

Exact calculation of the mean first-passage time of continuous-time random walks by nonhomogeneous Wiener–Hopf integral equations

M Dahlenburg^{1,2}, G Pagnini^{1,3}

¹ BCAM–Basque Center for Applied Mathematics, Alameda de Mazarredo 14, 48009 Bilbao, Basque Country – Spain

² Institute for Physics & Astronomy, University of Potsdam, 14476 Potsdam, Germany

³ Ikerbasque–Basque Foundation for Science, Plaza Euskadi 5, 48009 Bilbao, Basque Country – Spain

E-mail: mdahlenburg@bcamath.org

Received 17 January 2022

Accepted 12 December 2022

Abstract. We study the mean first-passage time (MFPT) for asymmetric continuous-time random walks in continuous-space characterised by waiting-times with finite mean and by jump-sizes with both finite mean and finite variance. In the asymptotic limit, this well-controlled process is governed by an advection-diffusion equation and the MFPT results to be finite when the advecting velocity is in the direction of the boundary. We derive a nonhomogeneous Wiener–Hopf integral equation that allows for the exact calculation of the MFPT by avoiding asymptotic limits and it emerges to depend on the whole distribution of the jump-sizes and on the mean-value only of the waiting-times, thus it holds for general non-Markovian random walks. Through the case study of a quite general family of asymmetric distributions of the jump-sizes that is exponential towards the boundary and arbitrary in the opposite direction, we show that the MFPT is indeed independent of the jump-sizes distribution in the opposite direction to the boundary. Moreover, we show also that there exists a length-scale, which depends only on the features of the distribution of jumps in the direction of the boundary, such that for starting points near the boundary the MFPT depends on the specific whole distribution of jump-sizes, in opposition to the universality emerging for starting points far-away from the boundary.

Keywords: mean first-passage time, continuous-time random walk, Wiener–Hopf integral equation

Submitted to: *J. Phys. A: Math. Theor.*

1. Introduction

The mean first-passage time (MFPT) statistics are important, in general, for many diffusive processes [1, 2, 3] and, in particular, for living [4, 5, 6] and reacting chemical systems [7, 8, 9]. A large number of results have been derived [10, 2, 11], together with recent findings both for Brownian [12, 13, 14, 15, 16, 17] and non-Brownian motions as the diffusing-diffusivity approach [18, 19, 20] or Lévy-like motions [21, 22, 23, 24]. Beside established findings, the quantitative experimental analysis of first-passage time distributions is becoming accessible in biology only recently [25, 26] and further advancements are still of high interest because of new emerging applications and results [27, 28, 29, 30, 31].

Here, we focus on the MFPT problem in the well-controlled setting of a one-dimensional continuous-time random walk (CTRW) [32] in a continuous-space when the waiting-times distribution has a finite mean and the jump-sizes distribution has both finite mean and finite variance. If the mean value of the jump-sizes is different from zero, in the asymptotic limit, the probability density function (PDF) of such random walker is governed by an advection-diffusion equation and the corresponding MFPT results to be finite when the advecting velocity is in the direction of the boundary. In particular, in the present study, we derive a nonhomogeneous Wiener–Hopf integral equation [33] for the determination of the MFPT that allows indeed for its exact calculation and then for avoiding asymptotic limits. This formula emerges to depend on the whole distribution of the jump-sizes and on the mean-value only of the waiting-times. Thus, it holds for general non-Markovian random walks since, in CTRW, Markovianity intended as a local governing equation of the particle distribution (in opposition to integral equation in time) emerges solely with an exponential distribution of the waiting-times, see, [34] and discussion around equation (2.6) in [35]. In this setting, we consider the quite general case study of a family of asymmetric distributions of the jump-sizes that is exponential towards the boundary and arbitrary in the opposite direction. By using the derived nonhomogeneous Wiener–Hopf equation, we can show that the MFPT is indeed independent of the jump-sizes distribution in the opposite direction to the boundary and this finding leads to the generalization of existing results as, for example, those based on an asymmetric double-exponential jump-sizes distribution [36, 37]. Moreover, we show that there exists a length-scale distinguishing near-boundary and far-boundary starting points that depends only on the features of the distribution of jumps in the direction of the boundary. Therefore, if for initial positions far-away from the boundary a universal behaviour is observed, namely the MFPT does not depend on the specific distribution of the jump-sizes, for initial positions near the boundary the emerging length-scale is indeed dependent on the specific distribution of the jump-sizes towards the boundary and then the universality is lost.

The advection-diffusion framework applies strictly to the case when the arrival location is fixed, but it can be assumed also as approximation when the mobility of the walker, e.g., the predator, is much greater than the mobility of the endpoint, e.g., the

pray [38, 39]. Actually, such approximation for animal movement works in practice, for example, for modelling fish population [40] or when red foxes preying on duck nests [41] and, in general, it is the best strategy for a prey to survive when the motion is on a lattice [42, 43].

However, beside searching processes in the advection-diffusion setting, we want to highlight that the derived formalism and results play a role indeed also in the recent studies on stochastic resetting (SR) [44, 45, 46] and in particular in its generalisation as random amplitude stochastic resetting (RASR) [47]. SR refers to diffusive processes that are interrupted by a step-back to the origin while RASR is its generalisation in the sense that the step-back to the origin is replaced indeed by a step-back with random amplitude. Recently, SR has been experimentally investigated [48], too. Actually, a relation exists between processes with resetting and optimal search strategies, in particular with certain intermittent search processes [44, 45].

A remarkable feature of resetting events on the diffusive dynamics is that non-stationary processes turn into stationary and the corresponding MFPT from infinite turns into finite [46, 49]. Therefore, for such processes, it is of high interest the derivation of a formalism for the determination of the survival probability as well as of the MFPT [47, 50, 49]. To this aim, on the basis of the so-called *first renewal picture*, an approach was introduced in the framework of the SR for determining the survival probability of Brownian diffusion in the presence of resetting interval lengths between subsequent resetting events that are exponentially distributed with time-dependent resetting rate [50]. The same approach was later extended to more general random motions and arbitrary distribution of resetting interval lengths [49].

A similar approach for determining the survival probability as well as the MFPT in the case of RASR [47] is desired and not available yet. We report that RASR can be divided into dependent RASR and independent RASR and that the simplest process of independent RASR is the CTRW. Therefore, the Wiener–Hopf equation here derived and the finding of a critical length-scale is a first-step in this direction. Moreover, we report that the CTRW under resetting was considered for studying the MFPT in the frame of anomalous diffusion with restarts [51, 52].

The remainder of the paper is organised as follows. In the next Section 2 we provide the mathematical setting for a general random walk that in the asymptotic limit is governed by an advection-diffusion equation and we discuss how such a system displays a finite MFPT. Then we derive in Section 3 the Wiener–Hopf equation for the exact computation of the MFPT and in Section 4 we put in action this formula for the case study of a quite general family of asymmetric distributions of the jump-sizes that is exponential towards the boundary and arbitrary in the opposite direction. Conclusions are finally reported in Section 5.

2. Finite-mean first-passage time

Let \mathbb{R} be the set of real numbers, we denote by \mathbb{R}^+ and by \mathbb{R}_0^+ the set of positive real numbers and the set of non-negative real numbers, i.e., $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$ and $\mathbb{R}_0^+ = \{t \in \mathbb{R} | t \geq 0\}$, analogously we denote by \mathbb{R}^- and by \mathbb{R}_0^- the set of negative real numbers and the set of non-positive real numbers.

We consider a one-dimensional random walk in continuous-space $x \in \mathbb{R}$ and continuous-time $t \in \mathbb{R}_0^+$. In particular, we study a CTRW characterised by a jump-sizes distribution $q : \mathbb{R} \rightarrow \mathbb{R}_0^+$ and by a waiting-times distribution between consecutive jumps $\psi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, which normalise according to $\int_{\mathbb{R}} q(\xi) d\xi = 1$ and $\int_{\mathbb{R}_0^+} \psi(\tau) d\tau = 1$, respectively. If the distribution of the waiting-times is exponential then the process is Markovian. We assume that drawings of the jump-sizes and of the waiting-times are statistically independent, and both are independent and identically distributed (iid) random variables.

