A New Accurate Approximation of the Gaussian Q-Function with Relative Error Less Than 1 Thousandth in a Significant Domain

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***Abstract*— The approximations of the Gaussian Q-function found in the literature have been often developed with the goal of obtaining high estimation accuracies in deriving the error probability for digital modulation schemes. Unfortunately, the obtained mathematical expressions are often too complex, even difficultly tractable. A new approximation for the Gaussian Q-function is presented in the form of the standard normal density multiplied by a rational function. The rational function is simply a linear combination of the first 5 integer negative powers of the same term, linear in *x*, using only 4 decimal constants. In this paper we make some considerations about the significant interval where to consider the Q-function in telecommunication theory. The relative error in absolute value of the given approximation is less than 0.06% in the considered significant interval.**

I. Introduction

The Gaussian Q-function is very widely used in communication and information theory, since it plays a major role in the performance analysis of many communication systems. Usually, the bit error probability or the symbol error probability in many communication systems are expressed in terms of it. However, since there is no known closed-form expression for $Q(x)$ [1], the numerical values for this function have been tabulated and often made available as built-in functions in mathematical software tools. On the other hand, the analytical problems associated with it have raised a great interest in finding its closed-form bounds or approximations for decades [2]-[9], in order to simplify the handling of the mathematical expressions involving it [5], [6]. In fact, exponential-type bounds or approximations are often useful in the bit-error probability evaluation for the most common communication and information theory problems, such those involving coding (see, e.g., [10]-[12], addressing low density parity check (LDPC) codes), fading (see, e.g., [13], considering the FPGA implementation of a burst error and burst erasure channel emulator, being the error burst the result of a temporary reduction in the power of the received signal (fading), leading to a demodulation failure of a certain number of symbols), and multichannel reception (see, e.g., [14], considering a joint spectrum and energy efficient resource allocation algorithm for D2D communications). These approximations for the Gaussian Q-function have been developed with the objective of obtaining high estimation accuracies, to derive the error probability for digital modulation schemes. Unfortunately, the obtained mathematical expressions are often too complex and difficultly tractable.

Purpose of this paper is to modify two classical approximations due to Cooper [15] and to Hastings [4], respectively, to obtain a simpler and more precise approximation for relative errors. The proposed new approximation of the Gaussian Q-function is in the form of the standard normal density multiplied by a rational function. The rational function is simply a linear combination of the first 5 integer negative powers of the same term, linear in *x*, using only 4 decimal constants. Moreover, we make some considerations about the significant interval where to consider the Q-function in telecommunication theory. The relative error in absolute value of the given approximation is less than 0.06% in the considered significant interval.

The paper is organized as follows. In Section II we recall the Q-function and its several equivalent definitions, we recall the domain of practical interest of the function $Q(x)$ in information and communications theory, and we present some conjectures on the utility of absolute and relative errors as means of evaluation of an approximation. In Section III we recall some known approximations and bounds on $Q(x)$ and we present our new one, and in Section IV we make some comparisons between our new approximation and the classical ones in terms of complexity and error performance. Finally, Section V summarizes the results of the paper.

II. The several equivalent definitions of $Q(x)$

Basically the Q-function expresses the integral of the right tail of the standard normal density

$$\phi(x) := \frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{-\frac{x^2}{2}}$$

whose graph is the classical Gaussian bell curve, and then its meaning is essentially the probability that a standard normal random variable X assumes a value greater than x.

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There are several equivalent definitions of $Q(x)$, expressed as integrals, series, and in terms of the function $\Phi(x)$, as these:

$$Q(x) := \int_{x}^{+\infty} \phi(t)\,dt$$

(see Formula 26.2.3 of [4])

$$Q(x) := \frac{1}{2} - \int_{0}^{x} \phi(t)\,dt$$

(and notice that this integral is on a bounded interval)

$$Q(x) := 1 - \int_{-\infty}^{x} \phi(t)\,dt = 1 - \Phi(x) \tag{1}$$

where $\Phi(x)$ is the well known *normal cumulative distribution function*, of fundamental importance in statistics, for which you may find online lots of tables of numerical values by searching images for "`normal table`".

