as in [3].

*Proof.* Let $\Delta xyz$ be a triangle. It has sides of length $p^{-\nu_p(x-z)}$, $p^{-\nu_p(x-y)}$, and $p^{-\nu_p(y-z)}$.

If any two of $\nu_p(x-z)$, $\nu_p(x-y)$, and $\nu_p(y-z)$ are equal, we're done.

WLOG, suppose that $\nu_p(x-y) \neq \nu_p(y-z)$. Then using the property (3) of the p-adic valuation, defined above, we have that $\nu_p(x-z) = min\Big(\nu_p(x-y), \nu_p(y-z)\Big)$. Ergo, at least two of the three sides must be of equal length. $\square$

The following definition and theorem are the main reason why we're interested in the p-adic absolute value, since it's one of the non-trivial absolute values.

**Definition**. We say that two absolute values, $|\cdot|$ and $|\cdot|'$, are *equivalent* if $|\cdot|' = |\cdot|^r$, for some $r > 0$.

**Ostrowski Theorem**. [5] Every non-trivial absolute value on $\mathbb{Q}$ is equivalent to either $|\cdot|_p$ or $|\cdot|_\infty$ , where $p$ is a prime number.

## 2.3 Two-Dimensional p-Adic Absolute Value

We can extend the ordinary distance on $\mathbb{Q}$ to $\mathbb{Q}^2$. Thus, as in [1], we want to construct a distance function on $\mathbb{Q}^2$, and we would like it to be ultrametric. First, as a non-example, we consider the following map:

$$D_p : \mathbb{Q}^2 \times \mathbb{Q}^2 \longrightarrow \mathbb{R}_{\geq 0}$$

such that for $A(a_1, a_2), B(b_1, b_2) \in \mathbb{Q}^2$,

$$D_p(A,B) = \sqrt{\left(d_p(a_1,b_1)\right)^2 + \left(d_p(a_2,b_2)\right)^2}.$$

**Example** : Let $A = \left(\frac{1}{4}, \frac{1}{4}\right)$; $B = (\frac{1}{6}, \frac{1}{3})$; $C = \left(0, \frac{1}{8}\right)$.

$$\begin{aligned}
D_2(A,C) &= \sqrt{\left(d_2\left(\frac{1}{4},0\right)\right)^2 + \left(d_2\left(\frac{1}{4},\frac{1}{8}\right)\right)^2} \\
&= \sqrt{\left(\left|\frac{1}{4}-0\right|_2\right)^2 + \left(\left|\frac{1}{4}-\frac{1}{8}\right|_2\right)^2} \\
&= \sqrt{\left(\left|\frac{1}{4}\right|_2\right)^2 + \left(\left|\frac{1}{8}\right|_2\right)^2} \\
&= \sqrt{\left(\left|2^{-2}\right|_2\right)^2 + \left(\left|2^{-3}\right|_2\right)^2} \\
&= \sqrt{(2^2)^2 + (2^3)^2} \\
&= \sqrt{80},
\end{aligned}$$

$$D_2(A,B) = \sqrt{\left(d_2\left(\frac{1}{4},\frac{1}{6}\right)\right)^2 + \left(d_2\left(\frac{1}{4},\frac{1}{3}\right)\right)^2}$$

$$= \sqrt{\left(\left|\frac{1}{4}-\frac{1}{6}\right|_2\right)^2+\left(\left|\frac{1}{4}-\frac{1}{3}\right|_2\right)^2}$$

$$= \sqrt{\left(\left|\frac{1}{12}\right|_2\right)^2+\left(\left|\frac{-1}{12}\right|_2\right)^2}$$

$$= \sqrt{\left(\left|\frac{1}{3}\cdot 2^{-2}\right|_2\right)^2+\left(\left|\frac{-1}{3}\cdot 2^{-2}\right|_2\right)^2}$$

$$= \sqrt{(2^2)^2+(2^2)^2}$$

$$= \sqrt{32},$$

$$D_2(B,C) = \sqrt{\left(d_2\left(\frac{1}{6},0\right)\right)^2+\left(d_2\left(\frac{1}{3},\frac{1}{8}\right)\right)^2}$$

$$= \sqrt{\left(\left|\frac{1}{6}-0\right|_2\right)^2+\left(\left|\frac{1}{3}-\frac{1}{8}\right|_2\right)^2}$$

$$= \sqrt{\left(\left|\frac{1}{6}\right|_2\right)^2+\left(\left|\frac{5}{24}\right|_2\right)^2}$$

$$= \sqrt{\left(\left|\frac{1}{3}\cdot 2^{-1}\right|_2\right)^2+\left(\left|\frac{5}{3}\cdot 2^{-3}\right|_2\right)^2}$$

$$= \sqrt{(2)^2+(2^3)^2}$$

$$= \sqrt{68}.$$

So, $D_2(A,C) \nleq max(D_2(A,B), D_2(B,C))$. That is $(\mathbb{Q}^2, D_2)$ is not an ultrametric space.

**Theorem**. [1] Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. A function on $\mathbb{Q}^2 \times \mathbb{Q}^2$, that does satisfy the ultrametric inequality, is given by $\underline{D(A,B) = max\left\{d(a_1,b_1), d(a_2,b_2)\right\}}$, where $d$ is an ultrametric distance.

