Suggestions for Interdisciplinary Teaching in Mathematics Education: The Case of the History of the Concept of Group

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ABSTRACT
The role of the history of mathematics in mathematics education, as a support in teaching, is widely acknowledged. It can serve to arouse students' interest in mathematics as well as to stimulate reflection on mathematical concepts and methods. This work focuses on the group as "cross-concept", in relation to its history from its origins in algebra to its applications in other areas. Given its very nature, the group should be adequately re-evaluated in an interdisciplinary approach, which may prove particularly captivating in linking the history of mathematics to the history of the psychology of perception. This historical interdisciplinary perspective may be useful in the context of university mathematics teaching, particularly teacher training.
Keywords: education, mathematics education, history of mathematics, history of psychology of perception, group, teacher training

[Hereinafter, the translations from the original texts are made by the authors of this paper.]

1. INTRODUCTION

Promoting historical reflection in mathematics, apart from being remarkable in its own right, takes on particular significance when one considers its possible effects on the improvement of present-day mathematics education, particularly in the field of mathematics teacher education and training. Regarding the interaction of the domains of mathematics education and the history of mathematics in the process of mathematics teaching, see, e.g., Furinghetti & Radford (2008), Fenaroli, Furinghetti & Somaglia (2014), Furinghetti (2020).

In this paper, we focus on the concept of group as a "cross-concept" (Zudini, Antonelli & Stucchi, 2011), proposing an analysis in reference to its history from its origins in algebra to its applications in other areas, especially the psychology of perception. This is done by linking the history of mathematics and the history of the psychology of perception, following a boundary-crossing approach, which makes it possible to see mathematics from a "new" point of view and in a "new" working environment: a "new" perspective, different from traditional approaches, which is particularly captivating and engaging, insofar as it concerns our human nature as perceiving beings.

The present work seeks to combine, on the one hand, the use in teaching of history as a motivating factor for learning, arousing the students' interest in mathematics as well as stimulating the reflection on mathematical concepts and methods; on the other hand, the interdisciplinary approach, in particular, the connection between the history of mathematics and the history of the psychology of perception, which is a particularly suitable example to show that mathematics is indeed useful. In other words, reasoning in mathematical terms even in areas other than mathematics is not wrong; on the contrary, it is potentially fruitful for understanding ourselves and our ability to perceive the world around us. The whole thing could be, for a (future) teacher who wants to teach mathematics and science in the secondary school, a link to a classroom discussion of the sense organs.

We have chosen, specifically, to expound the topic in a simple and perceptively effective manner, according to presentations already made by the authors of the paper recently, on various occasions (a talk addressed to an audience of historians of mathematics, in one case, and a seminar for a group of school mathematics teachers, in the other). This is to "see", with examples, objects and phenomena of perceptual experience.

The treatment proposed here is in line with the work of one of the authors, in which mathematics approaches psychology and cognitive science, in terms of a historical-epistemological study, with spin-offs at the level of university mathematics teaching, particularly teacher training (see, e.g., Zudini, 2014a, 2014b, 2018; Zudini & Zuccheri, 2016; Caprin & Zudini, 2015; Morganti & Zudini, 2021).

2. THE HISTORY OF A "CROSS-CONCEPT"

Group theory is one of the branches of mathematics that has proved to be one of the richest in developments throughout history, from its origins in relation to the problem of finding formulas for solutions of algebraic equations, to its first models ("groups of permutations"), to its extension in view of the classification of various geometries - which became necessary after the advent of non-Euclidean geometries -, to its various applications in the most diverse areas. The group, as a "cross-concept" (Zudini, Antonelli & Stucchi, 2011), has, in fact, found application in the most varied disciplines, from physics to the natural sciences, from the visual arts to music.

The concept of the group was introduced by the French mathematician Évariste Galois (1811-1832), in connection with the problem of finding formulas for solutions of algebraic equations (see Toti Rigatelli, 1989). Galois had the idea of associating each equation with a group, the group of permutations of its roots, and of linking the solvability of the equation by radicals to the nature of this group, obtaining the result that an algebraic equation, with real or complex coefficients, of any degree n, is solvable by radicals when and only when the group associated with it (which will later be called the "Galois group") enjoys the property of being, as we say today, "solvable".

