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# Levi–Civita Connections on Quantum Spheres

**Joakim Arnlind[1] · Kwalombota Ilwale[1] · Giovanni Landi[2,3,4]**

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**Abstract**
We introduce $q$-deformed connections on the quantum 2-sphere and 3-sphere, satisfying a twisted Leibniz rule in analogy with $q$-deformed derivations. We show that such connections always exist on projective modules. Furthermore, a condition for metric compatibility is introduced, and an explicit formula is given, parametrizing all metric connections on a free module. On the quantum 3-sphere, a $q$-deformed torsion freeness condition is introduced and we derive explicit expressions for the Christoffel symbols of a Levi–Civita connection for a general class of metrics. We also give metric connections on a class of projective modules over the quantum 2-sphere. Finally, we outline a generalization to any Hopf algebra with a (left) covariant calculus and associated quantum tangent space.

**Keyword** Noncommutative geometry · Noncommutative Levi–Civita connection · Quantum groups

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## 1 Introduction

In recent years, a lot of progress has been made in understanding Riemannian aspects of noncommutative geometry. These are not only mathematically interesting, but also

✉ Joakim Arnlind
joakim.arnlind@liu.se

Kwalombota Ilwale
kwalombota.ilwale@liu.se

Giovanni Landi
landi@units.it

1 Department of Mathematics, Linköping University, 581 83 Linköping, Sweden

2 Mathematics, Università di Trieste, 34127 Trieste, Italy

3 Institute for Geometry and Physics (IGAP), Trieste, Italy

4 INFN, Trieste, Italy

important in physics where noncommutative geometry is expected to play a key role, notably in a theory of quantum gravity. In Riemannian geometry the Levi–Civita connection and its curvature have a central role, and it turns out that there are several different ways of approaching these objects in the noncommutative setting (see e.g. [2, 4, 5, 7–10, 12, 14, 18, 21]).

From an algebraic perspective, the set of vector fields and the set of differential forms are (finitely generated projective) modules over the algebra of functions, a viewpoint which is also adopted in noncommutative geometry. However, considering vector fields as derivations does not immediately carry over to noncommutative geometry, since the set of derivations of a (noncommutative) algebra is in general not a module over the algebra but only a module over the center of the algebra. Therefore, one is led naturally to focus on differential forms and define a connection on a general module as taking values in the tensor product of the module with the module of differential forms. More precisely, let $M$ be a (right) $\mathcal{A}$-module and let $\Omega^1(\mathcal{A})$ denote a module of differential forms together with a differential $d : \mathcal{A} \to \Omega^1(\mathcal{A})$. A connection on $M$ is a linear map $\nabla : M \to M \otimes \Omega^1(\mathcal{A})$ satisfying a version of Leibniz rule

$$\nabla(mf) = (\nabla m)f + m \otimes df \tag{1.1}$$

for $f \in \mathcal{A}$ and $m \in M$. In differential geometry, for a vector field $X$ one obtains a covariant derivative $\nabla_X : M \to M$, by pairing differential forms with $X$ (as differential forms are dual to vector fields). In a noncommutative version of the above, there is in general no canonical way of obtaining a "covariant derivative" $\nabla_X : M \to M$. In a derivation based approach to noncommutative geometry (see e.g. [13, 14]), one puts emphasis on the choice of a Lie algebra $\mathfrak{g}$ of derivations of the algebra $\mathcal{A}$. Given a (right) $\mathcal{A}$-module $M$ one defines a connection as a map $\nabla : \mathfrak{g} \times M \to M$, usually writing $\nabla(\partial, m) = \nabla_\partial m$ for $\partial \in \mathfrak{g}$ and $m \in M$, satisfying

$$\nabla_\partial(mf) = (\nabla_\partial m)f + m\,\partial(f)$$

for $f \in \mathcal{A}$ and $m \in M$, in parallel with (1.1). We stress that in general $\mathfrak{g}$ is not a module over $\mathcal{A}$ when $\mathcal{A}$ is not commutative. Thus we do not require $\mathcal{A}$-linearity in the argument $\partial$ of $\nabla_\partial$. This is in contrast with the braided geometry framework [4, 6] where for a braided commutative algebra the braided Lie algebra of its braided derivations is a module over the algebra and such a $\mathcal{A}$-linearity on the connection can be stated. Braided commutativity of a Hopf algebra is a feature of its being cotriangular (and not just coquasitriangular).

For quantum groups, it turns out that natural analogues of vector fields are not quite derivations, but rather maps satisfying a twisted Leibniz rule. For instance, as we shall see, for the quantum 3-sphere $S_q^3$ one defines maps $X_a : S_q^3 \to S_q^3$ satisfying

$$X_a(fg) = X_a(f)\sigma_a(g) + f X_a(g) \tag{1.2}$$

for $f, g \in S_q^3$, and $\sigma_a : S_q^3 \to S_q^3$, for $a = 1, 2, 3$, are algebra morphisms. In this note we explore the possibility of introducing a corresponding $q$-affine connection on a (right) $S_q^3$-module $M$. Motivated by (1.2) we introduce a covariant derivative $\nabla_{X_a} : M \to M$ such that

$$\nabla_{X_a}(mf) = (\nabla_{X_a} m)\sigma_a(f) + mX_a(f)$$

for $f \in S_q^3$ and $m \in M$. In the following, we make these ideas precise and prove that there exist $q$-affine connections on projective modules. Again, we will not ask for $S_q^3$-linearity in the argument $X$ of $\nabla_X$. Furthermore, we introduce a condition for metric compatibility, and in the particular case of a left covariant calculus over $S_q^3$, we investigate a derivation based definition of torsion. Then we explicitly construct a Levi–Civita connection, that is a torsion free and metric compatible connection. Moreover, we construct metric connections on a class of projective modules over the quantum 2-sphere. We mention that the Riemannian geometry of quantum spheres was studied [7] from the point of view of a bimodule connection on differential forms satisfying (1.1) as well as a right Leibniz rule twisted by a braiding map. In a final section we sketch a way to generalise (some of) the constructions of the present paper to any Hopf algebra with a (left) covariant differential calculus and corresponding quantum tangent space of twisted derivations.

The present paper is an alternative and extended version of the paper [1] where the left module structure of differential forms was used to construct $q$-affine connections, rather than the right module structure considered in the following.

## 2 The Quantum 3-Sphere

In this section we recall a few basic properties of the quantum 3-sphere [22]. The algebra $S_q^3$ is a unital $*$-algebra generated by $a, a^*, c, c^*$ fulfilling

$$\begin{aligned}
&ac = qca \quad c^*a^* = qa^*c^* \quad ac^* = qc^*a, \\
&ca^* = qa^*c \quad cc^* = c^*c \quad a^*a + c^*c = aa^* + q^2cc^* = \mathbb{1},
\end{aligned}$$

for a real parameter $q$. The identification of $S_q^3$ with the quantum group $SU_q(2)$ is via the Hopf algebra structure given by

$$\begin{aligned}
&\Delta(a) = a \otimes a - qc^* \otimes c \quad \Delta(c) = c \otimes a + a^* \otimes c, \\
&\Delta(a^*) = -qc \otimes c^* + a^* \otimes a^* \quad \Delta(c^*) = a \otimes c^* + c^* \otimes a^*,
\end{aligned}$$

with antipode and counit

$$\begin{aligned}
&S(a) = a^* \quad S(c) = -qc \quad \epsilon(a) = 1 \quad \epsilon(c) = 0, \\
&S(a^*) = a \quad S(c^*) = -q^{-1}c^* \quad \epsilon(a^*) = 1 \quad \epsilon(c^*) = 0.
\end{aligned}$$

We also need the dual quantum enveloping algebra $\mathcal{U}_q(\mathrm{su}(2))$, which is the $*$-algebra with generators $E, F, K, K^{-1}$ satisfying

$$K^{\pm 1}E = q^{\pm 1}EK^{\pm 1} \quad K^{\pm 1}F = q^{\mp 1}FK^{\pm 1} \quad [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}.$$

The corresponding Hopf algebra structure is given by the coproduct,

$$\Delta(E) = E \otimes K + K^{-1} \otimes E \quad \Delta(F) = F \otimes K + K^{-1} \otimes F \quad \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}$$

together with antipode and counit

$$\begin{aligned} &S(K) = K^{-1} \quad S(E) = -qE \quad S(F) = -q^{-1}F, \\ &\epsilon(K) = 1 \quad \epsilon(E) = 0 \quad \epsilon(F) = 0. \end{aligned}$$

We recall that there is a unique bilinear pairing between $\mathcal{U}_q(\mathrm{su}(2))$ and $S_q^3$ given by

$$\left\langle K^{\pm 1}, a \right\rangle = q^{\mp 1/2}, \quad \left\langle K^{\pm 1}, a^* \right\rangle = q^{\mp 1/2}, \quad \langle E, c \rangle = 1, \quad \left\langle F, c^* \right\rangle = -q^{-1},$$

with the remaining pairings being zero.

The algebra $S_q^3$ is a noncommutative algebra which is not quasi-commutative. This stems from the Hopf algebra $SU_q(2)$ being coquasitriangular and not simply cotriangular. Dually, the Hopf algebra $\mathcal{U}_q(\mathrm{su}(2))$ is quasitriangular and not triangular [16, §8, §10].