*Proof.* We let $A = (a_1,a_2), B = (b_1,b_2)$, and $C = (c_1,c_2)$. Then,

$$\begin{aligned}
D(A,C) &= max\left\{d(a_1,c_1), d(a_2,c_2)\right\} \\
&\leq max\left\{max\left(d(a_1,b_1), d(b_1,c_1)\right), max\left(d(a_2,b_2), d(b_2,c_2)\right)\right\} \\
&= max\left\{d(a_1,b_1), d(b_1,c_1), d(a_2,b_2), d(b_2,c_2)\right\} \\
&= max\left\{max\left(d(a_1,b_1), d(a_2,b_2)\right), max\left(d(b_1,c_1), d(b_2,c_2)\right)\right\} \\
&= max\left\{D(A,B), D(B,C)\right\}.
\end{aligned}$$

$\square$

# 3 CONIC SECTIONS

In this section, we will compare conic sections in Euclidean space and conic sections in the ultrametric space.

## 3.1 Conic Sections in Euclidean Space

**Definition**. The conic sections are the curves generated by the intersection of a plane with one or two nappes of a cone, as mentioned in [13].

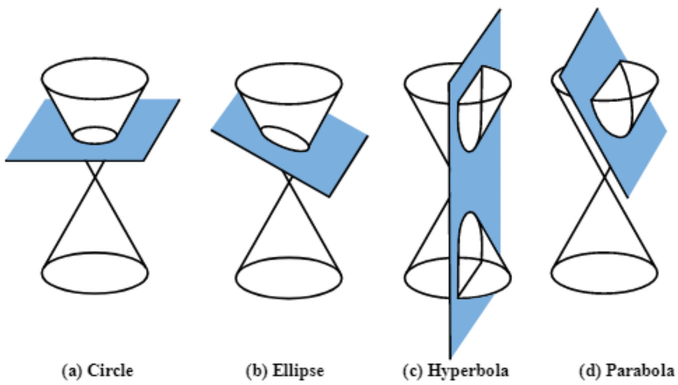


Figure 1: Conic Sections [14]

**Eccentricity**. The eccentricity $e$, of a conic section, as defined in [4], is the constant ratio of the distance of the point on the conic section from the focus and directrix, which is a fixed line that does not contain the focus. As eccentricity increases, the conic section deviates more and more from the shape of the circle. The value of $e$ for different conic sections is as follows:

- For circle, $e = 0$.
- For ellipse, $0 < e < 1$.
- For parabola, $e = 1$.
- For hyperbola, $e > 1$.

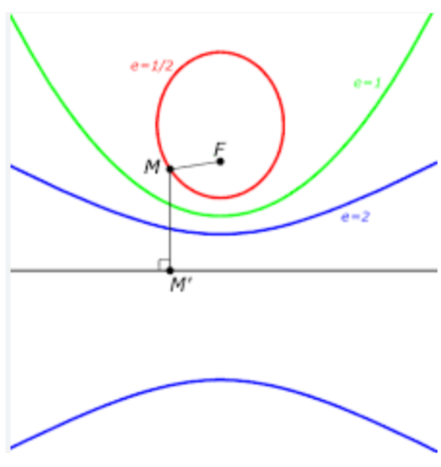


Figure 2: Eccentricity of Conic Sections [14]

**Circle**. The circle is a conic section that is formed when the cutting plane is parallel to the base of the cone, as shown in Figure 1($a$). The center is the focus point. The locus of the points have a fixed distance from the focus (center) and it is the radius of the circle. So, the circle, shown in Figure 3, with a radius $r$ and centered at (h,k) is the set:

$$\left\{(x,y) \in \mathbb{R}^2 : \quad \sqrt{(x-h)^2+(y-k)^2} = r, \;\; where\ r \in \mathbb{R}_{>0}\right\}$$

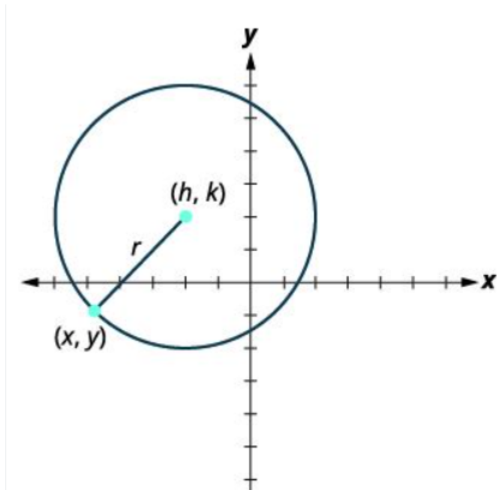


Figure 3: Circle [14]

**Ellipse**. The ellipse is a conic section that is formed when a plane intersects with the cone at one nappe and not parallel to the base, as defined in [15], and as shown in Figure 1($b$). Consider the ellipse centered at the origin, with two foci points $F(c,0)$ and $F'(-c,0)$, a minor axis of length $2b$, and a major axis of length $2a$, that is parallel to the $x$-axis.