Moreover, as it is reported in the Introduction, the present research aims also to be a preliminary study for further advancements in SR [44, 45, 46] and, more in general, in RASR theory [47], then within those frameworks the walker's trajectory can be interpreted as follows. Let $X_t : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ be the walker's position at time $t \geq 0$, and let the notation $X_{t-\tau} | y$ denoting the actual walker's position at time t provided that the walker was previously in $y \in \mathbb{R}$ at $\tau \in [0, t]$, which is the duration of the first random waiting-time. By using the idea of the *first renewal picture* [47, 50], the walker may stay until time t in the initial position $x_0 \in \mathbb{R}$ with probability $\Psi(t) = 1 - \int_0^t \psi(\tau) d\tau$ or, it may be at time t in $X_{t-\tau}$ starting from the new initial-like position $x_0 + \xi$, $\xi \in \mathbb{R}$, with the complementary probability $\int_0^t \psi(\tau) d\tau$, where ξ denotes a random jump amplitude that may occurs at the new initial datum $\tau \in [0, t]$. In formulae, for a statistically homogeneous process, the conditional PDF $p(x, t; x_0)$ of the walker emerges to be [53]

$$p(x, t; x_0) = \Psi(t)\delta(x - x_0) + \int_0^t \psi(\tau) \int_{-\infty}^{\infty} q(\xi)p(x - \xi, t - \tau; x_0) d\xi d\tau, \quad (1)$$

or

$$p(x, t; x_0) = \Psi(t)\delta(x - x_0) + \int_0^t \psi(\tau) \int_{-\infty}^{\infty} q(\xi - x_0)p(x, t - \tau; \xi) d\xi d\tau. \quad (2)$$

In equation (1), variable ξ represents the size of the first jump, while in (2) it represents the starting position immediately after the first jump.

We consider a CTRW with finite-mean waiting-times, i.e.,

$$0 < \langle \tau \rangle = \int_0^{\infty} \tau \psi(\tau) d\tau < +\infty, \quad (3)$$

and finite-mean and finite-variance jump-sizes, i.e.,

$$-\infty < \langle \xi \rangle = \int_{-\infty}^{\infty} \xi q(\xi) d\xi < +\infty, \quad (4)$$

$$0 < \langle \xi^2 \rangle = \int_{-\infty}^{\infty} \xi^2 q(\xi) d\xi < +\infty. \quad (5)$$

Therefore, the diffusive limit $t \rightarrow \infty$ of the process can be obtained in the corresponding limit $s \rightarrow 0$ of $\tilde{\psi}(s)$, i.e., $\tilde{\psi}(s) \simeq 1 - s\langle\tau\rangle + \mathcal{O}(s^2)$, and, analogously, the tails $|x| \rightarrow +\infty$ of the walker's PDF in the corresponding limit $\kappa \rightarrow 0$ of $\hat{q}(\kappa)$, i.e., $\hat{q}(\kappa) \simeq 1 + i\kappa\langle\xi\rangle - \kappa^2\langle\xi^2\rangle/2 + \mathcal{O}(\kappa^3)$, where $\tilde{\psi}(s)$ and $\hat{q}(\kappa)$ are the Laplace transform of $\psi(t)$ and the Fourier transform of $q(x)$, respectively. Thus the marginal distribution, namely $P(x, t) = \int_{-\infty}^{\infty} p(x - y, t)P_0(y) dy$, with $P(x, 0) = P_0(x)$, results to be governed by the advection-diffusion equation [10]

$$\frac{\partial P(x, t)}{\partial t} = -v \frac{\partial P(x, t)}{\partial x} + D \frac{\partial^2 P(x, t)}{\partial x^2}, \quad (6)$$

where

$$v = \frac{\langle\xi\rangle}{\langle\tau\rangle}, \quad D = \frac{\langle\xi^2\rangle}{2\langle\tau\rangle}. \quad (7)$$

The solution of (6) is [10, page 16, formula (1.3.30)]

$$P(x, t) = \int_{-\infty}^{\infty} \frac{P_0(y)}{\sqrt{4\pi Dt}} \exp\left\{-\frac{(x - y - vt)^2}{4Dt}\right\} dy. \quad (8)$$

Without loss in generality, we restrict the initial position in $x_0 \in \Omega \subseteq \mathbb{R}^+$ and locate the target in $x = 0$. This means that the starting position of the walker can not be located on the boundary. Let $\lambda : \Omega \times \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$ be PDF of the first arrival at the absorbing boundary in $x = 0$ at time $t \in \mathbb{R}_0^+$ for a given initial position in $x_0 \in \Omega$, then the survival probability, namely the probability that the walker is not absorbed, up to the time t is

$$\Lambda(x_0, t) = 1 - \int_0^t \lambda(x_0, \tau) d\tau, \quad t \in \mathbb{R}_0^+, \quad x_0 \in \Omega \subseteq \mathbb{R}^+. \quad (9)$$

Actually, the distribution $\lambda(x_0, t)$ and the conditional PDF $p(x, t; x_0)$ are related. In particular, if we assume that a walker, with initial position x_0 , is arrived in $x = 0$ at some instant $t - \tau$, with $\tau \in [0, t]$, then $\lambda(x_0, t)$ provides the probability for the walker to return in $x = 0$ after a time-interval τ , namely at time t , and this gives the convolution integral [54, 32, 10]

$$p(0, t; x_0) = \int_0^t \lambda(x_0, \tau) p(0, t - \tau; 0) d\tau. \quad (10)$$

Formula (10) can be derived also by including a sink $-\lambda(x_0, t)\delta(x)$ in the advection-diffusion equation (6) [52, 55].

To conclude, by combining (9) and (10) in the Laplace domain and by using (8), we have [10]

$$\begin{aligned} \tilde{\Lambda}(x_0, s) &= \frac{1}{s} \left[1 - \frac{\tilde{p}(0, s; x_0)}{\tilde{p}(0, s; 0)} \right] \\ &= \frac{1}{s} \left[1 - \exp\left\{-\frac{[v + \sqrt{v^2 + 4Ds}]x_0}{2D}\right\} \right]. \end{aligned} \quad (11)$$

Thus, by applying the L'Hôpital's rule, the MFPT results to be

$$T(x_0) = \lim_{s \rightarrow 0} \tilde{\Lambda}(x_0, s) = -\frac{x_0}{v} > 0, \quad \text{with } \langle \xi \rangle < 0, \quad (12)$$

by reminding from (7) that $\text{sgn}v = \text{sgn}\langle \xi \rangle$. Actually, the linear dependence of the MFPT on x_0 , as stated in formula (12), is indeed universal, namely it is independent of any specific distribution of the jump-sizes whenever the limit leading to (6) is met.

3. A nonhomogeneous Wiener–Hopf integral equation for the MFPT

The finite MFPT (12) derived in Section 2 holds in the limits $t, |x| \rightarrow +\infty$, or analogously $s, \kappa \rightarrow 0$, in which the behaviour is governed by an advection-diffusion equation (6). This behaviour can be observed only for initial position $x_0 \rightarrow +\infty$, namely far from the absorbing boundary. As a matter of fact, such asymptotic result is independent of the specific distributions of the waiting-times and of the jump-sizes, provided that (3) and (4,5) hold.

An alternative definition of the survival probability is indeed

$$\Lambda(x_0, t) = \int_0^\infty p_{\text{abs}}(x, t; x_0) dx, \quad \forall t \in \mathbb{R}_0^+, \quad x_0 \in \mathbb{R}^+, \quad (13)$$

where $p_{\text{abs}}(x, t; x_0)$ is the conditional distribution with an absorbing boundary at $x = 0$. From equation (2) we have

$$p_{\text{abs}}(x, t; x_0) = \Psi(t)\delta(x-x_0) + \int_0^t \psi(\tau) \int_0^\infty q(\xi-x_0)p_{\text{abs}}(x, t-\tau; \xi) d\xi d\tau, \quad (14)$$

such that it holds

$$\Lambda(x_0, t) = \Psi(t) + \int_0^t \psi(\tau) \int_0^\infty q(\xi-x_0)\Lambda(\xi, t-\tau) d\xi d\tau. \quad (15)$$

Hence, by passing through the Laplace transform, we obtain

$$\tilde{\Lambda}(x_0, s) = \tilde{\Psi}(s) + \tilde{\psi}(s) \int_0^\infty q(\xi-x_0)\tilde{\Lambda}(\xi, s) d\xi, \quad \forall x_0 \in \mathbb{R}^+. \quad (16)$$

By plugging (9) into (16), we observe that, for each waiting-times distribution with finite mean, the first passage time density $\lambda(x_0, t)$ is dependent on the specific waiting-times distribution. Through the limit $s \rightarrow 0$ in (16), we have that the MFPT $T(x_0)$ is

$$T(x_0) = \langle \tau \rangle + \int_0^\infty q(\xi-x_0)T(\xi) d\xi, \quad \forall x_0 \in \mathbb{R}^+, \quad (17)$$

which is a *nonhomogenous* Wiener–Hopf integral equation [33] whose kernel is a PDF [56, 57]. The condition $x_0 \neq 0$ allows for avoiding to step into distribution theory, which requires the definition of the kernel in such a way that the equation holds at the endpoints [33, Chapter 8]. An inhomogeneous Fredholm equation for the MFPT was indeed derived under different assumptions by using an unbiased random walk for modelling animal movement and searching [38].