An equivalent definition of $Q(x)$ based on continued fractions is (see Formula (26.2.14) of [4])

$$Q(x) = \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\,\pi}} \left\{ \frac{1}{x+} \frac{1}{x+} \frac{2}{x+} \frac{3}{x+} \frac{4}{x+} \dots \right\} \tag{2}$$

There are 2 other functions, classically considered, strictly related to $Q(x)$ and $\Phi(x)$:

1) *Error function*:

$$\mathrm{erf}(x) := \int_{-\infty}^{x} \frac{2}{\sqrt{\pi}}\,\mathrm{e}^{-t^2}\,dt = 1 - 2\,Q(x\sqrt{2})$$

2) *Complementary error function*:

$$\mathrm{erfc}(x) := \int_{x}^{+\infty} \frac{2}{\sqrt{\pi}}\,\mathrm{e}^{-t^2}\,dt = 2\,Q(x\sqrt{2})$$

A. Domain of practical interest of the function $Q(x)$ in Information and Communications Theory

The Q-function is defined over $\mathbb{R}$. However, in Information and Communication Theory, only positive real numbers $\mathbb{R}_{>0}$ are significant.

As done in [16], since the bit error probability of the binary phase shift keying (BPSK) modulation scheme over the additive white Gaussian noise (AWGN) channel can be expressed as

$$P_b(e) = Q(\sqrt{2\,\gamma})$$

being γ the signal-to-noise ratio (SNR), usually expressed in dB, we can assume that a significant interval for the argument γ in dB could be $[-10\,\mathrm{dB}, 10\,\mathrm{dB}]$.

Taking $\gamma = -10\,\mathrm{dB}$, in absolute value we get $\gamma = 10^{-1} = \frac{1}{10}$ and

$$\sqrt{2\,\gamma} = \sqrt{2 \cdot \frac{1}{10}} \approx 0.45$$

Taking, at the other interval extreme, $\gamma = 10\,\mathrm{dB}$, in absolute value we get $\gamma = 10^1 = 10$ and

$$\sqrt{2\,\gamma} = \sqrt{2 \cdot 10} \approx 4.47 \approx 4.5$$

Thus, we concentrate on the approximation for $Q(x)$ on the interval

$$I_{\text{significant}} := [0.45, 4.5]$$

B. Notes on absolute and relative errors

As observed in [16], in general, it may be said that when comparing the accuracy of 2 approximations of a function, the consideration of the relative error is more appropriate if that function has a zero limit. In fact, for example, it is of little interest to say that an approximate value of about 10^{-5} has an absolute error less than 10^{-4}.

All the 4 functions $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, $\Phi(x)$ and $Q(x)$ have limits 0 in their domain, but, if we consider $\Phi(x)$ for $x \geq 0$, its limits become

$$\frac{1}{2} \leq \Phi(x) < 1$$

and hence, for this function restricted to $x \geq 0$, the consideration of the absolute error is appropriate and generally used in literature. (The approximation of $\Phi(x)$ for $x \leq 0$ is essentially the problem of approximating $Q(x)$ for $x \geq 0$ because $\Phi(-x) = Q(x)$.)

Instead, for $\mathrm{erfc}(x)$ and $Q(x)$ the consideration of relative errors is more appropriate and generally used in literature.

III. Some known approximations and bounds of $Q(x)$ and our new one

A. 2 classical bounds and 2 classical approximations of $Q(x)$

The classical bounds

$$Q(x) < \frac{1}{\sqrt{2\,\pi}} \frac{\mathrm{e}^{\frac{-x^2}{2}}}{x} \tag{3}$$

and

$$Q(x) > \frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{\frac{-x^2}{2}} \left(\frac{1}{x} - \frac{1}{x^3} \right) \tag{4}$$

have been published in [17] (Formula (2.121)).