Let $p(x,y)$ be any point on the ellipse. By definition of an ellipse, the sum of distances of this point to the two foci points $F$ and $F'$ is constant. That is $d(p,F)+d(p,F') = C$, where $C$ is a constant. In fact, $C = 2a$. Hence, the ellipse, in Figure 4, is the set:

$$\left\{(x,y) \in \mathbb{R}^2 : \quad \sqrt{(x-c)^2+(y-0)^2} + \sqrt{(x+c)^2+(y-0)^2} = 2a\right\}.$$

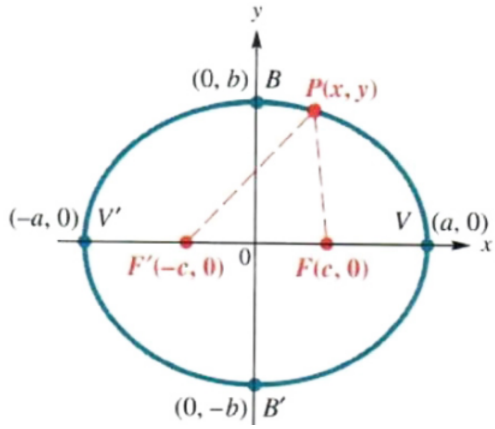


Figure 4: Ellipse [14]

**Parabola**.

The parabola is a conic section that is formed when the intersecting plane is at an angle to the surface of the cone, as shown in Figure 1($d$). It is a curve where any point of coordinates $(x, y)$ is at the same distance from the focus and the directrix, which is a straight line that is perpendicular to the axis of symmetry. Consider the parabola with vertex $(0, 0)$, a directrix $y = -p$, and a focus $(0, p)$. The distance $d$ from a point $(x, y)$ to the point $(x, -p)$ on the directrix is the difference of the $y$-values. That is $d = y + p$. The distance from the point $(x, y)$ and the point $(0, p)$ is $\sqrt{(x-0)^2 + (y-p)^2}$, and it's the same as the distance $d$. That is $y + p = \sqrt{(x-0)^2 + (y-p)^2}$. It follows that $x^2 = (y+p)^2 - (y-p)^2$. So, this parabola, shown in Figure 5, is the set:

$$\left\{(x, y) \in \mathbb{R}^2 : x^2 = 4yp\right\}.$$

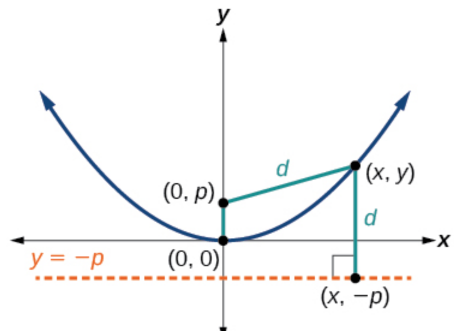


Figure 5: Parabola [14]

**Hyperbola**.

A hyperbola is formed when a plane is parallel to the axis of the cone, and intersects with both the nappes of the double cone, as shown in Figure 4(c). The two unconnected sections of the hyperbola are called branches. They are mirror images of each other, and their diagonally opposite arms approach the limit to a line. Consider the hyperbola centered at the origin, with the points $(-c,0)$ and $(c,0)$ as foci, and $2a$ as the distance between the two vertices $(a,0)$ and $(-a,0)$. Let $(x,y)$ be any point on this hyperbola. Then, by definition of a hyperbola, the difference of the distances from (x,y) to the foci is constant. That is $d\Big((x,y),(c,0)\Big) - d\Big((x,y),(-c,0)\Big) = C$, where $C$ is a constant and $C = 2a$. Hence, the hyperbola, in Figure 6, is the set:

$$\left\{(x,y) \in \mathbb{R}^2 : \quad \sqrt{(x-c)^2 + (y-0)^2} - \sqrt{(x+c)^2 + (y-0)^2} = 2a\right\}.$$

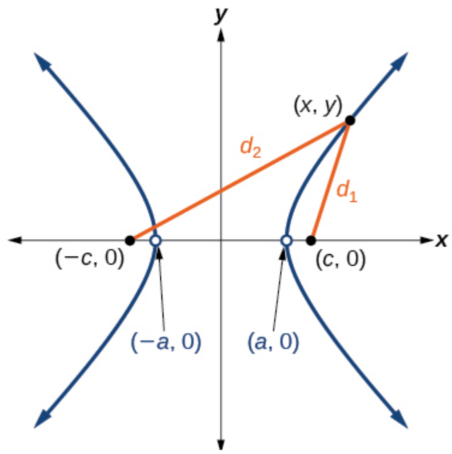


Figure 6: Hyperbola [14]

## 3.2 Ultrametric Circles

What is the circle, using the p-adic distance on $\mathbb{Q}$, of radius $r$, where $r \in \mathbb{Q}_{>0}$, and centered at $(0,0)$, for any prime $p$ ?

### 3.2.1 Circles Centered at the Origin

Define $C_r$ as the set of points of the same distance from the origin $(0,0)$, i.e.,
$C_r = \Big\{(x,y) \in \mathbb{Q}^2 : D_p\Big((x,y),(0,0)\Big) = r, \ \ where \ \ r \in \mathbb{Q}_{>0}\Big\}.$

**Lemma**. If $r \neq p^{-\nu}$, then $C_r = \emptyset$. That is:

$$\Big\{(x,y) \in \mathbb{Q}^2 : D_p\Big((x,y),(0,0)\Big) = ap^{-\nu}, \ \ p \nmid a \ and \ a \neq 1\Big\} = \emptyset.$$

*Proof.* Let $(x,y) \in \mathbb{Q}^2$. Suppose that $p^{\nu_1} \| x$ and $p^{\nu_2} \| y$, where $\nu_1$ and $\nu_2$ are the valuations of $x$ and $y$, respectively.

