GENERALIZED PLANE OFFSETS AND RATIONAL PARAMETERIZATIONS

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Abstract. In the first part of the paper a planar generalization of offset curves is introduced and some properties are derived. In particular, it is seen that these curves exhibit good regularity properties and a study on self-intersection avoidance is performed. The representation of a rational curve as the envelope of its tangent lines, following the approach of Pottmann, is revisited to give the explicit expression of all rational generalized offsets. Other famous shapes, such as constant width curves, bicycle tire-tracks curves and Zindler curves are related to these generalized offsets. This gives rise to the second part of the paper, where the particular case of rational parameterizations by a support function is considered and explicit families of rational constant width curves, rational bicycle tire-track curves and rational Zindler curves are generated and some examples are shown.

1. Introduction

Pythagorean-hodograph (PH) curves were introduced by Farouki and Sakkalis (1990) as polynomial curves with polynomial velocity. The curves that satisfy the analogous definition for rational functions are called rational PH curves, which were first studied by Fiorot and Gensane (1994) and Pottmann (1995). The interested reader can see (Farouki, 2008) and (Kosinka and Lávička, 2014) to get a wide spectrum of Pythagorean-hodograph curve theory and their applications. One of the most important properties of rational PH curves is that their offset curves are rational, so that they can be easily computed in an exact representation by CAD systems and no approximations are needed.

This paper is structured in two parts. In the first part, a planar generalization of offset curves, which we call (ω, d)-offset curves, is introduced. These curves are defined as those which are at a distance d from an initial curve in a direction making a constant angle ω with its tangent. The classical offset curves correspond to the particular case of $\omega = \pm\frac{\pi}{2}$.

Another interesting particular case of (ω, d)-offset curves are the curves generated by front wheel tire-tracks of a bicycle (case $\omega = 0$). The usual model of a bicycle is given by an oriented segment of constant length where the front endpoint is on the tangent direction of the rear endpoint. These endpoints describe the wheel tire-track curves in the motion of a bicycle (see e.g. Tabachnikov (2006) or Bor et al. (2020)).

A property that makes general (ω, d)-offset curves very interesting is what is shown in Proposition 1: if the initial curve is regular, then these offsets are singularity-free. In fact, singularities

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can only happen for the classical offset curves ($\omega = \pm\frac{\pi}{2}$). As another property, in Theorem 1 a formula relating the area of an (ω, d)-offset to a closed curve α with the area and the length of α is provided. This corresponds to a generalization of Steiner's formula for the area of offset curves.

In Section 2.2 the representation of all rational Pythagorean-hodograph curves given by Pottmann (1995) is revisited. Thanks to this representation we provide an explicit expression of all rational (ω, d)-offsets coming from a rational curve (Theorem 3). Of course, this result generalizes the expression given by Pottmann for offset curves. In addition, it provides as another interesting particular case the explicit expression of all rational front wheel bicycle curves which come from a rational rear wheel curve (Theorem 4).

Although generalized offsets according to a non-right angle are regular, self-intersections could still happen. Given a spine curve that does not self-intersect, we provide a bound on the offset distance d to avoid (global and local) self-intersections on its (ω, d)-offset curve (Proposition 2 and Theorem 5). Afterwards, the local self-intersection problem on generalized offsets is studied in terms of the curvature function of the spine curve (Propositions 3, 4 and Theorem 6). The study of global and local self-intersections of generalized offsets is the main contribution of the first part of the paper.

The second part of the paper focuses on curves parameterized by their inverse Gauss map, namely hedgehogs or any convex curve (Martinez-Maure, 1999). As seen by Šír et al. (2008) and Gravesen et al. (2008), any curve or surface defined by its inverse Gauss map with a polynomial/rational support function admits a rational parameterization. Actually, we can derive this result in the plane as a particular case of Pottmann's representation of rational curves (Section 3.2). The aim here is to use this particular representation by a support function to provide an explicit family of rational constant width curves and Zindler curves.

Constant width curves are those with the property that any pair of parallel supporting lines to the curve are a constant distance apart (Martini et al., 2019). In Section 4, Proposition 7, we provide a generic support function depending on some parameters which can be used to generate rational constant width curves.

Zindler curves are considered in Section 5. Zindler curves are plane curves in which a constant length chord is allowed to move with its endpoints over the curve such that it always divides the perimeter (or area) of the figure in a half (Zindler, 1921). Zindler curves are also solutions to the 2D floating body problem in equilibrium, so that they have important physical implications. Similarly as in constant width curves, we provide an explicit family of rational Zindler curves (Proposition 8).

The main contribution of this part are Propositions 7 and 8. These results are the actual motivation to introduce rationally parameterized curves following Pottmann's representation and generalized offsets. The explicit families of rational curves given in these propositions allows the user to produce infinitely many examples of rational curves of constant width and rational Zindler curves by changing the value of some free parameters.

2. A generalization of offset curves

A natural generalization of offset curves is given by allowing in their construction any constant angle instead of a right one (see Figure 1). In the plane we can refer to Cooker (1999) where this kind of curves was considered. In constant curvature surfaces we have the old works by Vidal Abascal (1948, 1947c,a) in Spanish, and the recent article (Monterde and Rochera, 2020) in which these curves are seen as a particular case of a more general kind of curves (where both the angle and the distance are arbitrary functions).

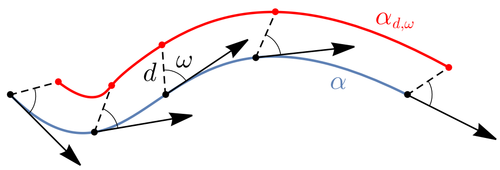

FIGURE 1. The offset curve $\alpha_{d,\omega}$ to α at a distance d according to a constant angle ω.

Some other works (Arrondo et al., 1997, 1999; Sendra and Sendra, 2000) also consider these generalized offsets (even for surfaces) and they are studied with a fairly algebraic approach. Other works such as (Chen and Lin, 2014) are also related.

2.1. **Definition and some properties.** The parametric definition of a generalized (or skewed) offset in the plane is the following.

Definition 1 (Generalized offset curve)**.** Let $\alpha : I \to \mathbb{R}^2$ be a regular curve, $d > 0$ and $\omega \in \,]-\pi, \pi]$. The *parallel* or *offset curve to α at a distance d according to a constant angle ω* is defined as the curve $\alpha_{d,\omega} : I \to \mathbb{R}^2$ given by

$$\alpha_{d,\omega}(t) = \alpha(t) + d\,\big(\cos\omega\,\mathbf{t}(t) + \sin\omega\,\mathbf{n}(t)\big),$$

where $\mathbf{t}$ and $\mathbf{n}$ are the tangent and normal vectors, respectively, of α. To shorten, we will say that $\alpha_{d,\omega}$ is the *(ω, d)-offset to α*.

Of course, offsets according to the angles $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ correspond to the two sides of classical parametric offset curves.

The following result is on the regularity of these generalized offsets. From it we conclude that all offsets to a regular curve according to a non-right angle are regular.

Proposition 1. *Let $\alpha : I \to \mathbb{R}^2$ be a regular curve, $d > 0$ and $\omega \in \,]-\pi, \pi]$. Any offset curve to α at a distance d according to the angle ω is always regular except for orthogonal (inner) offsets, where a sufficient condition for its regularity is to have*

$$d < \frac{1}{\kappa_{max}},$$

with $\kappa_{max} = \sup_{s\in I}\big|\kappa(s)\big|$.

Proof. We can suppose that α is parameterized by arc length. Thus, since

$$\alpha'_{d,\omega}(s) = \big(1 - d\,\kappa(s)\sin\omega\big)\,\mathbf{t}(s) + d\,\kappa(s)\cos\omega\,\mathbf{n}(s),$$

it follows that

$$\begin{aligned}\big\|\alpha'_{d,\omega}(s)\big\|^2 &= \big(1 - d\,\kappa(s)\sin\omega\big)^2 + d^2\,\kappa^2(s)\cos^2\omega \\ &= 1 - 2\,d\,\kappa(s)\sin\omega + d^2\,\kappa^2(s) = \cos^2\omega + \big(\sin\omega - d\,\kappa(s)\big)^2.\end{aligned}$$

Therefore, if $\omega \neq \pm\pi/2$, then $\cos\omega \neq 0$, so that

$$\left\|\alpha'_{d,\omega}(s)\right\| \neq 0$$

and $\alpha_{d,\omega}$ is regular. The second part of the statement for orthogonal offsets ($\omega = \pm\pi/2$) is well known (see for instance Farouki and Neff (1990b) or Patrikalakis and Maekawa (2002)). □

We must remark that we can avoid singularities by Proposition 1 but self-intersections may still happen. Let us consider now the classical example of the parabola to construct some of these generalized offsets.

