KATO–PONCE ESTIMATES FOR FRACTIONAL SUBLAPLACIANS IN THE HEISENBERG GROUP

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ABSTRACT. We give a proof of commutator estimates for fractional powers of the sublaplacian on the Heisenberg group. Our approach is based on pointwise and L^p estimates involving square fractional integrals and Littlewood–Paley square functions.

1. INTRODUCTION

In [32], Kato and Ponce proved the well known commutator estimate

$$\|J^s(fg) - fJ^s g\|_{L^p(\mathbb{R}^n)} \lesssim \|J^s f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)} + \|\partial f\|_{L^\infty(\mathbb{R}^n)}\|J^{s-1}g\|_{L^\infty(\mathbb{R}^n)},$$

for $1 < p < \infty$, and $s > 0$, where $J^s := (1-\Delta)^{s/2}$, $\partial = (\partial_1, \cdots, \partial_n)$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. Closely related to this, we have the following estimate by Kenig, Ponce, and Vega in [33]

$$\|(-\Delta)^{s/2}(fg) - f(-\Delta)^{s/2}g - g(-\Delta)^{s/2}f\|_{L^p(\mathbb{R}^n)} \lesssim \|(-\Delta)^{s_1/2}f\|_{L^{p_1}(\mathbb{R}^n)}\|(-\Delta)^{s_2/2}g\|_{L^{p_2}(\mathbb{R}^n)},$$

where $s = s_1 + s_2$, $0 < s, s_1, s_2 < 1$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $1 < p, p_1, p_2 < \infty$. The above estimates naturally arise in several different contexts. In particular, they turn out to be fundamental to close fixed point arguments in Sobolev spaces for some nonlinear dispersive PDE's. This motivates the investigation about the validity of commutator estimates in different geometries than the Euclidean setting.

In the recent paper [37] Maalaoui, Pinamonti, and Speight showed several L^p estimates for commutators, in particular Kato–Ponce type estimates involving fractional powers of the sublaplacian on Carnot groups (see [37, Theorem 6.5]). We also point out [36, Corollary 1], where Maalaoui provided a Chanillo-type pointwise estimate for 3-commutators involving fractional powers of the sublaplacian on Carnot groups. The methods in [37] rely on the use of the harmonic extension (2), integration by parts, and classical square functions, and follow the ideas by Lenzmann and Schikorra [35] in the Euclidean setting.

In the present manuscript, we prove Kato–Ponce type L^p estimates involving conformally invariant fractional powers of the sublaplacian on the Heisenberg group. Our approach, which is inspired by the proof by D'Ancona in [18] for the Euclidean case, is based on the study of nontangential square functions as crucial tools for the proof. In addition, our strategy makes use of both nonconformal and conformal harmonic extensions associated to the sublaplacian. The main feature, compared to [37, Theorem 6.5], is that the method allows us to provide weighted versions of the result.

Before stating our main results, we need to introduce the geometric and functional setting. A remarkable way to characterize nonlocal operators such as $(-\Delta)^{s/2}$ is via a weighted *Dirichlet-to-Neumann* map of an extension problem. This approach has been present in the literature since the 1950's, with the paper by Huber [31]. Closely related and containing the same circle of ideas, we find the work by Muckenhoupt and Stein [39]. We also mention the extension procedure introduced

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by Molchanov and Ostrovskii in [38] within a context of probability, see also the related work by Spitzer [48] and the more recent by Kolsrud [34]. In particular, the landmark work by Caffarelli and Silvestre [10], in which they studied the extension problem associated to the Laplacian on $\mathbb{R}^n$, and realized the fractional power $(-\Delta)^{s/2}$ as the map taking Dirichlet data to Neumann data, has been a rich source of development in the study of nonlocal operators in the last few years, especially from the point of view of partial differential equations.

Fractional powers of Laplacians also occur naturally in conformal geometry and scattering theory. Chang–González [11] showed that the fractional order Paneitz operators P_γ arising in the work of Graham and Zworski [29] in conformal geometry coincide with $(-\Delta)^{s/2}$ when the conformally compact Einstein manifold is taken to be the hyperbolic space. Later, Frank, González, Monticelli, and Tan [25] studied the extension problem associated to the sublaplacian $\mathcal{L}$ on the Heisenberg group $\mathbb{H}^n$. Unlike the case of $\mathbb{R}^n$, where $(-\Delta)^{s/2}$ are conformally invariant, in the context of Heisenberg groups $\mathcal{L}^s$, defined as the map taking Dirichlet to Neumann data in (2) below, are not. Hence, conformally invariant fractional powers of the sublaplacian, denoted by $\mathcal{L}_s$, are more relevant from a geometrical point of view than the pure fractional powers $\mathcal{L}^s$, see [4, 9, 21].

Let $\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R}$ denote the $(2n+1)$ dimensional Heisenberg group (see Section 2 for a brief review of the group structure). For $s > 0$, given a function $g \in C_0^\infty(\mathbb{H}^n \times \mathbb{R}^+)$, the extension problem for $\mathcal{L}^s$ consists of finding $U \in C_0^\infty(\mathbb{H}^n \times \mathbb{R}^+)$ such that

$$\begin{cases} \big(\partial_{\rho\rho} + \frac{1-2s}{\rho}\partial_\rho - \mathcal{L}\big)U((z,t),\rho) = 0 & ((z,t),\rho) \in \mathbb{H}^n \times \mathbb{R}^+, \\ U((z,t),0) = f(z,t), & (z,t) \in \mathbb{H}^n. \end{cases} \tag{1}$$

The extension problem for general second order partial differential operators has been studied by Stinga–Torrea [51]. The sublaplacian on $\mathbb{H}^n$ lies within this general theory and then it is shown that

$$\mathcal{L}^s g = c_s \lim_{\rho\to 0} \rho^{1-2s}\partial_\rho U. \tag{2}$$

We mention that when we consider $-\Delta$ and $\mathbb{R}^n$ instead of $\mathcal{L}$ and $\mathbb{H}^n$, then we are dealing with the extension problem for $(-\Delta)^s$ as in [10].

For $s > 0$, the extension problem for the fractional conformal sublaplacian $\mathcal{L}_s$ on $\mathbb{H}^n$ consists of finding $U \in C_0^\infty(\mathbb{H}^n \times \mathbb{R}^+)$ such that

$$\begin{cases} \big(\partial_{\rho\rho} + \frac{1-2s}{\rho}\partial_\rho + \frac{1}{4}\rho^2\partial_{tt} - \mathcal{L}\big)U((z,t),\rho) = 0 & ((z,t),\rho) \in \mathbb{H}^n \times \mathbb{R}^+, \\ U((z,t),0) = f(z,t), & (z,t) \in \mathbb{H}^n. \end{cases} \tag{3}$$

Note that the latter extension problem is different from the problem (1) due to the appearance of the extra term $\frac{1}{4}\rho^2\partial_{tt}$. The extension problem (3) occurs naturally if we consider $\mathbb{H}^n$ as the boundary of the Siegel's upper half space Ω_{n+1}. Using this connection, it was shown in [25] that for $f \in C_0^\infty(\mathbb{H}^n)$ there is a unique solution of the above equation which satisfies

$$\mathcal{L}_s f = c_s \lim_{\rho\to 0} \rho^{1-2s}\partial_\rho U. \tag{4}$$

We have defined the conformally and non conformally invariant fractional powers $\mathcal{L}_s$ and $\mathcal{L}^s$, respectively, via the corresponding extension problems. Other equivalent definitions are available, see [45, Theorem 1.2] for $\mathcal{L}_s$ and [51] for $\mathcal{L}^s$ (the latter reference concerns fractional powers of second order partial differential operators) and moreover it can be checked, see Subsection 2.4, that the operators $\mathcal{L}_s$ and $\mathcal{L}^s$ are equivalent in $L^p(\mathbb{H}^n)$, i.e., there exist $c, C > 0$ such that

$$c\|\mathcal{L}^s f\|_{L^p} \le \|\mathcal{L}_s f\|_{L^p} \le C\|\mathcal{L}^s f\|_{L^p}, \qquad 1 < p < \infty.$$

A weight w will be a positive function on $\mathbb{H}^n$ and for $1 < p < \infty$ we will consider weights w belonging to the Muckenhoupt class A_p, see Subsection 2.4 for the definition. Let $Q = 2n+2$ be the *homogeneous dimension* of $\mathbb{H}^n$. We refer to [27] for the theory of the Schwartz space over $\mathbb{H}^n$, namely $\mathcal{S}(\mathbb{H}^n)$. Our main result is the following.

Theorem 1.1. *Let $n \geq 1$. Assume that s, s_1, s_2 and p, p_1, p_2 satisfy*

$$s = s_1 + s_2, \qquad s \in (0, 1/2), \qquad s_j \in (0, 1/2), \qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{2Q}{Q + 4s_j} < p_j < \infty.$$

Then for all $u, v \in \mathcal{S}(\mathbb{H}^n)$ we have

$$\|\mathcal{L}_s(uv) - u\mathcal{L}_s v - v\mathcal{L}_s u\|_{L^p} \lesssim \|\mathcal{L}_{s_1} u\|_{L^{p_1}} \|\mathcal{L}_{s_2} v\|_{L^{p_2}}. \tag{5}$$

Moreover, for $w_j \in A_{q_j}$, where $1 < q_j = p_j\big(\frac{1}{2} + \frac{2s_j}{Q}\big)$,

$$\|\mathcal{L}_s(uv) - u\mathcal{L}_s v - v\mathcal{L}_s u\|_{L^p(w_1^{p/p_1} w_2^{p/p_2})} \lesssim \|\mathcal{L}_{s_1} u\|_{L^{p_1}(w_1)} \|\mathcal{L}_{s_2} v\|_{L^{p_2}(w_2)}. \tag{6}$$

Remark 1.2. *We stress that Theorem 1.1 is providing a weighted version of the Kato–Ponce inequality for fractional sublaplacians in the Heisenberg group, which is not present in* [37]*.*

We will follow the ideas in [18], which in turn are largely inspired by the strategy suggested by Stein in [50, Chapter V, §6.12]. The proof of our theorem will use analogous tools to the ones utilized in the Euclidean case. Nevertheless, in the Heisenberg group, such tools will be sometimes not already available in the literature and we will therefore have to produce them ourselves. We define the *square fractional integral* as

$$\mathcal{D}_s u(x) := \Big(\int_{\mathbb{H}^n} \frac{|u(xy^{-1}) - u(x)|^2}{|y|^{Q+4s}}\, dy \Big)^{1/2}, \quad 0 < s < 1/2, \quad x \in \mathbb{H}^n,$$

where xy^{-1} is the right translation by y^{-1} on the Heisenberg group, see Subsection 2.1. One of the crucial steps in the proof is a pointwise estimate for the square fractional integrals $\mathcal{D}_s$ by the so-called g_λ^*-function, defined in terms of the Poisson semigroup associated to the non-conformally invariant harmonic extension, i.e., to the problem (1) for $s = 1/2$. Let $X_1, \dots, X_n, Y_1, \dots, Y_n, T$, be a basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^n$ (see Subsection 2.3). Let

$$\nabla = (X_1, \dots, X_n, Y_1, \dots, Y_n, T, \partial_\rho), \tag{7}$$

we define the *Littlewood nontangential square function* g_λ^* as

$$g_\lambda^*(u)(x) := \Big(\int_0^\infty \int_{\mathbb{H}^n} \Big(\frac{\rho}{\rho + |y|} \Big)^{\lambda Q} \rho^{1-Q} |\nabla U(xy^{-1}, \rho)|^2\, dy\, d\rho \Big)^{1/2}, \quad x \in \mathbb{H}^n,$$

where $U(x, \rho)$ is the non-conformal harmonic extension of $u(x)$ in the upper half space. We will prove the following.