WLOG, let $\nu_1 \geq \nu_2$. Then, $D_p\Big((x,y),(0,0)\Big) = p^{-\nu_2} \neq ap^{-\nu}$, for any $\nu$. Thus, $(x,y) \notin C_r$, if $a \neq 1$. $\square$

Now, we look at the unit circle case, where the radius is 1. Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$, and let $\nu_1 = \nu_p(x)$ and $\nu_2 = \nu_p(y)$. So,

$$D_p\Big((x,y),(0,0)\Big) = p^{-min(\nu_1,\nu_2)} = 1 = p^0.$$

Then, $min(\nu_1,\nu_2) = 0$, and we consider two cases:

1. $\underline{\nu_1 = 0 \text{ and } \nu_2 \geq 0}$. That is $p \nmid a, p \nmid b$, and $p|c$.

2. $\underline{\nu_1 \geq 0 \text{ and } \nu_2 = 0}$. That is $p \nmid c, p \nmid d$, and $p|a$.

Hence, the circle of radius 1, centered at $(0,0)$ is the set:

$$\left\{\left(\frac{a}{b},\frac{c}{d}\right) : p \nmid ab, p|c \ \ OR \ \ p \nmid cd, p|a\right\}.$$

Consider the point $\left(\frac{3}{5},\frac{4}{5}\right)$. Then:

$$\begin{aligned}
D\left(\left(\frac{3}{5},\frac{4}{5}\right),(0,0)\right) &= max\left(d_2\left(\frac{3}{5},0\right),d_2\left(\frac{4}{5},0\right)\right)\\
&= max\left(\left|\frac{3}{5}\right|_2,\left|\frac{4}{5}\right|_2\right)\\
&= max\left(\left|\frac{3}{5}\cdot 2^0\right|_2,\left|\frac{1}{5}\cdot 2^2\right|_2\right)\\
&= max\left(1,\frac{1}{4}\right)
\end{aligned}$$

$$= 2^{-min(0,2)}$$

$$= 1.$$

So, the point $\left(\frac{3}{5}, \frac{4}{5}\right)$ is on the 2-adic unit circle.

### 3.2.2 Circles with Arbitrary Centers

Consider the points $x, y, h, k, r \in \mathbb{Q}$, such that, $|x|_p = p^{-\nu_1}$, $|y|_p = p^{-\nu_2}$, $|h|_p = p^{-\nu_3}$, $|k|_p = p^{-\nu_4}$, and $r = \frac{a}{b}p^{\nu_5}$, for some prime number $p$ and $a, b \in \mathbb{Q}_{>0}$.

We're interested in the set of points $(x,y) \in \mathbb{Q}^2$ of the same distance $r$ from the center $(h,k)$. That is $D\Big((x,y),(h,k)\Big) = r$. Then,

$$\begin{aligned} D\Big((x,y),(h,k)\Big) &= max\Big(d(x,h), d(y,k)\Big) \\ &= max\left(p^{-min(\nu_1,\nu_3)}, p^{-min(\nu_2,\nu_4)}\right) \\ &= p^{-min\Big(min(\nu_1,\nu_3), min(\nu_2,\nu_4)\Big)} \\ &= p^{-min(\nu_1,\nu_2,\nu_3,\nu_4)} \\ &= \frac{a}{b}p^{\nu_5}. \end{aligned}$$

Let $\nu = min\Big(\nu_1, \nu_2, \nu_3, \nu_3\Big)$. Then, $\frac{a}{b}p^{\nu_5} = p^{-\nu}$. Therefore, the p-adic circle centered at the point $(h,k) \in \mathbb{Q}^2$ and of radius $r \in \mathbb{Q}_{>0}$ is the set:

$$\Big\{(x,y) \in \mathbb{Q}^2 : \ r = p^{-\nu}, \ where\ \nu = min\Big(\nu_1, \nu_2, \nu_3, \nu_4\Big)\Big\}.$$

## 3.3 Ultrametric Ellipses

Unlike the circle, a point on an ellipse will not always be at the same distance from the two foci. However, the sum of distances between any point on the ellipse and the foci will always be the same.

### 3.3.1 Ellipses Centered at the Origin

Consider the ellipse $E$ with the $x-axis$ as the major axis and two foci $(-h,0)$ and $(h,0)$ with $h \in \mathbb{Q}_{>0}$. We let $(x,y) \in \mathbb{Q}^2$ be any point on $E$ with $x,y \in \mathbb{Q}_{>0}$. Then, the ellipse $E$ is the set:

$$E = \Big\{(x,y) \in \mathbb{Q}^2 : \ D\Big((x,y),(-h,0)\Big) + D\Big((x,y),(h,0)\Big) = C, \ where\ C \in \mathbb{Q}_{>0}\Big\}.$$

So,

$$\begin{aligned} C &= D\Big((x,y),(-h,0)\Big) + D\Big((x,y),(h,0)\Big) \\ &= max\Big(d_p(x,-h), d_p(y,0)\Big) + max\Big(d_p(x,h), d_p(y,0)\Big) \\ &= 2max\Big(d_p(x,h), d_p(y,0)\Big). \end{aligned}$$