Example 1. Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ be defined by $\alpha(t) = \big(t,\, t^2\big)$. The offset curve to α at a distance d according to an angle ω can be easily computed:

$$\alpha_{d,\omega}(t) = \left(t + d\,\frac{\cos\omega - 2\,t\sin\omega}{\sqrt{1+4\,t^2}},\; t^2 + d\,\frac{2\,t\cos\omega + \sin\omega}{\sqrt{1+4\,t^2}}\right).$$

As seen in Proposition 1, singularities can only happen for $\alpha_{d,\omega}$ if $\omega = \frac{\pi}{2}$, where a sufficient condition to avoid them is to take a distance

$$d < \frac{1}{\kappa_{\max}} = \frac{1}{2}.$$

To illustrate the regularity property of generalized offsets see Figure 2, where we consider a distance $d = 1$ and some generalized offsets according to different angles.

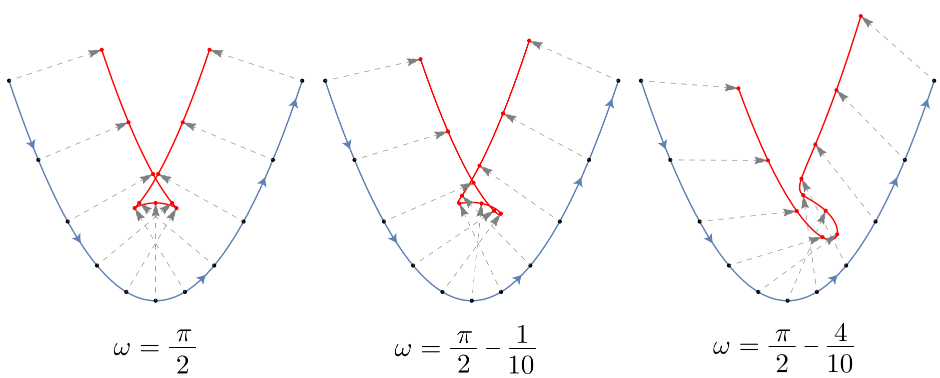

FIGURE 2. Some $(\omega, 1)$-offsets to a parabola for different angles ω.

As a remark, note that the definition of a generalized offset is not equivalent to the smooth 2D version of a canal surface, which are rationally parameterizable if the spine curve and the radius function are rational (Peternell and Pottmann, 1997). Clearly, the generalized offset of Figure 2-right, cannot be generated as an envelope of a family of circles centered at the parabola and with a smooth radius function.

For classical offset curves α_d at a distance d to a closed curve α, there are simple formulas relating their areas and lengths, namely,

$$\mathcal{A}(\alpha_d) = \mathcal{A}(\alpha) \pm \mathcal{L}(\alpha)\, d + \pi\, d^2,$$
$$\mathcal{L}(\alpha_d) = \mathcal{L}(\alpha) \pm 2\,\pi\, d,$$

where the sign $\pm$ is negative for inner offsets and positive for outer offsets. These formulas, due to Steiner (1840), are known as *Steiner's formulae for the offset to a curve* (see Chapters 1 and 10 of Gray (2004) for a more recent reference).

In the following result we give the corresponding formula for the area of generalized offsets. This formula is known for constant curvature surfaces (Vidal Abascal, 1948, 1947b; Monterde and Rochera, 2020), however, we will state its planar version here and give its proof using elementary differential geometry. We will not assume convexity (as Steiner did) but we will take some assumptions instead.

Theorem 1 (Steiner's formula for the area of generalized offsets). *Let $d > 0$, $\omega \in\,]-\pi, \pi]$ and let α be a positively oriented regular closed planar curve. Let $\alpha_{d,\omega}$ be the offset curve to α at a distance d according to the angle ω. Then*

$$\mathcal{A}(\alpha_d) = \mathcal{A}(\alpha) - \mathcal{L}(\alpha)\, d \sin\omega + \pi\, n\, d^2, \tag{1}$$

where n is the number of chord revolutions in the generation of $\alpha_{d,\omega}$.

Proof. Let $\alpha : I \to \mathbb{R}^2$, $\alpha(s) = \big(x(s),\, y(s)\big)$ be arc-length parameterized. Thus, $I = [0, \mathcal{L}(\alpha)]$. The (ω, d)-offset curve to α is

$$\alpha_{d,\omega} = \alpha + d\,(\cos\omega\,\mathbf{t} + \sin\omega\,\mathbf{n}).$$

In terms of coordinates $x = x(s)$ and $y = y(s)$ we can write

$$\alpha_{d,\omega} = \big(x + d\,(\cos\omega\, x' - \sin\omega\, y'),\; y + d\,(\cos\omega\, y' + \sin\omega\, x')\big).$$

Therefore, the area of $\alpha_{d,\omega}$ is

$$\begin{aligned} \mathcal{A}(\alpha_{d,\omega}) &= \int_I \big(x + d\,(\cos\omega\, x' - \sin\omega\, y')\big)\,\big(y' + d\,(\cos\omega\, y'' + \sin\omega\, x'')\big)\,\mathrm{d}s \\ &= \int_I x\, y'\,\mathrm{d}s + d\int_I \Big(\cos\omega\,(y'x' + xy'') + \sin\omega\,\big(xx'' - y'^2\big)\Big)\,\mathrm{d}s \\ &\quad + d^2 \int_I \big(-x''y' + \cos^2\omega\,(x'y'' + x''y') + \sin\omega\cos\omega\,(x'x'' - y'y'')\big)\,\mathrm{d}s. \end{aligned} \tag{2}$$

Now, since x and y define a closed curve, we have that

$$\int_I (y'x' + x\,y'')\,\mathrm{d}s = \int_I (x\,y')'\,\mathrm{d}s = 0$$

and

$$\int_I (x'y'' + x''y')\,\mathrm{d}s = \int_I (x'y')'\,\mathrm{d}s = 0.$$

Moreover, since $x'^2 + y'^2 = 1$, we deduce $x'x'' + y'y'' = 0$, so that

$$\int_I (x'x'' - y'y'')\,\mathrm{d}s = \int_I 2\,x'x''\,\mathrm{d}s = \int_I \big(x'^2\big)'\,\mathrm{d}s = 0.$$

In addition, notice that

$$-\int_I x''y'\,\mathrm{d}s = \int_I x'y''\,\mathrm{d}s = \pi\, n,$$

because it is the area of the tangent indicatrix of α, which is a n-traced unit circle. Therefore, the expression (2) above turns into

$$\mathcal{A}(\alpha_{d,\omega}) = \mathcal{A}(\alpha) + d\sin\omega \int_I \left(xx'' - y'^2\right) \mathrm{d}s + \pi\, n\, d^2.$$

Finally, notice that

$$\int_I (xx'' - y'^2)\, \mathrm{d}s = \int_I \left(-1 + x'^2 + xx''\right) \mathrm{d}s = \int_I \left(-1 + (xx')'\right) \mathrm{d}s = -\mathcal{L}(\alpha).$$

Thus, the expression of the statement is found. □

Remark 1. If the curves of Theorem 1 are not simple (they present self-intersections), then the areas must be counted by sign and multiplicity as it is pointed out in (Rochera, 2022c), where notice that the previous result can be seen as a particular case of Lemma 4.1 stated in the same paper. In order to have pairs of simple curves, the self-intersection avoidance of generalized offsets will be studied in Section 2.4.

Remark 2. The classical Steiner formula for the area of an offset curve is deduced simply by taking an angle $\omega = \pm\frac{\pi}{2}$ in Equation (1).

As seen in the proof of Proposition 1, for an arc-length parameterized curve α, we have

$$\left\|\alpha'_{d,\omega}(s)\right\|^2 = \cos^2\omega + \big(\sin\omega - d\,\kappa(s)\big)^2.$$

In general it is not possible to write the right-hand side as a perfect square, so that an easy analogous formula for the length of an (ω, d)-offset curve may in general not be possible except for the case $\omega = \pm\frac{\pi}{2}$.

2.2. **Representation of a curve as the envelope of its tangent lines.** Pottmann (1995) found an explicit expression of all rational curves whose offsets are rational. Actually, his method, based on the representation of a curve as the envelope of its tangent lines, is interesting by itself. We will reproduce this representation here, as we will use it later in Section 3.2 to give explicit rational expressions for generalized offsets and to rationally parameterized curves by a support function.

The main observation of Pottmann's method is that the usual stereographic projection map from the point $(0, 1)$ onto the OX axis induces a correspondence between rational parameterizations of $\mathbb{R}$ and rational parameterizations of the unit circle $\mathbb{S}^1$. This produces the following representation of a normal vector:

$$\mathbf{N}(t) = \big(n_1(t),\, n_2(t)\big) = \left(\frac{2\,a(t)\,b(t)}{a^2(t) + b^2(t)},\, \frac{a^2(t) - b^2(t)}{a^2(t) + b^2(t)}\right),$$

where $a = a(t)$ and $b = b(t)$ are polynomials.