Theorem 1.3. *Let $n \geq 1$, $0 < s < 1/2$ and $\lambda < 1 + \frac{4s}{Q}$. Then*

$$\mathcal{D}_s u(x) \leq \Lambda(n, s) g_\lambda^*(\mathcal{L}^s u)(x)$$

uniformly on $u \in \mathcal{S}(\mathbb{H}^n)$ and $x \in \mathbb{H}^n$, where $\Lambda(n, s) > 0$ is a constant depending only on n and s.

The proof of Theorem 1.1 will be concluded by combining Theorem 1.3 with the L^p estimates for g_λ^* proven in Subsection 3.2.

Structure of the paper. We start gathering some well known facts about the Heisenberg group and fractional powers of the sublaplacian in Section 2. In Section 3 we provide some technical results that will be needed to prove the main results. In particular, mapping properties for the square function and the nontangential square function are shown, and a mean value theorem for solutions of equations involving the sublaplacian and a potential on $\mathbb{H}^n \times \mathbb{R}^+$ is stated. Finally, the proofs of Theorems 1.3 and 1.1 are presented, respectively, in Sections 4 and 5.

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2. The Heisenberg group and fractional powers of the sublaplacian

Let us first introduce some definitions and set up notations concerning the Heisenberg group. We refer the reader to the book of G. B. Folland [24], although we closely follow the notations used in [52]. We also warn the reader that our notation and certain definitions may be slightly different from those used by others.

2.1. **Fourier transform on the Heisenberg group.** Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the $(2n+1)$ dimensional Heisenberg group, which is the nilpotent Lie group of step two whose underlying manifold is $\mathbb{R}^{2n+1}$ equipped with the group law

$$(z,t)(z',t') = \Big(z+z', t+t' + \frac{1}{2}\operatorname{Im} z\cdot\overline{z'}\Big),$$

where $z, z' \in \mathbb{C}^n$ and $t, t' \in \mathbb{R}$. Identifying $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$ and considering coordinates (x,u,t) we can write the group law as

$$(x,y,t)(x',y',t') = \Big(x+x', y+y', t+t' + \frac{1}{2}(x\cdot y' - x'\cdot y)\Big), \tag{8}$$

where $x, x', y, y' \in \mathbb{R}^n$ and $t, t' \in \mathbb{R}$. Note that $\operatorname{Im}\big((x+iy)\cdot(x'-iy')\big) = y\cdot x' - y'\cdot x = [(x,y)(x',y')]$ is the symplectic form on $\mathbb{R}^{2n}$.

For each $\lambda \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$, we have an irreducible unitary representation π_λ of $\mathbb{H}^n$ realized on $L^2(\mathbb{R}^n)$. The action of $\pi_\lambda(z,t)$ on $L^2(\mathbb{R}^n)$ is explicitly given by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t} e^{i(x\cdot\xi + \frac{1}{2}x\cdot y)}\varphi(\xi+y)$$

where $\varphi \in L^2(\mathbb{R}^n)$ and $z = x+iy$. By a theorem of Stone and Von Neumann, any irreducible unitary representation of $\mathbb{H}^n$ which acts as $e^{i\lambda t}\,\mathrm{Id}$ at the center of the Heisenberg group is unitarily equivalent to π_λ. In view of this, there are representations of $\mathbb{H}^n$ which are realized on the Fock spaces and equivalent to π_λ. We will not use these representations and refer the reader to [24] for details.

The group Fourier transform of a function $f \in L^1(\mathbb{H}^n)$ is the operator-valued function defined, for each $\lambda \in \mathbb{R}^*$, by

$$\widehat{f}(\lambda) := \pi_\lambda(f) = \int_{\mathbb{H}^n} f(z,w)\pi_\lambda(z,w)\,dz\,dw.$$

With an abuse of language, we will call the group Fourier transform just the Fourier transform. Observe that for each λ, $\widehat{f}(\lambda)$ is an operator acting on $L^2(\mathbb{R}^n)$. When $f \in L^1 \cap L^2(\mathbb{H}^n)$, it can be shown that $\widehat{f}(\lambda)$ is a Hilbert-Schmidt operator and the Plancherel theorem holds

$$\int_{\mathbb{H}^n} |f(z,t)|^2\,dz\,dt = \frac{2^{n-1}}{\pi^{n+1}}\int_{-\infty}^{\infty} \|\widehat{f}(\lambda)\|_{\mathrm{HS}}^2 |\lambda|^n\,d\lambda, \tag{9}$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm given by $\|T\|^2_{\mathrm{HS}} = \mathrm{tr}(T^*T)$, for T a bounded operator, T^* being the adjoint operator of T. By polarizing the Plancherel identity we get the Parseval formula

$$\int_{\mathbb{H}^n} f(z,t)\overline{g(z,t)}dzdt = \frac{2^{n-1}}{\pi^{n+1}}\int_{-\infty}^{\infty} \mathrm{tr}(\widehat{f}(\lambda)\widehat{g}(\lambda)^*)|\lambda|^n\, d\lambda.$$

Let f^λ stand for the inverse Fourier transform of f in the *central variable* t

$$f^\lambda(z) = \int_{-\infty}^{\infty} f(z,t)e^{i\lambda t}\, dt.$$

By taking the Euclidean Fourier transform of $f^\lambda(z)$ in the variable λ, we obtain

$$f(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t} f^\lambda(z)\, d\lambda. \tag{10}$$

By the definition of $\pi_\lambda(z,t)$ and $\widehat{f}(\lambda)$ it is easy to see that

$$\widehat{f}(\lambda) = \int_{\mathbb{C}^n} f^\lambda(z)\pi_\lambda(z,0)dz.$$

The operator which takes a function g on $\mathbb{C}^n$ into the operator

$$\int_{\mathbb{C}^n} g(z)\pi_\lambda(z,0)dz$$

is called the Weyl transform of g and is denoted by $W_\lambda(g)$. Thus $\widehat{f}(\lambda) = W_\lambda(f^\lambda)$.

Let us recall that the convolution of f with g on $\mathbb{H}^n$ is defined by

$$f * g(x) = \int_{\mathbb{H}^n} f(xy^{-1})g(y)\, dy, \quad x, y \in \mathbb{H}^n.$$

With $x = (z,t)$ and $y = (z',t')$, in view of (8), we have that $(z',t')^{-1} = (-z',-t')$ and the above takes the form

$$f * g(z,t) = \int_{\mathbb{H}^n} f\big((z,t)(-z',-t')\big)g(z',t')\, dz'\, dt'.$$

A simple computation shows that

$$(f*g)^\lambda(z) = \int_{\mathbb{C}^n} f^\lambda(z-z')g^\lambda(z')e^{\frac{i}{2}\lambda\,\mathrm{Im}(z\cdot\bar{z'})}\, dz'.$$

The convolution appearing on the right hand side is called the λ-twisted convolution and is denoted by $f^\lambda *_\lambda g^\lambda(z)$.

2.2. **Spectral theory of the Heisenberg group.** For $\lambda \in \mathbb{R}^*$ and $\alpha \in \mathbb{N}^n$, we introduce the family of modified Hermite functions

$$\Phi^\lambda_\alpha(x) = |\lambda|^{\frac{n}{4}}\Phi_\alpha(\sqrt{|\lambda|}x), \quad x \in \mathbb{R}^n.$$

Here, Φ_α is the normalized Hermite function on $\mathbb{R}^n$ which is an eigenfunction of the Hermite operator $H = -\Delta + |x|^2$ with eigenvalue $(2|\alpha| + n)$, see [53, Chapter 1.2]. The system is an orthonormal basis for $L^2(\mathbb{R}^n)$. In terms of Φ^λ_α we have the identity

$$\|\widehat{f}(\lambda)\|^2_{\mathrm{HS}} = \sum_{\alpha\in\mathbb{N}^n} \|\widehat{f}(\lambda)\Phi^\lambda_\alpha\|^2_{L^2}$$

and hence Plancherel (9) takes the form

$$\int_{\mathbb{H}^n} |f(z,t)|^2\, dz\, dt = \frac{2^{n-1}}{\pi^{n+1}}\int_{-\infty}^{\infty}\sum_{\alpha\in\mathbb{N}^n} \|\widehat{f}(\lambda)\Phi^\lambda_\alpha\|^2_{L^2}|\lambda|^n\, d\lambda.$$

We can write the spectral decomposition of the scaled Hermite operator $H(\lambda) = -\Delta + |\lambda|^2|x|^2$ as

$$H(\lambda) = \sum_{k=0}^{\infty}(2k+n)|\lambda|P_k(\lambda), \quad \lambda \in \mathbb{R}^*, \tag{11}$$

where $P_k(\lambda)$ are the (finite-dimensional) orthogonal projections defined on $L^2(\mathbb{R}^n)$ by

$$P_k(\lambda)\varphi = \sum_{|\alpha|=k} (\varphi, \Phi_\alpha^\lambda)\Phi_\alpha^\lambda,$$

where $\varphi \in L^2(\mathbb{R}^n)$ and $(\cdot,\cdot)$ is the inner product in $L^2(\mathbb{R}^n)$.

On the other hand, we define the scaled Laguerre functions of type $(n-1)$

$$\varphi_k^\lambda(z) = L_k^{n-1}\Big(\frac{1}{2}|\lambda||z|^2\Big)e^{-\frac{1}{4}|\lambda||z|^2}.$$

Here L_k^{n-1} are the Laguerre polynomials of type $(n-1)$, see [52, Chapter 1.4] for the definition and properties. It is the case that $\{\varphi_k^\lambda\}_{k=0}^\infty$ forms an orthogonal basis for the subspace consisting of radial functions in $L^2(\mathbb{C}^n)$, i.e., functions invariant under the action of $SO(2n)$, the group of all orthogonal matrices of order $2n$ and determinant 1.

The so-called special Hermite expansion of a function g defined on $\mathbb{C}^n$ written in its compact form reads as

$$g(z) = (2\pi)^{-n}|\lambda|^n \sum_{k=0}^{\infty} g *_\lambda \varphi_k^\lambda(z).$$

The connection betweeen the Hermite projections $P_k(\lambda)$ and the Laguerre functions φ_k^λ, via the Weyl transform, is given by the following important formula

$$W_\lambda(\varphi_k^\lambda) = (2\pi)^n|\lambda|^{-n}P_k(\lambda). \tag{12}$$

Observe that, in particular, for any function f on $\mathbb{H}^n$, we have the expansion

$$f^\lambda(z) = (2\pi)^{-n}|\lambda|^n \sum_{k=0}^{\infty} f^\lambda *_\lambda \varphi_k^\lambda(z).$$

2.3. **The sublaplacian.** Let us now define the sublaplacian on the Heisenberg group. A basis for the Lie algebra $\mathfrak{h}_n$ of left-invariant vector fields on $\mathbb{H}^n$ is given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}y_j\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2}x_j\frac{\partial}{\partial t}, \quad j = 1, 2, \dots, n, \qquad T = \frac{\partial}{\partial t}. \tag{13}$$

It is easily checked that the only non-trivial Lie brackets in $\mathfrak{h}_n$ are given by $[X_j, Y_j] = -T$ as all other brackets vanish. The Kohn-Laplacian on $\mathbb{H}^n$ is the second order operator

$$\mathcal{L} = -\sum_{j=1}^{n}(X_j^2 + Y_j^2),$$

known as the sublaplacian. It falls in the class of operators of the type *sums of squares of vector fields*. Though not elliptic, this operator shares several properties with its counterpart Δ on $\mathbb{R}^n$.

The group $\mathbb{H}^n$ admits a family of automorphisms indexed by $\mathbb{R}_+$ and given by the non-isotropic *Heisenberg dilations*

$$\delta_\lambda(z,t) = (\lambda z, \lambda^2 t), \quad \lambda > 0, \quad (z,t) \in \mathbb{H}^n.$$

A function $u : \mathbb{H}^n \to \mathbb{R}$ is said *homogenous of degree* $k \in \mathbb{Z}$ if for every $\lambda > 0$

$$u \circ \delta_\lambda = \lambda^k u.$$

With respect to these dilations, the vector fields X_j, Y_j, T are homogeneous of degree one and $\mathcal{L}$ is homogeneous of degree two. A fundamental solution $\Gamma(z,t)$ for $\mathcal{L}$ is given by

$$\Gamma(z,t) = c_Q |(z,t)|^{-Q+2}$$

where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$, $|(z,t)|^4 = |z|^4 + 16t^2$, and $c_Q > 0$ is a number depending only on Q. This was found by Folland [22], see also [49]. The gauge function

$$d : (z,t) \mapsto |(z,t)|$$

is the Koranyi norm, which is homogeneous of degree one with respect to the dilations δ_λ.