The pairing above induces a $\mathcal{U}_q(\mathrm{su}(2))$-bimodule structure (left and right actions) on $S_q^3$:

$$h \triangleleft f = f_{(1)} \left\langle h, f_{(2)} \right\rangle \quad \text{and} \quad f \triangleleft h = \left\langle h, f_{(1)} \right\rangle f_{(2)} \tag{2.1}$$

for $h \in \mathcal{U}_q(\mathrm{su}(2))$ and $f \in S_q^3$, with Sweedler's notation $\Delta(f) = f_{(1)} \otimes f_{(2)}$ (and implicit sum). The $*$-structure on $\mathcal{U}_q(\mathrm{su}(2))$, denoted here by $\dagger$ (to distinguish it from the $*$-structure of the algebra), is given by $(K^{\pm 1})^\dagger = K^{\pm 1}$ and $E^\dagger = F$. The action of $\mathcal{U}_q(\mathrm{su}(2))$ is compatible with the $*$-algebra structures in the following sense

$$h \triangleright f^* = \left(S(h)^\dagger \triangleright f\right)^* \quad f^* \triangleleft h = \left(f \triangleleft S(h)^\dagger\right)^*. \tag{2.2}$$

Let us for convenience list the left and right actions of the generators:

$$
\begin{aligned}
&K^{\pm 1} \triangleright a^n = q^{\mp\frac{n}{2}}\, a^n \quad K^{\pm 1} \triangleright c^n = q^{\mp\frac{n}{2}}\, c^n,\\
&K^{\pm 1} \triangleright a^{*\,n} = q^{\pm\frac{n}{2}} (a^*)^n \quad K^{\pm 1} \triangleright c^{*\,n} = q^{\pm\frac{n}{2}} (c^*)^n,\\
&E \triangleright a^n = -q^{(3-n)/2}[n]a^{n-1}c^* \quad E \triangleright c^n = q^{(1-n)/2}[n]c^{n-1}a^*,\\
&E \triangleright (a^*)^n = 0 \quad E \triangleright (c^*)^n = 0,\\
&F \triangleright a^n = 0 \quad F \triangleright c^n = 0,\\
&F \triangleright (a^*)^n = q^{(1-n)/2}[n]c(a^*)^{n-1} \quad F \triangleright (c^*)^n = -q^{-(1+n)/2}[n]a(c^*)^{n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
&a^n \triangleleft K^{\pm 1} = q^{\mp\frac{n}{2}} a^n \quad (a^*)^n \triangleleft K^{\pm 1} = q^{\pm\frac{n}{2}} (a^*)^n,\\
&c^n \triangleleft K^{\pm 1} = q^{\pm\frac{n}{2}} c^n \quad (c^*)^n \triangleleft K^{\pm 1} = q^{\mp\frac{n}{2}} (c^*)^n,\\
&a^n \triangleleft F = q^{\frac{n-1}{2}} [n]ca^{n-1} \quad (a^*)^n \triangleleft F = 0,\\
&c^n \triangleleft F = 0 \quad (c^*)^n \triangleleft F = -q^{\frac{n-3}{2}} [n]a^*(c^*)^{n-1},\\
&a^n \triangleleft E = 0 \quad (a^*)^n \triangleleft E = -q^{\frac{n-3}{2}} [n]c^*(a^*)^{n-1},\\
&c^n \triangleleft E = q^{\frac{n-1}{2}} [n]c^{n-1}a \quad (c^*)^n \triangleleft E = 0,
\end{aligned}
$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$.

### 2.1 The Covariant Calculus and the Quantum Tangent Space

It is well known [22] that there is a left covariant (first order) differential calculus on $S^3_q$, denoted by $\Omega^1(S^3_q)$, generated as a left $S^3_q$-module by

$$
\omega_1 = \omega_+ = a\,dc - qc\,da \quad \omega_2 = \omega_- = c^*da^* - qa^*dc^* \quad \omega_3 = \omega_z = a^*da + c^*dc.
$$

In fact, $\Omega^1(S^3_q)$ is a free left module with a basis given by $\{\omega_+, \omega_-, \omega_z\}$. Moreover, $\Omega^1(S^3_q)$ is a bimodule with respect to the relations

$$
\begin{aligned}
&\omega_z a = q^{-2} a\omega_z \quad \omega_z a^* = q^2 a^*\omega_z \quad \omega_z c = q^{-2} c\omega_z \quad \omega_z c^* = q^2 c^*\omega_z,\\
&\omega_\pm a = q^{-1} a\omega_\pm \quad \omega_\pm a^* = qa^*\omega_\pm \quad \omega_\pm c = q^{-1} c\omega_\pm \quad \omega_\pm c^* = qc^*\omega_\pm,
\end{aligned}
$$

and, furthermore, $\Omega^1(S^3_q)$ is a $*$-bimodule with

$$
\omega_+^\dagger = -\omega_- \quad \omega_z^\dagger = -\omega_z
$$

satisfying $(f\omega g)^\dagger = g^*\omega^\dagger f^*$ for $f, g \in S^3_q$ and $\omega \in \Omega^1(S^3_q)$.

The differential $d : S^3_q \to \Omega^1(S^3_q)$ is computed using a dual basis $\{X_+, X_-, X_z\}$ of twisted derivations (the corresponding quantum tangent space [16, §14.1.2]),

$$df = (X_+ \triangleright f)\omega_+ + (X_- \triangleright f)\omega_- + (X_z \triangleright f)\omega_z, \quad f \in S^3_q, \tag{2.3}$$

with explicitly,

$$X_+ = \sqrt{q}EK \quad X_- = \frac{1}{\sqrt{q}}FK \quad X_z = \frac{1-K^4}{1-q^{-2}}.$$

Their twisted derivation properties are easily found. For $f, g \in S^3_q$, and $a = \pm, z$ one has,

$$X_a \triangleright fg = f(X_a \triangleright g) + (X_a \triangleright f)(\sigma_a \triangleright g)$$

(and similarly for the right action), with

$$\sigma_+ = \sigma_- = K^2 \quad \text{and} \quad \sigma_z = K^4.$$

Furthermore, these maps satisfy the following $q$-deformed commutation relations

$$X_-X_+ - q^2X_+X_- = X_z, \tag{2.4}$$
$$q^2X_zX_- - q^{-2}X_-X_z = (1+q^2)X_-, \tag{2.5}$$
$$q^2X_+X_z - q^{-2}X_zX_+ = (1+q^2)X_+. \tag{2.6}$$

As for the $*$-structures, one checks that $X_\pm^\dagger = X_\mp$ and $K^\dagger = K$. From this, using (2.2) one computes, for $f \in S^3_q$, that

$$\begin{aligned} X_\pm \triangleleft f^* &= -(K^{-2}X_\mp \triangleleft f)^* = -K^2 \triangleleft (X_\mp \triangleleft f)^* \\ X_z \triangleleft f^* &= -(K^{-4}X_z \triangleleft f)^* = -K^4 \triangleleft (X_z \triangleleft f)^*. \end{aligned} \tag{2.7}$$

In the classical limit of $q = 1$, the above reduces to the Lie algebra of su(2) and the calculus is the usual calculus on the sphere $S^3$ given in terms of left invariant one-forms.

## 3 *q*-Affine Connections

In differential geometry, a connection extends the action of derivatives to vector fields, and for $S^3_q$ a natural set of ($q$-deformed) derivations is given by $\{X_+, X_-, X_z\}$. In this section, we will introduce a framework extending the action of $X_a$ to a connection on $S^3_q$-modules. Let us first define the set of $q$-deformed derivations we shall be interested in.

**Definition 3.1** The quantum tangent space of $S_q^3$ is defined as

$$TS_q^3 = \mathbb{C}\left\langle X_+, X_-, X_z \right\rangle,$$

that is the complex vector space generated by $X_a$ for $a = \pm, z$.

We point out that $TS_q^3$ is not a module over $S_q^3$.

Considering $TS_q^3$ to be the analogue of a (complexified) tangent space of $S_q^3$, we would like to introduce a covariant derivative $\nabla_X$ on a (right) $S_q^3$-module $M$, for $X \in TS_q^3$. Since the basis elements of $TS_q^3$ act as $q$-deformed derivations, the connection should obey an analogous $q$-deformed Leibniz rule. The motivating example is when $M = S_q^3$ and the action of $TS_q^3$ is simply $\nabla_X f = X < (f) = X(f)$ for $X \in TS_q^3$ and $f \in S_q^3$. (To lighten notation, in the following we shall drop the symbol $<$ for the left action when there is no risk of ambiguities.)

In fact, let us be slightly more general and consider the action on a free module of rank $n$. Thus, we let $M$ be a free right $S_q^3$-module with basis $\{e_i\}_{i=1}^n$, and write an arbitrary element $m \in M$ as $m = e_i m^i$ for $m^i \in S_q^3$, implicitly assuming a summation over $i$ from 1 to $n$.

Let us define $\nabla^0 : TS_q^3 \times M \to M$ by setting

$$\nabla^0_{X_a}(m) = e_i X_a(m^i) \tag{3.1}$$

for $m = e_i m^i \in M$ (and extending it linearly to all of $TS_q^3$). Now, it is easy to check that

$$\nabla^0_{X_a}(mf) = (\nabla^0_{X_a} m)\sigma_a(f) + mX_a(f)$$

for $f \in S_q^3$ and $m \in M$. Let us generalize these concepts to arbitrary right $S_q^3$-modules.