A curve α can be represented as the envelope of its tangents lines:

$$n_1(t)\, X + n_2(t)\, Y = h(t), \tag{3}$$

where $h(t) = \big\langle \alpha(t),\, \mathbf{N}(t)\big\rangle$ is called the *support function of* α, which we assume to be rational. Notice that $\mathbf{N}$ is actually an orthogonal vector defining the tangent lines (which may be coincident or opposite to the normal vector of α depending on t). The envelope of these tangent lines can be computed as usual, solving for X and Y the linear system formed by Equation (3) and its derivative with respect t:

$$n_1'(t)\,X + n_2'(t)\,Y = h'(t).$$

This produces the following parameterization of α:

$$\alpha = h\left(\frac{2\,a\,b}{a^2+b^2}, \frac{a^2-b^2}{a^2+b^2}\right) + \frac{(a^2+b^2)\,h'}{2\,(b\,a'-a\,b')}\left(-\frac{a^2-b^2}{a^2+b^2}, \frac{2\,a\,b}{a^2+b^2}\right). \tag{4}$$

The simplified version of this equation,

$$\alpha = h\left(\frac{2\,a\,b}{a^2+b^2}, \frac{a^2-b^2}{a^2+b^2}\right) + h'\left(-\frac{a^2-b^2}{2\,(b\,a'-a\,b')}, \frac{a\,b}{b\,a'-a\,b'}\right), \tag{5}$$

is Equation (2.4) given in (Pottmann, 1995). However, we prefer to write it as in (4) to emphasize that h can be realized as a support function of α, as we will detail in Section 3.

Although Pottmann looked for rational curves with rational offsets, his method yields a way to describe all curves which are PH. In fact, the resulting expression is in the form of the well-known characterization of Pythagorean triples of polynomials (see Theorem 17.1 of Farouki (2008)). Thus, the result can be stated as follows.

Theorem 2 (Pottmann)**.** *All rational curves $\alpha(t)$ which are Pythagorean-hodograph can be written in the form (5), where $a(t)$ and $b(t)$ are relatively prime polynomials and $h(t)$ is a rational function.*

From now on, a parameterization of a rational Pythagorean-hodograph curve α as in Theorem 2 will be called a *standard parameterization of α*.

It must be remarked that, as noted by Lü, having a Pythagorean hodograph is a sufficient but not necessary condition for a curve to possess rational offsets. This is because rational offsets can also be found by appropriate rational reparameterizations (see Lü (1995) and Farouki and Sederberg (1995)).

2.3. **Rational curves with rational (ω, d)-offsets.** From the procedure above it is easy to find the explicit expression of all continuous (ω, d)-offsets to a rational curve by its standard parameterization. Notice that we understand a continuous offset here as parameterized by a support function (Rochera, 2022a).

Theorem 3. *Let α be a rational Pythagorean-hodograph curve parameterized in the standard form (5). Then the (ω, d)-offset to α is rational and has the form*

$$\begin{aligned}\alpha_{d,\omega} &= (h + d\sin\omega)\left(\frac{2\,a\,b}{a^2+b^2}, \frac{a^2-b^2}{a^2+b^2}\right)\\ &\quad + \left(d\cos\omega + \frac{(a^2+b^2)\,h'}{2\,(b\,a'-a\,b')}\right)\left(-\frac{a^2-b^2}{a^2+b^2}, \frac{2\,a\,b}{a^2+b^2}\right).\end{aligned} \tag{6}$$

Of course, the representation of rational offsets given by Pottmann (1995) is a particular case of Theorem 3 when $\omega = \frac{\pi}{2}$. Another interesting particular case is the case of bicycle tire-track curves ($\omega = 0$):

Theorem 4. *Let α be a rational Pythagorean-hodograph parameterized in the standard form (5). If α represents the rear wheel tire-track curve of a bicycle of length d, then the front wheel curve is rational and has the form*

$$\alpha_{d,\omega} = h\left(\frac{2\,a\,b}{a^2+b^2},\, \frac{a^2-b^2}{a^2+b^2}\right) + \left(\frac{d}{a^2+b^2} + \frac{h'}{2\,(b\,a' - a\,b')}\right)\big(b^2-a^2,\, 2\,a\,b\big).$$

In particular, Zindler curves generated from a rational middle hedgehog will also be rational. We will see this in detail in Section 5.

2.4. **Global and local self-intersections of generalized offsets.** We have seen that generalized offsets according to a non-right angle are always regular (Proposition 1). However, self-intersections could happen, which motivates us to study how to control or avoid them. The study of self-intersections can be done algebraically if the algebraic equation that describes the generalized offset is known. Elimination methods to find algebraic equations of classical double-offsets to polynomial and rational curves were studied by Farouki and Neff (1990a). Analogous methods are possible for generalized offsets. Nevertheless, the complexity of the algebraic equation is high and, in the rational case, the handling of extraneous factors in the algebraic equation can be tedious.

There are methods to detect self-intersections for classical offsets based on tangents or on distance maps (see e.g. Elber (2003) or Seong et al. (2006)). However, the obvious extension of these methods to generalized offsets does not seem feasible.

Next we will provide a different method to detect and avoid global and local self-intersections in the generalized offsets based on a bound on the offset distance. Of course, in particular the same results are also applicable to classical offsets.

Proposition 2. *Let $\alpha : I \to \mathbb{R}^2$ be a regular simple curve. Let $\omega \in\,]-\pi, \pi]$ and $d > 0$ be such that*

$$d \neq \frac{\big\|\alpha(t) - \alpha(u)\big\|}{\big\|\mathbf{t}(t) - \mathbf{t}(u)\big\|},$$

for all $t, u \in I$ such that $\mathbf{t}(t) \neq \mathbf{t}(u)$. Then the (ω, d)-offset to α is simple.

Proof. Let $\alpha_{d,\omega}$ be an (ω, d)-offset to α. If $\alpha_{d,\omega}$ is not simple, then there exist $t, u \in I$, $t \neq u$, such that $\alpha_{d,\omega}(t) = \alpha_{d,\omega}(u)$. This implies that

$$\alpha(t) - \alpha(u) = d\Big(\cos\omega\,\big(\mathbf{t}(u) - \mathbf{t}(t)\big) + \sin\omega\,\big(\mathbf{n}(u) - \mathbf{n}(t)\big)\Big).$$

Therefore,

$$\big\|\alpha(t) - \alpha(u)\big\| = d\,\Big\|\cos\omega\,\big(\mathbf{t}(u) - \mathbf{t}(t)\big) + \sin\omega\,\big(\mathbf{n}(u) - \mathbf{n}(t)\big)\Big\|. \tag{7}$$

Now, notice that

$$\Big\|\cos\omega\,\big(\mathbf{t}(u) - \mathbf{t}(t)\big) + \sin\omega\,\big(\mathbf{n}(u) - \mathbf{n}(t)\big)\Big\|^2$$
$$= \cos^2\omega\,\big\|\mathbf{t}(u) - \mathbf{t}(t)\big\|^2 + \sin^2\omega\,\big\|\mathbf{n}(u) - \mathbf{n}(t)\big\|^2 + 2\sin\omega\cos\omega\,\big\langle \mathbf{t}(u) - \mathbf{t}(t),\, \mathbf{n}(u) - \mathbf{n}(t)\big\rangle.$$

Since the application $J : \mathbb{R}^2 \to \mathbb{R}^2$, $J(a, b) = (-b, a)$ is linear,

$$\big\langle \mathbf{t}(u) - \mathbf{t}(t),\, \mathbf{n}(u) - \mathbf{n}(t)\big\rangle = 0.$$

It also satisfies that $\|Jv\| = \|v\|$, for any $v \in \mathbb{R}^2$. Hence, we deduce

$$\Big\|\cos\omega\,\big(\mathbf{t}(u) - \mathbf{t}(t)\big) + \sin\omega\,\big(\mathbf{n}(u) - \mathbf{n}(t)\big)\Big\| = \big\|\mathbf{t}(u) - \mathbf{t}(t)\big\|,$$

and thus Equation (7) can be written as

$$\big\|\alpha(t) - \alpha(u)\big\| = d\,\big\|\mathbf{t}(t) - \mathbf{t}(u)\big\|.$$

Since α is simple, we have that $\alpha(t) \neq \alpha(u)$, which implies that $\big\|\mathbf{t}(t) - \mathbf{t}(u)\big\| \neq 0$, so that $\mathbf{t}(t) \neq \mathbf{t}(u)$. From this we get

$$d = \frac{\big\|\alpha(t) - \alpha(u)\big\|}{\big\|\mathbf{t}(t) - \mathbf{t}(u)\big\|}. \qquad \square$$

Given a regular simple curve $\alpha : I \to \mathbb{R}^2$, define

$$D_\alpha := \left\{ \frac{\big\|\alpha(t) - \alpha(u)\big\|}{\big\|\mathbf{t}(t) - \mathbf{t}(u)\big\|} \ : \ t, u \in I,\ \ \mathbf{t}(t) \neq \mathbf{t}(u) \right\}.$$

By Proposition 2, global and local self-intersections in (ω, d)-offsets to α are avoided if $d \notin D_\alpha$. In general, the set D_α is disconnected and is a union of real intervals. Thus, a sufficient condition to avoid self-intersections can be written as follows.