The theory, developed by Galois at a young age, between eighteen and twenty, and known today as "Galois theory", led to a genuine revolution in mathematics and was fundamental to the birth of a "new algebra".

The concept of group was later studied by the Norwegian Sophus Lie (1842-1899) (also in collaboration with the German Felix Klein). Lie indicated a new direction in the study of group theory, aiming to do for differential equations what Galois had done for algebraic ones; he thus came to conceive, in the context of multidimensional geometry, the so-called "finite continuous groups" ("Lie groups"). The "Lie theory" would, in turn, find important applications during the 20th century, particularly in quantum physics.

Felix Klein (1849-1925) put the concept of group at the base of the unifying vision (and systematic organisation) proposed for geometry in the "Erlanger Programm" (1872) (see Rowe, 1985; see also Hawkins, 1984, Ihmig, 1999, and, for further references on Klein's figure and work, Zudini, Antonelli & Stucchi, 2011).

Klein used this to show that every geometry can be characterised by a group of transformations and that the true object of geometry are the invariant properties with respect to this group of transformations (and thus no longer the geometric space derived from the observation of reality). Every geometry is nothing other than the study of the invariant properties with respect to a group of transformations from the space to itself.

If we consider the following groups of transformations in the plane (not all examined by Klein):

G_m (group of rigid motions or congruences)
G_s (group of similarities)
G_a (group of affinities)
G_p (group of projectivities)
G_t (group of topological transformations),

the following chain of inclusions holds:

$$G_m \subset G_s \subset G_a \subset G_p \subset G_t$$

The necessity of the group structure arises from the need to define, from time to time, an equivalence relation between the figures in the geometry considered.
In general, the following criterion of equivalence applies:

Given a group of transformations, G, two figures A and B are G-equivalent when there exists a transformation f belonging to G such that f(A) = B.

In G-geometry, only concepts that are invariant due to transformations of G are admissible. If, for example, we compare Euclidean geometry and affine geometry, we see that the group of rigid motions or congruences is contained in the group of affinities; therefore, everything that is invariant in the latter group of transformations is also invariant in the former.

On the basis of the concept of the group of transformations, we thus move in Klein's system from an ontological problem of truth to a consideration of (ideal) spaces according to their reciprocal relations.

This problem is clearly detached from the question of the true empirical space, which must therefore be decided on in accordance with other criteria, and not on the basis of an intrinsic consistency of geometry.

The criterion underlying ideal geometry is of logical origin, different from that of true physical space.

3. THE MATHEMATICAL GROUP THEORY

The study of groups falls within the mathematical field of algebra, which deals with so-called "algebraic systems" or "algebraic structures", understood, in general, as sets of objects that combine by means of operations.

The group, as a structure, is a fundamental constituent part of the discipline now called "abstract algebra", which also includes other structures such as rings, fields, etc.

In every algebraic treatise, one traditionally starts with groups for natural and profound reasons: firstly, groups, being structures with only one operation (or law of composition), can be described formally in a very simple way; on the other hand, despite this simplicity of description, they are characterised by the same fundamental algebraic concepts, such as homomorphism, quotient, etc., that one encounters in all algebraic structures and in all mathematics and that appear in group theory in an already clear and defined form.

The algebraic systems considered worthy of analysis are those which had been treated in numerous particular cases, until it was seen what the general phenomenon that included them was: as far as groups were concerned, between the end of the 19th and the beginning of the 20th century they were known through particular examples, and only later, during the 20th century, the notion of "abstract group" was given.

In mathematics, a non-empty set of elements G is called a "group" if an operation (or law of composition) is defined in G, whereby the composition of two elements of G is also an element of G (*closure property*), the operation is associative (*associative property*), there exists in G (and is unique) the identity or neutral element (*existence - and uniqueness - of the identity or neutral element in G*), and for each element of G there exists in G (and is unique) the inverse element (*existence - and uniqueness - of the inverse element in G*). A "subgroup" of a group G is a non-empty subset of G, for example H, which is itself a group with respect to the same operation as G. For a more detailed discussion of the mathematical theory of groups, see, for example, Herstein (1975).