The spectral decomposition of the sublaplacian is achieved via the special Hermite expansion introduced in the previous subsection. The action of the Fourier transform on functions of the form $\mathcal{L}f$ and Tf are given by

$$(\mathcal{L}f)\widehat{\ }(\lambda) = \widehat{f}(\lambda)H(\lambda), \qquad (Tf)\widehat{\ }(\lambda) = -i\lambda\widehat{f}(\lambda).$$

If L_λ is the operator defined by the relation $(\mathcal{L}f)^\lambda = L_\lambda f^\lambda$ then it follows that

$$W_\lambda(L_\lambda f^\lambda) = W_\lambda(f^\lambda)H(\lambda).$$

Recalling the spectral decomposition of $H(\lambda)$ given in (11) and the identity (12) we obtain

$$L_\lambda f^\lambda(z) = (2\pi)^{-n}\sum_{k=0}^{\infty}(2k+n)|\lambda| f^\lambda *_\lambda \varphi_k^\lambda(z).$$

Thus, by taking the Fourier transform in the variable λ in (10), the spectral decomposition of the sublaplacian is given by

$$\mathcal{L}f(z,t) = (2\pi)^{-n-1}\int_{-\infty}^{\infty}\Big(\sum_{k=0}^{\infty}(2k+n)|\lambda| f^\lambda *_\lambda \varphi_k^\lambda(z)\Big)e^{-i\lambda t}|\lambda|^n d\lambda.$$

2.4. **Fractional powers of the sublaplacian.** The fractional powers of the sublaplacian $\mathcal{L}^s$ defined in the introduction via the extension problem (1) can be equivalently defined via the spectral decomposition

$$\mathcal{L}^s f(z,t) = (2\pi)^{-n-1}\int_{-\infty}^{\infty}\Big(\sum_{k=0}^{\infty}\big((2k+n)|\lambda|\big)^s f^\lambda *_\lambda \varphi_k^\lambda(z)\Big)e^{-i\lambda t}|\lambda|^n d\lambda.$$

Note that $(\widehat{\mathcal{L}^s f})(\lambda) = \widehat{f}(\lambda)H(\lambda)^s$.

On the other hand, the operators $\mathcal{L}_s$ are also defined for $0 \le s < (n+1)$ by

$$\mathcal{L}_s f(z,t) = (2\pi)^{-n-1}\int_{-\infty}^{\infty}\Big(\sum_{k=0}^{\infty}(2|\lambda|)^s\frac{\Gamma(\frac{2k+n}{2}+\frac{1+s}{2})}{\Gamma(\frac{2k+n}{2}+\frac{1-s}{2})} f^\lambda *_\lambda \varphi_k^\lambda(z)\Big)e^{-i\lambda t}|\lambda|^n d\lambda. \tag{14}$$

As mentioned in the introduction, the operators $\mathcal{L}_s$ occur naturally in the context of CR geometry and scattering theory on the Heisenberg group: when we identify $\mathbb{H}^n$ as the boundary of the Siegel's upper half space in $\mathbb{C}^{n+1}$, they have the important property of being conformally invariant. In short, (14) means that $\mathcal{L}_s$ is the operator (see [9, (1.33)])

$$\mathcal{L}_s := (2|T|)^s\frac{\Gamma\big(\frac{\mathcal{L}}{2|T|}+\frac{1+s}{2}\big)}{\Gamma\big(\frac{\mathcal{L}}{2|T|}+\frac{1-s}{2}\big)}.$$

Thus $\mathcal{L}_s$ corresponds to the spectral multiplier

$$(2|\lambda|)^s\frac{\Gamma\big(\frac{2k+n}{2}+\frac{1+s}{2}\big)}{\Gamma\big(\frac{2k+n}{2}+\frac{1-s}{2}\big)}, \quad k \in \mathbb{N}.$$

Note that $\mathcal{L}_1 = \mathcal{L}$. It is known that $\mathcal{L}_s$ also has an explicit fundamental solution, see e.g. [17, page 530], in contrast with $\mathcal{L}^s$, for which an explicit expression for the fundamental solution is not known.

It can be checked that the operators $\mathcal{L}_s$ and $\mathcal{L}^s$ are equivalent in $L^p(\mathbb{H}^n)$, $1 < p < \infty$, i.e.,

$$c\|\mathcal{L}^s f\|_{L^p} \leq \|\mathcal{L}_s f\|_{L^p} \leq C\|\mathcal{L}^s f\|_{L^p}, \tag{15}$$

for some $c, C > 0$. Indeed, it suffices to prove that the operator $\mathcal{L}_s\mathcal{L}^{-s}$ is bounded on $L^p(\mathbb{H}^n)$. In order to conclude the latter, all we need to do is to check that the multiplier

$$M = \sum_{k=0}^{\infty}(2k+n)^{-s}\frac{\Gamma((2k+n+1+s)/2)}{\Gamma((2k+n+1-s)/2)}P_k$$

is a Fourier multiplier on $L^p(\mathbb{H}^n)$. In view of the known multiplier theorems ([41], [52, Theorem 2.6.1]) this amounts to check that the function (as a function of k)

$$(2k+n)^{-s}\frac{\Gamma\big(\frac{2k+n+1+s}{2}\big)}{\Gamma\big(\frac{2k+n+1-s}{2}\big)}$$

and its jth-derivatives up to order $(n+1)$ are bounded by $C_j k^{-j}$ for $j = 0, 1, \dots, (n+1)$, which is true in view of the known asymptotics for the ratio of gamma functions (see for instance [43])

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \quad \text{as } z \to \infty, \quad |\arg(z)| < \pi,$$

and the asymptotics of the polygamma function, involved in the derivatives of the Gamma function.

We can equip $\mathbb{H}^n$ with a metric induced by the Koranyi norm which makes it a space of homogeneous type. On such spaces there is a well defined notion of dyadic cubes and grids with properties similar to their counterparts in the Euclidean setting, we refer for instance to the construction by Christ [13, Section 3]. Given $1 < p < \infty$, by A_p we will denote the Muckenhoupt class of weights in $\mathbb{H}^n$, namely all nonnegative functions $w \in L^1_{\text{loc}}(\mathbb{H}^n)$ such that

$$[w]_{A_p} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w(x)\,dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-p'/p}\,dx\Big)^{p/p'} < \infty$$

where the supremum is taken over all cubes $Q \in \mathbb{H}^n$.

Weighted versions for Fourier multipliers on the Heisenberg group are also available in the literature, see for instance [3, Theorem 1.3]. A more general result concerning spectral multipliers in spaces of homogeneous type whose derivatives up to any order are bounded is contained in the important work [5, Theorem 1.1]. A simplified version adapted to our setting reads as follows. Let us denote $T := \mathcal{L}^{-s}\mathcal{L}_s$.

Theorem 2.1 ([5, Theorem 1.1])**.** *Let $0 < s < 1$. For $1 < p < \infty$, there exists a constant c_p such that, for every weight $w \in A_p$,*

$$\|T\|_{L^p(w)\to L^p(w)} \leq c_p[w]_{A_p}^{1/(p-1)}.$$

From Theorem 2.1 we can deduce a weighted version of (15), namely, for $w \in A_p$ with $1 < p < \infty$,

$$c\|\mathcal{L}^s f\|_{L^p(w)} \leq \|\mathcal{L}_s f\|_{L^p(w)} \leq C\|\mathcal{L}^s f\|_{L^p(w)}, \tag{16}$$

for some $c, C > 0$.

3. Toolbox

In this section we collect and study several ingredients that will be used to prove the main theorems, namely the extension problem and the integral representation associated with $\mathcal{L}_s$, mapping properties for the square function and the nontangential square function, and a mean value theorem for subsolutions of hypoelliptic equations with a potential on $\mathbb{H}^n \times \mathbb{R}^+$.

3.1. **The extension problem and a bilinear form associated with $\mathcal{L}_s$.** Let

$$\varphi_{s,\rho}(z,t) = \big((\rho^2+|z|^2)^2+16t^2\big)^{-\frac{n+1+s}{2}}, \tag{17}$$

which is integrable on $\mathbb{H}^n$ for all $s>0$, and

$$C(n,s) = \frac{4}{\pi^{n+1/2}}\frac{\Gamma(n+s)\Gamma\big(\frac{n+1+s}{2}\big)}{\Gamma(s)\Gamma(\frac{n+s}{2})}. \tag{18}$$

The following theorem, which provides a solution to the extension problem (3), realizes $\mathcal{L}_s f$ as the Dirichlet-to-Neumann map associated to the extension problem, and shows a pointwise representation for $\mathcal{L}_s$, can be found in [45, Theorem 1.2] (actually, here we are stating a reduced version of the result therein), see also [44].

Theorem 3.1. [45, Theorem 1.2] *Let $s>0$. Let $f\in L^p(\mathbb{H}^n)$, $1\le p<\infty$. Then, as $\rho\to 0^+$,*

$$w = C(n,s)\rho^{2s} f*\varphi_{s,\rho}\to f \qquad \text{in } L^p(\mathbb{H}^n), \tag{19}$$

where $\varphi_{s,\rho}$ is defined in (17) *and $C(n,s)$ is the one in* (18). *If we further assume that $\mathcal{L}_s f\in L^p(\mathbb{H}^n)$ then*

$$-\lim_{\rho\to 0^+}\rho^{1-2s}\partial_\rho(w(z,t,\rho)) = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}\mathcal{L}_s f(z,t).$$

Moreover, when $0<s<1/2$, we also have the pointwise representation

$$\mathcal{L}_s f(x) = b(n,s)\int_{\mathbb{H}^n}\frac{f(x)-f(y)}{|xy^{-1}|^{Q+2s}}\,dy \tag{20}$$

for all $f\in C^1(\mathbb{H}^n)$ such that $X_jf, Y_jf, Tf\in L^\infty(\mathbb{H}^n)$, $j=1,\dots,n$, where

$$b(n,s) := \frac{4^{1+s}}{\pi^{n+1/2}}\frac{\Gamma(n+s)\Gamma\big(\frac{n+1+s}{2}\big)}{\Gamma\big(\frac{n+s}{2}\big)|\Gamma(-s)|}. \tag{21}$$

In view of Theorem 3.1, the function $C(n,s)\varphi_{s,\rho}$ is understood as a generalized conformal Poisson kernel, which is a solution to a generalized conformal harmonic extension. Observe that, when $s=1/2$ in (3), we are reduced to the (conformal) harmonic extension

$$\big(\partial_{\rho\rho}+\frac14\rho^2\partial_{tt}-\mathcal{L}\big)U(z,t,\rho)=0 \qquad \lim_{\rho\to 0}U(z,t,\rho)=f(z,t) \quad \text{in } \mathbb{H}^n\times\mathbb{R}^+,$$

so that we write $U = C(n,1/2)f*\varphi_{1/2,\rho}$ in (19).