Theorem 5. *Let $\alpha : I \to \mathbb{R}^2$ be a regular simple curve. If $d > 0$ is such that*

$$d < \inf D_\alpha = \inf \left\{ \frac{\big\|\alpha(t) - \alpha(u)\big\|}{\big\|\mathbf{t}(t) - \mathbf{t}(u)\big\|} \ : \ t, u \in I,\ \ \mathbf{t}(t) \neq \mathbf{t}(u) \right\},$$

then any (ω, d)-offset to α is simple, where $\omega \in \,]-\pi, \pi]$.

In general, the analytic computation of the infimum of Theorem 5 is very difficult even for simple examples like the parabola. Therefore, numerical methods are needed for its computation. We should not forget that this method prevents global self-intersections as well. Sometimes there are no such global issues and a local analysis, that is easier, can be sufficient (as for the parabola). The aim of the following results is to relate the bound of Theorem 5 to a local self-intersection avoidance.

Proposition 3. *Let $\alpha : I \to \mathbb{R}^2$ be a regular simple curve. If $d > 0$ is such that $d < \inf D_\alpha$ (so self-intersections are avoided in any (ω, d)-offset curve to α), then*

$$d < \frac{1}{\kappa_{max}},$$

where $\kappa_{max} = \sup_{s \in I}\big|\kappa(s)\big|$.

Proof. First, let us show that

$$\lim_{u \to t} \frac{\big\|\alpha(t) - \alpha(u)\big\|}{\big\|\mathbf{t}(t) - \mathbf{t}(u)\big\|} = \frac{1}{\big|\kappa(t)\big|}.$$

By L'Hôpital's rule, we have

$$\lim_{u \to t} \frac{\big\|\alpha(t) - \alpha(u)\big\|^2}{\big\|\mathbf{t}(t) - \mathbf{t}(u)\big\|^2} \lim_{u \to t} \frac{\big\langle \mathbf{t}(u),\, \alpha(t) - \alpha(u)\big\rangle}{\kappa(u)\,\big\langle \mathbf{n}(u),\, \mathbf{t}(t)\big\rangle}.$$

Since κ is a continuous function, it is only left to compute the limit

$$\lim_{u\to t} \frac{\left\langle \mathbf{t}(u),\, \alpha(t)-\alpha(u)\right\rangle}{\left\langle \mathbf{n}(u),\, \mathbf{t}(t)\right\rangle},$$

for which we can apply L'Hôpital's rule again to find that

$$\lim_{u\to t} \frac{\kappa(u)\left\langle \mathbf{n}(u),\, \alpha(t)-\alpha(u)\right\rangle - 1}{-\kappa(u)\left\langle \mathbf{t}(u),\, \mathbf{t}(t)\right\rangle} = \frac{1}{\kappa(t)}.$$

Therefore, we can conclude that

$$\lim_{u\to t} \frac{\left\|\alpha(t)-\alpha(u)\right\|^2}{\left\|\mathbf{t}(t)-\mathbf{t}(u)\right\|^2} = \frac{1}{\kappa^2(t)},$$

which implies what we wanted to show.

Now, let $d>0$ such that $d<\inf D_\alpha$. In particular,

$$d < \lim_{u\to t} \frac{\left\|\alpha(t)-\alpha(u)\right\|}{\left\|\mathbf{t}(t)-\mathbf{t}(u)\right\|} = \frac{1}{\left|\kappa(t)\right|},$$

for all $t\in I$. □

The bound of Proposition 3 is not by chance, as it actually tells us that we also perform a local self-intersection avoidance. We know that such a bound constitutes the local bound for classical offsets. We will see that it is also the local bound for generalized offsets in the following results. Notice that the usual proof for classical offsets (which actually studies offset singularities) cannot be repeated in this setting because generalized offsets according to a non-right angle are always regular. Let

$$f(t,u) = \frac{\left\|\alpha(t)-\alpha(u)\right\|^2}{\left\|\mathbf{t}(t)-\mathbf{t}(u)\right\|^2}$$

and consider now the function

$$\tilde{f}(t,u) = \begin{cases} f(t,u), & \text{if } t\neq u,\\ \frac{1}{\kappa^2(t)}, & \text{if } t=u, \end{cases}$$

where it can be defined.

Proposition 4. *The function $\tilde{f}$ is continuous in its domain.*

Proof. If we prove that given $t_0\in I$ such that $\kappa(t_0)\neq 0$, we have

$$\lim_{(t,u)\to(t_0,t_0)} \tilde{f}(t,u) = \frac{1}{\kappa^2(t_0)},$$

then we will have that $\tilde{f}$ is continuous at (t_0,t_0), if $\kappa(t_0)\neq 0$. Iterated limits are easy to compute (with the same idea as in the proof of Proposition 3), but their value do not ensure the existence of the double limit.

If $t \neq u$, the idea is to simplify the expression of $f(t,u)$ to avoid an indeterminate limit form. This can be done using the local canonical form of α in a neighborhood of u (assume α is arc-length parameterized):

$$\begin{aligned}\alpha(t)-\alpha(u) = &\left((t-u) - \frac{(t-u)^3}{6}\,\kappa^2(u) - \frac{(t-u)^4}{8}\,\kappa(u)\,\kappa'(u) + \cdots\right)\mathbf{t}(u)\\ &+ \left(\frac{(t-u)^2}{2}\,\kappa(u) + \frac{(t-u)^3}{6}\,\kappa'(u) + \frac{(t-u)^4}{24}\,\big(\kappa''(u)-\kappa^3(u)\big) + \cdots\right)\mathbf{n}(u),\end{aligned}$$

and the same for $\mathbf{t}$:

$$\begin{aligned}\mathbf{t}(t)-\mathbf{t}(u) = &\left(-\frac{1}{2}\,(t-u)^2\,\kappa^2(u) - \frac{1}{2}\,(t-u)^3\,\kappa(u)\,\kappa'(u) + \cdots\right)\mathbf{t}(u)\\ &+ \left(\kappa(u)\,(t-u) + \frac{1}{2}\,\kappa'(u)\,(t-u)^2 + \frac{1}{6}\,\big(\kappa''(u)-\kappa^3(u)\big)\,(t-u)^3 + \cdots\right)\mathbf{n}(u).\end{aligned}$$

From this we can write

$$f(t,u) = \frac{\big\|\alpha(t)-\alpha(u)\big\|^2}{\big\|\mathbf{t}(t)-\mathbf{t}(u)\big\|^2} = \frac{(t-u)^2\,g(t,u)}{(t-u)^2\,h(t,u)} = \frac{g(t,u)}{h(t,u)},$$

where

$$\begin{aligned}g(t,u) &= 1 + (t-u)^2\,\bar{g}(t,u),\\ h(t,u) &= \kappa^2(u) + (t-u)\,\bar{h}(t,u),\end{aligned}$$

with $\bar{g}$ and $\bar{h}$ being the following continuous functions:

$$\begin{aligned}\bar{g}(t,u) &= -\frac{1}{12}\,\kappa^2(u) - \frac{1}{12}\,\kappa(u)\,\kappa'(u)\,(t-u) + \cdots\\ \bar{h}(t,u) &= \kappa(u)\,\kappa'(u) + \frac{1}{12}\,\big(4\,\kappa(u)\,\kappa''(u) + 3\,\kappa'^2(u) - \kappa^4(u)\big)\,(t-u) + \cdots\end{aligned}$$

Notice that $h(t,u) \neq 0$ (it can only be zero if $\kappa(u) = 0$ and $t = u$, which is not the case). Therefore,

$$\lim_{(t,u)\to(t_0,t_0)} f(t,u) = \lim_{(t,u)\to(t_0,t_0)} \frac{g(t,u)}{h(t,u)} = \frac{g(t_0,t_0)}{h(t_0,t_0)} = \frac{1}{\kappa^2(t_0)}. \qquad \square$$

Theorem 6. *The function $\tilde{f}$ is differentiable in its domain. Moreover, given t_0 such that $\kappa(t_0) \neq 0$ and $\kappa''(t_0) \neq 0$, the function $\tilde{f}$ attains a local minimum at a point (t_0,t_0) if and only if κ attains a local maximum at t_0.*

Proof. For points (t,u) of the domain with $t \neq u$, the function $\tilde{f}$ is trivially differentiable. Let us see now that it is also differentiable at a point (t_0,t_0) such that $\kappa(t_0) \neq 0$.