We observe that for discrete time-steps, i.e., $t \mapsto n \in \mathbb{N}$, we have $\psi(\tau) = \delta(\tau - 1) \mapsto \delta_{n-1}$ and $\Psi(t) = \Theta(1 - t) \mapsto \delta_{n0}$, such that formula (15) reduces to the well-known homogeneous Wiener–Hopf integral equation for Markovian processes [1, equation (10)]

$$\Lambda(x_0, n) = \int_0^\infty q(\xi - x_0) \Lambda(\xi, n - 1) d\xi, \quad (18)$$

with boundary condition $\Lambda(x_0, 0) = 1, \forall x_0 \in \mathbb{R}^+$. Clearly, discrete-time random walks are a special case of CTRW. In particular, we stress that while equation (18) is a straightforward special case of (15), equation (15) cannot be derived from (18). In fact, the nonhomogeneous nature of formula (15) embodied by the probability of no jump in $[0, t]$ is a feature that does not belong to a discrete-time setting. Formula (15) can not be derived from formula (18) and this allows for a more detailed analysis of the MFPT with respect to the existing literature, as it discussed in the following. Moreover, from (17), we observe that the MFPT $T(x_0)$ depends on the whole distribution of the jump-sizes, that in general is asymmetric, and on the mean-value only of the waiting-times, while it is independent of the whole distribution of these last. From the independence of (17) of the waiting-times distribution, we have that (17) holds for general non-Markovian CTRWs, in fact it is known that solely in the special case of an exponential distribution of waiting-times a CTRW is Markovian [34, 35].

Equation (17) can indeed be extended to the d -dimensional case. As a matter of fact, formula (2) can be extended to a CTRW in d -dimensions such that, by defining $q : \mathbb{R}^d \rightarrow \mathbb{R}_0^+$, for all $t \in \mathbb{R}_0^+$ and $\mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^d$, we have that

$$p(\mathbf{x}, t; \mathbf{x}_0) = \Psi(t) \delta^d(\mathbf{x} - \mathbf{x}_0) + \int_0^t \psi(\tau) \int_{\mathbb{R}^d} q(\boldsymbol{\xi} - \mathbf{x}_0) p(\mathbf{x}, t - \tau; \boldsymbol{\xi}) d^d \boldsymbol{\xi} d\tau, \quad (19)$$

which, in presence of an absorbing boundaries, turns into

$$p_{\text{abs}}(\mathbf{x}, t; \mathbf{x}_0) = \Psi(t) \delta^d(\mathbf{x} - \mathbf{x}_0) + \int_0^t \psi(\tau) \int_{\mathbb{R}^{d+}} q(\boldsymbol{\xi} - \mathbf{x}_0) p_{\text{abs}}(\mathbf{x}, t - \tau; \boldsymbol{\xi}) d^d \boldsymbol{\xi} d\tau, \quad (20)$$

where $\mathbb{R}^{d+} = \{(x_1, \dots, x_d)^T \in \mathbb{R}^d | x_i > 0, \forall i \in \{1, \dots, d\}\}$. Finally, by repeating the same steps from (2) to (17), we obtain the following d -dimensional Wiener–Hopf equation for computing the MFPT of a d -dimensional CTRW

$$T(\mathbf{x}_0) = \langle \tau \rangle + \int_{\mathbb{R}^{d+}} q(\boldsymbol{\xi} - \mathbf{x}_0) T(\boldsymbol{\xi}) d^d \boldsymbol{\xi}, \quad \forall \mathbf{x}_0 \in \mathbb{R}^{d+}. \quad (21)$$

4. The role of the distributions of the jump-sizes towards and away from the boundary

We introduce now dimensional spatial-quantities ξ^D and x_0^D through a length-scale ℓ :

$$\xi^D = \ell \xi, \quad x_0^D = \ell x_0, \quad (22)$$

such that, consistently with (7), it holds

$$\langle \xi^D \rangle = \ell \langle \xi \rangle = \ell v \langle \tau \rangle = v^D \langle \tau \rangle. \quad (23)$$

Moreover, we consider the following quite general family of distributions of the jump-sizes:

$$\rho(\xi^D) = q(-\xi^D) = \begin{cases} \frac{(1-b)}{\ell} \exp\left(-\frac{\xi^D}{\ell}\right); & \text{if } \xi^D \in \mathbb{R}_0^+, \\ b \rho_{<}(\xi^D); & \text{if } \xi^D \in \mathbb{R}^-, \end{cases} \quad (24)$$

that generalizes the double-exponential distribution studied in literature [36, 37]. In (24), the length-scale ℓ represents the mean value of the jump-sizes toward the boundary and parameter b denotes the probability to jump away from the absorbing boundary. Thus, we distinguish between exponentially-distributed jumps in the direction of the absorbing boundary and arbitrary jump-distributions in the opposite direction, such that

$$\int_{-\infty}^0 \rho_{<}(\xi^D) d\xi^D = 1, \quad \int_{-\infty}^0 \xi^D \rho_{<}(\xi^D) d\xi^D = -\langle \xi^+ \rangle = -\frac{\ell}{a}, \quad a > 0, \quad (25)$$

The above formula (25) defines the parameter a that is the absolute value of the ratio between the mean value of the jump-sizes towards the boundary, i.e., $-\ell$, and the mean value of the jump-sizes in the opposite direction to the boundary, i.e., $\langle \xi^+ \rangle$. Thus, if $a > 1$ the jump-sizes towards the boundary are on average larger than the jump-sizes away from the boundary and vice versa if $a < 1$. Indeed, if $a = 1$ the mean value of the jump-sizes is the same in both directions.

For the average of the jump-sizes, we obtain

$$\begin{aligned} \langle \xi^D \rangle &= \int_{-\infty}^{\infty} \xi^D q(\xi^D) d\xi^D = - \int_{-\infty}^{\infty} \xi^D \rho(\xi^D) d\xi^D, \\ &= -b \int_{-\infty}^0 \xi^D \rho_{<}(\xi^D) d\xi^D - \frac{(1-b)}{\ell} \int_0^{\infty} \xi^D \exp\left(-\frac{\xi^D}{\ell}\right) d\xi^D, \\ &= b \langle \xi^+ \rangle - (1-b) \ell = \frac{b\ell}{a} - (1-b) \ell = \ell \left[\frac{1+a}{a} b - 1 \right], \\ &= \ell \langle \xi \rangle, \quad \langle \xi \rangle = \frac{1+a}{a} b - 1, \end{aligned} \quad (26)$$

which is in agreement with formula (23) and, for any arbitrary length-scale $\ell \in \mathbb{R}^+$, it fulfils the condition $\langle \xi^D \rangle < 0$ if $a > b/(1-b)$ or, equivalently, $b < a/(1+a)$, and we remark that this last establishes also an upper bound for b at which the MFPT becomes infinite, see figure 1c.

In order to calculate the solution of (17) with (24), in the spirit of the Wiener–Hopf technique [58, 59, 60], we first generalise the MFPT to $T : \mathbb{R} \rightarrow \mathbb{R}$ and equation (17) reads

$$T(x_0) = f(x_0) + \int_0^{\infty} \rho(x_0 - \xi) T(\xi) d\xi, \quad x_0 \in \mathbb{R}, \quad (27)$$

where

$$T(x_0) = \begin{cases} T_+(x_0) : \mathbb{R}^+ \rightarrow \mathbb{R}, \\ T_-(x_0) : \mathbb{R}_0^- \rightarrow \mathbb{R}, \end{cases} \quad (28)$$

$$f(x_0) = \begin{cases} f_+(x_0) : \mathbb{R}^+ \rightarrow \mathbb{R}, \\ f_-(x_0) : \mathbb{R}_0^- \rightarrow \mathbb{R}. \end{cases} \quad (29)$$

For our purposes, and without loss of generality, we set $f_-(x_0) = 0$, with $x_0 \in \mathbb{R}_0^-$, and then we have

$$T_-(x_0) = \int_0^\infty \rho(x_0 - \xi) T_+(\xi) d\xi, \quad x_0 \in \mathbb{R}_0^-. \quad (30)$$

We introduce the generalised Fourier transform with $\kappa \in \mathbb{C}$

$$\widehat{T}(k) = \int_{-\infty}^\infty \exp(ikx_0) T(x_0) dx_0, \quad (31)$$

that leads to the following pairs

$$\widehat{T}_\pm(\kappa) = \pm \int_0^{\pm\infty} \exp(+i\kappa x_0) T_\pm(x_0) dx_0, \quad (32)$$

$$T_\pm(x_0) = \frac{1}{2\pi} \int_{L_\pm} \exp(-i\kappa x_0) \widehat{T}_\pm(\kappa) d\kappa, \quad (33)$$

where L_\pm are proper integration paths in the complex plane.