On the other hand, when $s=1/2$, the extension problem (1) takes the form

$$\big(\partial_\rho^2-\mathcal{L}\big)U=0 \quad \text{in } \mathbb{H}^n\times\mathbb{R}^+, \qquad U(z,t,0)=u(z,t), \qquad \text{in } \mathbb{H}^n. \tag{22}$$

We will denote $U(x,\rho):=e^{-\rho\mathcal{L}^{1/2}}u(x)$. An explicit expression is not known for the solution of this problem, however the subordination formula

$$e^{-\rho\mathcal{L}^{1/2}} = \rho\int_0^\infty (4\pi w)^{-\frac12}e^{-\frac{\rho^2}{4w}}e^{-w\mathcal{L}}\,dw$$

allows to write the non-conformal Poisson kernel $P_\rho(x)$ as

$$P_\rho(x) = \rho\int_0^\infty (4\pi w)^{-\frac12}e^{-\frac{\rho^2}{4w}}q_w(x)\,dw$$

and the known sharp estimates for the heat kernel $q_w(z,t)$ (see e.g. [53, Proposition 2.8.2]) yield the following sharp estimates for the Poisson kernel (see also [2, Theorem 6.11 (i)])

$$P_\rho(x)\le C_n\frac{\rho}{(\rho^2+|x|^2)^{\frac{Q+1}{2}}}. \tag{23}$$

The latter estimate will be crucial to prove mapping properties of the square function operators given in the next subsection. It is also important to notice that $e^{-\rho\mathcal{L}^{1/2}}$ is indeed a semigroup, in contrast with the fact that the operator $f\mapsto f*\varphi_{1/2,\rho}$ is not.

3.2. **The square functions g, g^*_λ.** Recall that we are letting

$$\nabla=(X_1,\dots,X_n,Y_1,\dots,Y_n,T,\partial_\rho).$$

We define the *square g-function* by

$$g(u)(x)=\Big(\int_0^\infty|\nabla U(x,\rho)|^2\rho\,d\rho\Big)^{1/2}$$

and the *Littlewood nontangential square function* g^*_λ as

$$g^*_\lambda(u)(x):=\Big(\int_0^\infty\int_{\mathbb{H}^n}\Big(\frac{\rho}{\rho+|y|}\Big)^{\lambda Q}\rho^{1-Q}|\nabla U(xy^{-1},\rho)|^2\,dy\,d\rho\Big)^{1/2}, \tag{24}$$

where $U(x,\rho)$ is the non-conformal harmonic extension of $u(x)$ in the upper half space in (22). In [52, Chapter 2.6], Thangavelu defined g and g^* functions in terms of the heat semigroup and proved L^p mapping properties for these operators.

For g, the following basic result can be proven as in the Euclidean case, see [50, Chapter 4, §1].

Theorem 3.2. *Let $n\geq1$ and $w\in A_p$. For any $u\in L^p(w):=L^p(\mathbb{H}^n,w)$ we have, for $1<p<\infty$,*

$$c_p\|u\|_{L^p(w)}\leq\|g(u)\|_{L^p(w)}\leq C_p\|u\|_{L^p(w)}.$$

Proof. The proof follows classical arguments, we point out the main steps. Observe that $|\nabla U(x,\rho)|^2=|\partial_\rho U|^2+|\nabla_x U(x,\rho)|^2$, where $|\nabla_x U(x,\rho)|^2=\sum_{j=1}^n(|X_jU|^2+|Y_jU|^2+|TU|^2)$. It will be appropriate to introduce the following two partial g-functions, namely

$$g_1(u)(x)=\Big(\int_0^\infty|\partial_\rho U(x,\rho)|^2\rho\,d\rho\Big)^{1/2},\qquad g_x(u)(x)=\Big(\int_0^\infty|\nabla_x U(x,\rho)|^2\rho\,d\rho\Big)^{1/2}.$$

Note that $g^2=g_1^2+g_x^2$.

Let us focus on the L^2 estimate for g_1. Applying Plancherel theorem for the Fourier transform on $\mathbb{H}^n$, we get (call $U(x,\rho)=:U_\rho(x)$)

$$\|g_1(u)\|_{L^2}^2=\int_0^\infty\int_{\mathbb{H}^n}|\partial_\rho U(x,\rho)|^2\rho\,dx\,d\rho=\frac{2^{n-1}}{\pi^{n+1}}\int_0^\infty\int_{-\infty}^\infty\|\widehat{(\partial_\rho U_\rho)}(\lambda)\|_{\mathrm{HS}}^2|\lambda|^n\,d\lambda\,\rho\,d\rho.$$

We also have

$$\widehat{(\partial_\rho U_\rho)}(\lambda)=-\widehat{u}(\lambda)H(\lambda)^{1/2}e^{-\rho H(\lambda)^{1/2}}$$

and so the squared Hilbert-Schmidt norm is given by the sum

$$\sum_{\alpha\in\mathbb{N}^n}(2|\alpha|+n)|\lambda|e^{-2\rho((2|\alpha|+n)|\lambda|)^{1/2}}\|\widehat{u}(\lambda)\Phi_\alpha^\lambda\|_{L^2}^2.$$

Integrating the above with respect to $\rho\,d\rho$, we get

$$\|g_1(u)\|_{L^2}^2=\frac14\frac{2^{n-1}}{\pi^{n+1}}\int_{-\infty}^\infty\|\widehat{u}(\lambda)\|_{\mathrm{HS}}^2|\lambda|^n\,d\lambda=\frac14\|u\|_{L^2}^2. \tag{25}$$

Now, for $p\neq2$, the converse inequality $\|u\|_{L^p(w)}\leq c\|g(u)\|_{L^p(w)}$ can be derived with a polarization argument from the L^2 identity (25) involving the weight and its dual, as in [14, Section 6] and the fact that $g_1(x)\leq g(x)$ implies $\|g_1\|_{L^p(w)}\leq\|g\|_{L^p(w)}$.

The inequality $\|g(u)\|_{L^p(w)}\leq C\|u\|_{L^p(w)}$ for $1<p<\infty$ follows as in [50, Chapter 4, §1] by using the estimates for the Poisson kernel (23) and the theory of vector-valued operators in spaces of homogeneous type [28, 46]. □

We will also need to prove estimates for g^*_λ.

Theorem 3.3. *Let $n \geq 1$ and $\lambda > 1$. For any $u \in L^p(\mathbb{H}^n)$ we have, for $1 < p < \infty$,*

$$\|g_\lambda^*(u)\|_{L^p} \lesssim \|u\|_{L^p}$$

for $\lambda > \max\left\{1, \frac{2}{p}\right\}$.

Proof. The proof of Theorem 3.3 also follows the lines of the corresponding Euclidean result, see [50, Chapter 4, §2]. We detail the pertinent ingredients.

We start with the case $p \geq 2$, where only the hypothesis $\lambda > 1$ is relevant. We can write

$$\|g_\lambda^*(u)\|_{L^p}^2 = \sup_{\|\psi\|_{L^{(p/2)'}} \leq 1} \int_0^\infty \int_{\mathbb{H}^n} \rho |\nabla U(y,\rho)|^2 \Big(\int_{\mathbb{H}^n} \Big(\frac{\rho}{\rho + |x^{-1}y|} \Big)^{\lambda Q} \rho^{-Q} \psi(x)\, dx \Big)\, dy\, d\rho.$$

On the other hand

$$\sup_{\rho>0} \int_{\mathbb{H}^n} \Big(\frac{\rho}{\rho + |x^{-1}y|} \Big)^{\lambda Q} \rho^{-Q} \psi(x)\, dx \leq C \sup_{\rho>0} (\psi * \varphi_\rho)(y),$$

where $\varphi_\rho(x) = \rho^{-Q}\varphi(x/\rho)$ with $\varphi(x) = (1+|x|)^{-\lambda Q}$. It can be proved, analogously as in [50, Theorem 2, §2.2 of Chapter III], that

$$\sup_{\rho>0} (\psi * \varphi_\rho)(y) \leq C M\psi(y)$$

where, for $f \in L^1_{\text{loc}}(\mathbb{H}^n)$, M is the centered Hardy–Littlewood maximal function given by

$$Mf(z,t) = \sup_{r>0} \frac{1}{|B((z,t),r)|} \int_{B((z,t),r)} |f(z',t')|\, dz', \quad (z,t) \in \mathbb{H}^n.$$

Here $B((z,t),r)$ denotes the open ball with center (z,t) and radius r induced by the Korányi norm d and, for a measurable set A, we denote the volume by $|A|$. Observe that the Hardy–Littlewood maximal operator is bounded in $L^p(\mathbb{H}^n)$ for $1 < p < \infty$. This follows from general results on a space of homogeneous type in the sense of Coifman and Weiss [16]. Therefore,

$$\|g_\lambda^*(u)\|_{L^p}^2 \leq C_\lambda \int_{\mathbb{H}^n} |g(u)(y)|^2 M\psi(y)\, dy \leq C_\lambda \|g(u)\|_{L^p}^2 \|M\psi\|_{L^{(p/2)'}} \leq C_\lambda \|f\|_{L^p}^2.$$

Here we have used Theorem 3.2 and the boundedness of the maximal function.

Let us move to the case $p < 2$. Let $\mu \geq 1$ and write the following variant of the maximal function

$$M_\mu f(z,t) = \Big(\sup_{r>0} \frac{1}{|B((z,t),r)|} \int_{B((z,t),r)} |f(z',t')|^\mu\, dz' \Big)^{1/\mu}, \quad (z,t) \in \mathbb{H}^n$$

for which the following holds

$$\|M_\mu f\|_{L^p} \leq C_{p,\mu} \|f\|_{L^p}, \quad p > \mu.$$

We also have the estimate

$$|U(x^{-1}y,\rho)| \leq C_\mu \Big(1 + \frac{|y|}{\rho}\Big)^{Q/\mu} M_\mu u(x).$$

The above follows analogously as in [50, Chapter IV, Lemma 4] with the help of the bound for the Poisson kernel in (23). With this, proceeding as in [50, Chapter IV, §2.1], we have

$$\begin{aligned}(g_\lambda^*(u)(x))^2 &= \frac{1}{p(p-1)} \int_0^\infty \int_{\mathbb{H}^n} \Big(\frac{\rho}{\rho + |y|} \Big)^{\lambda Q} \rho^{1-Q} (U(x^{-1}y,\rho))^{2-p} |\mathcal{L}U^p(x^{-1}y,\rho)|\, dy\, d\rho \\ &\leq C_\mu^{2-p} (M_\mu u(x))^{2-p} I^*(x),\end{aligned}$$

with

$$I^*(x) = \int_0^\infty \int_{\mathbb{H}^n} \Big(\frac{\rho}{\rho + |y|} \Big)^{\lambda' Q} \rho^{1-Q} \mathcal{L}U^p(x^{-1}y,\rho)\, dy\, d\rho.$$

Observe that

$$\int_{\mathbb{H}^n} I^*(x)\,dx = \int_0^\infty \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \Big(\frac{\rho}{\rho+|xy^{-1}|}\Big)^{\lambda' Q} \rho^{1-Q}\mathcal{L}U^p(y,\rho)\,dx\,dy\,d\rho$$
$$= C_{\lambda'}\int_0^\infty \int_{\mathbb{H}^n} \rho\mathcal{L}U^p(y,\rho)\,dy\,d\rho$$

where in the last step we used that, for $\lambda'>1$,

$$\rho^{-Q}\int_{\mathbb{H}^n}\Big(\frac{\rho}{\rho+|x|}\Big)^{\lambda' Q}dx = \int_{\mathbb{H}^n}\Big(\frac{1}{1+|x|}\Big)^{\lambda' Q}dx = C_{\lambda'}<\infty.$$

It is easy to check that an analogous identity to [50, Chapter IV, Lemma 2] also holds in our context, namely

$$\int_0^\infty\int_{\mathbb{H}^n}\rho\mathcal{L}U^p(y,\rho)\,dy\,d\rho = \int_{\mathbb{H}^n}U^p(y,0)\,dy.$$

Gathering all the ingredients above, we infer that

$$\int_{\mathbb{H}^n} I^*(x)\,dx = C_{\lambda'}\|u\|^p_{L^p}. \tag{26}$$

Finally, by Hölder's inequality with exponents r and r', $1<r<2$ (which is possible by the fact that $\big(\frac{2-p}{2}\big)pr'=p$ and $rp/2=1$ if $r=2/p$)

$$\int_{\mathbb{H}^n}(g^*_\lambda(u)(x))^p\,dx \le C\int_{\mathbb{H}^n}(M_\mu u(x))^{p(2-p)/2}(I^*(x))^{p/2}\,dx$$
$$\le C\Big(\int_{\mathbb{H}^n}(M_\mu u(x))^p\,dx\Big)^{1/r'}\Big(\int_{\mathbb{H}^n}I^*(x)\,dx\Big)^{1/r}.$$

The conclusion follows in view of the boundedness of the maximal operator M_μ and (26). □

With the corresponding modifications in the proof, a weighted version of Theorem 3.3 can be also obtained, (or by adapting the proof in [40, Corollary, p. 110] to an space of homogeneous type, taking into account the estimates for the Poisson kernel (23)), using the weighted boundedness for the Hardy–Littlewood maximal operator on spaces of homogeneous type (e.g. [1]).