With the same idea as in the proof of Proposition 4 (using the local form of α and $\mathbf{t}$), we can compute the partial derivatives of $f(t,u)$ locally (for t in a neighborhood of u). Since

$$\bar{h}(t,t) = \kappa(t)\,\kappa'(t),$$

with a straightforward computation we get

$$f_t(t_0,t_0) = -\frac{\bar h(t_0,t_0)}{\kappa^4(t_0)} = -\frac{\kappa'(t_0)}{\kappa^3(t_0)},$$
$$f_u(t_0,t_0) = -\frac{2\,\kappa(t_0)\,\kappa'(t_0) - \bar h(t_0,t_0)}{\kappa^4(t_0)} = -\frac{\kappa'(t_0)}{\kappa^3(t_0)}.$$

As these partial derivatives exist in an open neighborhood of (t_0,t_0) and are continuous, we can conclude that $\tilde f$ is differentiable at (t_0,t_0). Thus, $\tilde f$ is differentiable in its domain.

Given $t_0 \in I$ such that $\kappa(t_0) \neq 0$, notice that $f_t(t_0,t_0) = f_u(t_0,t_0) = 0$ if and only if $\kappa'(t_0) = 0$ so that, we have that (t_0,t_0) is a critical point of $\tilde f$ if and only if t_0 is a critical point of κ^2.

Now, if $\tilde f$ has a local minimum at (t_0,t_0), then there exists $r > 0$ such that $f(t_0,t_0) \le f(t,u)$ for all $(t,u) \in B\big((t_0,t_0),r\big)$. In particular,

$$\frac{1}{\kappa^2(t_0)} = f(t_0,t_0) \le f(t,t) = \frac{1}{\kappa^2(t)},$$

for any $t \in I$ such that $|t - t_0| < \frac{r}{\sqrt{2}}$. This proves that $\kappa^2(t) \le \kappa^2(t_0)$ locally, which means that κ^2 attains a local maximum at t_0.

The other implication can be proved with the second partial derivative test. Since $\bar g(t,t) = -\frac{1}{12}\,\kappa^2(t)$ and

$$\bar h_t(t,t) = \frac{1}{12}\big(4\,\kappa(t)\,\kappa''(t) + 3\,\kappa'^2(t) - \kappa^4(t)\big),$$
$$\bar h_u(t,t) = \frac{1}{12}\big(8\,\kappa(t)\,\kappa''(t) + 9\,\kappa'^2(t) + \kappa^4(t)\big),$$

we can also compute analogously

$$f_{tt}(t_0,t_0) = \frac{9\,\kappa'^2(t_0) - 4\,\kappa(t_0)\,\kappa''(t_0)}{6\,\kappa^4(t_0)},$$
$$f_{tu}(t_0,t_0) = f_{ut}(t_0,t_0) = \frac{9\,\kappa'^2(t_0) - 2\,\kappa(t_0)\,\kappa''(t_0)}{6\,\kappa^4(t_0)},$$
$$f_{uu}(t_0,t_0) = \frac{9\,\kappa'^2(t_0) - 4\,\kappa(t_0)\,\kappa''(t_0)}{6\,\kappa^4(t_0)},$$

and conclude that, in fact, $\tilde f$ is of class $\mathcal{C}^2$ in its domain.

Suppose that $t_0 \in I$ is a critical point of κ^2. Since $\kappa(t_0) \neq 0$, this means that $\kappa'(t_0) = 0$. Under the assumption $\kappa''(t_0) \neq 0$, we have

$$f_{tt}(t_0,t_0)\,f_{uu}(t_0,t_0) - f_{tu}(t_0,t_0)^2 = \frac{\kappa''(t_0)^2}{3\,\kappa(t_0)^6} > 0$$

and the sign of

$$f_{tt}(t_0,t_0) = -\frac{2\,\kappa''(t_0)}{3\,\kappa^3(t_0)}$$

is the opposite of the sign of $\kappa(t_0)\,\kappa''(t_0)$. From this, we can deduce that the function $\tilde f$ attains a local minimum at (t_0,t_0) if and only if the function κ^2 attains a local maximum at t_0. $\square$

Example 2. Consider again the parabola from Example 1. In this case, we have

$$f(t,u) = \frac{\left\|\alpha(t)-\alpha(u)\right\|^2}{\left\|\mathbf{t}(t)-\mathbf{t}(u)\right\|^2} = \frac{\sqrt{1+4\,t^2}\,\sqrt{1+4\,u^2}\,(t-u)^2\left(1+(t+u)^2\right)}{2\sqrt{1+4\,t^2}\,\sqrt{1+4\,u^2}-8\,t\,u-2}.$$

The function is not defined in the diagonal $t=u$.

To avoid self-intersections in the generalized offset $\alpha_{d,\omega}$ we must compute the infimum of this function. It can be done numerically (we have used built-in functions of Mathematica to approximate such a value). The approximation we get of its infimum is 0.25 (see Figure 3), which means that $\inf D_\alpha \approx 0.5$.

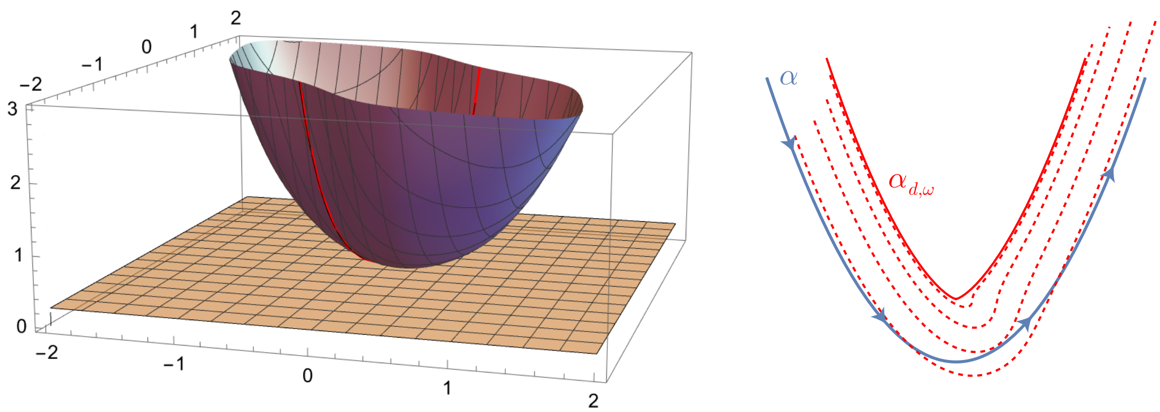

Figure 3. On the left, the graph of the function $f(t,u)$ extended continuously with the (red) curve $\left(t,t,1/\kappa^2(t)\right)$, where κ is the curvature function of the parabola α, and a plot of the plane $z=0.25$. On the right, some $(\omega,0.5)$-offsets to α for different values of $\omega \in \left[0,\frac{\pi}{2}\right]$. These do not present self-intersections.

In this case, a local analysis using Theorem 6 is helpful. In fact, we can also compute the value of the infimum analytically by computing the local minimum of the function $\frac{1}{\kappa^2(t)}$, which is indeed $\frac{1}{\kappa^2_{\max}} = \frac{1}{4}$. The corresponding value $\frac{1}{2}$ for the offset distance is the bound to avoid local self-intersections.

Example 3. Consider the curve $\alpha : [0,1] \to \mathbb{R}^2$ given by

$$\alpha(t) = \Bigg(\frac{1}{2}\left(-405\,t^7+1092\,t^6-2079\,t^5+3150\,t^4-2555\,t^3+882\,t^2-91\,t+6\right),$$
$$-22\,t^7+385\,t^6-945\,t^5+910\,t^4-420\,t^3+84\,t^2+7\,t+1\Bigg).$$

Let us control first local self-intersections. In this case, we have five critical points of κ. From these we can compute $1/\kappa_{\max} \approx 0.398923$. However, this bound on the offset distance does not prevent global self-intersections (see Figure 4).

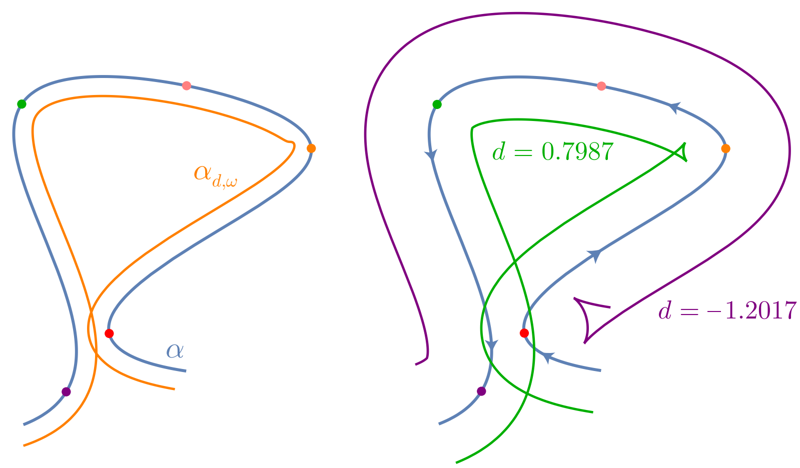

FIGURE 4. The curve α and its critical points. On the left, its $(0.3989, 3\pi/8)$-offset, which self-intersects globally. On the right, two classical offsets to α at local bound distances given by two critical points.