**Example**. We consider the case where $C = 2$. Then,

$$\begin{aligned} 2p^0 &= 2max\Big(d_p(x,h), d_p(y,0)\Big) \\ p^0 &= max\Big(d_p(x,h), d_p(y,0)\Big) \\ &= max\Big(d_p(x,h), |y|_p\Big). \end{aligned}$$

Recall that $x, y$, and $h$ are positive rational numbers. Then, $x = \frac{a}{b}, y = \frac{s}{t}$, and $h = \frac{c}{d}$, where $a, b, c, d, s, t \in \mathbb{Z}_{>0}$, and all fractions are in lowest terms. Hence,

$$p^0 = max\left( \left| \frac{ad-bc}{bd} \right|_p, \ \left| \frac{s}{t} \right|_p \right).$$

Thus, the ellipse $E$ is the set:

$$E = \left\{ \begin{array}{ll} \left(\frac{a}{b}, \frac{s}{t}\right) \in \mathbb{Q}^2 : & p \nmid s, \ p \nmid t, \ and \ p|(ad-bc) \\ \\ & \text{OR} \ \ p \nmid (ad-bc), \ p \nmid bd, \ and \ p|s \\ \\ & \text{OR} \ \ p \nmid s, \ p \nmid t, \ p \nmid (ad-bc), \ and \ p \nmid bd \\ \\ \end{array} \right\}.$$

**General Case**. For any $C \in \mathbb{Q}_{>0}$, with $C \neq 2$, we have:

$$2max\Big(d_p(x,h), d_p(y,0)\Big) = C,$$

$$max\Big(d_p(x,h), d_p(y,0)\Big) = \frac{C}{2}.$$

We know that $max\Big(d_p(x,h), d_p(y,0)\Big) = p^{-\nu}$, where $\nu = min(\nu_1, \nu_2)$, with $p^{-\nu_1} = d_p(x,h)$ and $p^{-\nu_2} = d_p(y,0)$. So, $C = 2p^{-\nu}$. Then,

$$C = \begin{cases} 2p^{-\nu}, & \text{if p } is \ odd \\ \\ 2^{1-\nu}, & \text{if p} = 2. \end{cases}$$

Now, and without loss of generality, we consider $d_p(y,0) = p^{-\nu}$. That means $y = \frac{s'}{t'}p^{\nu}$, where $p \nmid s'$, $p \nmid t'$, and $\frac{s'}{t'}$ is in lowest term. So, $d_p(x,h) = \left|\frac{ad-bc}{bd}\right|_p \leq p^{-\nu}$.

Since $\frac{ad-bc}{bd}$ can be written as $\frac{\alpha}{\beta}p^{\nu'}$, with $p \nmid \alpha$, $p \nmid \beta$ and $\frac{\alpha}{\beta}$ is in lowest term, then $\left|\frac{ad-bc}{bd}\right|_p = p^{-\nu'}$. Thus, we must have $\nu' \geq \nu$.

So the ellipse is the set:

$$E = \left\{ \begin{array}{ll} (x,y) \in \mathbb{Q}^2 : & |y|_p = p^{-\nu}, d_p(x,h) = p^{-\nu'},\ \ with\ \nu' \geq \nu,\ \ and\ p\ is\ odd \\ & |y|_p = 2^{-\nu}, d_p(x,h) = p^{-\nu'},\ \ with\ \nu' \geq \nu,\ \ and\ p\ = 2 \\ & \\ & \qquad\qquad OR \\ & \\ & \left|\frac{ad-bc}{bd}\right|_p = p^{-\nu},\ |y|_p = p^{-\nu'},\ with\ \nu' \geq \nu,\ \ and\ p\ is\ odd \\ & \left|\frac{ad-bc}{bd}\right|_p = 2^{-\nu},\ |y|_p = p^{-\nu'},\ with\ \nu' \geq \nu,\ \ and\ p\ = 2 \end{array} \right\}.$$

## 4 THE P-ADIC RATIONAL NUMBERS

In this section, we introduce some results concerning sequences and series, with respect to the p-adic distance. This will lead to a discussion on p-adic expansions. Then we move on to do arithmetic in $\mathbb{Q}_p$.

**Definition**. [5] Consider the field $\mathbb{F}$, and let $|\cdot|$ be an absolute value on $\mathbb{F}$.

1. A sequence $\{a_n\}$ is said to be a *Cauchy sequence*, if for every $\epsilon > 0$ , there is a $N \in \mathbb{Z}$ such that $|a_n - a_m| < \epsilon$, whenever $n, m \geq N$.

2. If every Cauchy sequence of elements of $\mathbb{F}$ has a limit, then the field $\mathbb{F}$ is said to be *complete*.

3. A subset $S \subset \mathbb{F}$ is said to be dense in $\mathbb{F}$, if every open ball around every element of $\mathbb{F}$ contains an element of $S$. That is $B(x, \epsilon) \cap S \neq \emptyset$, for every $x \in \mathbb{F}$ and every $\epsilon > 0$.