Theorem 3.4. *Let $n\ge 1$ and $\lambda>1$. For any $u\in L^p(\mathbb{H}^n)$ we have, for $1<p<\infty$, and $w\in A_{\min\{p,\frac{p\lambda}{2}\}}$,*

$$\|g^*_\lambda(u)\|_{L^p(w)}\lesssim\|u\|_{L^p(w)}.$$

for $\lambda>\max\big\{1,\frac{2}{p}\big\}$.

3.3. **A mean value theorem on $\mathbb{H}^n\times\mathbb{R}^+$.** Let us write $E:=-\mathcal{L}+\partial_{\rho\rho}$. It turns out that E is homogeneous of degree 2, hypoelliptic[1] and formally self-adjoint. These facts imply that it possesses a fundamental solution Γ which is C^∞ off the diagonal in $\mathbb{H}^n\times\mathbb{R}^+\times\mathbb{H}^n\times\mathbb{R}^+$, see [23, 47]. Recall that $d(z,t)=|(z,t)|=(|z|^4+16t^2)^{1/4}$. Let us denote $\widetilde{d}((z,t),\rho)=(\rho^4+|z|^4+16t^2)^{1/4}$,

$$B_r=\{((z,t),\rho)\in\mathbb{H}^n\times\mathbb{R}^+ : \widetilde{d}((z,t),\rho)<r\} \tag{27}$$

and call this set the *extended Heisenberg ball* centered at the origin with radius r. Balls centered at points other than the origin are defined by left-translation and the usual Euclidean distance. We let $d((z,t),(z',t'))=d((z',t')^{-1}(z,t))$ denote the distance between (z,t) and (z',t'). Then the ball $B_r((z',t'),\rho')=B(((z',t'),\rho'),r)$ centered at (z',t',ρ') with radius r is obtained by replacing

[1]A differential operator D is hypoelliptic if the solutions of the equation $Df=g$ with $g\in C^\infty$, are also C^∞. In our case, by a theorem of Hörmander [30], since X_j,Y_j, $j=1,\dots,n$ are vector fields with the property that their commutators up to a certain order span the tangent space at every point, then $\sum_{j=1}^n\big(X_j^2+Y_j^2\big)$ is hypoelliptic, and hence E is.

$(\rho^4+|z|^4+16t^2)^{1/4}$ in (27) with $\widetilde{d}(((z,t),\rho),((z',t'),\rho')) = \big(d((z,t),(z',t'))^4+|\rho-\rho'|^4\big)^{1/4}$. For simplicity, below we will denote $(x,\rho)=((z,t),\rho)$, $(y,\rho')=((z',t'),\rho')$.

In the remarkable work [15], the authors establish a uniform Harnack's inequality for nonnegative subsolutions of equations involving a general hypoelliptic operator which is sum of squares of vector fields plus a potential V which is a measurable function belonging to a certain analogue of the local Kato class. Rewriting the results in [15] specified to our context, such a Harnack's inequality concerns equations of the type $(E-V)u \geq 0$, where V satisfies

$$\lim_{r\to 0} \sup_{(x,\rho)\in\Omega} \int_{B_r(x,\rho)} |V(y,\rho)|\Gamma((x,\rho),(y,\rho'))\,dy\,d\rho' = 0, \tag{28}$$

where $\Omega \subset \mathbb{H}^n \times \mathbb{R}^+$ is an open set. Observe that in the definition above we took the discussion on [15, page 703] into account to replace the E-balls by the balls $B_r(x,\rho)$ in [15, (1.5)].

Given an open set $\Omega \subset \mathbb{H}^n \times \mathbb{R}^+$, let $\widetilde{W}(\Omega)$ be the completion of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{\widetilde{W}(\Omega)} = \int_\Omega (|\nabla u|^2 + u^2)\,dx\,d\rho,$$

where ∇ is the one in (7). We also denote $\widetilde{W}_{\mathrm{loc}}(\Omega)$ as the class of those $u \in L^2_{\mathrm{loc}}$ such $\varphi u \in \widetilde{W}(\Omega)$ for every $\varphi \in C_0^\infty(\Omega)$. Following [15, Definition 3.3], we say that $u \in \widetilde{W}_{\mathrm{loc}}(\Omega)$ is a solution of $Eu - Vu \geq 0$ in U if

$$\int_\Omega (\nabla u \cdot \nabla\varphi + Vu\varphi)\,dx\,d\rho \leq 0 \quad \text{for every } \varphi \in \widetilde{W}_{\mathrm{loc}}(\Omega), \quad \varphi \geq 0 \text{ a.e. in } \Omega.$$

We will prove the following mean value inequality, which is inspired by the Harnack inequality in [15, Lemma 4.1] adapted to the present setting.

Lemma 3.5. *There exist $r_0>0$ and for every $p>0$ a constant $C>0$, such that if $u \in \widetilde{W}_{\mathrm{loc}}(\Omega)$ is a nonnegative solution of $Eu - Vu \geq 0$ in Ω, with V satisfying* (28)*, then*

$$u^p(x,\rho) \leq \frac{C}{|B_r(x,\rho)|}\int_{B_r(x,\rho)} u^p(y,\rho')\,dy\,d\rho' \tag{29}$$

for every $r \leq r_0$ provided $B_{4r}(x,\rho) \subset \Omega$.

In order to prove Lemma 3.5, we will need two results from [15] adapted to the current setting.

Theorem 3.6 ([15, Theorem 3.9])**.** *There exist $C=C(\Omega)>0$ and $R=R(\Omega)>0$ such that for every $r<R$ such that $B_{3r}(x,\rho)\subset\Omega$,*

$$u(x,\rho) \leq C\Big(\fint_{B_r(x,\rho)} u^2\Big)^{1/2} \qquad \text{a.e. in } \Omega$$

for every nonnegative function $u \in \widetilde{W}_{\mathrm{loc}}(\Omega)$ such that $Eu - Vu \geq 0$.

Corollary 3.7 ([15, Corollary 3.10])**.** *There exist positive constants $C=C(\Omega)$, $R=R(\Omega)$, and $M=M(\Omega)$ such that if $u \in \widetilde{W}_{\mathrm{loc}}(\Omega)$, $u\geq 0$ satisfies $Eu-Vu\geq 0$ in Ω, $r<R$ and $B_{4r}(x,\rho)\subset\Omega$, then*

$$\sup_{B_{sr}(x,\rho)} u^2 \leq \frac{C}{(t-s)^M|B_r(x,\rho)|}\int_{B_{tr}(x,\rho)} u^2\,dy\,d\rho$$

for every s,t, with $1/2 \leq s < t \leq 1$.

Proof of Lemma 3.5. The proof follows the lines of [15, Proof of Lemma 4.1], but slightly twisting the argument therein. Let us assume, without loss of generality, that $0<p<2$. Set

$$J(s) = \frac{1}{|B_r(x,\rho)|^{1-2/p}}\Big(\int_{B_{sr}(x,\rho)} u^2\,dy\,d\rho\Big)\Big(\int_{B_r(x,\rho)} u^p\,dy\,d\rho\Big)^{-2/p}.$$

By Theorem 3.6 we have

$$\begin{aligned} u(x,\rho)^p &\le C\Big(\fint_{B_{r/2}(x,\rho)} u^2\Big)^{p/2} \\ &= \frac{C}{|B_{r/2}(x,\rho)|^{p/2}}\Big[J(1/2)|B_r(x,\rho)|^{1-2/p}\Big(\int_{B_r(x,\rho)} u^p\Big)^{2/p}\Big]^{p/2} \\ &\simeq \frac{1}{|B_r(x,\rho)|}J(1/2)^{p/2}\int_{B_r(x,\rho)} u^p, \end{aligned}$$

where the above $\simeq$ holds in view of the doubling condition for the balls $B_\delta(x,\rho)$, namely $|B_\delta(x,\rho)| \le |B_{2\delta}(x,\rho)| \le C|B_\delta(x,\rho)|$ for some $C > 0$, see [42] and [15, (1.12)].

At this point we have to show that $J(1/2) \le C$. We get, for $1/2 \le s < t \le 1$,

$$\begin{aligned} J(s) &= \frac{1}{|B_r(x,\rho)|^{1-2/p}}\Big(\int_{B_{sr}(x,\rho)} u^{2-p}u^p\,dy\,d\rho\Big)\Big(\int_{B_r(x,\rho)} u^p\,dy\,d\rho\Big)^{-2/p} \\ &\le \frac{1}{|B_r(x,\rho)|^{1-2/p}}\Big(\sup_{B_{sr}(x,\rho)} u^2\Big)^{(2-p)/2}\int_{B_{sr}(x,\rho)} u^p\,dy\,d\rho\Big(\int_{B_r(x,\rho)} u^p\,dy\,d\rho\Big)^{-2/p} \\ &\underset{\text{Corollary 3.7}}{\le} \frac{1}{|B_r(x,\rho)|^{1-2/p}}\Big(\frac{C}{(t-s)^M|B_r(x,\rho)|}\int_{B_{tr}(x,\rho)} u^2\,dy\,d\rho\Big)^{(2-p)/2} \\ &\qquad\times\int_{B_{sr}(x,\rho)} u^p\,dy\,d\rho\Big(\int_{B_r(x,\rho)} u^p\,dy\,d\rho\Big)^{-2/p} \\ &\underset{B_{sr}(x,\rho)\subset B_r(x,\rho)}{\le} \Big[\frac{1}{|B_r(x,\rho)|^{-2/p}}\Big(\int_{B_r(x,\rho)} u^p\,dy\,d\rho\Big)^{-2/p}\Big]^{(2-p)/2} \\ &\qquad\times\Big(\frac{C}{(t-s)^M|B_r(x,\rho)|}\int_{B_{tr}(x,\rho)} u^2\,dy\,d\rho\Big)^{(2-p)/2} \\ &= \Big(\frac{C}{(t-s)^M}\Big)^{(2-p)/2}J(t)^{(2-p)/2}. \end{aligned}$$

From this inequality it can be proved, see [12, pp. 417–418], [15, Proof of Lemma 4.1], or [20, pp. 1004–1005] (by a reasoning originally by Dahlberg and Kenig), that $J(1/2) \le C$, uniformly in r. This concludes the proof. □

Remark 3.8. *We observe that, in the proof of Lemma 3.5, the local result Corollary 3.7 is only used to prove the inequality involving $J(s)$. Lemma 3.5 extends the mean value inequality of Theorem 3.6 to allow L^p-averages on the right hand side.*

We also recall the following result proved in [47], stated in our context.

Theorem 3.9 ([47]). *For every bounded set $\Omega \subset \mathbb{H}^n \times \mathbb{R}^+$ there exist $C_1, C_2, r_0 > 0$ such that for every $(x,\rho) \in \Omega$ and every $(y,\rho') \in \Omega \setminus \{(x,\rho)\}$ with $\widetilde{d}((x,\rho),(y,\rho')) \le r_0$ we have*

$$C_1\frac{\widetilde{d}((x,\rho),(y,\rho'))^2}{|B_{\widetilde{d}((x,\rho),(y,\rho'))}(x,\rho)|} \le \Gamma((x,\rho),(y,\rho')) \le C_2\frac{\widetilde{d}((x,\rho),(y,\rho'))^2}{|B_{\widetilde{d}((x,\rho),(y,\rho'))}(x,\rho)|}.$$

Observe that, if V is constant on a bounded open set $\Omega \subset \mathbb{H}^n \times \mathbb{R}^+$ then the condition (28) holds, in virtue of Theorem 3.9.