Thus, in this case it is necessary to compute the infimum of the function $f(t, u)$. An approximation of this infimum is 0.0997674, so that its square root provides the bound $d < 0.31586$ on the offset distance to prevent global self-intersections as well (see Figure 5). Graphically, it corresponds to a local minimum of f that is attained outside the diagonal $t = u$.

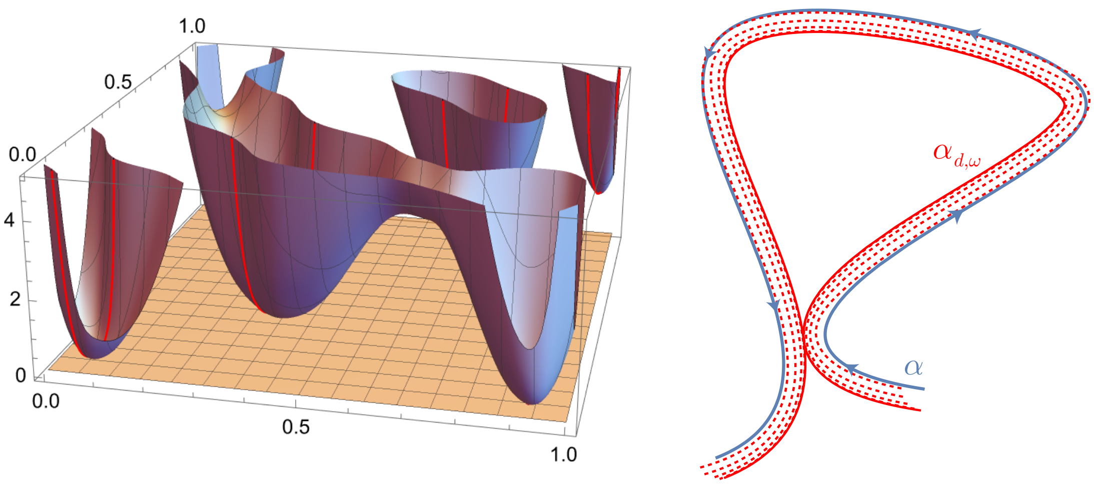

FIGURE 5. On the left, the graph of the function $f(t, u)$ extended continuously with the (red) curve $\big(t, t, 1/\kappa^2(t)\big)$, where κ is the curvature function of α, and a plot of the plane $z = 0.0997674$. On the right, some $(\omega, 0.31586)$-offsets to α for different values of $\omega \in \left[0, \frac{\pi}{2}\right]$. These do not present self-intersections.

2.5. **A remark on a related problem.** Curves of constant width can also be defined as self-parallel curves, in the sense that its (classical) offset at a distance equal to the constant width describes the same curve.

Once generalized offsets have been defined, there is a natural extension of constant width curves by measuring a kind of *skewed width* according to a non-right angle. That is, closed curves in which the endpoints of a constant length chord can travel along maintaining a constant angle with the tangent to the curve at the first endpoint.

In the papers (Santaló, 1944; Vidal Abascal, 1947a), some results involving this kind of curves are given both in the plane and in constant curvature surfaces. The same results can be deduced from the setting presented by Combes (1944). Nevertheless, no example of such curves is found in the literature but the circle. In a spring seminar of 1977, Norman F. Lindquist from the Western Washington University raised the question, which can be found in (Broman, 1979), about if the circle is the only curve of constant *skewed* width (according to a non-right angle).

Related questions were already asked by Lyusternik (1946). Finally, Kovalev (1980) provided an answer to the planar problem (in Russian, an English version is available: Kovalev (1982)) making reference to Lyusternik but not to the other authors. The conclusion is that the circle is the only convex curve that satisfies this notion of *constant skewed width*, so that the concept becomes trivial except for the case of a right angle, where we have the classical constant width curves.

3. Parameterization of curves by a support function

3.1. **Trigonometric classical approach.** In plane differential geometry, the most common way to parameterize a unit circle c is using a trigonometric parameterization, for instance,

$$c(\theta) = (\sin\theta,\, -\cos\theta), \quad \theta \in [-\pi, \pi].$$

If we consider that this parameterization describes a normal vector defining a set of tangent lines to a closed curve α, then a parameterization for α appears naturally by means of what is called a support function.

Given a smooth function $p : [-\pi, \pi] \to \mathbb{R}$, consider the family of lines defined by

$$\sin(\theta)\, X - \cos(\theta)\, Y = p(\theta). \tag{8}$$

The function $p(\theta)$ is the signed distance from the origin to the tangent line given by the direction $(\cos\theta, \sin\theta)$. But note that $p(\theta)$ is not necessarily trigonometric.

Consider the envelope α of the family of lines (8). This produces the following parameterization of the envelope:

$$\alpha(\theta) = p(\theta)\,(\sin\theta,\, -\cos\theta) + p'(\theta)\,(\cos\theta,\, \sin\theta). \tag{9}$$

Notice that for each $\theta \in [-\pi, \pi]$, we have one and only one normal direction to α. If α is closed, then α is called the *hedgehog defined by the support function* p (Martinez-Maure, 1999).

We have that

$$\left\|\alpha'(\theta)\right\| = p(\theta) + p''(\theta).$$

Notice that the curvature κ of α satisfies $\left\|\alpha'(\theta)\right\| \kappa(\theta) = 1$. Thus, if

$$p(\theta) + p''(\theta) > 0, \tag{10}$$

then α is a convex curve parameterized as the envelope of its family of tangent lines. Thus, convex curves are singularity-free hedgehogs.

3.2. **Rational parameterizations by a support function.** As the reader can easily notice, Pottmann's method described in Section 2.2 is analogous to the trigonometric classical approach of Section 3.1 but for rational functions. Therefore, it provides a way to find a rational parameterization of a hedgehog α by a rational support function $h : \mathbb{R} \to \mathbb{R}$.

If we take $a(t) = t$ and $b(t) = 1$ in (4), we get

$$\alpha(t) = h(t)\left(\frac{2\,t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) + \frac{1}{2}(t^2+1)\,h'(t)\left(-\frac{t^2-1}{t^2+1}, \frac{2\,t}{t^2+1}\right). \tag{11}$$

Any curve α that can be written as in (11) will be said to be *rationally parameterized by a (rational) support function h*.

The expression (11) can also be obtained by "translating" the trigonometric form (9) by the corresponding rational expressions. The "translation" from trigonometric to rational expressions is achieved by

$$\cos(\theta) = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin(\theta) = \frac{2\,t}{1+t^2}.$$

The relation between these two parameters is

$$t = \tan\frac{\theta}{2}$$

which is the well-known parameter change to transform trigonometric expressions into rational forms when solving integrals. The resulting curve will be rational as long as the support function was trigonometric.

Therefore, the following parameter change allows us to "translate" trigonometric expressions into the corresponding rational forms:

$$\theta(t) = 2\arctan(t).$$

In such a case, notice that

$$p'\big(\theta(t)\big) = \frac{(p\circ\theta)'(t)}{\theta'(t)} = \frac{1}{2}(1+t^2)\,h'(t)$$

is the factor that appears together with the tangent vector in (11). As in the trigonometric case, this quantity also represents a signed distance (see Figure 6).

Remark 3. In Figure 6 and in the rest of figures of the paper, the rational closed curves which are plotted have a gap just because the plot range parameter is, of course, not the entire real line, but a finite domain $[-a, a]$ for some $a \in \mathbb{R}$.

If α is wanted to be closed (as the curve of Figure 6), then the condition is that

$$\lim_{t\to-\infty} h(t) = \lim_{t\to+\infty} h(t)$$

and that these limits are finite.

As a direct consequence of Theorem 2 we have the following statement.

Proposition 5. *Any rationally parameterized curve by a support function is Pythagorean-hodograph.*

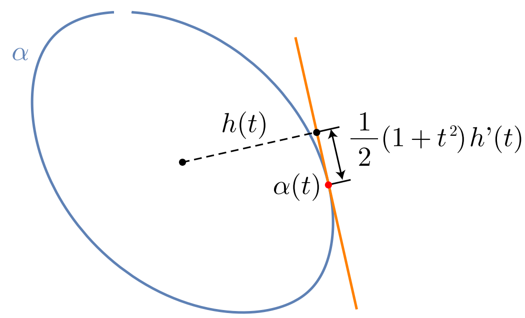

FIGURE 6. A rationally parameterized convex curve α by a support function h.