4. GROUP THEORY AND THE PSYCHOLOGY OF PERCEPTION

The concept of group has found application in the most diverse disciplines: from physics to the natural sciences, from the visual arts to music.

In particular, psychology has used it to represent its objects and the structures that bind them. The areas of use have been and are diverse (see Zudini, Antonelli & Stucchi, 2011): in social psychology, the concept of subgroup (derived from that of group) has been used in the representation of "social networks", as part of the so-called "relational algebra" or "algebra of social networks" (see Pattison, 1993; Wassermann & Faust, 1994); in the theory of cognitive development developed by Jean Piaget (1896-1980), with outcomes also in the "modern mathematics" movement (see, e.g., Bolondi, 2007), reference is made, in relation to the domain of "propositional operations", to the concept of the "INRC group" (acronym obtained from the words "identity", "negation", "reciprocity", "correlativity"), understood as a system of possible forms of thought mobility of which a subject proves to be capable in performing a task (see Piaget, 1972; see also Ascher, 1984); in the psychology of perception, group theory is used to deal with invariants (e.g., perceptual constants).

In general, one speaks of a "transformational approach to vision" for all areas of the psychology of perception where the algebraic concept of the group of transformations is used, for problems that may specifically concern both the descriptive psychology of perception (understood as the study of the results of perceptual activity) and the genetic one (i.e., the study of the processes of formation of the results of perceptual activity).

5. PERCEPTUAL CONSTANTS

Perceptual constants (of shape, size, and colour) can be interpreted as an invariance with respect to the group of transformations that leave metric properties invariant, i.e., rigid motions in Euclidean three-dimensional space.

Cesare L. Musatti (1897-1989), a psychologist and psychoanalyst who played an important role in Italian science and culture of the last century, speaks in these terms (see Musatti, 1957, 1958a, 1958b).

It is important to remember, in this context, that Musatti knew mathematics well. Here are some biographical notes of interest for our discussion (see Adamo & Zudini, 2006a).

After an initial period of education at home, under the guidance of his mother, Musatti enrolled at the Liceo Classico Foscarini in Venice and began to take an interest in mathematics and philosophy.

After graduating from high school in 1915, he chose to study mathematics at the Faculty of Science at the University of Padua, where prominent figures, such as Gregorio Ricci Curbastro, Francesco Severi, and Tullio Levi-Civita were teaching at the time.

"Dissatisfied with the dogmatic way in which mathematical disciplines are taught" (Reichmann, 1996-1999, p. 39), Musatti soon decided to move to the Faculty of Humanities, a "livelier" (Reichmann, 1996-1999, p. 41) environment than Science. Decisive in this sense was his meeting with Antonio Aliotta (1881-1964), a pupil of Francesco De Sarlo (1864-1937) in Florence and then professor of theoretical philosophy in Padua, whose course on experimental psychology Musatti, a first-year mathematics student, attended in the academic year 1915-1916.
In 1916 Musatti interrupted his studies to participate as a volunteer in the First World War. At the end of the conflict, he returned to Padua, to the university, still undecided between mathematics and philosophy.

On Aliotta's advice, he resolved to complete his philosophy studies with a thesis, assigned to him by Aliotta himself, on non-Euclidean geometries.

In 1919, Aliotta moved to Naples and Musatti completed his thesis under the guidance of Vittorio Benussi (1878-1927), who had moved from Graz to Padua at that time and was assigned to teach experimental psychology (on the life and figure of Benussi, see, e.g., Adamo & Zudini, 2006b; Zudini, 2011, 2012). Musatti graduated with him on 3 November 1921, although the thesis does not, on closer inspection, have much to do with psychology.

The thesis, entitled "Geometrie non-euclidee e problema della conoscenza"("Non-Euclidean Geometries and the Problem of Knowledge"), to which Musatti devoted himself for almost two years, is divided along the lines of Bertrand Russell's *An Essay on the Foundations of Geometry* (1897) into a "Saggio storico-critico intorno alle interpretazioni gnoseologiche delle geometrie non-euclidee" ("Historical-critical essay on the gnoseological interpretations of non-Euclidean geometries") and a "Saggio di una teoria generale della conoscenza spaziale geometrica" ("Essay on a general theory of geometric spatial knowledge").