**Definition 3.2** Let $M$ be a right $S_q^3$-module. A *right $q$-affine connection on $M$* is a map $\nabla : TS_q^3 \times M \to M$ such that

(1) $\nabla_X(\lambda_1 m_1 + \lambda_2 m_2) = \lambda_1 \nabla_X m_1 + \lambda_2 \nabla_X m_2$,
(2) $\nabla_{\lambda_1 X + \lambda_2 Y} m = \lambda_1 \nabla_X m + \lambda_2 \nabla_Y m$,
(3) $\nabla_{X_a}(mf) = (\nabla_{X_a} m)\sigma_a(f) + mX_a(f), \quad a = \pm, z$,

for $m, m_1, m_2 \in M$, $f \in S_q^3$, $X \in TS_q^3$ and $\lambda_1, \lambda_2 \in \mathbb{C}$.

***Remark 3.3*** As mentioned previously, the space $TS_q^3$ is not a module over $S_q^3$. Thus we are not requiring $S_q^3$-linearity 'in the first argument', that is we are not requiring the connection to satisfy the relation $\nabla_{Xf}\, m = (\nabla_X\, m) f)$ for $f \in S_q^3$. This is in contrast with what happens in braided geometry [4, 6] where for a braided commutative algebra the braided Lie algebra of its braided derivations is a module over the algebra and such a relation on the connection can be stated. Braided commutativity of a Hopf algebra is a feature of its being cotriangular.

**Definition 3.4** A *Hermitian form on a right $S_q^3$-module $M$ is a map $h : M \times M \to S_q^3$ such that*

$$
\begin{aligned}
&h(m_1, m_2 f) = h(m_1, m_2) f \quad h(m_1, m_2)^* = h(m_2, m_1),\\
&h(m_1 + m_2, m_3) = h(m_1, m_3) + h(m_2, m_3),
\end{aligned}
$$

for $f \in S_q^3$ and $m_1, m_2, m_3 \in M$. Moreover, $h$ is said to be *invertible* if the induced map $\hat{h} : M \to M^*$, defined by $\hat{h}(m_1)(m_2) = h(m_1, m_2)$, is bijective.

On a free module with basis $\{e_i\}_{i=1}^n$, a Hermitian form is given by $h_{ij} = h_{ji}^* \in S_q^3$ by setting

$$h(m_1, m_2) = (m_1^i)^* h_{ij} m_2^j$$

for $m_1 = e_i m_1^i \in (S_q^3)^n$ and $m_2 = e_i m_2^i \in (S_q^3)^n$. Moreover, if $h$ is invertible, then there exist $h^{ij} \in S_q^3$ such that $h^{ij} h_{jk} = \delta_k^i \mathbb{1}$. In case the module is projective (but not necessarily free) and generated by $\{e_i\}_{i=1}^n$, one can find $h^{ij} \in S_q^3$ such that $e_i h^{ij} h_{jk} = e_k$ if the Hermitian form is invertible (see e.g. [3]).

Next, we will introduce a notion of compatibility between a $q$-affine connection and a Hermitian form. To motivate Definition 3.5, let us study the case of free modules. For the $q$-affine connection $\nabla^0$ in (3.1), one finds that

$$
\begin{aligned}
X_+\big(h(m_1, m_2)\big) &= X_+\big((m_1^i)^* h_{ij} m_2^j\big)\\
&= (m_1^i)^* X_+(h_{ij} m_2^j) + X_+\big((m_1^i)^*\big) K^2(h_{ij} m_2^j)\\
&= (m_1^i)^* h_{ij} X_+(m_2^j) + (m_1^i)^* X_+(h_{ij}) K^2(m_2^j)\\
&\quad + X_+\big((m_1^i)^*\big) K^2(h_{ij} m_2^j).
\end{aligned}
$$

For the connection $\nabla^0$, a natural requirement for the compatibility with $h$ is to demand that $X_+(h_{ij}) = 0$. Then, from (2.7) $X_+(f^*) = -(K^{-2} X_-(f))^* = -K^2(X_-(f))^*$, and one has,

$$
\begin{aligned}
X_+\big(h(m_1, m_2)\big) &= (m_1^i)^* h_{ij} X_+(m_2^j) + X_+\big((m_1^i)^*\big) K^2(h_{ij} m_2^j)\\
&= (m_1^i)^* h_{ij} X_+(m_2^j) - \big(K^{-2} X_-(m_1)\big)^* K^2(h_{ij} m_2^j)\\
&= (m_1^i)^* h_{ij} X_+(m_2^j) - K^2\big(X_-(m_1)^*\big) K^2(h_{ij} m_2^j)\\
&= (m_1^i)^* h_{ij} X_+(m_2^j) - K^2\big(X_-(m_1)^* h_{ij} m_2^j\big)\\
&= h\big(m_1, \nabla_{X_+}^0 m_2\big) - K^2\big(h(\nabla_{X_-}^0 m_1, m_2)\big).
\end{aligned}
$$

Corresponding formulas are easily worked out for $\nabla_{X_-}^0, \nabla_{X_z}^0$, and we shall take this as a motivation for the following definition.

**Definition 3.5** A $q$-affine connection $\nabla$ on a right $S_q^3$-module $M$ is compatible with the Hermitian form $h : M \times M \to S_q^3$ if

$$X_+\big(h(m_1, m_2)\big) = h\big(m_1, \nabla_{X_+} m_2\big) - K^2\big(h(\nabla_{X_-} m_1, m_2)\big), \tag{3.2}$$
$$X_-\big(h(m_1, m_2)\big) = h\big(m_1, \nabla_{X_-} m_2\big) - K^2\big(h(\nabla_{X_+} m_1, m_2)\big), \tag{3.3}$$
$$X_z\big(h(m_1, m_2)\big) = h\big(m_1, \nabla_{X_z} m_2\big) - K^4\big(h(\nabla_{X_z} m_1, m_2)\big), \tag{3.4}$$

for $m_1, m_2 \in M$.

Note that (3.2) and (3.3) are equivalent since

$$\Big(X_+\big(h(m_2, m_1)\big) - h(m_2, \nabla_{X_+} m_1) + K^2\big(h(\nabla_{X_-} m_2, m_1)\big)\Big)^* \\ = -K^{-2}\Big(X_-\big(h(m_1, m_2)\big) + K^2\big(h(\nabla_{X_+} m_1, m_2)\big) - h(m_1, \nabla_{X_-} m_2)\Big).$$

In the case of a $q$-affine connection on a free module, one can derive a convenient parametrization of all connections that are compatible with a given Hermitian form. To this end, let us introduce some notation. Let $(S_q^3)^n$ be a free right $S_q^3$-module with basis $\{e_i\}_{i=1}^n$. A $q$-affine connection $\nabla$ on $(S_q^3)^n$ can be determined by specifying the Christoffel symbols

$$\nabla_{X_a} e_i = e_j \Gamma^j_{ai},$$

with $\Gamma^j_{ai} \in S_q^3$ for $a = \pm, z$ and $i, j = 1, \ldots, n$, and setting

$$\nabla_{X_a}(e_i m^i) = (\nabla_{X_a} e_i)\sigma_a(m^i) + e_i X_a(m^i) = e_j\big(\Gamma^j_{ai}\sigma_a(m^i) + X_a(m^j)\big).$$

The next result gives the form of the Christoffel symbols for a $q$-affine connection compatible with an invertible Hermitian form on a free module.

**Proposition 3.6** *Let $(S_q^3)^n$ be a free right $S_q^3$-module with a basis $\{e_i\}_{i=1}^n$ and let $\nabla$ be a $q$-affine connection on $(S_q^3)^n$ given by the Christoffel symbols $\nabla_a e_i = e_j \Gamma^j_{ai}$. Furthermore, assume that $h$ is an invertible Hermitian form on $(S_q^3)^n$ and set $h_{ij} = h(e_i, e_j)$. Then $\nabla$ is compatible with $h$ if and only if there exist $\gamma_{ij}, \rho_{ij} \in S_q^3$ such that $\rho_{ij}^* = \rho_{ji}$ and*

$$\Gamma^i_{+j} = h^{ik}\big(\tfrac{1}{2} X_+(h_{kj}) + K(\gamma_{kj})\big), \tag{3.5}$$
$$\Gamma^i_{-j} = h^{ik}\big(\tfrac{1}{2} X_-(h_{kj}) + K(\gamma^*_{jk})\big), \tag{3.6}$$
$$\Gamma^i_{zj} = h^{ik}\big(\tfrac{1}{2} X_z(h_{kj}) + K^2(\rho_{kj})\big). \tag{3.7}$$