In fact, in this case we have

$$\|\alpha'(t)\| = \frac{\left|4\,h(t) + 2\,t\,(1+t^2)\,h'(t) + (1+t^2)^2\,h''(t)\right|}{2\,(1+t^2)}.$$

From this we can also set the analogous condition to (10) for rational parameterizations:

Proposition 6. *Let α be a rationally parameterized curve by a support function h. The curve α is regular if and only if*

$$4\,h(t) + 2\,t\,(1+t^2)\,h'(t) + (1+t^2)^2\,h''(t)$$

has no zero.

Remark 4. It is straightforward to check that the curvature κ of a rationally parameterized curve α satisfies

$$\|\alpha'(t)\|\,\kappa(t) = \frac{2}{1+t^2}.$$

Hence, from Proposition 6, if the support function h for a rationally parameterized closed curve α satisfies

$$4\,h(t) + 2\,t\,(1+t^2)\,h'(t) + (1+t^2)^2\,h''(t) > 0, \tag{12}$$

for all $t \in \mathbb{R}$, then the curve α is convex (and regular).

Example 4. Let us give an example of a rationally parameterized curve by a support function which is not a closed curve. Consider the support function

$$h(t) = a\,(1+t^2),$$

for some $a \in \mathbb{R} \setminus \{0\}$. The parametric expression (11) reduces to

$$\alpha(t) = a\,\big(1 - 3\,t^2,\, t\,(3 - t^2)\big),$$

which is Tschirnhausen's cubic (see Figure 7). Indeed, as we expect from Proposition 5, it is a classic example of a PH curve (that in fact is the unique PH cubic, see Farouki (2008)):

$$\|\alpha'(t)\| = 3\,|a|\,(1+t^2).$$

Indeed, it is a regular curve for all $t \in \mathbb{R}$.

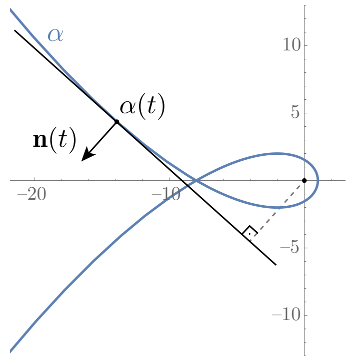

Figure 7. Tschirnhausen's cubic. Notice how it is parameterized by the inverse Gauss map.

4. Rational curves of constant width

In the classical trigonometric case it is very easy to generate constant width curves by taking appropriate support functions. In particular, support functions of the kind

$$h(\theta) = \frac{d}{2} + \sum_{k=0}^{n} \Big(a_{2k+1} \cos\big((2\,k+1)\,\theta\big) + b_{2k+1} \sin\big((2\,k+1)\,\theta\big) \Big) \tag{13}$$

satisfy that $h(\theta) + h(\theta + \pi) = d$ and, therefore, if d is chosen such that $h(\theta) + h''(\theta) > 0$, then they describe convex curves of constant width d.

There are other methods to construct curves of constant width, the interested reader can see, for instance, (Yaglom and Boltyanskiĭ, 1960; Martini and Mustafaev, 2008; Ait-Haddou et al., 2008). Removing the convexity constraint, the support functions (13) actually generate hedgehogs of constant width whose offsets, for a sufficiently large distance, become convex curves of constant width (Rochera, 2022a,b).

In this section, let us see what happens with closed curves rationally parameterized by a support function.

Consider a pair of parallel supporting lines to a convex curve α, say at $\alpha(t)$ and $\alpha(u)$. It must be satisfied that the normals are opposite at these points, i.e.

$$\left(\frac{2\,t}{t^2+1}, \frac{t^2-1}{t^2+1} \right) + \left(\frac{2\,u}{u^2+1}, \frac{u^2-1}{u^2+1} \right) = (0,\,0).$$

The unique solution of the equation above is $u = -1/t$. That is, the width of the figure measured by these two supporting lines is equal to $h(t) + h(-1/t)$.

Thus, a rationally parameterized curve α is of constant width d if and only if

$$h(t) + h(-1/t) = d,$$

for all $t \in \mathbb{R} \setminus \{0\}$. It is known that if α is a curve of constant width, then the chord joining the contact points of the parallel supporting lines is orthogonal to those lines (Yaglom and Boltyanskiĭ, 1960). This means that the chord which measures the constant width has endpoints $\alpha(t)$ and

$\alpha(-1/t)$:

$$\big\|\alpha(t) - \alpha(-1/t)\big\| = d,$$

for all $t \in \mathbb{R} \setminus \{0\}$.

The most easy example of a curve of constant width is a circle. A rational circle of radius r can be generated from a constant support function $h(t) = r$:

$$\alpha(t) = r\left(\frac{2\,t}{t^2+1}, \frac{t^2-1}{t^2+1}\right).$$

Now, let us construct the rational support functions corresponding to (13) to get rational constant width curves. The "translation" is easy by the properties of Chebyshev polynomials. Recall that a Chebyshev polynomial T_n of degree n can be defined recursively by

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_{n+1}(x) &= 2\,x\,T_n(x) - T_{n-1}(x), \quad n \geq 1. \end{aligned}$$

If $n = 2\,k+1$, these polynomials satisfy

$$\cos\big((2\,k+1)\,t\big) = T_{2k+1}(\cos t)$$

and

$$\sin\big((2\,k+1)\,t\big) = (-1)^k\,T_{2k+1}(\sin t).$$

This gives rise to the following result, for which Chebyshev polynomials are useful to write the explicit expression in a compact manner.

Proposition 7. *Any rationally parameterized curve by a support function of the kind*

$$h(t) = \frac{d}{2} + \sum_{k=0}^{n}\left(a_{2k+1}\,T_{2k+1}\left(\frac{1-t^2}{1+t^2}\right) + b_{2k+1}\,T_{2k+1}\left(\frac{2\,t}{1+t^2}\right)\right) \tag{14}$$

is a hedgehog of constant width d. If, in addition, d is such that (12) is satisfied, then it is a convex curve of constant width d.

Proof. The parity of Chebyshev polynomials is given by:

$$T_n(-x) = (-1)^n\,T_n(x).$$

Hence, for $n = 2\,k+1$, the Chebyshev polynomial T_{2k+1} is an odd function. This is actually enough to conclude the result (Gravesen et al., 2008). Explicitly, we have

$$T_{2k+1}\left(\frac{1-(-1/t)^2}{1+(-1/t)^2}\right) = T_{2k+1}\left(\frac{t^2-1}{t^2+1}\right) = -T_{2k+1}\left(\frac{1-t^2}{1+t^2}\right)$$

and

$$T_{2k+1}\left(\frac{2\,(-1/t)}{1+(-1/t)^2}\right) = T_{2k+1}\left(\frac{-2\,t}{t^2+1}\right) = -T_{2k+1}\left(\frac{2\,t}{1+t^2}\right).$$

Therefore, the support functions (14) satisfy

$$h(t) + h(-1/t) = d,$$

so that the corresponding rationally parameterized curve α is of constant width d. □

Remark 5. Notice that all the curves of constant width provided by Proposition 7 are PH as a consequence of Proposition 5.

Example 5. The rational support function (14) for $n = 0$ takes the form

$$h(t) = \frac{d}{2} + a_1 \frac{1 - t^2}{1 + t^2} + b_1 \frac{2\,t}{1 + t^2},$$

which produces a parameterization of a circle centered at $(b_1, -a_1)$ with diameter the constant width d:

$$\alpha(t) = (b_1, -a_1) + \frac{d}{2}\left(\frac{2\,t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1}\right).$$

Consider now a rational support function of the kind (14) for $n = 1$. The explicit expression is

$$\begin{aligned} h(t) = \frac{d}{2} + a_1 \frac{1 - t^2}{1 + t^2} + b_1 \frac{2\,t}{1 + t^2} \\ - a_3 \frac{(t^6 - 15\,t^4 + 15\,t^2 - 1)}{(1 + t^2)^3} - b_3 \frac{2\,t\,(3\,t^4 - 10\,t^2 + 3)}{(1 + t^2)^3}. \end{aligned}$$

The corresponding rational parameterization is

$$\begin{aligned} \alpha(t) = (b_1, -a_1) + \frac{d}{2}\left(\frac{2\,t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1}\right) \\ + a_3 \left(\frac{16\,t\,(t^2 - 1)^3}{(1 + t^2)^4}, -\frac{t^8 + 20\,t^6 - 90\,t^4 + 20\,t^2 + 1}{(1 + t^2)^4}\right) \\ + b_3 \left(\frac{-3\,t^8 + 36\,t^6 - 50\,t^4 + 36\,t^2 - 3}{(1 + t^2)^4}, \frac{64\,t^3\,(1 - t^2)}{(1 + t^2)^4}\right), \end{aligned}$$

which is of degree 8. The free parameters a_1 and b_1 perform just a translation. The parameters a_3 and b_3 change the figure shape (they also rotate and change the figure size), see in Figure 8 some examples for different values of a_3.