By applying Fourier transform (31) to equation (27) and by using formulae (32) and (33), we observe that

$$\begin{aligned} \widehat{T}(\kappa) &= \widehat{T}_-(\kappa) + \widehat{T}_+(\kappa), \\ \int_{-\infty}^\infty \exp(+i\kappa x_0) f(x_0) dx_0 &= \int_0^\infty \exp(+i\kappa x_0) f_+(x_0) dx_0 = \widehat{f}_+(\kappa), \\ \int_{\mathbb{R}} \exp(+i\kappa x_0) \left\{ \int_{\mathbb{R}^+} \rho(x_0 - \xi) T_+(\xi) d\xi \right\} dx_0 &= \int_{\mathbb{R}^+} \left\{ \int_{\mathbb{R}} \exp(+i\kappa x_0) \rho(x_0 - \xi) dx_0 \right\} T_+(\xi) d\xi, \\ &= \int_{\mathbb{R}^+} \exp(+i\kappa \xi) T_+(\xi) d\xi \int_{\mathbb{R}} \exp(+i\kappa y) \rho(y) dy, \end{aligned}$$

and we obtain

$$[1 - \widehat{\rho}(\kappa)] \widehat{T}_+(\kappa) = -\widehat{T}_-(\kappa) + \widehat{f}_+(\kappa). \quad (34)$$

Moreover, since by comparing (17) and (27) we have that $f_+(x_0) = \langle \tau \rangle = \text{const.} > 0$ for $x_0 \in \mathbb{R}^+$, then from the formula

$$\frac{1}{2\pi i} \int_{L_+} \frac{\exp(-i\kappa x_0)}{\kappa} d\kappa = \text{Res} \frac{\exp(-i\kappa x_0)}{\kappa} = \lim_{\kappa \rightarrow 0} \kappa \frac{\exp(-i\kappa x_0)}{\kappa} = 1, \quad (35)$$

it holds

$$\widehat{f}_+(\kappa) = -i \frac{\langle \tau \rangle}{\kappa}. \quad (36)$$

By following the above procedure and by using the standard Wiener–Hopf method [58, 59, 60], we have calculated the MFPT of CTRW models in continuous-space with a paradigmatic asymmetric double-exponential distribution of jump-sizes and mean value $\langle \xi \rangle$ (see Appendix A). The emerging non-uniqueness issues have been solved by imposing constraints on physical consistency as: non-negativity, increasing monotonicity and the

convergence to the expected asymptotic behaviour for starting-points far-away from the boundary. Finally, the MFPT results to be

$$T_+(x_0) = -\frac{\langle \tau \rangle}{\langle \xi \rangle} (1 + x_0), \quad \forall x_0 \in \mathbb{R}^+, \langle \xi \rangle \in \mathbb{R}^-, \langle \tau \rangle \in \mathbb{R}^+, \quad (37)$$

that is in agreement with the literature [37, page 513, Corollary 3.2].

By reversing the dimensional to non-dimensional relations (22) and (23), the ratio $x_0/\langle \xi \rangle$ in terms of non-dimensional quantities becomes in dimensional quantities

$$\frac{x_0}{\langle \xi \rangle} = \frac{x_0^D}{\ell} \frac{\ell}{v^D \langle \tau \rangle} = \frac{x_0^D}{v^D \langle \tau \rangle}, \quad \frac{1}{\langle \xi \rangle} = \frac{\ell}{\langle \xi^D \rangle} = \frac{\ell}{v^D \langle \tau \rangle}, \quad (38)$$

and, in dimensional form, formula (37) reads

$$T_+(x_0^D | \langle \xi^D \rangle) = -\frac{1}{v^D} (\ell + x_0^D), \quad \forall x_0^D \in \mathbb{R}^+, v^D \in \mathbb{R}^-, \ell \in \mathbb{R}^+. \quad (39)$$

We remind that, the same asymmetric exponential-distribution of jumps that we considered here (A.1) as a special case of (24), it was considered joined with a diffusion process and a constant drift with an exponential waiting-times distribution [37], while a different family of asymmetric exponential-distribution of jumps was indeed used in a partially related study on a one-dimensional lattice [36].

We show now that the MFPT of a CTRW with exponentially distributed jumps towards the boundary is independent of the distribution of the jumps in the opposite direction. In particular, if through the Wiener–Hopf integral equation (27) we have that formula (39) is the MFPT for an asymmetric double-exponential distribution, we can show, again through (27), that it is indeed also the solution when the jump-sizes are distributed according to (24).

Equation (27) in dimensional form is

$$T_+(x_0^D | \langle \xi^D \rangle) = \langle \tau \rangle + \int_0^\infty \rho(x_0^D - \xi^D) T_+(\xi^D | \langle \xi^D \rangle) d\xi^D, \quad x_0^D \in \mathbb{R}^+, \quad (40)$$

and by plugging (39) into (40) we have

$$\begin{aligned} -\frac{1}{v^D} (\ell + x_0^D) &= \langle \tau \rangle - \frac{1-b}{v^D \ell} \int_0^{x_0^D} \exp\left(\frac{\xi^D - x_0^D}{\ell}\right) (\ell + \xi^D) d\xi^D \\ &\quad - \frac{b}{v^D} \int_{-\infty}^0 \rho_{<}(\xi^D) (\ell + x_0^D - \xi^D) d\xi^D, \\ &= \langle \tau \rangle - \frac{(1-b)x_0^D}{v^D} - \frac{b(\ell + x_0^D + \langle \xi^+ \rangle)}{v^D}, \\ &= \frac{b\langle \xi^+ \rangle - (1-b)\ell}{v^D} - \frac{x_0^D}{v^D} - \frac{b(\ell + \langle \xi^+ \rangle)}{v^D}, \\ &= -\frac{1}{v^D} (\ell + x_0^D), \quad x_0^D \in \mathbb{R}^+. \end{aligned} \quad (41)$$

Thus, the MFPT of an asymmetric CTRW in continuous-space with exponentially distributed jumps in the direction of the boundary is indeed independent of the distribution of the jumps in the opposite direction. This result generalizes those based on the double-exponential jump distribution, see, e.g., [36, 37], and it is new in literature.

The proof 10 has been checked against numerical simulations of the corresponding CTRW models for different jump-size distributions, see Appendix B for details. The plots of the comparisons are shown in figure 1 where, beside the linear growing of the MFPT $T(x_0)$ with respect to x_0 with varying $\langle \xi \rangle$ or varying $\langle \tau \rangle$, the dependence of the MFPT on the involved parameters a and b is also displayed.

From formula (39), we have that the asymptotic universal behaviour (12) is attained when x_0^D is much more far from the boundary than the length-scale ℓ and this last depends only on the features of the jump-size distribution towards the boundary (24, 40-41), i.e.,

$$T_+(x_0^D) \sim -\frac{x_0^D}{v^D}, \quad v^D \in \mathbb{R}^-, \quad \forall x_0^D \gg \ell, \quad (42)$$

with $0 \leq b < a/(1+a)$ and $a \in \mathbb{R}^+$.

Moreover, from (39-41) in the setting (24), we have that there exists a self-similarity property between the jump-sizes and the initial position because the process scales through the same length-scale ℓ with respect to both. By introducing a time-scale of the MFPT that is $\tau_{\text{MFPT}} = -\ell/v^D = -1/v \in \mathbb{R}^+$, then the MFPT scales according to

$$T_+(x_0) = \tau_{\text{MFPT}} \mathcal{T}_+\left(\frac{x_0^D}{\ell}\right), \quad \mathcal{T}_+(z) = 1 + z, \quad \forall x_0^D, \ell, \tau_{\text{MFPT}} \in \mathbb{R}^+. \quad (43)$$

Hence, from formula (43) we may see that if the jump-sizes ξ^D scales with respect to a larger (smaller) length-scale ℓ then also the resulting MFPT with an initial position x_0^D scales with respect to a larger (smaller) length-scale.

5. Conclusions

We studied the problem of a finite MFPT when the diffusion is ruled by a CTRW model characterised by waiting-times with finite mean and by jump-sizes with both finite mean and finite variance.