Strongly imbued with Aliotta's lecture, the treatment, starting from a purely mathematical problem, then becomes a philosophical and epistemological study on the conditions of possibility of spatial experience (see Zudini, Antonelli & Stucchi, 2011). For further details on the figure and life of Musatti, see, for example, Adamo & Zudini (2006a).

6. OBJECT PROPERTIES AND RELATIONAL PROPERTIES

Musatti distinguishes between object properties (size, shape, colour) and relational properties of objects (distance, position in space, illumination).

What is the relationship between object properties and relational properties of objects in our perceptual experience?

Object properties and relational properties are interdependent, interacting: changes in shape and size (deformations) covary with changes in distance and position (movement), changes in colour covary with changes in illumination.

What remains constant and what changes?

Let us look at some examples of interdependence between object properties and relational properties:

1. colour and illumination (Adelson's illusion and shadows);
2. Shepard's monsters (a location in space determines the perceived size);
3. depth kinetic effect (a two-dimensional deformation determines the perception of a three-dimensional rotation);

4. stereokinetic phenomena (movement determines the perception of a three-dimensional object).

6.1 ADELSON'S ILLUSION AND SHADOWS
6.1.1 ADELSON'S ILLUSION

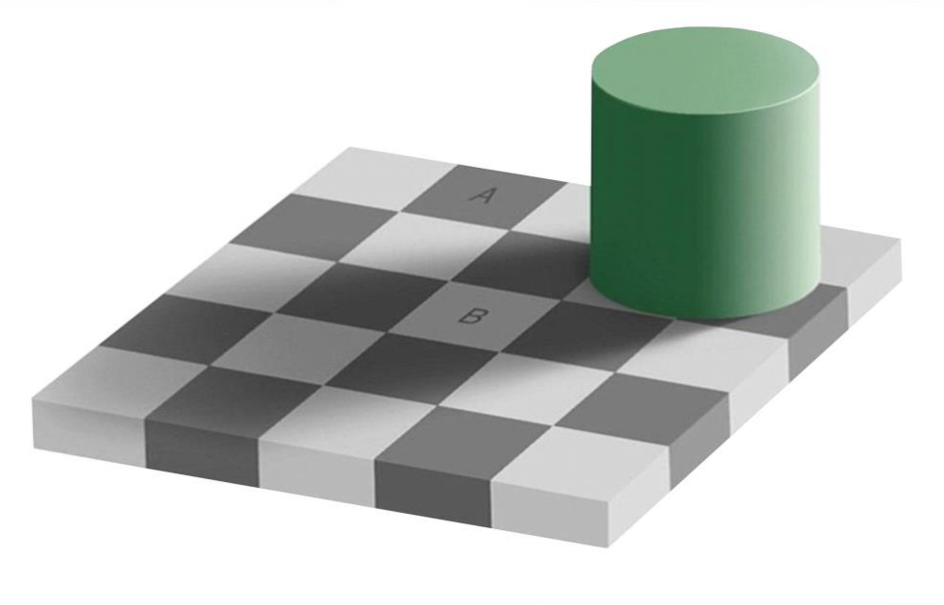

Figure 1. Adelson's checker-shadow illusion (2005)
(Source: public domain, modified)

Interaction between colour and illumination
The effect (Figure 1) depends on the fact that tile A is surrounded by lighter tiles and tile B is surrounded by darker tiles and is also in the shadow cast by the green cylinder. These are typically relational properties that determine the perceived object colour of the tile.

But this is not always the case. In other relational contexts, objectively different colours are perceived as the same colour. This is the case with shadows. See Figure 2.

6.1.2 THE COLOUR OF SHADOWS

Figure 2. The colour of shadows
(Source: public domain, modified)

So, to sum up (Figure 3):
The physical stimulus is the same, but we see a different colour (checker-shadow illusion).
The physical stimulus is different, but we see the same colour (sunlit vs shady).

a) b)

Figure 3. a) The physical colours of A and B are the same, but in the context of the Adelson's illusion their colour is perceived as different. b) Even if the physical colours of sunlit and shady parts are different, we perceptually assign to them the same colour.