It may be observed that the first convergent (truncation to the first term) of (2) is exactly the upper bound (3) and the second convergent

$$\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\,\pi}} \frac{1}{x + \frac{1}{x}}$$

may be developed, for $x > 1$, as

$$\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\,\pi}} \frac{1}{x} \left(1 - \frac{1}{x^2} + \dots \right)$$

to be compared with the lower bound (4).

The approximation [15]

$$Q(x) \approx \frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{\frac{-x^2}{2}} \left(\frac{1}{x} - \frac{1}{2\,x^3} \right) \tag{5}$$

is the arithmetic mean of the 2 bounds (3) and (4).

A high precision approximation of $Q(x)$ is immediately obtained by complementation $1-$ (applying (1)) from a very classical approximation of $\Phi(x)$ reported (with absolute error less than $7.5 \cdot 10^{-8}$, which is the same for $Q(x)$) in the classical book of Abramowitz and Stegun [4] (see Formula (26.2.17)) in a chapter authored by M. Zelen and N.C. Severo, and credited to C. Hastings, Jr., *Approximations for digital computers*, (1955) Princeton Univ. Press, Princeton, N. J.

$$\begin{array}{l} Q(x) \approx \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 \right) \\ t := \frac{1}{1+p\,x} \qquad p = 0.231\,641\,9 \\ b_1 = +0.319\,381\,53 \quad b_2 = -0.356\,563\,782 \\ b_3 = +1.781\,477\,937 \quad b_4 = -1.821\,255\,978 \\ b_5 = +1.330\,274\,429 \end{array} \tag{6}$$

This approximation is so widely used in practice, that the advanced search tool of Google declares (June 2021) about 2380 results searching (contemporarily) the 3 strings 0.2316419 0.31938153 1.781477937 (and at a glance the results refers just to that formula).

B. New approximation with 4 decimal constants

Purpose of this paper is to modify (5) and (6) to obtain an approximation

- more simple than (6);
- more precise than (6) for relative error in $I_{\mathrm{significant}}$.

This new approximation may be compared with the very classical approximation (6) of Hastings, originally given for $\Phi(x)$), from which one immediately obtains the corresponding approximation for Q(x).

Modifying the classical approximations (5) and (6) we propose

$$Q(x) \approx \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \left(\frac{1}{\left(x+\frac{\pi}{4}\right)} + \frac{a}{\left(x+\frac{\pi}{4}\right)^2} + \right.$$
$$\left. + \frac{b}{\left(x+\frac{\pi}{4}\right)^3} + \frac{c}{\left(x+\frac{\pi}{4}\right)^4} + \frac{d}{\left(x+\frac{\pi}{4}\right)^5} \right)$$
$$a = 0.85512 \qquad b = -1.07$$
$$c = -0.02568 \qquad d = 0.32955$$

with maximum relative error in absolute value $|\varepsilon_r| < 5.9 \cdot 10^{-4}$ in $[0.45, 4.5]$.

To get a better approximation, Hastings' philosophy was that of introducing a displacement p, so as to obtain an approximation (6) that does not diverge in $x = 0$ as (5), thanks to the term $\frac{1}{1+p\,x}$. However, this approximation was *not* designed to lower the relative error for $Q(x)$, but to lower the absolute error of $\Phi(x)$. Thus, on the basis of (6), a new search of the coefficients was performed to obtain the minimization of the relative error in absolute value, more appropriate to evaluate the performance of a Q-function approximation, as explained in Section II-B.

In details, in $I_{\mathrm{significant}} = [0.45, 4.5]$, the absolute error $|\varepsilon|$ and the relative error in absolute value $|\varepsilon_r|$ of the new approximation are:

$$|\varepsilon| < 2.0 \cdot 10^{-4}$$
$$|\varepsilon_r| < 5.9 \cdot 10^{-4}$$

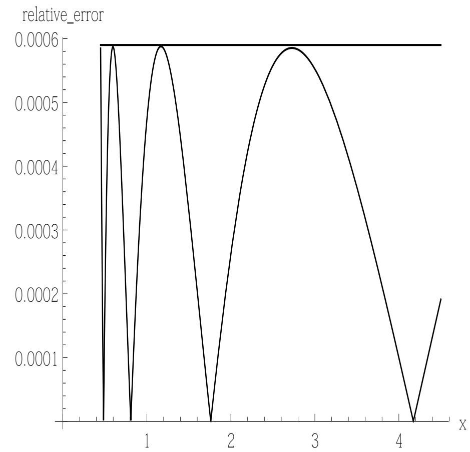

Fig. 1. The relative error in absolute value $|\varepsilon_r| < 5.9 \cdot 10^{-4}$ of the new approximation in the significant interval $I_{\mathrm{significant}} = [0.45, 4.5]$.

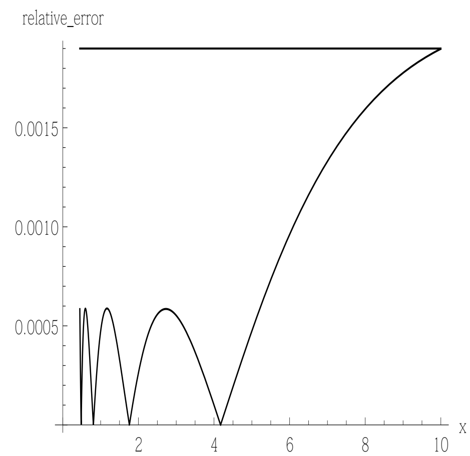

Fig. 2. The relative error in absolute value $|\varepsilon_r| < 1.9 \cdot 10^{-3}$ of the new approximation in the interval $I_{10} = [0.45, 10]$.

and also in the interval $[0.45, 10]$ the performance is quite good, $|\varepsilon_r| < 1.9 \cdot 10^{-3}$, and even in the interval $[0.45, 100]$, with $|\varepsilon_r| < 2.1 \cdot 10^{-3}$. The upper limit 10 for the domain of $Q(x)$ has been considered in telecommunication theory, for example in [18].

In Figs. 1, 2 and 3 is reported the relative error in absolute value $|\varepsilon_r|$ of the new approximation in the intervals $I_{\mathrm{significant}} = [0.45, 4.5]$, $I_{10} = [0.45, 10]$ and $I_{100} = [0.45, 100]$, respectively.

In Fig. 4 is reported the absolute error $|\varepsilon|$ of the new approximation in the interval $[0.45, 10]$.

In Fig. 5 is reported the comparison between the $Q(x)$ complete distribution and the new approximation of the Q-function in log scale of ordinate: the two curves are perfectly superimposed.

IV. Comparison of the New Approximation with the Two Classical Approximations (5) and (6)

The computational complexity may be evaluated by counting the number of non-rational functions applied to the ar-

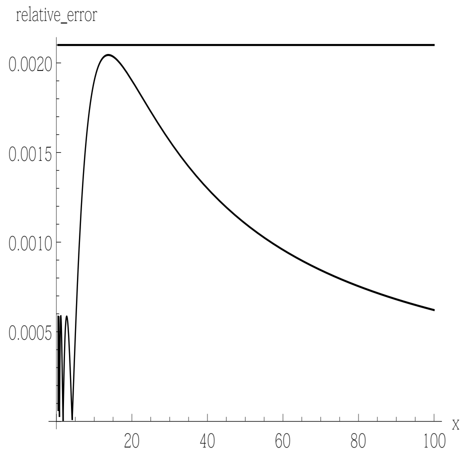

Fig. 3. The relative error in absolute value $|\varepsilon_r| < 2.1 \cdot 10^{-3}$ of the new approximation in the interval $I_{100} = [0.45, 100]$.