In particular, we obtain a nonhomogeneous Wiener–Hopf integral equation (17) that allows for an exact calculation of the MFPT by avoiding asymptotic limits. This formula results to depend on the whole distribution of the jump-sizes and on the mean-value only of the waiting-times, thus it holds for general non-Markovian CTRWs. The derived Wiener–Hopf integral equation (17) has been used for the paradigmatic case of an asymmetric double-exponential distribution of the jump-sizes and also for a more general family of asymmetric distributions of the jump-sizes, namely exponential towards the boundary and arbitrary in the opposite direction, and we derived two main results that are: *i*) when the jumps towards the boundary are exponentially distributed then the MFPT is indeed independent of the jumps distribution in the opposite direction and *ii*) a length-scale emerges, which depends only on the features of the distribution of jump-sizes in the direction of the boundary, that establishes a criterion for distinguishing when the starting point is near and when it is far-away from the boundary. As a matter of fact, in opposition to the universal MFPT for starting points that are far-away from

the boundary, for starting points that are near the boundary this universality is lost, specifically because of the dependence of such emerging length-scale on the specific distribution of the jump-sizes towards the boundary. A scaling-law for the solution also emerges.

Moreover, the derived Wiener–Hopf equation is supported by the comparison of the MFPT as calculated by using formula (17) against the MFPT obtained as output of the simulations of the corresponding CTRW models with different jump-sizes distributions.

These findings can be viewed as an extension, when the two processes are comparable, of the results derived by Kou and Wang [37], who discussed a more general jump diffusion process including both a Brownian motion and jumps, together with a constant drift, but with an asymmetric double-exponential distribution of jump-sizes and in the Markovian setting by adopting exponentially-distributed waiting times.

To conclude, we observe that, in the considered case (24), the limit of the exact MFPT for initial positions approaching the boundary is not zero but determined by the emerged length-scale, which is a parameter of the jump-sizes distribution towards the boundary. Hence, this non-zero limit provides also an indirect estimation of the jump distribution towards the boundary as exponential when the MFPT is known, for example, from data or from molecular simulations.

Concerning the application of the derived result, a finite MFPT is proper of some searching models that lead to advection-diffusion equations [38, 39] but, beside this, we would like to highlight that a finite MFPT is indeed proper also of diffusive processes with stochastic resetting [44, 45, 46]. In this respect, we report that the Wiener–Hopf equation here derived and the corresponding approach constitute a first step for determining the survival probability as well as the MFPT in the generalised stochastic resetting RASR [47], which are not available yet. Therefore, an extension of the present approach for fulfilling this purpose embodies a future perspective if this research.

To conclude, we would like to remind that in spite of the fact that random walks, or at least their classical settings, seem to be fully understood, some general features are still under investigation. In this respect, we have in mind the analysis concerning the fact that diffusive models meet the Galilean invariance, in the best cases, solely weakly [61] and also the proof that exponential tails of walkers’ PDF are indeed a universal property of diffusing particles at finite time, as well as at short time [62]. The result here derived, despite obtained in a classical setting for the CTRW approach, joins with those lasts.

Acknowledgments

This research is supported by the Basque Government through the BERC 2018–2021 program; by the Spanish Ministry of Science, Innovation and Universities through the BCAM Severo Ochoa excellence accreditation SEV-2017-0718 and through the project PID2019-107685RB-I00 and the Predoc Severo Ochoa 2018 grant PRE2018-084427. The authors acknowledge the three anonymous referees for their useful and constructive

criticism that guided us to an improvement of the research.

Appendix A.

We analyse the paradigmatic case study of a double-exponential distribution for the jump-sizes:

$$q(\xi) = \begin{cases} (1-b)\exp(\xi); & \text{if } \xi \in \mathbb{R}_0^-, \\ ab\exp(-a\xi); & \text{if } \xi \in \mathbb{R}^+, \end{cases} \quad (\text{A.1})$$

where $b \in [0, 1]$, $a \in \mathbb{R}^+$ and, for lightening the notation, the length-scale ℓ has been dropped for a while. The mean value of the non-dimensional ξ results to be

$$\langle \xi \rangle = \left(\frac{1+a}{a} \right) b - 1, \quad (\text{A.2})$$

that is consistent with (26).

$$\rho(\xi) = q(-\xi) = \begin{cases} (1-b)\exp(-\xi); & \text{if } \xi \in \mathbb{R}_0^+, \\ ab\exp(a\xi); & \text{if } \xi \in \mathbb{R}^-. \end{cases} \quad (\text{A.3})$$

Since in the considered case (A.1) it results $\hat{\rho}(k) = ab/(a+ik) + (1-b)/(1-ik)$, formula (34) reduces to

$$\hat{T}_+(\kappa) = \hat{T}_+^0(\kappa) - i\langle \tau \rangle \left(\frac{a+i\kappa(1-a)+\kappa^2}{\kappa^3+i\kappa^2(ab+b-a)} \right), \quad (\text{A.4})$$

where $\hat{T}_+^0(\kappa)$ is the solution of the homogeneous case, i.e., $f_+(x_0) = 0$. Solution $\hat{T}_+^0(\kappa)$ can be determined, by definition, up to a multiplicative constant (see formula (17) with $\langle \tau \rangle = 0$) that here we denote by C :

$$\hat{T}_+^0(\kappa) = -C \frac{(a+i\kappa)(1-i\kappa)}{\kappa^2+i\kappa(ab+b-a)} \hat{T}_-(\kappa). \quad (\text{A.5})$$

By remembering the definition of $T_-(x_0)$ in (30), that holds for $x \in \mathbb{R}_0^-$, we have that

$$\begin{aligned} \hat{T}_-(\kappa) &= - \int_0^{-\infty} \exp[+i\kappa x_0] \left\{ \int_0^{\infty} ab \exp(a(x_0 - \xi)) T_+(\xi) d\xi \right\} dx_0 \\ &= - \int_0^{-\infty} \exp(+i\kappa x_0 + ax_0) dx_0 \int_0^{\infty} ab \exp(-a\xi) T_+(\xi) d\xi \\ &= \frac{T_-(0)}{a+i\kappa}, \end{aligned} \quad (\text{A.6})$$

and then

$$\begin{aligned} T_+^0(x_0) &= \frac{1}{2\pi} \int_{L_-} \exp(-i\kappa x_0) \hat{T}_+^0(\kappa) d\kappa \\ &= -C \frac{T_-(0)}{2\pi i} \int_{L_-} \frac{(i+\kappa) \exp(-i\kappa x_0)}{\kappa^2+i\kappa(ab+b-a)} d\kappa \\ &= -C T_-(0) \left\{ \frac{1}{ab+b-a} + \frac{(1+a-ab-b) \exp[(a-ab-b)x_0]}{a-ab-b} \right\}, \end{aligned} \quad (\text{A.7})$$

by remembering that

$$\begin{aligned} \text{Res} \left\{ \frac{(i + \kappa) \exp(-i\kappa x_0)}{\kappa^2 + i\kappa(ab + b - a)} \right\} &= \lim_{\kappa \rightarrow 0} \left\{ \kappa \frac{(i + \kappa) \exp(-i\kappa x_0)}{\kappa^2 + i\kappa(ab + b - a)} \right\} + \\ &\quad \lim_{\kappa \rightarrow i(a-ab-b)} \left\{ [\kappa + i(ab + b - a)] \frac{(i + \kappa) \exp(-i\kappa x_0)}{\kappa^2 + i\kappa(ab + b - a)} \right\} \\ &= \frac{1}{ab + b - a} + \frac{(1 + a - ab - b) \exp((a - ab - b)x_0)}{a - ab - b}. \end{aligned} \quad (\text{A.8})$$