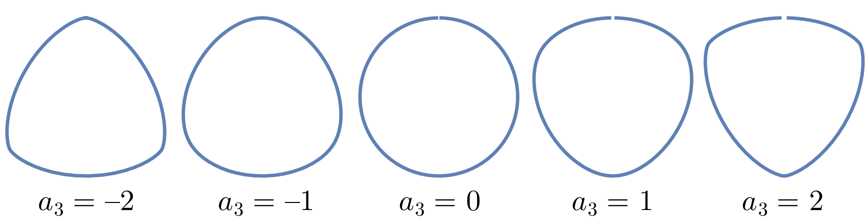

FIGURE 8. For fixed values $d = 36$, $a_1 = b_1 = b_3 = 0$, the shape of α varying the parameter a_3.

5. Tire tracks of a bicycle and rational Zindler curves

In this section we will give an explicit family of rationally parameterized Zindler curves.

Let β be the curve that is found by going a constant distance ℓ in the tangent direction from each point of α:

$$\beta(t) = \alpha(t) + \ell\, \mathbf{t}(t),$$

where $\mathbf{t}(t)$ is the tangent vector to α at $\alpha(t)$. The pair of curves $(\alpha,\, \beta)$ serve to parameterize the wheel tire-tracks of a bicycle of length ℓ, where α represents its rear wheel and β its front wheel. The two-sided front wheel curve to α is given piecewise by β and the analogous curve in the opposite direction of $\mathbf{t}(t)$.

It is known that there is a "duality" between constant width curves and Zindler curves (Martini et al., 2019; Oliveros, 1997; Rochera, 2022a,b). The relationship is given in terms of the middle hedgehog, which is the curve generated by the midpoint of the family of constant length chords. For a constant width curve, these chords are those that measure the constant width at each point. For Zindler curves, these chords are those which cut in a half the perimeter of the curve (and which are called *halving chords*).

A family of rational hedgehogs of "constant width 0" (projective hedgehogs) can be obtained from (14) by taking $d = 0$. On the one hand, we can generate curves of constant width d from this hedgehog by just considering continuous offset curves at the distance $\frac{d}{2}$. This is translated to adding this value to the support function of the middle hedgehog as done in Equation (14). On the other hand, Zindler curves can be generated as continuous front wheel tire-track curves of a bicycle of length $\frac{d}{2}$ having the middle hedgehog as the rear wheel.

As a direct consequence we have the following result, where we write the associated curves to those of Proposition 7.

Proposition 8. *Any parametric curve defined by*

$$p(t)\left(\frac{2\,t}{t^2+1},\, \frac{t^2-1}{t^2+1}\right) + \frac{1}{2}\left(\frac{d}{t^2+1} + p'(t)\right)(1-t^2,\, 2\,t), \tag{15}$$

where

$$p(t) = \sum_{k=0}^{n}\left(a_{2k+1}\, T_{2k+1}\left(\frac{1-t^2}{1+t^2}\right) + b_{2k+1}\, T_{2k+1}\left(\frac{2\,t}{1+t^2}\right)\right), \tag{16}$$

is a rational (generalized) Zindler curve that has its perimeter cut in half with a chord of length d. For a sufficiently large d, the curve becomes a classical Zindler curve (i.e., the halving chords touch the curve only at their endpoints).

Proof. The rationally parameterized hedgehog γ defined by the support function p is Pythagorean-hodograph (by Proposition 5). In particular, it is written in the standard form (5) for the support function p. Then by Theorem 4 particularized to curves rationally parameterized by a support function, i.e. $a(t) = t$ and $b(t) = 1$, we get the expression of the continuous front wheel curve of γ for a bicycle of length $d/2$, which is Equation (15) of the statement. By construction, it is a generalized Zindler curve. Taking d such that $h(t) = \frac{d}{2} + p(t)$ satisfies (12) is a sufficient condition to ensure that the family of chords of constant length d cut the curve at precisely two points (and not more). □

Example 6. Consider a support function of the kind (16) for the middle hedgehog with $n = 1$:

$$h(t) = a_1 \frac{1-t^2}{1+t^2} + b_1 \frac{2\,t}{1+t^2} - a_3 \frac{(t^6 - 15\,t^4 + 15\,t^2 - 1)}{(1+t^2)^3} - b_3 \frac{2\,t\,(3\,t^4 - 10\,t^2 + 3)}{(1+t^2)^3}.$$

Now, by Proposition 8, the parametric curve defined by the expression (15), namely,

$$\begin{aligned}\beta(t) = (b_1,\, -a_1) + \frac{d}{2}\left(\frac{1-t^2}{1+t^2}, \frac{2\,t}{1+t^2}\right) \\ + a_3\left(\frac{16\,t\,(t^2-1)^3}{(1+t^2)^4}, -\frac{t^8 + 20\,t^6 - 90\,t^4 + 20\,t^2 + 1}{(1+t^2)^4}\right) \\ + b_3\left(\frac{-3\,t^8 + 36\,t^6 - 50\,t^4 + 36\,t^2 - 3}{(1+t^2)^4}, \frac{64\,t^3\,(1-t^2)}{(1+t^2)^4}\right),\end{aligned}$$

is a rational Zindler curve (of degree 8). This Zindler curve is the one associated with the constant width curve considered in Example 5 for $n = 1$. Again, the parameters a_1 and b_1 perform just a translation and the values a_3 and b_3 change the figure shape. See in Figure 9 the resulting curve β for different values of a_3.

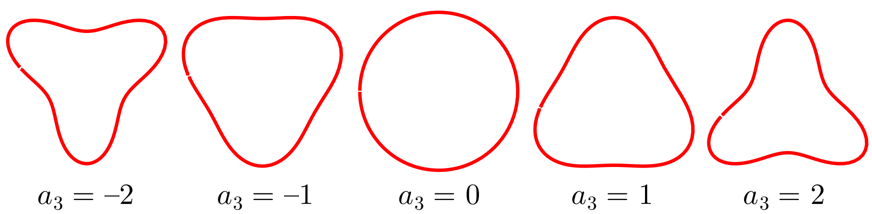

FIGURE 9. For fixed values $d = 36$, $a_1 = b_1 = b_3 = 0$, the shape of β varying the parameter a_3.

Example 7. We can construct more versatile rational examples with Propositions 7 and 8 taking higher degrees. Hedgehogs of constant width and Zindler curves which are not rotationally symmetric can be generated changing the parameter values, and that results in less typical examples for these types of curves (see Figure 10). If one wants to get convex curves of constant width, it is enough to choose a larger parameter value for the distance.

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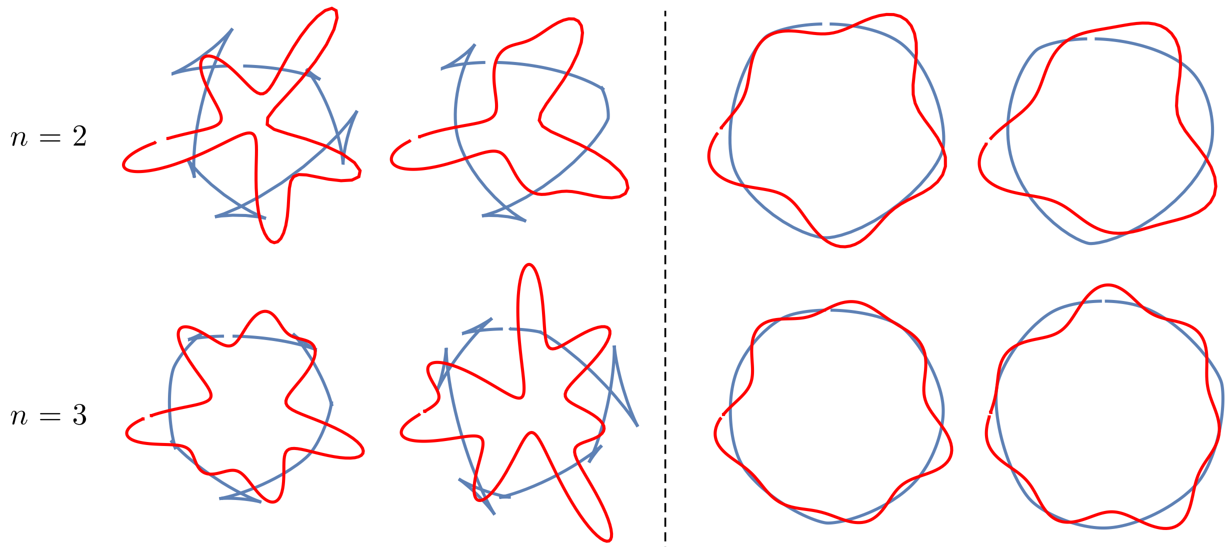

Figure 10. Examples of hedgehogs of constant width and their associated Zindler curves for $n = 2$ and $n = 3$ and for different values of a_i and b_i. The figures on the right are found by taking a higher d value in the corresponding curves on the left.

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