***Proof*** Let us start by showing that if (3.2)–(3.4) in Definition 3.5 hold for a set of generators of the module, then the equations hold for all elements of the module. Thus, for $m_1 = e_i m_1^i$ and $m_2 = e_j m_2^j$, one computes

$$\begin{aligned}
&h\big(e_i m_1^i, \nabla_{X_+}(e_j m_2^j)\big) - K^2\big(h(\nabla_{X_-}(e_i m_1^i), e_j m_2^j)\big)\\
&\quad= (m_1^i)^* h\big(e_i, (\nabla_{X_+} e_j)\sigma_+(m_2^j) + e_j X_+(m_2^j)\big)\\
&\qquad - K^2\big(h((\nabla_{X_-} e_i)\sigma_-(m_1^i) + e_i X_-(m_1^i), e_j)m_2^j\big)\\
&\quad= (m_1^i)^* h\big(e_i, \nabla_{X_+} e_j\big)\sigma_+(m_2^j) + (m_1^i)^* h\big(e_i, e_j\big)X_+(m_2^j)\\
&\qquad - K^2\big(\sigma_-(m_1^i)^* h(\nabla_{X_-} e_i, e_j)m_2^j\big) - K^2\big(X_-(m_1^i)^* h(e_i, e_j)m_2^j\big),
\end{aligned}$$

and using that $\sigma_+ = \sigma_- = K^2$, $K(f)^* = K^{-1}(f^*)$ and $X_-(f)^* = -K^{-2}X_+(f^*)$ one may rewrite the above expression as

$$\begin{aligned}
&h\big(e_i m_1^i, \nabla_{X_+}(e_j m_2^j)\big) - K^2\big(h(\nabla_{X_-}(e_i m_1^i), e_j m_2^j)\big)\\
&\quad= (m_1^i)^* \Big(h(e_i, \nabla_{X_+} e_j) - K^2\big(h(\nabla_{X_-} e_i, e_j)\big)\Big)K^2(m_2^j)\\
&\qquad + (m_1^i)^* h\big(e_i, e_j\big)X_+(m_2^j) + X_+(m_1^*)K^2\big(h(e_i, e_j)m_2^j\big).
\end{aligned}$$

Now, assuming that (3.2) holds for $m_1 = e_i$ and $m_2 = e_j$, i.e.

$$X_+(h(e_i, e_j)) = h(e_i, \nabla_{X_+} e_j) - K^2\big(h(\nabla_{X_-} e_i, e_j)\big),$$

one obtains

$$\begin{aligned}
&h\big(e_i m_1^i, \nabla_{X_+}(e_j m_2^j)\big) - K^2\big(h(\nabla_{X_-}(e_i m_1^i), e_j m_2^j)\big)\\
&\quad= (m_1^i)^* X_+\big(h(e_i, e_j)\big)K^2(m_2^j) + (m_1^i)^* h\big(e_i, e_j\big)X_+(m_2^j) + X_+(m_1^*)K^2\big(h(e_i, e_j)m_2^j\big)\\
&\quad= (m_1^i)^* X_+\big(h(e_i, e_j)m_2^j\big) + X_+(m_1^*)K^2\big(h(e_i, e_j)m_2^j\big)\\
&\quad= X_+\big((m_1^i)^* h(e_i, e_j)m_2^j\big) = X_+\big(h(e_i m_1^i, e_j m_2^j)\big),
\end{aligned}$$

by using that $f X_+(g) + X_+(f)\sigma_+(g) = X_+(fg)$. An analogous computation corresponding to (3.4) shows that one indeed only needs to check (3.2)–(3.4) for a set of generators.

It is then straight forward to check that the $q$-affine connection $\nabla$, defined by

$$\nabla_{X_a} e_i = e_j \Gamma_{ai}^j$$

with $\Gamma^j_{+i}, \Gamma^j_{-i}, \Gamma^j_{zi}$ given by (3.5)–(3.7) defines a connection compatible with $h$. For instance,

$$\begin{aligned}
h\big(e_i, \nabla_{X_+} e_j\big) - K^2\big(h(\nabla_{X_-} e_i, e_j)\big) &= h_{ik}\Gamma^k_{+j} - K^2\big((\Gamma^k_{-i})^* h_{kj}\big)\\
&= h_{ik}\Gamma^k_{+j} - K^2\big((h_{jk}\Gamma^k_{-i})^*\big)\\
&= h_{ik}h^{kl}\big(\tfrac{1}{2}X_+(h_{lj}) + K(\gamma_{lj})\big) - K^2\Big(\big(h_{jk}h^{kl}\left(\tfrac{1}{2}X_-(h_{li}) + K(\gamma^*_{il})\right)\big)^*\Big)\\
&= \tfrac{1}{2}X_+(h_{ij}) + K(\gamma_{ij}) - K^2\big(\tfrac{1}{2}X_-(h_{ji})^* + K(\gamma^*_{ij})^*\big)\\
&= \tfrac{1}{2}X_+(h_{ij}) + K(\gamma_{ij}) - K^2\big(-\tfrac{1}{2}K^{-2}X_+(h_{ij}) + K^{-1}(\gamma_{ij})\big)\\
&= \tfrac{1}{2}X_+(h_{ij}) + K(\gamma_{ij}) + \tfrac{1}{2}X_+(h_{ij}) - K(\gamma_{ij}) = X_+(h_{ij}).
\end{aligned}$$

Conversely, assume that the connection $\nabla$ is compatible with $h$, and write $\nabla_{X_a} e_i = e_j\Gamma^j_{ai}$. From the compatibility condition (3.2) one finds that the Christoffel symbols satisfy

$$\begin{aligned}
X_+(h_{ij}) &= h_{ik}\Gamma^k_{+j} - K^2\big((\Gamma^k_{-i})^* h_{kj}\big)\\
&= h_{ik}\Gamma^k_{+j} - K^2\big((h_{jk}\Gamma^k_{-i})^*\big)\\
&= \Gamma_{+,ij} - K^2\big(\Gamma^*_{-,ji}\big),
\end{aligned}$$

with $\Gamma_{a,ij} = h_{ik}\Gamma^k_{aj}$, which can be written as

$$\Gamma_{+,ij} = X_+(h_{ij}) + K^2\big(\Gamma^*_{-,ji}\big). \tag{3.8}$$

Defining

$$\gamma_{ij} = K^{-1}(\Gamma_{-,ji})^* + \tfrac{1}{2}K^{-1}X_+(h_{ij})$$

it follows immediately that $\Gamma_{-,ij} = \frac{1}{2}X_-(h_{ij}) + K(\gamma^*_{ji})$, and (3.8) implies that

$$\Gamma_{+,ij} = \tfrac{1}{2}X_+(h_{ij}) + K(\gamma_{ij}),$$

giving (3.5) and (3.6) via $\Gamma^i_{aj} = h^{ik}\Gamma_{a,kj}$. Similarly, (3.4) implies that

$$X_z(h_{ij}) = \Gamma_{z,ij} - K^4\big((\Gamma_{z,ji})^*\big), \tag{3.9}$$

and defining

$$\rho_{ij} = K^{-2}(\Gamma_{z,ij}) - \tfrac{1}{2}K^{-2}X_z(h_{ij})$$

it follows immediately that $\Gamma_{z,ij} = \frac{1}{2}X_z(h_{ij}) + K^2(\rho_{ij})$, and (3.9) implies that $\rho_{ij} = \rho^*_{ji}$. $\square$

Thus the previous proposition gives the general class of $q$-affine connections which are compatible with an invertible Hermitian form on a free right $S_q^3$-module. Later on in Sect. 4, on the right free $S_q^3$-module $\Omega^1(S_q^3)$ we shall select a subclass of these that are also torsion free.

### 3.1 *q*-Affine Connections on Projective Modules

As expected, one can construct $q$-affine connections on projective modules. More precisely, one proves the following result.

**Proposition 3.7** *Let $M$ be a right $S_q^3$-module and let $\nabla$ be a $q$-affine connection on $M$. Given a projection on $M$, i.e. an endomorphism $p : M \to M$ such that $p^2 = p$, then $p \circ \nabla$ is a $q$-affine connection on the right $S_q^3$-module $p(M)$.*

***Proof*** Since $\nabla$ is a $q$-affine connection and $p$ is an endomorphism, it is immediate that $\widetilde{\nabla} = p \circ \nabla$ satisfies properties (3.2) and (3.2) in Definition 3.2. Moreover, for $m \in p(M)$

$$\begin{aligned}\widetilde{\nabla}_{X_a}(mf) &= p\big(\nabla_{X_a}(mf)\big) = p\big((\nabla_{X_a}m)\,\sigma_a(f)\big) + p\big(mX_a(f)\big)\\ &= (\widetilde{\nabla}_{X_a}m)\,\sigma_a(f) + mX_a(f),\end{aligned}$$

since $p(m) = m$ when $m \in p(M)$. We conclude that $\widetilde{\nabla}$ is a $q$-affine connection on $p(M)$. □

Since we have shown in the previous section that $q$-affine connections exist on free modules, Proposition 3.7 implies that every projective $S_q^3$-module can be equipped with a $q$-affine connection. Moreover, let $\nabla$ and $\widetilde{\nabla}$ be $q$-affine connections on a $S_q^3$-module $M$ and define

$$\alpha(X, m) = \nabla_X m - \widetilde{\nabla}_X m.$$

Then $\alpha : TS_q^3 \times M \to M$ satisfies

$$\alpha(\lambda X + \mu Y, m_1) = \lambda\alpha(X, m_1) + \mu\alpha(Y, m_1), \tag{3.10}$$
$$\alpha(X, m_1 f + m_2 g) = \alpha(X, m_1)f + \alpha(X, m_2)g, \tag{3.11}$$

for $m_1, m_2 \in M$, $X, Y \in TS_q^3$, $f, g \in S_q^3$ and $\lambda, \mu \in \mathbb{C}$. Conversely, every $q$-affine connection on a projective module $M$ can be written as

$$\nabla_X m = p(\nabla_X^0 m) + \alpha(X, m),$$

where $\nabla^0$ is the connection defined in (3.1) and $\alpha : TS_q^3 \times M \to M$ is an arbitrary map satisfying (3.10) and (3.11). Next, let us show that a connection on a projective module is compatible with the restricted metric if the projection is orthogonal.