6.2 SHEPARD'S MONSTERS

The size of an object (object property) depends on the context (relational property).
By playing on the interdependence between object properties and relational properties, curious illusory effects can be achieved, as in the following figure (Figure 4).

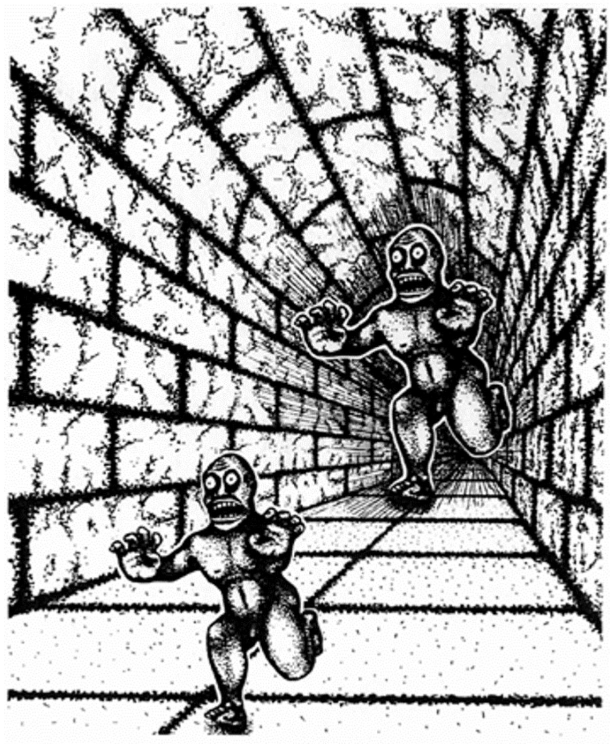

Figure 4. Shepard's monsters. The physical size of the two monsters is the same (they are the same picture drawn in different positions), but their size is perceived as different. (Source: Shepard, 1990)

Position in space determines perceived magnitude.

6.3 KINETIC DEPTH EFFECT

A two-dimensional deformation of the image results in the perception of a three-dimensional rotation (Figure 5) (see Wallach & O'Connell, 1953).

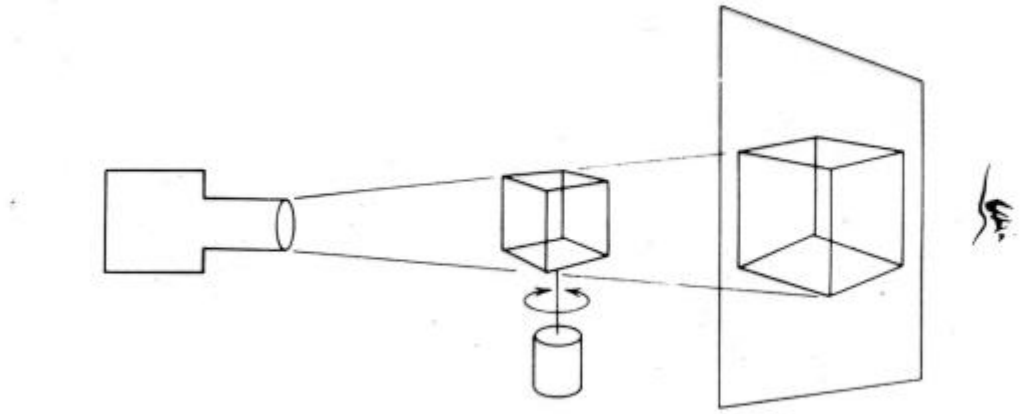

Figure 5. Kinetic depth effect
(Source: Schiffman, 1990, modified)

6.4 STEREOKINETIC PHENOMENA

Changes in the relational properties (position in space of the circle with the dot and their reciprocal position with respect to space) determine the perception of a three-dimensional object corporeity (Figure 6) (see Musatti, 1924, 1975; see also Wallach & O'Connell, 1953; Wallach, Weisz & Adams, 1956).