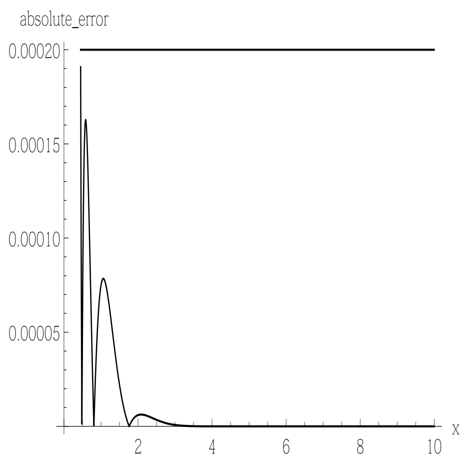

Fig. 4. The absolute error $|\varepsilon| < 2.0 \cdot 10^{-4}$ of the new approximation in the interval $I_{10} = [0.45, 10]$. The absolute error in x=0.45 is 0.0001908...

gument x. In this sense, the two classical approximations (5) and (6), and our new one, all have the same computational complexity equal to 1, since they all present only one non-rational expression (i.e., $\mathrm{e}^{-\frac{x^2}{2}}$). Moreover, to evaluate the complexity, it may be also useful to evaluate how many decimal constants are present in an approximation. In this sense, the approximation (6) is the most complicated, with 6 (longer) decimal constants, our new one has only 4 decimal constants, whereas the approximation (5) is the simplest since it does not present any decimal constant.

Clearly (5), despite its simplicity, is also the worst approximation from the point of view of the absolute error $|\varepsilon|$ and of the relative error in absolute value $|\varepsilon_r|$, presenting, in $I_{\text{significant}} = [0.45, 4.5]$:

$$|\varepsilon| < 1.6$$

$$|\varepsilon_r| < 4.6$$

i.e., a relative error in absolute value of the 461% (since it diverges in $x = 0$).

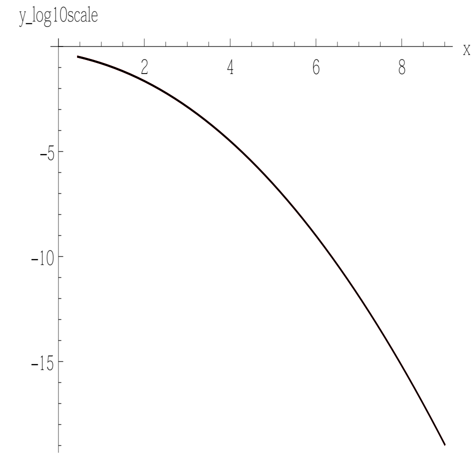

Fig. 5. Comparison between the $Q(x)$ complete distribution and the new approximation of the Q-function in log scale of ordinate.

On the other hand (6), by [4] published for $\Phi(x)$ (which tends to 1 and is always > 0.5) was designed to lower the absolute error of $\Phi(x)$, and then of $Q(x)$ (which is the same), on the whole real axis. But, as explained above, values near 0 of x, of great interest for $\Phi(x)$ in statistics, are of no interest in communication theory for $Q(x)$. Notice that the approximation was *not* designed to lower the relative error for $Q(x)$. Thus, as far as Hastings' approximation (6) is concerned, we may, in conclusion, observe that

- it is the most complicated, with 6 (longer) decimal constants;
- the relative error in absolute value for $Q(x)$ is $< 9.3 \cdot 10^{-4}$ in $I_{\text{significant}} = [0.45, 4.5]$, and $< 2.0 \cdot 10^{-2}$ in the interval $[0.45, 10]$, both higher/worse than those of the proposed new approximation;
- it grants the lowest/best absolute error $< 7.5 \cdot 10^{-8}$ when considered on the whole positive real axis, $x \geq 0$, for $\Phi(x)$ and correspondently for $Q(x)$, thanks to its higher complexity.

V. Conclusions

This paper proposes a new approximation for the Gaussian Q-function presented in the form of the standard normal density multiplied by a rational function. The rational function is simply a linear combination of the first 5 integer negative powers of the same term, linear in *x*, using only 4 decimal constants. In this paper we make some considerations about the significant interval where to consider the Q-function in telecommunication theory. The relative error in absolute value of the given approximation is less than 0.06% in the considered significant interval $[0.45, 4.5]$. Even moving the upper limit of the interval till 100, the proposed new approximation performs quite well with a relative error in absolute value less than 0.21%.