**Proposition 3.8** *Let $\nabla$ be a $q$-affine connection on the $S_q^3$-module $M$ and assume furthermore that $\nabla$ is compatible with a Hermitian form $h$ on $M$. If $p : M \to M$ is an orthogonal projection, i.e. $p$ is a projection such that, for all $m_1, m_2 \in M$,*

$$h\big(p(m_1), m_2\big) = h\big(m_1, p(m_2)\big)$$

*then $\widetilde{\nabla} = p \circ \nabla$ is a $q$-affine connection on $p(M)$ that is compatible with $h$ restricted to $p(M)$.*

***Proof*** First of all, it follows from Proposition 3.7 that $\widetilde{\nabla} = p \circ \nabla$ is a $q$-affine connection on $p(M)$. Since $p$ is an orthogonal projection, one finds that for $m_1, m_2 \in p(M)$

$$\begin{aligned}
&h\big(m_1, \widetilde{\nabla}_{X_+} m_2\big) - K^2\big(h(\widetilde{\nabla}_{X_-} m_1, m_2)\big)\\
&\quad = h\big(m_1, p(\nabla_{X_+} m_2)\big) - K^2\big(h(p(\nabla_{X_-} m_1), m_2)\big)\\
&\quad = h\big(p(m_1), \nabla_{X_+} m_2\big) - K^2\big(h(\nabla_{X_-} m_1, p(m_2))\big)\\
&\quad = h\big(m_1, \nabla_{X_+} m_2\big) - K^2\big(h(\nabla_{X_-} m_1, m_2)\big) = X_+\big(h(m_1, m_2)\big)
\end{aligned}$$

by using that $\nabla$ is compatible with $h$. A similar computation shows that

$$X_z\big(h(m_1, m_2)\big) = h\big(m_1, \widetilde{\nabla}_{X_z} m_2\big) - K^4\big(h(\widetilde{\nabla}_{X_z} m_1, m_2)\big),$$

from which we conclude that $\widetilde{\nabla}$ is compatible with $h$ restricted to $p(M)$. □

## 4 A $q$-Affine Levi–Civita Connection on $\Omega^1(S_q^3)$

In this section we shall construct a $q$-affine connection on $\Omega^1(S_q^3)$, compatible with an invertible Hermitian form $h$ and satisfying a certain torsion freeness condition. The module $\Omega^1(S_q^3)$ is a free $S_q^3$-module of rank 3 with basis $\omega_+, \omega_-, \omega_z$ which implies that the results of Proposition 3.6 may be used. Although $\Omega^1(S_q^3)$ has a bimodule structure, we shall only consider the right module structure of $\Omega^1(S_q^3)$ in what follows. In the case of a $q$-affine connection on $\Omega^1(S_q^3)$, there is a natural definition of torsion freeness, suggested by the relations (2.4)–(2.6).

We have already mentioned that those relations reduce to the Lie algebra of su(2) in the classical limit of $q = 1$. These relations are reflected in the notion of the torsion $T$. For instance one would have $T(X_-, X_+) := \nabla_- X_+ - \nabla_+ X_- - [X_-, X_+]$ and its vanishing is just the condition $\nabla_- X_+ - \nabla_+ X_- = [X_-, X_+] = X_z$; for dual forms this translates into $\nabla_- \omega_+ - \nabla_+ \omega_- = \omega_z$. There are similar expressions the other two cases. Given the duality between the derivations $X_a$ and the basis forms $\omega_a$, for $a = \pm, z$, we propose the following definition for a torsion freeness condition on the connection.

**Definition 4.1** A $q$-affine connection $\nabla$ on $\Omega^1(S_q^3)$ is *torsion free* if

$$\nabla_-\omega_+ - q^2\nabla_+\omega_- = \omega_z, \tag{4.1}$$
$$q^2\nabla_z\omega_- - q^{-2}\nabla_-\omega_z = (1+q^2)\omega_-, \tag{4.2}$$
$$q^2\nabla_+\omega_z - q^{-2}\nabla_z\omega_+ = (1+q^2)\omega_+. \tag{4.3}$$

In the following, we will construct a torsion free $q$-affine connection on $\Omega^1(S_q^3)$ that is compatible with a Hermitian form. We call a connection satisfying these conditions a $q$-affine Levi–Civita connection. As it turns out, for such connections to exist, the Hermitian form needs to satisfy a compatibility condition.

Deriving a family of metric and torsion free connections under some conditions undermines in general the classical uniqueness result for such a (Levi–Civita) connection. This seems to be a common feature of the study of linear connections in the framework of truly non commutative algebras (and not just braided-commutative). It is however interesting to see when and why metric and torsion free connections are unique or not. A natural question would then be under which additional 'natural' conditions is it possible to single out a canonical connection. This problem will be addressed elsewhere.

**Proposition 4.2** *Let $h$ be an invertible Hermitian form on the (right) $S_q^3$-module $\Omega^1(S_q^3)$ and write $h_{ab} = h(\omega_a, \omega_b)$. A $q$-affine Levi–Civita connection on $\Omega^1(S_q^3)$ exists if and only if*

$$X_z(h_{++} - q^2h_{--}) = K^2X_-(h_{z+}) - q^2X_-(h_{-z}) - q^2K^2X_+(h_{z-}) + X_+(h_{+z}). \tag{4.4}$$

***Proof*** Assume that $h$ is an invertible Hermitian form on $\Omega^1(S_q^3)$ and write $h_{ab} = h(\omega_a, \omega_b)$ with inverse $h^{ab}$. Furthermore, we write $\nabla_a = \nabla_{X_a}$ and

$$\nabla_a\omega_b = \omega_c\Gamma^c_{ab}$$

for $a, b = \pm, z$. In terms of $\Gamma_{a,bc} = h_{bp}\Gamma^p_{ac}$ the torsion free equations (4.1)–(4.3) become

$$\Gamma_{-,a+} - q^2\Gamma_{+,a-} = h_{az}, \tag{4.5}$$
$$q^2\Gamma_{z,a-} - q^{-2}\Gamma_{-,az} = (1+q^2)h_{a-}, \tag{4.6}$$
$$q^2\Gamma_{+,az} - q^{-2}\Gamma_{z,a+} = (1+q^2)h_{a+}. \tag{4.7}$$

Since $\Omega^1(S_q^3)$ is a free (right) module, one can apply the results of Proposition 3.6 to obtain

$$\Gamma_{+,ab} = \tfrac{1}{2}X_+(h_{ab}) + K(\gamma_{ab}),$$
$$\Gamma_{-,ab} = \tfrac{1}{2}X_-(h_{ab}) + K(\gamma^*_{ba}),$$
$$\Gamma_{z,ab} = \tfrac{1}{2}X_z(h_{ab}) + K(\rho_{ab}),$$

for "parameters" $(\gamma_{ab}, \rho_{ab} = \rho^*_{ba})$ in $S^3_q$, giving all $q$-affine connections compatible with $h$.
Inserting the above expressions into (4.5)–(4.7) gives

$$\gamma^*_{+a} - q^2\gamma_{a-} = K^{-1}(h_{az}) - \tfrac{1}{2}K^{-1}X_-(h_{a+}) + \tfrac{1}{2}q^2K^{-1}X_+(h_{a-}) \equiv A_a,$$
$$q^2K(\rho_{a-}) - q^{-2}\gamma^*_{za} = K^{-1}\Big[(1+q^2)h_{a-} - \tfrac{1}{2}q^2X_z(h_{a-}) + \tfrac{1}{2}q^{-2}X_-(h_{az})\Big] \equiv B_a,$$
$$q^2\gamma_{az} - q^{-2}K(\rho_{a+}) = K^{-1}\Big[(1+q^2)h_{a+} - \tfrac{1}{2}q^2X_+(h_{az}) + \tfrac{1}{2}q^{-2}X_z(h_{a+})\Big] \equiv C_a.$$

Note that the right hand sides $A_a$, $B_a$ and $C_a$ only depend on the metric components $h_{ab}$.
The above nine equations can be grouped into three independent sets:
Group 1

$$\gamma^*_{++} - q^2\gamma_{+-} = A_+, \tag{G1.1}$$

$$\gamma^*_{+-} - q^2\gamma_{--} = A_-, \tag{G1.2}$$

Group 2

$$q^2\tilde{\rho}_{+-} - q^{-2}\gamma^*_{z+} = B_+, \tag{G2.1}$$

$$q^2\tilde{\rho}_{z-} - q^{-2}\gamma^*_{zz} = B_z, \tag{G2.2}$$

$$q^2\gamma_{-z} - q^{-2}\tilde{\rho}_{-+} = C_-, \tag{G2.3}$$

$$q^2\gamma_{zz} - q^{-2}\tilde{\rho}_{z+} = C_z, \tag{G2.4}$$

Group 3

$$\gamma^*_{+z} - q^2\gamma_{z-} = A_z, \tag{G3.1}$$

$$q^2\tilde{\rho}_{--} - q^{-2}\gamma^*_{z-} = B_-, \tag{G3.2}$$

$$q^2\gamma_{+z} - q^{-2}\tilde{\rho}_{++} = C_+, \tag{G3.3}$$

where for notational convenience we denoted $\tilde{\rho}_{ab} = K(\rho_{ab})$.