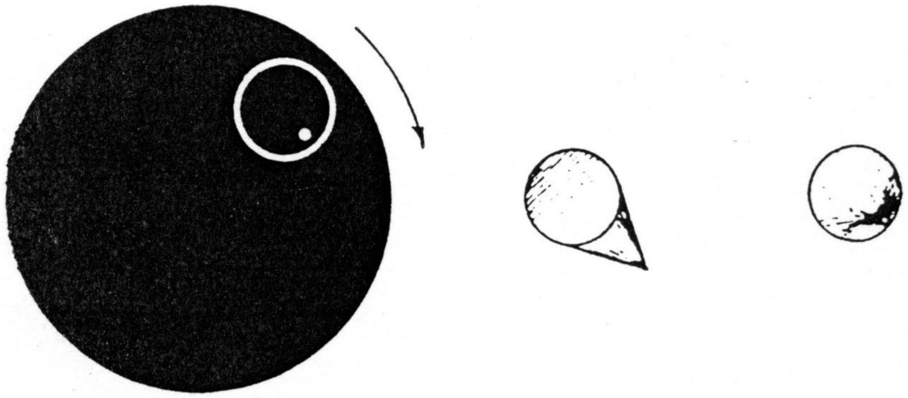

Figure 6. The stereokinetic effect
(Source: Musatti, 1924)

7. INTERACTION BETWEEN OBJECT PROPERTIES AND RELATIONAL PROPERTIES

What is the purpose of the interdependence between object properties and relational properties? Their covariation makes it possible to "extract a constancy [i.e., an *invariant*] from a situation of variability".

For example, from the covariation between object colour and illumination (light composition) we obtain the object's colour constancy (Figure 7).

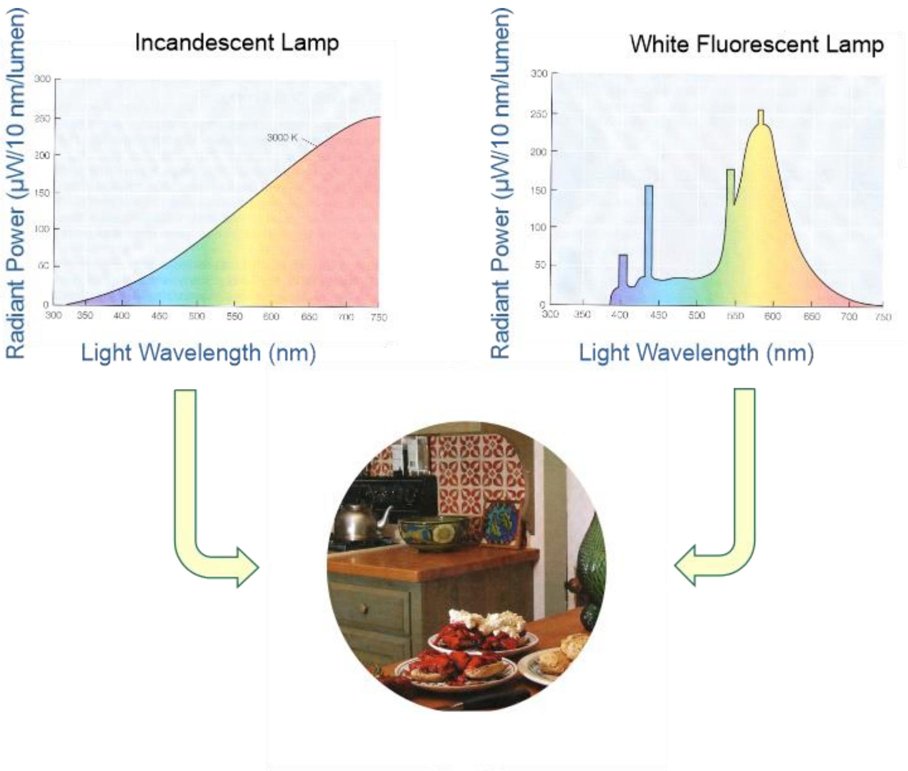

Figure 7. Chromatic constancy. The colours of the objects do not change even if the lighting sources are quite different. (Source: public domain, modified)

Musatti observes that objects appear to us as constant in colour as long as it is possible to convert variations in radiation coming from them into variations in illumination. When this is not possible, the variation is experienced as a variation in object colour.