We finally mention that other mean value inequalities have been proved in the literature for sublaplacians on Carnot groups (see [6, 7, 8]). In the following sections, it will be fundamental to involve such an inequality for solutions of equations involving a sublaplacian and a suitable potential.

4. Proof of Theorem 1.3

The proof follows the arguments in [18], which in turn are inspired by the strategy sketched in [50]. Without loss of generality, we will prove the inequality at $x = 0$. Let $u \in \mathcal{S}(\mathbb{H}^n)$ and recall that we denote by $U(x, \rho) := e^{-\rho\mathcal{L}^{1/2}}u(x)$ the Poisson semigroup associated with the non-conformal harmonic extension (22). In $\mathbb{H}^n \times \mathbb{R}^+$, let Γ be the path joining the points $(0, 0)$ and $(y, 0)$, consisting of the line segments joining $(0, 0)$ with $(0, |y|)$, $(0, |y|)$ with $(y, |y|)$, and $(y, |y|)$ with $(y, 0)$, where $y \in \mathbb{H}^n$. By Stokes' theorem we have

$$\int_\Gamma \nabla U \, d\lambda = U(y, 0) - U(0, 0) = u(y) - u(0),$$

which implies

$$|u(y) - u(0)| \leq \int_0^{|y|} \big(|\nabla U(y, \lambda)| + |\nabla U(0, \lambda)| + |\nabla U(\lambda\widehat{y}, |y|)|\big) \, d\lambda, \qquad \widehat{y} = \frac{y}{|y|}.$$

Let us denote by $F(x, \rho) := e^{-\rho\mathcal{L}^{1/2}}(\mathcal{L}^s u)$ the non-conformal harmonic extension of $\mathcal{L}^s u$.

We will need two lemmas. First, we have the following fundamental identity.

Lemma 4.1. *Let $u \in \mathcal{S}(\mathbb{H}^n)$. Then, for $0 < 2s < n + 1$,*

$$U(x, \rho) = \frac{1}{\Gamma(s)} \int_0^\infty F(x, \rho + \mu)\mu^{2s-1} \, d\mu.$$

Proof. By using the semigroup property of $e^{-\rho\mathcal{L}^{1/2}}u(x)$ and the fact that $\mathcal{L}^{-s}\mathcal{L}^s = \mathrm{Id}$, we get

$$\begin{aligned}
\int_0^\infty F(x, \rho + \mu)\mu^{2s-1} \, d\mu &= \int_0^\infty e^{-(\rho+\mu)\mathcal{L}^{1/2}}(\mathcal{L}^s u)(x)\mu^{2s-1} \, d\mu \\
&= e^{-\rho\mathcal{L}^{1/2}} \int_0^\infty e^{-\mu\mathcal{L}^{1/2}}(\mathcal{L}^s u)(x)\mu^{2s-1} \, d\mu \\
&= \Gamma(2s)e^{-\rho\mathcal{L}^{1/2}}\mathcal{L}^{-s}(\mathcal{L}^s)u(x) \\
&= \Gamma(2s)e^{-\rho\mathcal{L}^{1/2}}u(x) = \Gamma(2s)U(x, \rho),
\end{aligned}$$

as desired. □

It is important to remark that Lemma 4.1 is strongly based on the fact that, for $0 < 2s < n+1$, the Riesz potentials $\mathcal{L}^{-s}$ are defined via function calculus in terms of the (non-conformal) Poisson semigroup associated with $\mathcal{L}$ as

$$\mathcal{L}^{-s}f(z, t) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\mu\mathcal{L}^{1/2}}f(z, t)\frac{d\mu}{\mu^{1-2s}}.$$

There is no such an analogue representation for the Riesz potentials $\mathcal{L}_{-s}$ in terms of the solution $f \mapsto f * \varphi_{1/2,\rho}$ of the conformally invariant harmonic extension (see the interesting discussion in [26, Section 1, (1.3) and (1.11)]) which is the reason why we refrain from using a g_λ^*-function associated with $\mathcal{L}_{1/2}$.

The second lemma shows that $|\nabla F|^2$ is a solution to the inequality $(\partial_{\rho\rho} - \mathcal{L})u + 2u \geq 0$.

Lemma 4.2. *Let $F(x, \rho)$ be the non-conformal harmonic extension of $\mathcal{L}^s u$, i.e. $(\partial_{\rho\rho} - \mathcal{L})F = 0$, with $F(x, 0) = c_n\mathcal{L}^s u(x)$. Then*

$$(\partial_{\rho\rho} - \mathcal{L})|\nabla F|^2 + 2|\nabla F|^2 \geq 0.$$

Proof. Denote

$$f := |\nabla F|^2 = \sum_{i=1}^n \big((X_iF)^2 + (Y_iF)^2\big) + (TF)^2 + (\partial_\rho F)^2.$$

We also have the following commuting relations

$$[X_j, Y_j] = -T, \qquad [X_j^2, Y_j] = 2X_jT, \qquad [Y_j^2, X_j] = -2Y_jT \tag{30}$$

and all other brackets vanish. We have that

$$X_jf = 2\sum_{i=1}^{n}(X_iF)(X_jX_iF) + 2\sum_{i=1}^{n}(Y_iF)(X_jY_iF) + 2(TF)(X_jTF) + 2(\partial_\rho F)(X_j\partial_\rho F)$$

and

$$\begin{aligned} X_j^2f = 2\sum_{i=1}^{n}\big((X_jX_iF)^2 + (X_iF)(X_j^2X_iF)\big) + 2\sum_{i=1}^{n}\big((X_jY_iF)^2 + (Y_iF)(X_j^2Y_iF)\big)\\ + 2(X_jTF)^2 + 2(TF)(X_j^2TF) + 2(X_j\partial_\rho F)^2 + 2(\partial_\rho F)(X_j^2\partial_\rho F). \end{aligned}$$

Analogously,

$$\begin{aligned} Y_j^2f = 2\sum_{i=1}^{n}\big((Y_jX_iF)^2 + (X_iF)(Y_j^2X_iF)\big) + 2\sum_{i=1}^{n}\big((Y_jY_iF)^2 + (Y_iF)(Y_j^2Y_iF)\big)\\ + 2(Y_jTF)^2 + 2(TF)(Y_j^2TF) + 2(Y_j\partial_\rho F)^2 + 2(\partial_\rho F)(Y_j^2\partial_\rho F) \end{aligned}$$

and, finally

$$\begin{aligned} \partial_{\rho\rho}f = 2\sum_{i=1}^{n}\big((\partial_\rho X_iF)^2 + (X_iF)(\partial_{\rho\rho}X_iF)\big) + 2\sum_{i=1}^{n}\big((\partial_\rho Y_iF)^2 + (Y_iF)(\partial_{\rho\rho}Y_iF)\big)\\ + 2(\partial_\rho TF)^2 + 2(TF)(\partial_{\rho\rho}TF) + 2(\partial_{\rho\rho}F)^2 + 2(\partial_\rho F)(\partial_{\rho\rho\rho}F). \end{aligned}$$

Adding all the terms in j, and taking the commutation relations into account, we have

$$
\begin{aligned}
&(\partial_{\rho\rho} - \mathcal{L})|\nabla F|^2 \\
&= \underbrace{2\sum_{j=1}^{n}\sum_{i=1}^{n}\Big((X_jX_iF)^2 + (Y_jX_iF)^2 + (X_jY_iF)^2 + (Y_jY_iF)^2 + (X_j\partial_\rho F)^2 + (Y_j\partial_\rho F)^2\Big)}_{I} \\
&\quad \underbrace{+ 2\sum_{i=1}^{n}\Big((\partial_\rho X_iF)^2 + (\partial_\rho Y_iF)^2\Big)}_{II} \\
&\quad \underbrace{+ 2\sum_{j=1}^{n}\Big((X_jTF)^2 + (Y_jTF)^2 + (\partial_\rho TF)^2 + (X_j\partial_\rho F)^2 + (Y_j\partial_\rho F)^2 + (\partial_{\rho\rho}F)^2\Big)}_{III} \\
&\quad \underbrace{+ 2\sum_{i=1}^{n}(X_iF)\Big[\sum_{j=1}^{n}\Big((X_iX_j^2F) + (X_iY_j^2F)\Big) + (X_i\partial_{\rho\rho}F)\Big]}_{IV} \\
&\quad \underbrace{+ 2\sum_{i=1}^{n}(Y_iF)\Big[\sum_{j=1}^{n}\Big((Y_iX_j^2F) + (Y_iY_j^2F)\Big) + (Y_i\partial_{\rho\rho}F)\Big]}_{V} \\
&\quad \underbrace{+ 2(TF)\Big[\sum_{j=1}^{n}\Big((TX_j^2F) + (TY_j^2F)\Big) + (T\partial_{\rho\rho}F)\Big]}_{VI} \\
&\quad \underbrace{+ 2(\partial_\rho F)\Big[\sum_{j=1}^{n}\Big((\partial_\rho X_j^2F) + (\partial_\rho Y_j^2F)\Big) + (\partial_\rho\partial_{\rho\rho}F)\Big]}_{VII} \\
&\quad \underbrace{+ 2\sum_{j=1}^{n}\Big((X_jTF)^2 + (Y_jTF)^2 + (\partial_\rho TF)^2\Big) + 4\sum_{j=1}^{n}\Big((X_jF)(Y_jTF) - (Y_jF)(X_jTF)\Big)}_{VIII}.
\end{aligned}
$$

First notice that $I+II+III \geq 0$. Moreover, since F is harmonic, that is $\big(\partial_{\rho\rho}+\sum_{j=1}^{n}(X_j^2+Y_j^2)\big)F = 0$, we also have that $IV+V+VI+VII = 0$. Therefore, by the elementary inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ we can easily estimate

$$
\begin{aligned}
&(\partial_{\rho\rho} - \mathcal{L})|\nabla F|^2 \geq VIII \\
&\geq 2\sum_{j=1}^{n}\Big((X_jTF)^2 + (Y_jTF)^2 + (\partial_\rho TF)^2\Big) - 2\sum_{j=1}^{n}\Big((X_jF)^2 + (Y_jTF)^2 + (Y_jF)^2 + (X_jTF)^2\Big) \\
&\geq -2\sum_{j=1}^{n}\Big((X_jF)^2 + (Y_jF)^2\Big) \geq -2|\nabla F|^2,
\end{aligned}
$$

and this completes the proof of the lemma. □

We can now start with the proof of Theorem 1.3. Lemma 4.1 and a change of variables yield

$$|u(y)-u(0)| \leq \int_0^{|y|}\Big(\int_\lambda^\infty \big(|\nabla F(y,\mu)|+|\nabla F(0,\mu)|+|\nabla F(\lambda\widehat{y},\mu+|y|-\lambda)|\big)(\mu-\lambda)^{2s-1}\,d\mu\Big)\,d\lambda$$
$$=: I+II+III+IV,$$

where

$$I := \int_0^{|y|}\int_\lambda^{|y|} |\nabla F(y,\mu)|(\mu-\lambda)^{2s-1}\,d\mu\,d\lambda, \qquad II := \int_0^{|y|}\int_\lambda^{|y|} |\nabla F(0,\mu)|(\mu-\lambda)^{2s-1}\,d\mu\,d\lambda,$$
$$III := \int_0^{|y|}\int_\lambda^{|y|} |\nabla F(\lambda\widehat{y},\mu+|y|-\lambda)|(\mu-\lambda)^{2s-1}\,d\mu\,d\lambda,$$
$$IV := \int_0^{|y|}\int_{|y|}^\infty \big(|\nabla F(y,\mu)|+|\nabla F(0,\mu)|+|\nabla F(\lambda\widehat{y},\mu+|y|-\lambda)|\big)(\mu-\lambda)^{2s-1}\,d\mu\big)\,d\lambda.$$

Estimate of IV. Observe that, for $A>0$ and $2s<A<1$,

$$IV \lesssim \sup_{|x|\leq|y|<\rho} |\nabla F(x,\rho)|\rho^A \int_0^{|y|}\int_{|y|}^\infty \mu^{-A}(\mu-\lambda)^{2s-1}\,d\mu\,d\lambda \lesssim \sup_{|x|\leq|y|<\rho} |\nabla F(x,\rho)|\rho^A|y|^{1+2s-A}. \tag{31}$$

In view of Lemma 4.2 we have, by Lemma 3.5 with $p=1$,

$$|\nabla F|^2 \leq \frac{C}{|B(x,\rho)|}\int_{B_\rho(x,\rho)} |\nabla F|^2(y)\,dy \tag{32}$$

Moreover, we have that

$$D(x,\rho) := \big\{(\xi,\tau)\in\mathbb{H}^n\times\mathbb{R}^+ : |\xi-x|<\frac{\rho}{2}, |\tau-\rho|<\frac{\rho}{2}\big\}$$

has equivalent measure to $B_{\rho/2}(x,\rho)$.