The equations in Group 1 can be solved as

$$\gamma_{++} = A_+^* + q^2\gamma_{+-}^*, \tag{4.8}$$
$$\gamma_{--} = q^{-2}\gamma_{+-}^* - q^{-2}A_-, \tag{4.9}$$

and the equations in Group 2 can be solved as

$$\gamma_{z+} = q^4\tilde{\rho}_{+-}^* - q^2 B_+^*, \tag{4.10}$$
$$\rho_{z-} = q^{-2}K^{-1}(B_z) + q^{-4}K^{-1}(\gamma_{zz}^*), \tag{4.11}$$
$$\gamma_{-z} = q^{-2}C_- + q^{-4}\tilde{\rho}_{-+}, \tag{4.12}$$
$$\rho_{z+} = q^4K^{-1}(\gamma_{zz}) - q^2K^{-1}(C_z). \tag{4.13}$$

Note that the condition $\rho_{ab}^* = \rho_{ba}$ will not pose a problem here, since neither $\rho_{-z}$ nor $\rho_{+z}$ appear in any other equation, and may simply be defined as $\rho_{-z} = \rho_{z-}^*$ and $\rho_{+z} = \rho_{z+}^*$.

For the equations in Group 3, the fact that we require $\rho_{++}^* = \rho_{++}$ and $\rho_{--} = \rho_{--}^*$ gives a non-trivial condition for solutions to exist. From (G3.2) and (G3.3) one obtains

$$\gamma_{z-} = q^4K^{-2}(\tilde{\rho}_{--}) - q^2 B_-^*, \tag{4.14}$$
$$\gamma_{+z} = q^{-2}C_+ + q^{-4}\tilde{\rho}_{++}, \tag{4.15}$$

and inserted into (G3.1) this gives

$$\begin{aligned} &q^{-4}\rho_{++} - q^6\rho_{--} = K(A_z) - q^4K(B_-^*) - q^{-2}K(C_+^*) \quad \Leftrightarrow \\ &\rho_{++} = q^{10}\rho_{--} + q^4K(A_z) - q^8K(B_-^*) - q^2K(C_+^*). \end{aligned} \tag{4.16}$$

A necessary (and sufficient) condition for solutions to exist, is that the right hand side of the above equation is Hermitian. From

$$\begin{aligned} A_z &= K^{-1}\Big[h_{zz} - \tfrac{1}{2}X_-(h_{z+}) + \tfrac{1}{2}q^2X_+(h_{z-})\Big], \\ B_- &= K^{-1}\Big[(1+q^2)h_{--} - \tfrac{1}{2}q^2X_z(h_{--}) + \tfrac{1}{2}q^{-2}X_-(h_{-z})\Big], \\ C_+ &= K^{-1}\Big[(1+q^2)h_{++} - \tfrac{1}{2}q^2X_+(h_{+z}) + \tfrac{1}{2}q^{-2}X_z(h_{++})\Big], \end{aligned}$$

one obtains

$$
\begin{aligned}
K(A_z) &= h_{zz} - \tfrac{1}{2}X_-(h_{z+}) + \tfrac{1}{2}q^2X_+(h_{z-}),\\
K(B_-^*) &= (1+q^2)K^2(h_{--}) + \tfrac{1}{2}q^2K^{-2}X_z(h_{--}) - \tfrac{1}{2}q^{-2}X_+(h_{z-})\\
&= \tfrac{q^2}{2(1-q^{-2})}\big(K^2(h_{--}) + K^{-2}(h_{--})\big) - \tfrac{q^{-2}}{1-q^{-2}}K^2(h_{--}) - \tfrac{1}{2}q^{-2}X_+(h_{z-}),\\
K(C_+^*) &= (1+q^2)K^2(h_{++}) + \tfrac{1}{2}q^2X_-(h_{z+}) - \tfrac{1}{2}q^{-2}K^{-2}X_z(h_{++})\\
&= -\tfrac{q^{-2}}{2(1-q^{-2})}\big(K^2(h_{++}) + K^{-2}(h_{++})\big) + \tfrac{q^2}{1-q^{-2}}K^2(h_{++}), + \tfrac{1}{2}q^2X_-(h_{z+}),
\end{aligned}
$$

by using that $X_z = (1-K^4)/(1-q^{-2})$.
Since $\rho_{--}$ and $h_{zz}$, as well as $K^2(h_{--}) + K^{-2}(h_{--})$ and $K^2(h_{++}) + K^{-2}(h_{++})$, are Hermitian, the non-Hermitian terms of (4.16), which we denote by $S$, become

$$
S = q^6X_+(h_{z-}) - q^4X_-(h_{z+}) + \tfrac{q^6}{1-q^{-2}}K^2(h_{--}) - \tfrac{q^4}{1-q^{-2}}K^2(h_{++}).
$$

Thus, a necessary and sufficient condition for $\rho_{++}$ to be Hermitian is that

$$
\begin{aligned}
0 = S - S^* = {}& q^6X_+(h_{z-}) + q^6K^{-2}X_-(h_{-z}) - q^4X_-(h_{z+}) - q^4K^{-2}X_+(h_{+z})\\
&+ \tfrac{q^6}{1-q^{-2}}K^2(h_{--}) - \tfrac{q^6}{1-q^{-2}}K^{-2}(h_{--}) - \tfrac{q^4}{1-q^{-2}}K^2(h_{++}) + \tfrac{q^4}{1-q^{-2}}K^{-2}(h_{++}).
\end{aligned}
$$

By using that $X_z = (1-K^4)(1-q^{-2})^{-1}$, the above condition can be written as

$$
\begin{aligned}
&q^6X_+(h_{z-}) + q^6K^{-2}X_-(h_{-z})\\
&\quad - q^4X_-(h_{z+}) - q^4K^{-2}X_+(h_{+z}) + K^{-2}X_z\big(q^4h_{++} - q^6h_{--}\big) = 0,
\end{aligned}
$$

which is equivalent to (4.4). Hence, assuming the above relation to hold true, a solution to the torsion free equations, which is also compatible with $h$, is given by (4.8)–(4.16). The free parameters in this solution are $\gamma_{+-}$, $\gamma_{-+}$, $\gamma_{zz}$, $\rho_{+-}$ and $\rho_{--}^* = \rho_{--}$, $\rho_{++}^* = \rho_{++}$. □

Although the general $q$-affine Levi–Civita connection on $\Omega^1(S_q^3)$ may be written down, the expressions are rather lengthy and not particularly illuminating. However, let us explicitly write down a Levi–Civita connection in the particular case of a diagonal metric of the form

$$
h_{--} = h, \quad h_{++} = q^2h, \quad h_{zz} = h_z, \quad h_{ab} = 0 \quad \text{if } a \neq b,
$$

with $h$ and $h_z$ invertible elements of $S_q^3$; note that this choice clearly satisfies (4.4) in Proposition 4.2. Using the solution given by (4.8)–(4.16) in the proof of Proposition 4.4 one finds

$$
\begin{aligned}
\nabla_{X_+}\omega_a &= \omega_b h^{bc}\big(\tfrac{1}{2}X_+(h_{ca}) + K(\gamma_{ca})\big),\\
\nabla_{X_-}\omega_a &= \omega_b h^{bc}\big(\tfrac{1}{2}X_-(h_{ca}) + K(\gamma_{ac}^*)\big),\\
\nabla_{X_z}\omega_a &= \omega_b h^{bc}\big(\tfrac{1}{2}X_z(h_{ca}) + K^2(\rho_{ca})\big),
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma_{++} &= \tfrac{1}{2}q^2K^{-1}X_+(h) + q^2\gamma_{+-}^*,\\
\gamma_{--} &= -\tfrac{1}{2}K^{-1}X_+(h) + q^{-2}\gamma_{+-}^*,\\
\gamma_{+z} &= (1+q^2)K^{-1}(h) + \tfrac{1}{2}q^{-2}K^{-1}X_z(h),\\
\gamma_{z+} &= q^4K^{-1}(\rho_{-+}),\\
\gamma_{-z} &= q^{-4}K(\rho_{-+}),\\
\gamma_{z-} &= q^4K^{-1}(\rho_{--}) - q^2(1+q^2)K^{-1}(h) - \tfrac{1}{2}q^4K^{-3}X_z(h),
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_{z+} &= q^4K^{-1}(\gamma_{zz}) + q^4K^{-2}X_+(h_z),\\
\rho_{+z} &= \rho_{z+}^* = q^4K(\gamma_{zz}^*) - q^4X_-(h_z),\\
\rho_{z-} &= \tfrac{1}{2}q^{-4}K^{-2}X_-(h_z) + q^{-4}K^{-1}(\gamma_{zz}^*),\\
\rho_{-z} &= \rho_{z-}^* = -\tfrac{1}{2}q^{-4}X_+(h_z) + q^{-4}K(\gamma_{zz}),\\
\rho_{++} &= q^{10}\rho_{--} + q^4h_z - \tfrac{1}{2}q^4(1+q^2)(1+q^4)\big(K^2(h) + K^{-2}(h)\big).
\end{aligned}
$$