Hence, by applying anti-transformation (32) to (A.4) and by using (A.7), we obtain

$$\begin{aligned} T_+(x_0) &= -C T_-(0) \left(\frac{1}{ab + b - a} + \frac{(1 + a - ab - b) \exp((a - ab - b)x_0)}{a - ab - b} \right) \\ &\quad + \frac{\langle \tau \rangle}{2\pi i} \int_{L_-} \frac{(a + i\kappa(1 - a) + \kappa^2) \exp(-i\kappa x_0)}{\kappa^3 + i\kappa^2(ab + b - a)} d\kappa, \end{aligned} \quad (\text{A.9})$$

that, after computing

$$\begin{aligned} \text{Res} \left\{ \frac{(a + i\kappa(1 - a) + \kappa^2) \exp(-i\kappa x_0)}{\kappa^3 + i\kappa^2(ab + b - a)} \right\} &= \lim_{\kappa \rightarrow 0} \frac{d}{d\kappa} \left\{ \kappa^2 \frac{(a + i\kappa(1 - a) + \kappa^2) \exp(-i\kappa x_0)}{\kappa^3 + i\kappa^2(ab + b - a)} \right\} \\ &\quad + \lim_{\kappa \rightarrow i(a-ab-b)} \left\{ [\kappa + i(ab + b - a)] \frac{(a + i\kappa(1 - a) + \kappa^2) \exp(-i\kappa x_0)}{\kappa^3 + i\kappa^2(ab + b - a)} \right\} \\ &= \frac{ax_0(a - b - ab) - a^2b + b + a^2}{(a - ab - b)^2} \\ &\quad - \frac{(ab + b)(1 + a - ab - b) \exp((a - ab - b)x_0)}{(a - ab - b)^2} \\ &= \frac{ax_0}{a - ab - b} + \frac{a^2 + b - a^2b}{(a - ab - b)^2} \\ &\quad + \frac{(ab + b)(1 + a)(b - 1) \exp((a - ab - b)x_0)}{(a - ab - b)^2}, \end{aligned} \quad (\text{A.10})$$

becomes

$$\begin{aligned} T_+(x_0) &= -C T_-(0) \left\{ \frac{1}{ab + b - a} + \frac{(1 + a - ab - b) \exp[(a - ab - b)x_0]}{a - ab - b} \right\} \\ &\quad + \langle \tau \rangle \left\{ \frac{ax_0}{a - ab - b} + \frac{a^2 + b - a^2b}{(a - ab - b)^2} \right. \\ &\quad \left. + \frac{(ab + b)(1 + a)(b - 1) \exp((a - ab - b)x_0)}{(a - ab - b)^2} \right\}. \end{aligned} \quad (\text{A.11})$$

Moreover, constant C can be estimated by calculating $T_-(0)$ through definition (30) by using (A.11) and, by remembering that

$$\int_0^\infty \exp(-a\xi) d\xi = \frac{1}{a}, \quad \int_0^\infty \xi \exp(-a\xi) d\xi = \frac{1}{a^2}, \quad a > 0,$$

we obtain $C = -1$. Finally, the desired solution is

$$T_+(x_0) = T_-(0) \left\{ \frac{1}{ab + b - a} + \frac{(1 + a - ab - b) \exp((a - ab - b)x_0)}{a - ab - b} \right\}$$

$$+\langle\tau\rangle\left\{\frac{ax_0}{a-ab-b}+\frac{a^2+b-a^2b}{(a-ab-b)^2}+\frac{[(ab+b)(1+a)(b-1)]\exp[(a-ab-b)x_0]}{(a-ab-b)^2}\right\}\quad (\text{A.12})$$

that, by using (A.2), can be written in terms of $\langle\xi\rangle$:

$$T_+(x_0) = T_-(0)\left\{\frac{1}{a\langle\xi\rangle}-\frac{(1-a\langle\xi\rangle)\exp(-a\langle\xi\rangle x_0)}{a\langle\xi\rangle}\right\} + \langle\tau\rangle\left\{-\frac{x_0}{\langle\xi\rangle}+\frac{\langle\xi\rangle(1-a)+1}{a\langle\xi\rangle^2}-\frac{(\langle\xi\rangle+1)(1-a\langle\xi\rangle)\exp(-a\langle\xi\rangle x_0)}{a\langle\xi\rangle^2}\right\}.\quad (\text{A.13})$$

In the case $\langle\xi\rangle = 0$, formula (A.13) reduces to

$$T_+(x_0; \langle\xi\rangle = 0) = T_-(0)(x_0 + 1) + \langle\tau\rangle\left\{1 + (1-a)x_0 - \frac{ax_0^2}{2}\right\},\quad (\text{A.14})$$

which can be derived by using the series expansion of the exponential function, and in the symmetric case $a = 1$, from (A.14) it results

$$T_+(x_0; \langle\xi\rangle = 0, a = 1) = T_-(0)(1 + x_0) + \langle\tau\rangle\left\{1 - \frac{x_0^2}{2}\right\},\quad (\text{A.15})$$

that are both (A.14) and (A.15) not MFPT solutions.

In fact, any MFPT solution has to fulfil, by definition, the conditions

$$\frac{dT_+}{dx_0} > 0, \quad \forall x_0 \in \mathbb{R}^+, \quad (\text{A.16})$$

$$T_+(x_0) \geq 0, \quad \forall x_0 \in \mathbb{R}^+. \quad (\text{A.17})$$

Therefore, for solution (A.13), condition (A.16) is fulfilled if

$$\frac{dT_+(x_0)}{dx_0} = T_-(0)(1-a\langle\xi\rangle)\exp(-a\langle\xi\rangle x_0) + \langle\tau\rangle\left\{-\frac{1}{\langle\xi\rangle}+\frac{(1+\langle\xi\rangle)(1-a\langle\xi\rangle)\exp(-a\langle\xi\rangle x_0)}{\langle\xi\rangle}\right\} > 0, \quad \forall x_0 \in \mathbb{R}^+ \quad (\text{A.18})$$

which implies

$$T_-(0) > \langle\tau\rangle\left[\frac{\exp(a\langle\xi\rangle x_0)}{\langle\xi\rangle(1-a\langle\xi\rangle)}-\frac{1+\langle\xi\rangle}{\langle\xi\rangle}\right], \quad \forall x_0 \in \mathbb{R}^+, \quad (\text{A.19})$$

where $(1-a\langle\xi\rangle) \geq 0$ as it follows from (A.2). Hence, since it holds that

$$\sup_{x_0 \in \mathbb{R}^+} \left\{ \langle\tau\rangle \left[\frac{\exp(a\langle\xi\rangle x_0)}{\langle\xi\rangle(1-a\langle\xi\rangle)} - \frac{1+\langle\xi\rangle}{\langle\xi\rangle} \right] \right\} = \begin{cases} \infty, & \text{if } \langle\xi\rangle \geq 0, \\ -\langle\tau\rangle \frac{1+\langle\xi\rangle}{\langle\xi\rangle}, & \text{if } \langle\xi\rangle < 0, \end{cases} \quad (\text{A.20})$$

inequality (A.18), and then condition (A.16), is fulfilled if

$$T_-(0) = -\langle\tau\rangle \frac{1+\langle\xi\rangle}{\langle\xi\rangle} + c, \quad \forall c \in \mathbb{R}_0^+, \quad \langle\xi\rangle \in \mathbb{R}^-. \quad (\text{A.21})$$

When formula (A.21) is plugged into (A.13), we have that for $x_0 \in \mathbb{R}^+$, $c \in \mathbb{R}_0^+$, $\langle \xi \rangle \in \mathbb{R}^-$,

$$T_+(x_0) = -\frac{\langle \tau \rangle}{\langle \xi \rangle}(1 + x_0) + \frac{c}{a\langle \xi \rangle}[1 - (1 - a\langle \xi \rangle) \exp(-a\langle \xi \rangle x_0)], \quad (\text{A.22})$$

which fulfils condition (A.17), too. Regarding constant c , we observe that formula (A.22) meets the asymptotic linear growing with respect to x_0 (12) when it holds $c = 0$. To conclude, the MFPT of CTRW models in continuous-space is

$$T_+(x_0) = -\frac{\langle \tau \rangle}{\langle \xi \rangle}(1 + x_0), \quad \forall x_0 \in \mathbb{R}^+, \langle \xi \rangle \in \mathbb{R}^-, \langle \tau \rangle \in \mathbb{R}^+, \quad (\text{A.23})$$

that is formula (37).

Appendix B.

In figure 1, formula (41) is tested against the corresponding CTRW model. In particular, the trajectories of the CTRW are generated by the iterative procedure

$$\begin{cases} X_n = X_{n-1} + \xi_n, & X_0 = x_0 > 0, \quad n \in \mathbb{N}, \\ t_n = t_{n-1} + \tau_n, & t_0 = 0, \quad n \in \mathbb{N}, \end{cases} \quad (\text{B.1})$$

with the random jump-sizes drawn according to distribution (24) with $\ell = 1$, i.e.,

$$\xi_n = \begin{cases} \ln(1 - u_1^{(1)}), & \text{if } u_2 > b, \\ \frac{\chi_m}{m}, & m \in [1, \dots, 10], \text{ otherwise,} \end{cases} \quad (\text{B.2})$$

where χ_m is a Poisson distributed random variable with mean m/a ,

$$\chi_m = \chi_{m-1} - \frac{\ln(1 - u_1^{(m)})}{a}, \quad \chi_0 = 0, \quad m \in [1, \dots, 10], \quad (\text{B.3})$$

and the random waiting-times are drawn from one of the following three distributions

$$\tau_n = -\langle \tau \rangle \ln(1 - u_3), \quad \text{exponential distribution,} \quad (\text{B.4})$$

$$\tau_n = 2\langle \tau \rangle u_4, \quad \text{uniform distribution,} \quad (\text{B.5})$$

$$\tau_n = \langle \tau \rangle, \quad \text{delta distribution,} \quad (\text{B.6})$$

such that $u_1^{(j)}, u_2, u_3, u_4 \sim U(0, 1), \forall j \in [1, \dots, m], b \in [0, 1]$ and $a \in \mathbb{R}^+$. We obtain the same results with all the distributions of waiting-times and all the distributions of jump-sizes in the opposite direction to the boundary as expected from formula (41).