As we have seen, constancy of colour is the invariant that is maintained. However, this is not always the case. A case in which colour constancy is not maintained occurs when illumination is given by monochromatic light (this is the trick used, for example, in supermarket windows where meat is displayed: a potentially disgusting brown meat is seen in an attractive bright red colour).

8. INTERACTION BETWEEN SHAPE AND SIZE

The same reasoning applies to the covariation between shape-size and position-distance of objects: the variation in shape and size of objects appears to us as a constant object as long as it is possible to convert this variation (due, for example, to movement) into variations of their position and distance in space. When this is not possible, the variation is perceived as a change in the shape and size of the object.

The following figure (Figure 8) shows an example where the background changes the shape of the object.

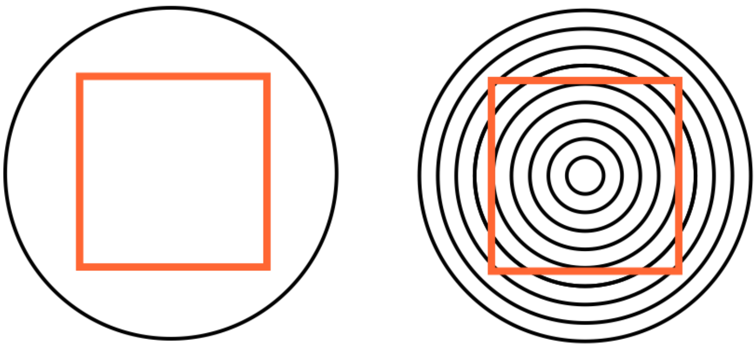

Figure 8. Orbison illusion (1939)

9. CONCLUSION

The question naturally arises at this point:
What remains constant and what changes?

As we have seen in the various examples, in general objects (shape, size, and colour) remain constant (invariant). But we have also seen that this is not always the case.
What is the rule that allows us to perceive a set of covariations between object properties and relational properties as constancy (invariance) of colour, shape, and size of the object?
According to Musatti, this rule can be traced back to the mathematical concept of group. In particular, to the group of transformations that leave metric properties invariant, i.e., rigid motions in Euclidean three-dimensional space.
Therefore, Musatti proposes that the invariances we observe in our perceptual experience are not due to a generic tendency toward constancy, but to a tendency to constancy as invariance with respect to a set of transformations constituting a group.

Thus, Musatti (1957, pp. 338-339):

[...] the perceptual situation determined by a variable retinal image, i.e., one that changes shape and size. Acting in this situation is a tendency to constancy, that is, a tendency to perceive an object invariable in size and shape. But this tendency can only be expressed if that constant object can be seen as undergoing a transformation that falls within that group of transformations that leave precisely the metric properties invariant, i.e., a rigid motion in Euclidean three-dimensional space. If, on the other hand, the deformation of the retinal image is of a different type, one can sometimes also have the perception of a motion in Euclidean three-dimensional space, but with a residual variation, experienced as a partial alteration in form and size. The limiting case is constituted by a null transformation of the group: the object seen does not undergo any dislocation in three-dimensional space, but only deforms; and this because there is no component of the retinal deformation that, given the structure of the group of rigid motions, succeeds in translating into a motion.

The problem is that things are always more complicated than we would like: the retinal image is not Euclidean, but rather describable within elliptic geometry.
Musatti assumes that we automatically transform all deformations that are incompatible with Euclidean geometry (elliptic geometry in the case of the retinal image) into Euclidean transformations (example of "Stratton-like" deformations).
In this way, Musatti assumes that the perceptual world is in fact described by the group of transformations that leave metric properties unchanged (Euclidean geometry). He is in good company, of course: the same assumption is present in scholars such as Hermann von Helmholtz, the aforementioned Felix Klein, Federigo Enriques, and Ernst Cassirer (see Cassirer, 1944, as well as Enriques, 1906).

However, it is legitimate to ask oneself - and to ask students - the "intriguing" question of whether this is really the best hypothesis to describe our perceptual experience of invariants.

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