Furthermore, setting $\gamma_{+-} = \rho_{-+} = \gamma_{zz} = \rho_{--} = 0$ one obtains

$$
\begin{aligned}
\nabla_+\omega_+ &= \omega_+h^{-1}X_+(h),\\
\nabla_+\omega_- &= \omega_z\frac{h_z^{-1}}{1-q^{-2}}\Big(\big(1-\tfrac{1}{2}q^4\big)K^2(h) - \tfrac{1}{2}q^4K^{-2}(h)\Big),\\
\nabla_+\omega_z &= \omega_+q^{-2}h^{-1}\Big(K^2(h_z) + \tfrac{1}{1-q^{-2}}\Big(\Big(q^2-\tfrac{1}{2}q^6\Big)h - \tfrac{1}{2}q^6K^4(h)\Big)\Big) + \omega_z\tfrac{1}{2}h_z^{-1}X_+(h_z),\\
\nabla_-\omega_+ &= \omega_z + \omega_z\frac{h_z^{-1}}{1-q^{-2}}\Big(\big(q^2-\tfrac{1}{2}q^6\big)K^2(h) - \tfrac{1}{2}q^6K^{-2}(h)\Big),\\
\nabla_-\omega_- &= \omega_-h^{-1}X_-(h),\\
\nabla_-\omega_z &= \omega_-\frac{h^{-1}}{1-q^{-2}}\Big(\Big(1-\tfrac{1}{2}q^4\Big)h - \tfrac{1}{2}q^4K^4(h)\Big) + \omega_z\tfrac{1}{2}h_z^{-1}X_-(h_z),\\
\nabla_z\omega_+ &= \omega_z\tfrac{1}{2}q^4h_z^{-1}X_+(h_z) + \omega_+q^2h^{-1}K^2(h_z) + \omega_+\frac{h^{-1}}{1-q^{-2}}\Big(\Big(1-\tfrac{1}{2}q^8\Big)h - \tfrac{1}{2}q^8K^4(h)\Big),\\
\nabla_z\omega_- &= \omega_z\tfrac{1}{2}h_z^{-1}q^{-4}X_-(h_z) + \omega_-\frac{h^{-1}}{1-q^{-2}}\big(\tfrac{1}{2}h - \tfrac{1}{2}K^4(h)\big),\\
\nabla_z\omega_z &= \omega_z\tfrac{1}{2}h_z^{-1}X_z(h_z) - \omega_+\tfrac{1}{2}q^2h^{-1}K^2X_-(h_z) - \omega_-\tfrac{1}{2}q^{-4}h^{-1}K^2X_+(h_z),
\end{aligned}
$$

giving a $q$-affine Levi–Civita connection on $\Omega^1(S_q^3)$ with respect to the Hermitian form $h$.

## 5 The Quantum 2-Sphere

The noncommutative (standard) Podleś sphere $S^2_q$ [20] can be considered as a subalgebra of $S^3_q$ by identifying the generators $B_0$, $B_+$, $B_-$ of $S^2_q$ as

$$B_0 = cc^* \quad B_+ = ca^* \quad B_- = ac^* = B_+^*,$$

satisfying then the relations

$$\begin{aligned} &B_- \, B_0 = q^2 \, B_0 \, B_- \quad B_+ \, B_0 = q^{-2} \, B_0 \, B_+, \\ &B_- \, B_+ = q^2 \, B_0 \left(\mathbb{1} - q^2 \, B_0\right) \quad B_+ \, B_- = B_0 \left(\mathbb{1} - B_0\right). \end{aligned}$$

These elements generate the fix-point algebra of the right $U(1)$-action

$$\alpha_z(a) = az \quad \alpha_z(a^*) = a^*\bar{z} \quad \alpha_z(c) = cz \quad \alpha_z(c^*) = c^*\bar{z} \tag{5.1}$$

for $z \in U(1)$ and $a \in S^3_q$, related to the $U(1)$-Hopf-fibration $S^2_q \hookrightarrow S^3_q$. Equivalently, the sphere $S^2_q$ is the invariant subalgebra of $S^3_q$ for the left action of $K$: $S^2_q = \{f \in S^3_q,\ K \triangleright f = f\}$. Then, the left action of the $X_a$ does not preserve the algebra $S^2_q$ (since their left action does not commute with that of $K$): one readily computes,

$$\begin{aligned} &X_+ \triangleright B_0 = qa^*c^* \quad X_- \triangleright B_0 = -q^{-1}ca \quad X_z \triangleright B_0 = 0, \\ &X_+ \triangleright B_+ = q(a^*)^2 \quad X_- \triangleright B_+ = c^2 \quad X_z \triangleright B_+ = 0, \\ &X_+ \triangleright B_- = q^2(c^*)^2 \quad X_- \triangleright B_- = -q^{-1}(a)^2 \quad X_z \triangleright B_- = 0. \end{aligned}$$

Note, however, that the right action of $X_a$ leaves $S^2_q$ invariant; i.e. $f \triangleleft X_a \in S^2_q$ for $f \in S^2_q$ and $a = \pm, z$. This is shown explicitly in Eq. (A.1) in the "Appendix".

### 5.1 A Left Covariant Calculus

Since the element $X_z$ acts trivially (on the left) on $S^2_q$, the differential (2.3) when restricted to $f \in S^2_q$ becomes

$$df = (X_- \triangleright f)\,\omega_- + (X_+ \triangleright f)\,\omega_+. \tag{5.2}$$

Moreover, when acting on $S^2_q$ both $X_+$ and $X_-$ are usual derivations since $K$ and then $\sigma_z$ are the identity on $S^2_q$. Classically, the form (5.2) of the differential that uses left invariant vector fields and forms can be seen as identifying the cotangent bundle of $S^2$ with the direct sum of the line bundles of 'charge' $\pm 2$, that is $\Omega^1(S^2) \simeq \mathcal{L}_{-2}\omega_- \oplus \mathcal{L}_{+2}\omega_+$. This identification can be used also for the quantum sphere $S^2_q$ with the line bundles defined as in (5.4).

In particular from (5.2) one finds

$$
\begin{aligned}
dB_+ &= q\,(a^*)^2\,\omega_+ + c^2\,\omega_-,\\
dB_- &= -q^2\,(c^*)^2\,\omega_+ - q^{-1}\,a^2\,\omega_-,\\
dB_0 &= c^*a^*\,\omega_+ - q^{-1}ca\,\omega_-,
\end{aligned}
$$

which can be inverted to yield

$$
\begin{aligned}
\omega_+ &= q^{-1}a^2\,dB_+ - q^2c^2\,dB_- + (1+q^2)ac\,dB_0,\\
\omega_- &= (c^*)^2\,dB_+ - q(a^*)^2\,dB_- - (1+q^2)c^*a^*\,dB_0,
\end{aligned}
$$

implying that the differential in (5.2) can be expressed as

$$
\begin{aligned}
df = &\big(q^{-1}(X_+ \triangleright f)\,a^2 + (X_- \triangleright f)\,(c^*)^2\big)dB_+\\
&- \big(q^2(X_+ \triangleright f)\,c^2 + q(X_- \triangleright f)\,(a^*)^2\big)dB_-\\
&+ (1+q^2)\big((X_+ \triangleright f)\,ac - (X_- \triangleright f)\,c^*a^*\big)dB_0.
\end{aligned}
\tag{5.3}
$$

In spite of the fact that $X_\pm \triangleright f \notin S_q^2$, from the commutation relations $K X_\pm = q^\mp X_\mp K$ one infer that all coefficients are in $S_q^2$. For instance: $K \triangleleft \big((X_+ \triangleright f)\,a^2\big) = \big((K X_+ \triangleright f)\,K \triangleleft a^2\big) = (q X_+ \triangleright f)\,q^{-1} \triangleleft a^2 = (X_+ \triangleright f)\,a^2$, and similarly for the other terms.

### 5.2 Connections on Projective Modules over $S_q^2$

The definition of $q$-affine connections applies equally well to the subalgebra $S_q^2$. The right actions of $X_\pm$, $X_z$ preserve $S_q^2$ [cf. (A.1) in the "Appendix"], and thus restrict to twisted derivations on $S_q^2$. However, even classically it is not possible to find two vector fields that span the tangent space of $S^2$ at each point. This is a consequence of the fact that the module of vector fields on $S^2$ is not a free module and one needs at least three vector fields to generate the module of vector fields. Analogously, for the quantum 2-sphere, even though the right actions of $X_\pm$, $X_z$ are related, as shown in (A.2), there is no global way of writing e.g $X_z$ as a $S_q^2$-linear combination of $X_\pm$. Hence, for a $q$-affine connection $\nabla$ on a $S_q^2$-module $M$ we still need three operators, that is a map

$$
\nabla : \mathbb{C}\langle X_+, X_-, X_z\rangle \times M \to M
$$

satisfying the conditions of Definition 3.2. Moreover, even if a $q$-affine connection is not $S_q^2$-linear in its first argument, one expects a relation among the covariant derivatives, although this needs not be immediately implied by the relation (A.2) on the derivations. In this section, we construct $q$-affine connections on a class of projective modules over $S_q^2$.