References

[1] M. K. Simon and M.-S. Alouini, *Digital Communication over Fading Channels*, John Wiley & Sons, Inc., Hoboken, New Jersey, 2002.

[2] Z. W. Birnbaum, “An inequality for Mills’ ratio”, *The Annals of Mathematical Statistics*, vol. 13, no. 2, 1942, pp. 245-246.

[3] H. Chernoff, “A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations”, *The Annals of Mathematical Statistics*, vol. 23, no. 4, Dec. 1952, pp. 493-507.

[4] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions. With Formulas, Graphs, and Mathematical Tables.*, National Bureau of Standards, Applied Mathematics Series, vol. 55, 10-th reprint, Dec. 1972, with corrections.

[5] P. O. Borjesson and C. E. Sundberg, “Simple approximations of the error function $Q(x)$ for communications applications”, *IEEE Transactions on Communications*, vol. 27, no. 3, Mar. 1979, pp. 639-643.

[6] N. C. Beaulieu, “A simple series for personal computer computation of the error function $Q(.)$”, *IEEE Transactions on Communications*, vol. 37, no. 9, Sept. 1989, pp. 989-991.

[7] C. Tellambura and A. Annamalai, “Efficient computation of erfc(x) for large arguments”, *IEEE Transactions on Communications*, vol. 48, no. 4, Apr. 2000, pp. 529-532.

[8] G. K. Karagiannidis and A. S. Lioumpas, “An improved approximation for the Gaussian Q-function”, *IEEE Communications Letters*, vol. 11, no. 8, Aug. 2007, pp. 644-646.

[9] Y. Isukapalli and B. D. Rao, “An analytically tractable approximation for the Gaussian Q-function”, *IEEE Communications Letters*, vol. 12, no. 9, Sept. 2008, pp. 669-671.

[10] F. Babich, M. Noschese. A. Soranzo, and F. Vatta, “Low complexity rate compatible puncturing patterns design for LDPC codes”, *Journal of Communication Software and Systems (JCOMSS)*, vol. 14, no. 4, December 2018, pp. 350-358.

[11] F. Vatta, A. Soranzo, M. Comisso, G. Buttazzoni, and F. Babich, “Performance study of a class of irregular LDPC codes through low complexity bounds on their belief-propagation decoding thresholds”, *Proceedings of the AEIT 2019 International Annual Conference*, Florence, Italy, September 18-20, 2019.

[12] F. Vatta, A. Soranzo, M. Comisso, G. Buttazzoni, and F. Babich, “New explicitly invertible approximation of the function involved in LDPC codes density evolution analysis using a Gaussian approximation”, *Electronics Letters*, vol. 55, no. 22, October 2019, pp. 1183-1186.

[13] C. Travan, F. Vatta, and F. Babich, *Journal of Communications Software and Systems (JCOMSS)*, vol. 16, no. 1, March 2020, pp. 19-29.

[14] S. Selmi and R. Bouallègue, *Journal of Communications Software and Systems (JCOMSS)*, vol. 16, no. 1, March 2020, pp. 57-65.

[15] G. R. Cooper and C. D. McGillem, *Probabilistic Methods of Signal and System Analysis*, 3rd ed. Oxford, U.K., 2012.

[16] A. Soranzo, F. Vatta, M. Comisso, G. Buttazzoni, and F. Babich, “New very simply explicitly invertible approximation of the Gaussian Q-function”, *Proceedings of the 2019 International Conference on Software, Telecommunications and Computer Networks, SoftCOM’19*, Split, Croatia, September 19-21, 2019.

[17] J. M. Wozencraft and I. M. Jacobs, *Principles of communication engineering*, Wiley, New York, NY, USA, 1965.

[18] S. Aggarwal, “A survey-cum-tutorial on approximations to Gaussian Q function for symbol error probability analysis over Nakagami-m fading channels”, *IEEE Communications Surveys & Tutorials*, vol. 21, no. 3, Third Quarter 2019, pp. 2195-2223.