The absorbing boundary located in $x = 0$ can be passed only by a jump-event and, thus, the first passage-time (FPT) can be numerically computed by

$$\text{FPT} = t_n, \quad \text{provided that } X_n < 0 \text{ and } X_i > 0, \quad \forall i = 1, \dots, n-1. \quad (\text{B.7})$$

The MFPT $T_+(x_0)$ is

$$T_+(x_0) = \frac{1}{N} \sum_{j=1}^N \text{FPT}_j, \quad N \in \mathbb{N}, \quad (\text{B.8})$$

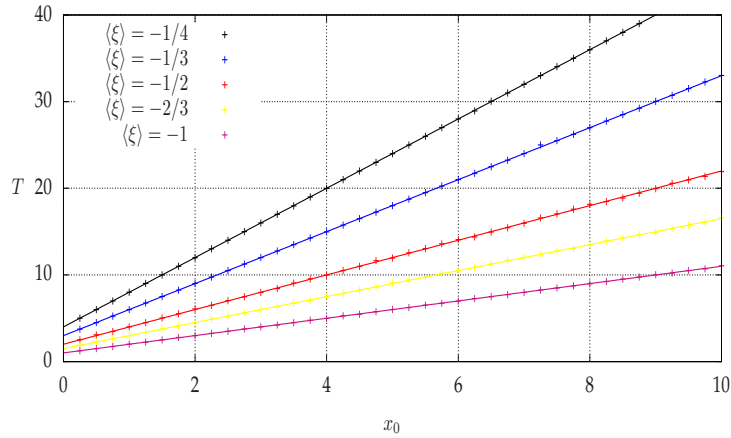
where N is the number of independent realizations of the iteration procedure (B.1).

References

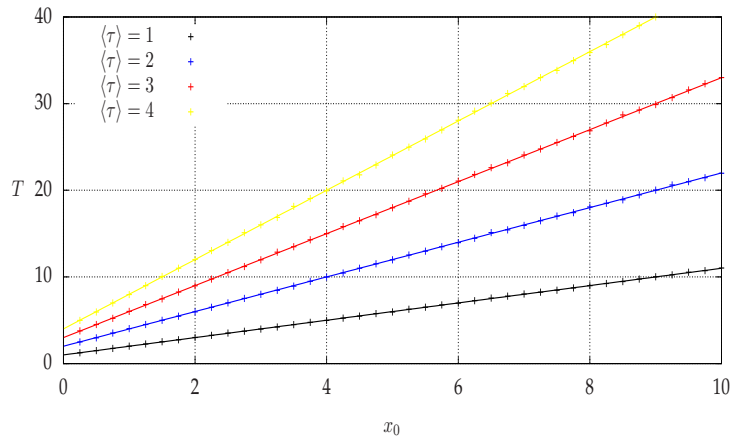
- [1] A. J. Bray, S. N. Majumdar, and G. Schehr. Persistence and first-passage properties in nonequilibrium systems. *Adv. Phys.*, 62:225–361, 2013.
- [2] R. Metzler, G. Oshanin, and S. Redner, editors. *First-Passage Phenomena and Their Applications*. World Scientific, Singapore, 2014.
- [3] D. S. Grebenkov, D. Holcman, and R. Metzler, editors. *New trends in first-passage methods and applications in the life sciences and engineering*. Number Special Issue in J. Phys. A: Math. Theor. IOP Publishing, 2020.
- [4] P. C. Bressloff and J. M. Newby. First passage time problems in biophysical jump processes with fast kinetics. In R. Metzler, G. Oshanin, and S. Redner, editors, *First-Passage Phenomena and Their Applications*, chapter 12, pages 277–305. World Scientific, Singapore, 2014.
- [5] T. Chou and M. R. D’Orsogna. First passage problems in biology. In R. Metzler, G. Oshanin, and S. Redner, editors, *First-Passage Phenomena and Their Applications*, chapter 13, pages 306–345. World Scientific, Singapore, 2014.
- [6] N. F. Polizzi, M. J. Therien, and D. N. Beratan. Mean first-passage times in biology. *Isr. J. Chem.*, 56:816–824, 2016.
- [7] G. H. Weiss. First passage time problems in chemical physics. In I. Prigogine, editor, *Advances in Chemical Physics*, volume 13, pages 1–18. John Wiley & Sons. Inc., 1967.
- [8] A. Szabo, K. Schulten, and Z. Schulten. First passage time approach to diffusion controlled reactions. *J. Chem. Phys.*, 72:4350–4357, 1980.
- [9] P. Hänggi, P. Talkner, and M. Borkovec. Reaction-rate theory: fifty years after Kramers. *Rev. Mod. Phys.*, 62, 1990.
- [10] S. Redner. *A Guide to First-Passage Processes*. Cambridge University Press, Cambridge, 2001.
- [11] M. Nyberg, T. Ambjörnsson, and L. Lizana. A simple method to calculate first-passage time densities with arbitrary initial conditions. *New J. Phys.*, 18:063019, 2016.
- [12] R. Metzler. Brownian motion and beyond: first-passage, power spectrum, non-gaussianity, and anomalous diffusion. *J. Stat. Mech.-Theory Exp.*, page 114003, 2019.
- [13] D. Hartich and A. Godec. Extreme value statistics of ergodic Markov processes from first passage times in the large deviation limit. *J. Phys. A: Math. Theor.*, 52:244001, 2019.
- [14] S. N. Majumdar and B. Meerson. Statistics of first-passage Brownian functionals. *J. Stat. Mech.-Theory Exp.*, 2020:023202, 2020. Corrigendum in J. Stat. Mech.: Theory Exp. (2021) 039801.
- [15] D. S. Grebenkov, R. Metzler, and G. Oshanin. From single-particle stochastic kinetics to macroscopic reaction rates: fastest first-passage time of N random walkers. *New J. Phys.*, 22:103004, 2020.
- [16] M. J. Kearney and R. J. Martin. Statistics of the first passage area functional for an Ornstein–Uhlenbeck process. *J. Phys. A: Math. Theor.*, 54:055002, 2021.
- [17] I. Eliazar. Selfsimilarity of diffusions’ first passage times. *J. Phys. A: Math. Theor.*, 54:055003, 2021.
- [18] Y. Lanoiselée, N. Moutal, and D. S. Grebenkov. Diffusion-limited reactions in dynamic heterogeneous media. *Nature communications*, 9(1):1–16, 2018.
- [19] V. Sposini, A. Chechkin, and R. Metzler. First passage statistics for diffusing diffusivity. *J. Phys. A: Math. Theor.*, 52:04LT01, 2019.
- [20] D. S. Grebenkov, V. Sposini, R. Metzler, G. Oshanin, and F. Seno. Exact first-passage time distributions for three random diffusivity models. *J. Phys. A: Math. Theor.*, 54:04LT01, 2021.
- [21] S. N. Majumdar, P. Mounaix, and G. Schehr. Survival probability of random walks and Lévy flights on a semi-infinite line. *J. Phys. A: Math. Theor.*, 50:465002, 2017.
- [22] A. Padash, A. V. Chechkin, B. Dybiec, I. Pavlyukevich, B. Shokri, and R. Metzler. First-passage properties of asymmetric Lévy flights. *J. Phys. A: Math. Theor.*, 52:454004, 2019.
- [23] A. Padash, A. V. Chechkin, B. Dybiec, M. Magdziarz, B. Shokri, and R. Metzler. First passage time moments of asymmetric Lévy flights. *J. Phys. A: Math. Theor.*, 53:275002, 2020.