**Proposition 4.1**. For any prime number $p$, $\lim_{n\to\infty} p^n = 0$ with respect to $|\cdot|_p$.

*Proof.* The Archimedean property says that for two given positive numbers $x$ and $y$, there is a number $N \in \mathbb{N}$ such that $Nx > y$. Let $\epsilon > 0$, and consider the case where $x = \epsilon$ and $y = 1$. Then, for some $N \in \mathbb{N}$, $N\epsilon > 1$. That is $\frac{1}{N} < \epsilon$. Because $N < p^N$, we have $\frac{1}{p^N} < \frac{1}{N} < \epsilon$. Now, let $n \geq N$. So, $\frac{1}{p^n} \leq \frac{1}{p^N} < \epsilon$. Note that $p^n \neq 0$ and $\nu_p(p^n) = n$. It follows that $|p^n|_p = p^{-n}$. Thus, $|p^n|_p < \epsilon$. As a result, $\lim_{n\to\infty} p^n = 0$. □

**Corollary 4.2**. [12] Let $\{a_n\}$ be a sequence in $\mathbb{Q}_p$. Then, $\sum_{n=0}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$.

*Proof.* Let $\{s_n\}$ be the sequence of partial sums for $\{a_n\}$, and let $\epsilon > 0$.

First, we assume that $\sum_{n=0}^{\infty} a_n$ converges. It implies that $\{s_n\}$ is Cauchy. So, by Definition (1) , there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $|s_{n+1} - s_n|_p < \epsilon$. Hence, $|a_{n+1}|_p < \epsilon$, for all $n \geq N$. If we let $N' = N + 1$, then $|a_n|_p < \epsilon$ for $n \geq N'$.

Next, we assume that $\lim_{n\to\infty} a_n = 0$. Then there is some $N \in \mathbb{N}$ such that for all $n \geq N$,$|a_n|_p < \epsilon$. It implies that $|a_{n+1}|_p < \epsilon$ for all $n \geq N$ . Similarly, $|s_{n+1} - s_n|_p < \epsilon$ for all $n \geq N$. By Definition (1), $\{s_n\}$ is Cauchy and hence converges. Thus, $\sum_{n=0}^{\infty} a_n$ converges. □

**Proposition 4.3**. [12] The p-adic series $\sum_{n=0}^{\infty} p^n$ converges, and its sum is $\frac{1}{1-p}$.

*Proof.* Let $\{s_n\}$ be the sequence of partial sums of $\sum_{n=0}^{\infty} p^n$. So, $s_n = \sum_{i=0}^{n-1} p^i$. We have that $1^n - p^n = (1-p)\sum_{i=0}^{n-1} p^i$. That is $s_n = \frac{1^n - p^n}{1-p} = \frac{1-p^n}{1-p}$. It follows that $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1-p^n}{1-p}\right)$. Then, $\lim_{n\to\infty} s_n = \dfrac{1 - \lim_{n\to\infty} p^n}{1-p}$. By Proposition 4.1, we know that $\lim_{n\to\infty} p^n = 0$. Hence, $\lim_{n\to\infty} s_n = \dfrac{1}{1-p}$. Consequently, $\sum_{n=0}^{\infty} p^n = \dfrac{1}{1-p}$. □

Note that using this result, for instance, in $\mathbb{Q}_2$, we have $-1 = 1 + 2 + 4 + 8 + \cdots$. We'll make sense of this, when we introduce the p-adic expansion.

**Proposition 4.4**. [7] Any series of the form $\sum_{n=k}^{\infty} a_n p^n$, with $a_n \in \{0, 1, \cdots, p-1\}$ and $k \in \mathbb{Z}$, converges in $\mathbb{Q}_p$.

*Proof.* By Proposition 4.1, we have that $\lim_{n\to\infty} p^n = 0$. We suppose that $0 \leq a_n < p$ for all $n \geq k$. It implies that $\lim_{n\to\infty} a_n p^n = 0$. So, by Corollary 4.2, $\sum_{n=k}^{\infty} a_n p^n$ converges in $\mathbb{Q}_p$. □

**Definition**. [2] The p-adic expansion of the number $\beta \in \mathbb{Q}_p$ is a series of the form:

$$\beta = \sum_{n=k}^{\infty} a_n p^n,$$

where $k \in \mathbb{Z} \cup \{\infty\}$, $a_k \neq 0$, and $0 \leq a_n \leq p-1$, for all $n \geq k$.

The integers $a_n$ are the *coefficients* of the expansion, as mentioned in [2].

**Examples**.

1. 172 in base 3 is $20101_3$, since $172 = 2 \cdot 3^4 + 0 \cdot 3^3 + 1 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0$. So, we write the 3-adic expansion as 10102 to designate $1 \cdot 3^0 + 0 \cdot 3^1 + 1 \cdot 3^2 + 0 \cdot 3^3 + 2 \cdot 3^4$.

2. $1,133,655$ in base 11 is $704806_{11}$, because $1,133,655 = 7 \cdot 11^5 + 0 \cdot 11^4 + 4 \cdot 11^3 + 8 \cdot 11^2 + 0 \cdot 11^1 + 6 \cdot 11^0$. That is, we can write the 11-adic expansion of $1,133,655$ as 608407. So, $1,133,655 = 6 \cdot 11^0 + 0 \cdot 11^1 + 8 \cdot 11^2 + 4 \cdot 11^3 + 0 \cdot 11^4 + 7 \cdot 11^5$.