Then, from (31) and (32), we get

$$(IV)^2 \lesssim \sup_{|x|\leq|y|<\rho} \rho^{2A}|y|^{2+4s-2A}\int_D |\nabla F(\xi,\tau)|^2\,d\xi\,d\tau\frac{1}{\rho^{1+Q}}.$$

Let

$$C := \{(\xi,\tau)\in\mathbb{H}^n\times\mathbb{R}^+ : |\xi|\leq 3\tau\}.$$

Notice that $D\subset\{(\xi,\tau)\in C : \tau\geq |y|/2\}$ and in D we have that $\rho/2\leq\tau\leq 3/2\rho$, see Figure 1, thus

$$(IV)^2 \lesssim |y|^{2+4s-2A}\int_{C,\tau\geq|y|/2} |\nabla F(\xi,\tau)|^2\tau^{2A-Q-1}\,d\xi\,d\tau.$$

We divide by $|y|^{Q+4s}$ and integrate in y, to obtain

$$\int_{\mathbb{H}^n}\frac{(IV)^2}{|y|^{Q+4s}}\,dy \lesssim \int_C\Big(\int_{|y|\leq 2\tau}|y|^{2-Q-2A}\,dy\Big)|\nabla F(\xi,\tau)|^2\tau^{2A-Q-1}\,d\xi\,d\tau \lesssim \int_C |\nabla F(\xi,\tau)|^2\tau^{1-Q}\,d\xi\,d\tau. \tag{33}$$

Estimate of III. In this case, $0\leq\lambda\leq\mu\leq|y|$, so that $\mu+|y|-\lambda\geq\lambda+|y|-\lambda=|y|\geq\lambda=|\lambda\widehat{y}|$, and then

$$III \lesssim \sup_{|x|\leq|y|\leq\rho}\rho^A|\nabla F(x,\rho)|\int_0^{|y|}\int_\lambda^{|y|}(|y|+\mu-\lambda)^{-A}(\mu-\lambda)^{2s-1}\,d\xi\,d\lambda$$
$$\lesssim \sup_{|x|\leq|y|\leq\rho}\rho^A|\nabla F(x,\rho)||y|^{-A+1+2s}.$$

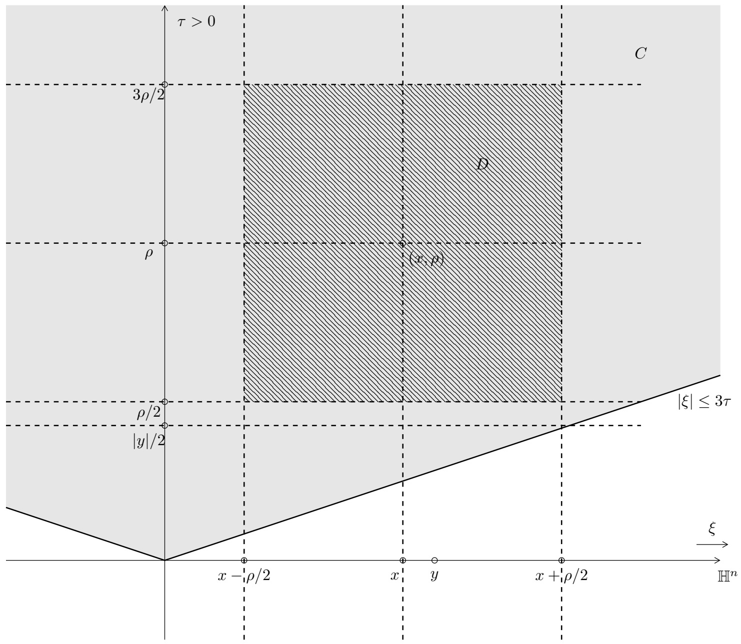

FIGURE 1. The sets D and C.

Hence, reasoning as in the case of IV, we conclude that III is also bounded by the right-hand side of (33), namely

$$\int_{\mathbb{H}^n} \frac{(III)^2}{|y|^{Q+4s}}\, dy \lesssim \int_C |\nabla F(\xi,\tau)|^2 \tau^{1-Q}\, d\xi\, d\tau. \tag{34}$$

Estimate of II. Let $0 < \varepsilon < 4s$. By Fubini and Cauchy–Schwarz, we have

$$\begin{aligned} II &= \int_0^{|y|} \Big(\int_0^{\mu} |\nabla F(0,\mu)|(\mu-\lambda)^{2s-1}\, d\lambda \Big)\, d\mu \\ &\simeq \int_0^{|y|} |\nabla F(0,\mu)| \mu^2 s\, d\mu \\ &\le \Big(\int_0^{|y|} |\nabla F(0,\mu)|^2 \mu^{1+4s-\varepsilon}\, d\mu \Big)^{1/2} \Big(\int_0^{|y|} \mu^{\varepsilon-1}\, d\mu \Big)^{1/2} \\ &\lesssim \Big(\int_0^{|y|} |\nabla F(0,\mu)|^2 \mu^{1+4s-\varepsilon}\, d\mu \Big)^{1/2} |y|^{\varepsilon/2}. \end{aligned}$$

From here,

$$(II)^2 \lesssim \int_0^{|y|} |\nabla F(0,\mu)|^2 \mu^{1+4s-\varepsilon}\, d\mu |y|^{\varepsilon},$$

thus, applying Fubini

$$\begin{aligned} \int_{\mathbb{H}^n} \frac{(II)^2}{|y|^{Q+4s}}\, dy &\lesssim \int_{\mathbb{H}^n} |y|^{\varepsilon-Q-4s} \Big(\int_0^{|y|} |\nabla F(0,\mu)|^2 \mu^{1+4s-\varepsilon}\, d\mu \Big)\, dy \\ &= \int_0^{\infty} \Big(\int_{|y|\ge\mu} |y|^{\varepsilon-Q-4s}\, dy \Big) |\nabla F(0,\mu)|^2 \mu^{1+4s-\varepsilon}\, d\mu \\ &\lesssim \int_0^{\infty} |\nabla F(0,\mu)|^2 \mu\, d\mu. \end{aligned}$$

Then by Lemma 3.5 with $p = 1$ we obtain

$$\int_{\mathbb{H}^n} \frac{(II)^2}{|y|^{Q+4s}}\,dy \lesssim \int_0^\infty \mu\mu^{-1-Q} \int_E |\nabla F(\xi,\tau)|^2\,d\xi\,d\tau\,d\mu$$

with $E := \{(\xi,\tau) : |\xi| \le \mu/2, |\tau-\mu| \le \mu/2\}$. Notice that, in E, $\mu/2 \le \tau \le 3/2\mu$, so $2/3\tau \le \mu \le 2\tau$ and $E \subset C$, where C is the cone defined above. By Fubini,

$$\int_{\mathbb{H}^n} \frac{(II)^2}{|y|^{Q+4s}}\,dy \lesssim \int_C |\nabla F(\xi,\tau)|^2 \Big(\int_{2/3\tau}^{2\tau} \mu^{-Q}\,d\mu\Big)\,d\xi\,d\tau \lesssim \int_C |\nabla F(\xi,\tau)|^2 \tau^{1-Q}\,d\xi\,d\tau. \tag{35}$$

Estimate of I. Let $0<\varepsilon<4s$. By Fubini and Cauchy–Schwarz, we get

$$\begin{aligned} I = \int_0^{|y|} \Big(\int_0^\mu |\nabla F(y,\mu)|(\mu-\lambda)^{2s-1}\,d\lambda\Big)\,d\mu &\simeq \int_0^{|y|} |\nabla F(y,\mu)|\mu^{2s}\,d\mu \\ &\lesssim \Big(\int_0^{|y|} |\nabla F(y,\mu)|^2 \mu^{1+4s-\varepsilon}\,d\mu\Big)^{1/2} |y|^{\varepsilon/2}. \end{aligned}$$

Therefore, by using Fubini again

$$\begin{aligned} \int_{\mathbb{H}^n} \frac{(I)^2}{|y|^{Q+4s}}\,dy &\lesssim \int_{\mathbb{H}^n} |y|^{\varepsilon-Q-4s} \Big(\int_0^{|y|} |\nabla F(y,\mu)|^2 \mu^{1+4s-\varepsilon}\,d\mu\Big)\,dy \\ &= \int_0^\infty \Big(\int_{|y|\ge\mu} |y|^{\varepsilon-Q-4s} |\nabla F(y,\mu)|^2\,dy\Big) \mu^{1+4s-\varepsilon}\,d\mu. \end{aligned}$$

Hence,

$$\int_{\mathbb{H}^n} \frac{(I)^2}{|y|^{Q+4s}}\,dy \lesssim \int_{|y|\ge\mu} |\nabla F(y,\mu)|^2 \frac{\mu^{1+4s-\varepsilon}}{|y|^{Q+4s-\varepsilon}}\,dy\,d\mu. \tag{36}$$

Gathering (33), (34), (35), and (36), we have that, for $\varepsilon \in (0,4s)$,

$$\int_{\mathbb{H}^n} \frac{|u(y)-u(0)|^2}{|y|^{Q+4s}}\,dy \lesssim \int_C |\nabla F(\xi,\tau)|^2 \tau^{1-Q}\,d\xi\,d\tau + \int_{|y|\le\mu} |\nabla F(y,\mu)|^2 \frac{\mu^{1+4s-\varepsilon}}{|y|^{Q+4s-\varepsilon}}\,dy\,d\mu. \tag{37}$$

Recalling the definition of g_λ^* in (24), then

$$g_\lambda^*(\mathcal{L}^s u)(0)^2 = \int_0^\infty \int_{\mathbb{H}^n} \Big(\frac{\tau}{\tau+|\xi|}\Big)^{\lambda Q} \tau^{1-Q} |\nabla F(\xi,\tau)|^2\,d\tau.$$

It is easy to see that the first integral in the right-hand side of (37) can be estimated by $g_\lambda^*(\mathcal{L}^s u)(0)^2$ for any λ. For the second one, observe that the restriction $\mu \le |y|$ implies that $\frac{1}{|y|} \le \frac{2}{\mu+|y|}$, therefore

$$\frac{\mu^{1+4s-\varepsilon}}{|y|^{Q+4s-\varepsilon}} \lesssim \frac{\mu^{1+4s-\varepsilon}}{(|y|+\mu)^{Q+4s-\varepsilon}}.$$

Then the second term is bounded by $g_\lambda^*(\mathcal{L}^s u)(0)^2$ with $\lambda = \frac{1}{Q}(Q+4s-\varepsilon)$. Since ε is arbitrarily small, the proof is finished.