- [24] V. V. Palyulin, G. Blackburn, M. A. Lomholt, N. W. Watkins, R. Metzler, R. Klages, and A. V. Chechkin. First passage and first hitting times of Lévy flights and Lévy walks. *New J. Phys.*, 21:103028, 2019.
- [25] A. L. Thorneywork, J. Gladrow, Y. Qing, M. Rico-Pasto, F. Ritort, H. Bayley, A. B. Kolomeisky, and U. F. Keyser. Direct detection of molecular intermediates from first-passage times. *Sci. Adv.*, 6:eaa4642, 2020.
- [26] D. W. B. Broadwater, A. W. Cook, and H. D. Kim. First passage time study of DNA strand displacement. *Biophys. J.*, 120, 2021. In press.
- [27] D. Hartich and A. Godec. Duality between relaxation and first passage in reversible Markov dynamics: rugged energy landscapes disentangled. *New J. Phys.*, 20:112002, 2018.
- [28] D. S. Grebenkov and A. T. Skvortsov. Mean first-passage time to a small absorbing target in an elongated planar domain. *New J. Phys.*, 22:113024, 2020.
- [29] D. S. Grebenkov, R. Metzler, and G. Oshanin. A molecular relay race: sequential first-passage events to the terminal reaction centre in a cascade of diffusion controlled processes. *New J. Phys.*, 23:093004, 2021.
- [30] D. S. Grebenkov, R. Metzler, and G. Oshanin. Distribution of first-reaction times with target regions on boundaries of shell-like domains. *New Journal of Physics*, 23(12):123049, 2021.
- [31] M. J. Simpson, D. J. VandenHeuvel, J. M. Wilson, S. W. McCue, and E. J. Carr. Mean exit time for diffusion on irregular domains. *New J. Phys.*, 23:043030, 2021.
- [32] E. W. Montroll and G. H. Weiss. Random walks on lattices. II. *J. Math. Phys.*, 6(2):167–181, 1965.
- [33] R. Estrada and R. P. Kanwal. *Singular Integral Equations*. Springer Science+Business Media, LLC, New York, 2000.
- [34] R. Zwanzig. From classical dynamics to continuous time random walks. *Journal of Statistical Physics*, 30(2):255–262, 1983.
- [35] F. Mainardi, M. Raberto, R. Gorenflo, and E. Scalas. Fractional calculus and continuous-time finance ii: the waiting-time distribution. *Physica A: Statistical Mechanics and its Applications*, 287(3-4):468–481, 2000.
- [36] D. Gutkowicz-Krusin, I. Procaccia, and J. Ross. Analytic results for asymmetric random walk with exponential transition probabilities. *J. Stat. Phys.*, 19:525–541, 1978.
- [37] S. G. Kou and H. Wang. First passage times of a jump diffusion process. *Adv. Appl. Probab.*, 35:504–531, 2003.
- [38] H. W. McKenzie, M. A. Lewis, and E. H. Merrill. First passage time analysis of animal movement and insights into the functional response. *Bull. Math. Biol.*, 71:107–129, 2009.
- [39] V. Kurella, J. C. Tzou, D. Coombs, and M. J. Ward. Asymptotic analysis of first passage time problems inspired by ecology. *Bull. Math. Biol.*, 77:83–125, 2015.
- [40] B. Faugeras and O. Maury. Modeling fish population movements: From an individual-based representation to an advection-diffusion equation. *J. Theor. Biol.*, 247:837–848, 2007.
- [41] M. A. Sovada, A. B. Sargeant, and J. W. Grier. Differential effects of coyotes and red foxes on duck nest success. *J. Wildl. Manag.*, 59:1–9, 1995.
- [42] M. Moreau, G. Oshanin, O. Benichou, and M. Coppey. Pascal principle for diffusion-controlled trapping reactions. *Phys. Rev. E*, 67:045104, 2003.
- [43] M. Moreau, G. Oshanin, O. Benichou, and M. Coppey. Lattice theory of trapping reactions with mobile species. *Phys. Rev. E*, 69:046101, 2004.
- [44] M. R. Evans and S. N. Majumdar. Diffusion with stochastic resetting. *Phys. Rev. Lett.*, 106:160601, 2011.
- [45] M. R. Evans and S. N. Majumdar. Diffusion with optimal resetting. *J. Phys. A: Math. Theor.*, 44:435001, 2011.
- [46] M. R. Evans, S. N. Majumdar, and G. Schehr. Stochastic resetting and applications. *J. Phys. A: Math. Theor.*, 53:193001, 2020.
- [47] M. Dahlenburg, A. V. Chechkin, R. Schumer, and R. Metzler. Stochastic resetting by a random

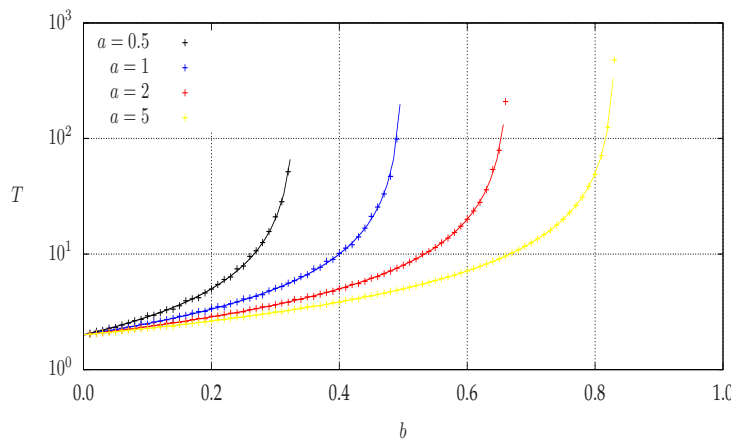
- amplitude. *Phys. Rev. E*, 103:052123, 2021.
- [48] O. Tal-Friedman, A. Pal, A. Sekhon, S. Reuveni, and Y. Roichman. Experimental realization of diffusion with stochastic resetting. *J. Phys. Chem. Lett.*, 11:7350, 2020.
- [49] A. Masó-Puigdellosas, D. Campos, and V. Méndez. Transport properties and first-arrival statistics of random motion with stochastic reset times. *Phys. Rev. E*, 99:012141, 2019.
- [50] A. Pal, A. Kundu, and M. R. Evans. Diffusion under time-dependent resetting. *J. Phys. A: Math. Theor.*, 49:225001, 2016.
- [51] A. S. Bodrova and I. M. Sokolov. Continuous-time random walks under power-law resetting. *Phys. Rev. E*, 101:062117, 2020.
- [52] V. Méndez, A. Masó-Puigdellosas, T. Sandev, and D. Campos. Continuous time random walks under Markovian resetting. *Phys. Rev. E*, 103:022103, 2021.
- [53] V. Zaburdaev, S. Denisov, and J. Klafter. Lévy walks. *Rev. Mod. Phys.*, 87:843–530, 2015.
- [54] E.W. Montroll. Random walks on lattices. *Proc. Symp. Appl. Math. (Am. Math. Soc.)*, 16:193–220, 1964.
- [55] A. V. Chechkin, R. Metzler, V. Y. Gonchar, J. Klafter, and L. V. Tanatarov. First passage and arrival time densities for Lévy flights and the failure of the method of images. *J. Phys. A: Math. Gen.*, 36:L537–L544, 2003.
- [56] F. Spitzer. The Wiener–Hopf equation whose kernel is a probability density. *Duke Math. J.*, 24:327–343, 1957.
- [57] F. Spitzer. The Wiener–Hopf equation whose kernel is a probability density. II. *Duke Math. J.*, 27:363–372, 1960.
- [58] I. D. Abrahams and G. R. Wickham. General Wiener–Hopf factorization of matrix kernels with exponential phase factors. *SIAM J. Appl. Math.*, 50:819–838, 1990.
- [59] F. G. Leppington. Wiener–Hopf technique. In D. G. Crighton, A. P. Dowling, J. E. Ffowcs Williams, M. Heckl, and F. G. Leppington, editors, *Modern Methods in Analytical Acoustics. Lecture Notes*, chapter 5, pages 148–167. Springer–Verlag GmbH, Berlin Heidelberg, 1992.
- [60] A. V. Kisil, I. D. Abrahams, G. Mishuris, and S. V. Rogosin. The Wiener–Hopf technique, its generalizations and applications: constructive and approximate methods. *Proc. R. Soc. A*, 477:20210533, 2021.
- [61] A. Cairoli, R. Klages, and A. Baule. Weak Galilean invariance as a selection principle for coarse-grained diffusive models. *Proc. Natl. Acad. Sci. USA*, 115:5714–5719, 2018.
- [62] E. Barkai and S. Burov. Packets of diffusing particles exhibit universal exponential tails. *Phys. Rev. Lett.*, 124:060603, 2020.



(a)



(b)



(c)

Figure 1: Comparison of the MFPT $T(x_0)$ computed by formula (41) (solid line) and by the corresponding CTRW model (cross symbols). Panel a) T vs x_0 with $\langle \tau \rangle = 1$ and varying $\langle \xi \rangle$; Panel b) T vs x_0 with $\langle \xi \rangle = -1$ and varying $\langle \tau \rangle$; Panel c) T vs b with $x_0 = \langle \tau \rangle = 1$ and varying a .