3. Here, we go back to the example in Proposition 4.3, where $-1 = 1 + 2 + 4 + \cdots$. We proceed by adding 1 to both sides of the equality. Then,

$$-1 + 1 = 1 + 2^0 + 2^1 + 2^2 + \cdots$$

$$0 = 0 \cdot 2^0 + 2^1 + 2^1 + 2^2 + \cdots$$

$$0 = 0 \cdot 2^0 + 0 \cdot 2^1 + 2^2 + 2^2 + \cdots$$

$$0 = 0 \cdot 2^0 + 0 \cdot 2^1 + 0 \cdot 2^2 + \cdots$$

If we continue, we end up having 0 on both sides of the equality. So, the series $1 + 2 + 4 + \cdots$ does represent the additive inverse of 1 in $\mathbb{Q}_2$.

**Arithmetic in $\mathbb{Q}_\mathbf{p}$**. The mechanics of adding, subtracting, multiplying, and dividing is quite similar to the corresponding elementary operations on decimals. The difference is that the "carrying" and "borrowing" etc, go from left to right instead of right to left, see [8].

**Examples**. We consider a few examples of basic arithmetic in $\mathbb{Q}_p$.

- Example of addition.

$$\begin{array}{cccccc} & 5\cdot 7^{-1}+ & 3\cdot 7^{0}+ & 6\cdot 7^{1}+ & 2\cdot 7^{2}+ & \cdots\cdots \\ + & 4\cdot 7^{-1}+ & 1\cdot 7^{0}+ & 3\cdot 7^{1}+ & 5\cdot 7^{2}+ & \cdots\cdots \\ \hline & 2\cdot 7^{-1}+ & 5\cdot 7^{0}+ & 2\cdot 7^{1}+ & 1\cdot 7^{2}+ & \cdots\cdots \end{array}$$

- Example of subtraction.

$$\begin{array}{cccccc} & 3\cdot 7^{-1}+ & 0\cdot 7^{0}+ & 4\cdot 7^{1}+ & 1\cdot 7^{2}+ & \cdots\cdots \\ - & 5\cdot 7^{-1}+ & 6\cdot 7^{0}+ & 4\cdot 7^{1}+ & 3\cdot 7^{2}+ & \cdots\cdots \\ \hline & 5\cdot 7^{-1}+ & 0\cdot 7^{0}+ & 6\cdot 7^{1}+ & 4\cdot 7^{2}+ & \cdots\cdots \end{array}$$

- Example of multiplication.

$$\begin{array}{cccccc} & 2\cdot 7^{-1}+ & 5\cdot 7^{0}+ & 3\cdot 7^{1}+ & 1\cdot 7^{2}+ & \cdots\cdots \\ \times & 3\cdot 7^{-1}+ & 4\cdot 7^{0}+ & 2\cdot 7^{1}+ & 6\cdot 7^{2}+ & \cdots\cdots \\ \hline 6\cdot 7^{-2}+ & 1\cdot 7^{-1}+ & 4\cdot 7^{0}+ & 4\cdot 7^{1}+ & \cdots\cdots & \cdots\cdots \\ & 1\cdot 7^{-1}+ & 0\cdot 7^{0}+ & 1\cdot 7^{1}+ & \cdots\cdots & \cdots\cdots \\ & & 4\cdot 7^{0}+ & 3\cdot 7^{1}+ & \cdots\cdots & \cdots\cdots \\ & & & 5\cdot 7^{1}+ & \cdots\cdots & \cdots\cdots \\ \hline 6\cdot 7^{-2}+ & 2\cdot 7^{-1}+ & 1\cdot 7^{0}+ & 0\cdot 7^{1}+ & \cdots\cdots & \cdots\cdots \end{array}$$

- Example of division.

$$
\begin{array}{r}
5\cdot 7^0+1\cdot 7^1+6\cdot 7^2+\cdots \\
3\cdot 7^0+5\cdot 7^1+1\cdot 7^2+\cdots\cdot\big)\overline{1\cdot 7^0+2\cdot 7^1+4\cdot 7^2+\cdots} \\
-\ \ (1\cdot 7^0+6\cdot 7^1+1\cdot 7^2+\cdots) \\
\hline
3\cdot 7^1+2\cdot 7^2+\cdots\cdots \\
-\ \ (3\cdot 7^1+5\cdot 7^2+\cdots\cdots) \\
\hline
4\cdot 7^2+\cdots\cdot \\
-\ \ (4\cdot 7^2+\cdots\cdot) \\
\hline
0\cdot 7^2+\cdots\cdot
\end{array}
$$

Note that we can use the long division to find the expansion of a rational number.

**Example**. We want to find the expansion of $\frac{1}{7}$ in $\mathbb{Q}_5$. That is $\sum_{n=k}^{\infty} a_n 5^n$, where $k \in \mathbb{Z} \cup \{\infty\}$, $a_k \neq 0$, and $0 \le a_n \le 4$, for all $n \ge k$.