5. Proof of Theorem 1.1

Let us denote by $x = (z,t)$. The pointwise representation (20) implies the identity

$$\mathcal{L}_s(uv) - u\mathcal{L}_s v - v\mathcal{L}_s u = b(n,s) T_s(u,v), \quad 0<s<1/2,$$

where $b(n,s)$ is the constant in (21) and $T_s(u,v)$ is the bilinear form

$$T_s(u,v)(x) = \int_{\mathbb{H}^n} \frac{[(u(xy^{-1})-u(x))(v(xy^{-1})-v(x))]}{|y|^{Q+2s}}\,dy, \qquad x \in \mathbb{H}^n, \quad 0<s<1/2.$$

Then, in order to prove Theorem 1.1, we are reduced to show that, for all $u, v \in \mathcal{S}(\mathbb{H}^n)$ we have

$$\|T_s(u,v)\|_{L^p} \lesssim \|\mathcal{L}_{s_1} u\|_{L^{p_1}} \|\mathcal{L}_{s_2} v\|_{L^{p_2}}.$$

Recall the definition of the square fractional integral

$$\mathcal{D}_s u(x) := \Big(\int_{\mathbb{H}^n} \frac{|u(xy^{-1}) - u(x)|^2}{|y|^{Q+4s}} \, dy \Big)^{1/2}, \quad 0 < s < 1/2.$$

Observe that, by Cauchy–Schwarz we have the poinwise estimate

(38)
$$|T_s(u,v)(x)| \le \mathcal{D}_{s_1} u(x) \mathcal{D}_{s_2} v(x), \qquad x \in \mathbb{H}^n, \qquad s = s_1 + s_2, \qquad s \in (0, 1/2) \qquad s_j \in (0, 1/2).$$

In view of (38), Hölder's inequality and Theorem 1.3 yield

$$\|T_s(u,v)\|_{L^p} \lesssim \|\mathcal{D}_{s_1} u\|_{L^{p_1}} \|\mathcal{D}_{s_2} u\|_{L^{p_2}} \le \Lambda(n, s_1)\Lambda(n, s_2) \|g^*_{\lambda_1}(\mathcal{L}^{s_1} u)\|_{L^{p_1}} \|g^*_{\lambda_2}(\mathcal{L}^{s_2} u)\|_{L^{p_2}},$$

for any $p, p_1, p_2 \in (0, \infty]$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, any $s_j \in (0, 1/2)$, $s_1 + s_2 = s < 1/2$ and any $\lambda_j < 1 + \frac{4s_j}{Q}$. By Theorem 3.3 and (15) we conclude (5), namely

$$\|T_s(u,v)\|_{L^p} \lesssim \|\mathcal{D}_{s_1} u\|_{L^{p_1}} \|\mathcal{D}_{s_2} u\|_{L^{p_2}} \lesssim \Lambda(n, s_1)\Lambda(n, s_2) \|\mathcal{L}_{s_1} u\|_{L^{p_1}} \|\mathcal{L}_{s_2} u\|_{L^{p_2}}$$

provided λ_j is such that $\max\big\{1, \frac{2}{p_j}\big\} < \lambda_j < 1 + \frac{4s_j}{Q}$.

The weighted estimate (6) is proved similarly. First, instead of Theorem 3.3, we use Theorem 3.4, which imposes the conditions $w_j \in A_{q_j}$, for $1 < q_j < \min\big\{p_j, p_j\big(\frac{1}{2} + \frac{2s_j}{Q}\big)\big\}$. Nevertheless, the self-improving property of Muckenhoupt weights (i.e., $A_q \subset A_p$, $1 \le q < p$, see [19, Proposition 7.2 (1)]) allows to relax the condition into $w_j \in A_{q_j}$, for $1 < q_j = \min\big\{p_j, p_j\big(\frac{1}{2} + \frac{2s_j}{Q}\big)\big\}$. Since we always have $2s_j/Q < 1/2$, we obtain that $w_j \in A_{q_j}$, for $1 < q_j = p_j\big(\frac{1}{2} + \frac{2s_j}{Q}\big)$, as desired. The conclusion follows by using (16).

References

[1] H. Aimar and R. A. Macías, Weighted norm inequalities for the Hardy-Littlewood maximal operator on spaces of homogeneous type, *Proc. Amer. Math. Soc.* **91** (1984), 213–216.

[2] J.–P. Anker, E. Damek, and C. Yacoub, Spherical analysis on harmonic *AN* groups, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **23** (1996), 643–679.

[3] S. Bagchi, Fourier multipliers on the Heisenberg group revisited, *Studia Math.* **260** (2021), 241–272.

[4] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality, *Ann. of Math. (2)* **138** (1993), 213–242.

[5] F. Bernicot, D. Frey, and S. Petermichl, Sharp weighted norm estimates beyond Calderón–Zygmund theory, *Anal. PDE* **9** (2016), 1079–1113.

[6] A. Bonfiglioli and E. Lanconelli, Liouville-type theorems for real sub-Laplacians, *Manuscripta Math.*, **105** (2001), 111–124.

[7] A. Bonfiglioli and E. Lanconelli, Subharmonic functions on Carnot groups, *Math. Ann.*, **325** (2003), 97–122.

[8] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics. Springer, Berlin, 2007.

[9] T. P. Branson, L. Fontana, and C. Morpurgo, Moser–Trudinger and Beckner–Onofri's inequalities on the CR sphere, *Ann. Math. (2)*, **177** (2013), 1–52.

[10] L. A. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007), 1245–1260.

[11] S-Y. A. Chang and M.d.M. González, Fractional Laplacian in conformal geometry, *Adv. Math.* **226** (2011), no. 2, 1410–1432.

[12] F. Chiarenza, E. Fabes, and N. Garofalo, Harnack's inequality for Schrödinger operators and the continuity of solutions, *Proc. Amer. Math. Soc.* **98** (1986), no. 3, 415–425.

[13] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, *Colloq. Math.* **60/61** (1990), no.2, 601–628.

[14] Ó. Ciaurri and L. Roncal, Littlewood-Paley-Stein g_k-functions for Fourier-Bessel expansions, *J. Funct. Anal.* **258** (2010), 2173–2204.

[15] G. Citti, N. Garofalo, and E. Lanconelli, Harnack's inequality for sums of squares of vector fields plus a potential, *Amer. J. Math.* **115** (1993), no. 3, 699–734.

[16] R. Coifman and G. Weiss, Analyse harmonique non-commutative sur certains espaces homogènes, Lecture Notes in Math. **242**. Springer-Verlag, Berlin-New York, 1971.

[17] M. Cowling and U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one, *Invent. Math.* **96** (1989), 507–549.

[18] P. D'Ancona, A short proof of commutator estimates, *J. Fourier Anal. Appl.* **25** (2019), no.3, 1134–1146.

[19] J. Duoandikoetxea, *Fourier analysis. Translated and revised from the 1995 Spanish original by David Cruz-Uribe*. Graduate Studies in Mathematics, 29. American Mathematical Society, Providence, RI, 2001.

[20] E. B. Fabes and D. W. Stroock, The L^p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations, *Duke Math. J.* **51** (1984), 997–1016.

[21] R. L. Frank and E. H. Lieb, Sharp constants in several inequalities on the Heisenberg group, *Ann. Math. (2)* **176** (2012), 349–381.

[22] G. B. Folland, A fundamental solution for a subelliptic operator, *Bull. Amer. Math. Soc.* **79** (1973), 373–376.

[23] G. B. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups *Ark. Mat.* **13** (1975), 161–207.

[24] G. B. Folland, *Harmonic Analysis in Phase Space*, Ann. Math. Stud. **122**. Princeton University Press, Princeton, N.J., 1989.

[25] R. L. Frank, M.d.M. González, D. D. Monticelli and J. Tan, An extension problem for the CR fractional Laplacian, *Adv. Math.* **270** (2015), 97–137.

[26] N. Garofalo and G. Tralli, Feeling the heat in a group of Heisenberg type, *Adv. Math.* **381** (2021), 107635, 42pp.

[27] D. Geller, Fourier analysis on the Heisenberg group. I. Schwartz space, *J. Functional Analysis* **36** (1980), 205–254.

[28] L. Grafakos, L. Liu, and D. Yang, Vector–valued singular integrals and maximal functions on spaces of homogeneous type, *Math. Scand.* **104** (2009), 296–310.

[29] C. R. Graham and M. Zworski, Scattering matrix in conformal geometry, *Invent. Math.* **152** (2003), 89–118.

[30] L. Hörmander, Hypoelliptic second order differential equations, *Acta Math.* **119** (1967), 147–171.

[31] A. Huber, On the uniqueness of generalized axially symmetric potentials, *Ann. of Math. (2)* **60** (1954), 351–358.

[32] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Commun. Pure Appl. Math.* **41** (1988), 891–907.

[33] C. E. Kenig, G. Ponce, and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, *Commun. Pure Appl. Math.* **46** (1993), 527–620.

[34] T. Kolsrud, Traces of harmonic functions, capacities, and traces of symmetric Markov processes, *J. Theoret. Probab.* **2** (1989), 399–418.

[35] E. Lenzmann and A. Schikorra, Sharp commutator estimates via harmonic extensions, *Nonlinear Anal.* **193** (2020), 111375, 37pp.

[36] A. Maalaoui, A note on commutators of the fractional sub-Laplacian on Carnot groups, *Commun. Pure Appl. Anal.* **18** (2019), 435–453.

[37] A. Maalaoui, A. Pinamonti, and G. Speight, Function spaces via fractional Poisson kernel on Carnot groups and applications, arXiv:2005.06363v1.

[38] S. A. Molchanov and E. Ostrovskii, Symmetric stable processes as traces of degenerate diffusion processes, *Theor. Probability Appl.* **14** (1969), 128–131.

[39] B. Muckenhoupt and E. M. Stein, Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.* **118** (1965), 17–92.

[40] B. Muckenhoupt and R. L. Wheeden, Norm inequalities for the Littlewood-Paley function g_λ^*, *Trans. Amer. Math. Soc.* **191** (1974), 95–111.

[41] D. Müller and E. M. Stein, On spectral multipliers for Heisenberg and related groups, *J. Math. Pures Appl.* **73** (1994), 413–440.

[42] A. Nagel, E. M. Stein, and S. Wainger, Balls and metrics defined by vector fields. I. Basic properties, *Acta Math.* **155** (1985), 103–147.

[43] F. W. J. Olver and L. C. Maximon, Bessel Functions, *NIST handbook of mathematical functions* (edited by F. W. F. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark), Chapter 5, National Institute of Standards and Technology, Washington, DC, and Cambridge University Press, Cambridge, 2010. Available online in `http://dlmf.nist.gov/5`.

[44] L. Roncal and S. Thangavelu, Hardy's inequality for fractional powers of the sublaplacian on the Heisenberg group, *Adv. Math.* **302** (2016), 106–158.

[45] L. Roncal and S. Thangavelu, An extension problem and trace Hardy inequality for the sublaplacian on H-type groups, *Int. Math. Res. Not. IMRN* **14** (2020), 4238–4294.

[46] F. J. Ruiz and J. L. Torrea, Vector-valued Calderón–Zygmund theory and Carleson measure on spaces of homogeneous nature, *Studia Math.* **88** (1988), 221–243.

[47] A. Sánchez–Calle, Fundamental solutions and geometry of sum of squares of vector fields, *Inv. Math.* **78** (1984), 143–160.
[48] F. Spitzer, Some theorems concerning 2-dimensional Brownian motions, *Trans. Amer. Math. Soc.* **87** (1958), 187–197.
[49] E. M. Stein, Some problems in harmonic analysis suggested by symmetric spaces and semi-simple groups, *Actes, Congrès Intern. Math.*, Nice, **1** (1970), 179–189.
[50] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series **30**. Princeton University Press, Princeton (1970).
[51] P. R. Stinga and J. L. Torrea, Extension problem and Harnack's inequality for some fractional operators, *Comm. Partial Differential Equations* **35** (2010), 2092–2122.
[52] S. Thangavelu, *Harmonic Analysis on the Heisenberg Group*, Progress in Mathematics **159**. Birkhäuser, Boston, MA, 1998.
[53] S. Thangavelu, *An introduction to the uncertainty principle. Hardy's theorem on Lie groups. With a foreword by Gerald B. Folland*, Progress in Mathematics **217**. Birkhäuser, Boston, MA, 2004.

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