First, notice that $\left(\frac{1}{7}\right)\cdot(7) = 1$. That can be rewritten as:

$\left(\sum_{n=k}^{\infty} a_n 5^n\right)\cdot\left(2\cdot 5^0+1\cdot 5^1\right) = 1\cdot 5^0+0\cdot 5^1+0\cdot 5^2+0\cdot 5^3+0\cdot 5^4+0\cdot 5^5+\cdots\cdot$

Hence, $\sum_{n=k}^{\infty} a_n 5^n = \frac{1\cdot 5^0+0\cdot 5^1+0\cdot 5^2+0\cdot 5^3+0\cdot 5^4+0\cdot 5^5+\cdots\cdot}{2\cdot 5^0+1\cdot 5^1}$. Then,

$$
\begin{array}{r}
3\cdot 5^0+3\cdot 5^1+0\cdot 5^2+2\cdot 5^3+1\cdot 5^4+4\cdot 5^5+\cdots\cdots\cdots\cdots \\
2\cdot 5^0+1\cdot 5^1\big)\overline{1\cdot 5^0+0\cdot 5^1+0\cdot 5^2+0\cdot 5^3+0\cdot 5^4+0\cdot 5^5+\cdots\cdots\cdots\cdots} \\
-\ \ (1\cdot 5^0+4\cdot 5^1) \\
\hline
1\cdot 5^1+4\cdot 5^2+4\cdot 5^3+4\cdot 5^4+4\cdot 5^5+\cdots\cdots\cdots\cdots \\
-\ \ (1\cdot 5^1+4\cdot 5^2) \\
\hline
4\cdot 5^3+4\cdot 5^4+4\cdot 5^5\cdots\cdots\cdots\cdots \\
-\ \ (4\cdot 5^3+2\cdot 5^4) \\
\hline
2\cdot 5^4+4\cdot 5^5\cdots\cdots\cdots\cdots \\
-\ \ (2\cdot 5^4+1\cdot 5^5) \\
\hline
3\cdot 5^5+4\cdot 5^6\cdots\cdots \\
-\ \ (3\cdot 5^5+0\cdot 5^6+1\cdot 5^7) \\
\hline
4\cdot 5^6+1\cdot 5^7\cdots
\end{array}
$$

As a result, the expansion of $\frac{1}{7}$ in $\mathbb{Q}_5$ is:

$$\sum_{n=k}^{\infty} a_n 5^n = 3 \cdot 5^0 + 3 \cdot 5^1 + 0 \cdot 5^2 + 2 \cdot 5^3 + 1 \cdot 5^4 + 4 \cdot 5^5 + \cdots .$$

# 5 FUTURE WORK

Probing deeper, and in order to develop this work further, we want to continue exploring conic sections, using the p-adic distance. So, for the future work, we'll be interested in the following:

- Remaining cases of circles and ellipses with arbitrary centers and arbitrary foci.
- Remaining conic sections. That is how parabolas and hyperbolas can be defined with respect to p-adic distances.
- General *equidistant sets*.

  **Definition**. Let ($\mathbb{X}$, d) be a metric space and let $A$ be a non-empty set of $\mathbb{X}$. If $x \in \mathbb{X}$, then the distance of $x$ to $A$ is: $dist(x, A) = inf\left\{d(x, a) : a \in A\right\}$.

  Now if $A$ and $B$ are both non-empty sets, then equidistant set determined by $A$ and $B$ and denoted $A = B$ is defined as: $\{A = B\} := \{x \in \mathbb{R}^n : dist(x, A) = dist(x, B)\}$.
- Can we do this in $\mathbb{Q}_p^2$ instead of $\mathbb{Q}^2$?

BIBLIOGRAPHY

[1] G. Akst. A Strange Ultrametric Geometry. *Mathematics Magazine*, 49(3):142–145, 1976.

[2] K. Conrad. The P-Adic Expansion of Rational Rumbers. 2019. Accessed: March 10, 2023.

[3] C. Crompton. Some Geometry of The P-Adic Rationals. *Rose-Hulman Undergraduate Mathematics Journal*, 8(1):2, 2007.

[4] M. Gill. *Eccentricity and The Cultural Imagination in Nineteenth-Century Paris*. OUP Oxford, 2009.

[5] F. Gouvêa. *P-Adic Numbers*. Springer, 1997.

[6] T. Herwig. The P-Adic Completion of Q and Hensel's Lemma, 2011.

[7] S. Katok. *P-Adic Analysis Compared With Real*, volume 37. American Mathematical Soc., 2007.

[8] N. Koblitz. *P-Adic Numbers, P-Adic Analysis, and Zeta-Functions*, volume 58. Springer Science & Business Media, 2012.

[9] D. Madore. A First Introduction to P-Adic Number. 7 december 2000, 2008.

[10] I. Niven, H.S. Zuckerman, and H.L. Montgomery. *An Introduction to The Theory of Numbers*. Wiley, 1991. Accessed: March 21, 2023.

[11] C. Perez-Garcia and W. Schikhof. *Locally Convex Spaces Over Non-Archimedean Valued Fields*. Cambridge University Press, first edition, 2010.

[12] A. Pomerantz. An Introduction to The P-Adic Numbers. 2020. Accessed: March 08, 2023.

[13] M. Ponce and P. Santibánez. On Equidistant Sets and Generalized Conics: The Old and The New. *The American Mathematical Monthly*, 121(1):18–32, 2014.

[14] G. Strang and E. Herman. Conic Sections, 2022. LibreTextsmathematics. Accessed: March 21, 2023.

[15] J. Wilker. Equidistant Sets and Their Connectivity Properties. *Proceedings of the American Mathematical Society*, 47(2):446–452, 1975.

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