The quantum Peter–Weyl theorem for $S_q^3$ results into an explicit (vector space) decomposition of the algebra $S_q^3$, that is $S_q^3 = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$, with

$$\mathcal{L}_n = \{f \in S_q^3 : \alpha_z(f) = \bar{z}^n f\}, \tag{5.4}$$

for the $U(1)$ action $\alpha_z$ in (5.1). Equivalently, $\mathcal{L}_n = \left\{f \in S_q^3\,,\ K \triangleright f = q^{-\frac{n}{2}} f\right\}$. It follows that $\mathcal{L}_0 = S_q^2$, as well as $\mathcal{L}_n\mathcal{L}_m \subseteq \mathcal{L}_{n+m}$. Clearly, the right action of $\mathcal{U}_q(\mathrm{su}(2))$ leaves each $\mathcal{L}_n$ invariant. On the other hand, for the left action one has $X_\pm \triangleright \mathcal{L}_n \subset \mathcal{L}_{n\mp 2}$. It is easy to see that $\mathcal{L}_n$ is a $S_q^2$-bimodule. For $f, g \in S_q^2$ and $\psi_n \in \mathcal{L}_n$,

$$\alpha_z(f\psi_n g) = \alpha_z(f)\alpha_z(\psi_n)\alpha_z(g) = \bar{z}^n(f\psi_n g),$$

which says that $\mathcal{L}_n$ is a $S_q^2$-bimodule. As a right (or equivalently left) module, each $\mathcal{L}_n$ can be realised as a finitely generated projective $S_q^2$-module as we now briefly recall (cf. [11, 15, 17]).

For $n \geq 0$ and $\mu = 0, 1, \ldots, n$, let $(\Psi_n)_\mu, (\Phi_n)_\mu \in S_q^3$ be given as

$$(\Phi_n)_\mu = \sqrt{\alpha_{n\mu}}c^{n-\mu}a^\mu \quad (\Psi_n)_\mu = \sqrt{\beta_{n\mu}}(c^*)^\mu(a^*)^{n-\mu}$$

with

$$\alpha_{n\mu} = \prod_{k=0}^{n-\mu-1} \frac{1-q^{2(n-k)}}{1-q^{2(k+1)}} \quad \beta_{n\mu} = q^{2\mu}\prod_{k=0}^{\mu-1} \frac{1-q^{-2(n-k)}}{1-q^{-2(k+1)}}.$$

It is straight-forward to check that

$$\sum_{\mu=0}^{n}(\Phi_n)^*_\mu(\Phi_n)_\mu = \sum_{\mu=0}^{n}(\Psi_n)^*_\mu(\Psi_n)_\mu = \mathbb{1},$$

implying that

$$\begin{aligned}(p_n)^\mu{}_\nu &= (\Psi_n)_\mu(\Psi_n)^*_\nu = \sqrt{\beta_{n\mu}\beta_{n\nu}}(c^*)^\mu(a^*)^{n-\mu}a^{n-\nu}c^\nu,\\ (p_{-n})^\mu{}_\nu &= (\Phi_n)_\mu(\Phi_n)^*_\nu = \sqrt{\alpha_{n\mu}\alpha_{n\nu}}c^{n-\mu}a^\mu(a^*)^\nu(c^*)^{n-\nu},\end{aligned}$$

satisfy $p_n^2 = p_n$ and $p_{-n}^2 = p_{-n}$. Moreover, it is easy to see that the entries $(p_n)^\mu{}_\nu$ and $(p_{-n})^\mu{}_\nu \in S_q^2$, which implies that one has finitely generated projective $S_q^2$-modules

$$M_n = \begin{cases} p_n(S_q^2)^{n+1} & \text{if } n \geq 0,\\ p_{-|n|}(S_q^2)^{|n|+1} & \text{if } n < 0.\end{cases}$$

These modules $M_n$ are isomorphic as right $S_q^2$-modules to $\mathcal{L}_n$ for each $n \in \mathbb{Z}$.

Now, let $\{e_\mu\}_{\mu=0}^{n}$ be a basis of $(S_q^2)^{n+1}$. Given an invertible Hermitian form $h$ on $(S_q^2)^{n+1}$, the proof of Proposition 3.6 (repeated verbatimly for the algebra $S_q^2$), gives a $q$-affine connection on $(S_q^2)^{n+1}$ compatible with $h$ as

$$\begin{aligned}
\widetilde{\nabla}_{X_+} e_\mu &= e_\nu \Gamma^{\nu}_{+\mu} = e_\nu h^{\nu\rho}\big(\tfrac{1}{2} h_{\rho\mu} \triangleleft X_+ + a_{\rho\mu} \triangleleft K\big),\\
\widetilde{\nabla}_{X_-} e_\mu &= e_\nu \Gamma^{\nu}_{-\mu} = e_\nu h^{\nu\rho}\big(\tfrac{1}{2} h_{\rho\mu} \triangleleft X_- + a^*_{\mu\rho} \triangleleft K\big),\\
\widetilde{\nabla}_{X_z} e_\mu &= e_\nu \Gamma^{\nu}_{z\mu} = e_\nu h^{\nu\rho}\big(\tfrac{1}{2} h_{\rho\mu} \triangleleft X_z + b_{\rho\mu} \triangleleft K^2\big),
\end{aligned}$$

for arbitrary $a_{\mu\nu}, b_{\mu\nu} \in S_q^2$ such that $b^*_{\mu\nu} = b_{\nu\mu}$. If $n \geq 0$ then $\hat{e}_\mu = e_\nu (p_n)^\nu_\mu$ are generators of $M_n = p_n (S_q^2)^{n+1}$ and Proposition 3.7 applied (mutatis mutandis) to $S_q^2$ implies that $\nabla = p_n \circ \widetilde{\nabla}$ is a $q$-affine connection on $M_n$ with

$$\begin{aligned}
\nabla_{X_+} \hat{e}_\mu &= p_n\big(\widetilde{\nabla}_{X_+} e_\nu (p_n)^{\nu}{}_{\mu}\big) = p_n(\widetilde{\nabla}_{X_+} e_\nu)\big((p_n)^{\nu}{}_{\mu} \triangleleft K^2\big) + \hat{e}_\nu\big((p_n)^{\nu}{}_{\mu} \triangleleft X_+\big)\\
&= \hat{e}_\gamma h^{\gamma\rho} q^{2(\mu-\nu)}\big(\tfrac{1}{2} h_{\rho\nu} \triangleleft X_+ + a_{\rho\nu} \triangleleft K\big)(p_n)^{\nu}{}_{\mu} + \hat{e}_\nu\big((p_n)^{\nu}{}_{\mu} \triangleleft X_+\big),\\
\nabla_{X_-} \hat{e}_\mu &= \hat{e}_\gamma h^{\gamma\rho} q^{2(\mu-\nu)}\big(\tfrac{1}{2} h_{\rho\nu} \triangleleft X_- + a^*_{\nu\rho} \triangleleft K\big)(p_n)^{\nu}{}_{\mu} + \hat{e}_\nu\big((p_n)^{\nu}{}_{\mu} \triangleleft X_-\big),\\
\nabla_{X_z} \hat{e}_\mu &= \hat{e}_\gamma h^{\gamma\rho} q^{4(\mu-\nu)}\big(\tfrac{1}{2} h_{\rho\nu} \triangleleft X_z + b_{\rho\nu} \triangleleft K^2\big)(p_n)^{\nu}{}_{\mu} + \hat{e}_\nu\big((p_n)^{\nu}{}_{\mu} \triangleleft X_z\big),
\end{aligned}$$

using that $(p_n)^{\mu}{}_{\nu} \triangleleft K = q^{\nu-\mu}(p_n)^{\mu}{}_{\nu}$. Moreover, if $p_n$ is orthogonal with respect to $h$, then $\nabla$ is compatible with the restriction of $h$ to $M_n$. A similar construction goes for $n < 0$.

## 6 Further Comments: Sketching a Generalization

As final section of comments we sketch a way to generalise (some of) the constructions above for any Hopf algebra with a left covariant differential calculus and corresponding quantum tangent space [23]. While referring to [16, 14.1] for details, we recall that a first order differential calculus $(\Gamma, d)$ over the Hopf algebra $(H, \Delta, S, \varepsilon)$ is called left-covariant if there is a linear map $\Delta_\Gamma : \Gamma \to H \otimes \Gamma$ such that, for all $f, g \in H$ it holds that

$$\Delta_\Gamma(f\, dg) = \Delta(f)(\mathrm{id} \otimes d))\Delta(g).$$

An element $\rho \in \Gamma$ is called left-invariant if $\Delta_\Gamma(\rho) = 1 \otimes \rho$ and we let ${}_{\mathrm{inv}}\Gamma$ denote the vector space of invariant elements. There is then a corresponding quantum tangent space $T_\Gamma \subset H^\circ$ (the dual Hopf algebra) with a unique bilinear form $\langle \cdot, \cdot \rangle : T_\Gamma \times \Gamma \to \mathbb{C}$ such that

$$\langle X, f\, dg \rangle = \varepsilon(f) X(g),$$

for $g, f \in H$, and $X \in T_\Gamma$. The vector spaces ${}_{\mathrm{inv}}\Gamma$ and $T_\Gamma$ form a non-degenerate dual pair with respect to this bilinear form. Also, the pairing induces a left action as in (2.1),

$$X \triangleleft f = f_{(1)} \langle X, f_{(2)} \rangle$$

for $X \in T_\Gamma$ and $f \in H$. Furthermore, one has dual bases $\{X_a,\ a = 1, 2, \dots, n\}$ of $T_\Gamma$ and $\{\omega_a,\ a = 1, 2, \dots, n\}$ of ${}_{\mathrm{inv}}\Gamma$ and a family of functionals $\{\sigma^a_b,\ a, b = 1, 2, \dots, n\}$ such that

$$df = \sum_a (X_a \triangleleft f)\, \omega_a,$$
$$X_a \triangleleft (fg) = f X_a \triangleleft (g) + X_b \triangleleft (f)\, \sigma^b_a \triangleleft (g). \tag{6.1}$$

In the dual Hopf algebra $H^\circ$ we have

$$\Delta\sigma^a_b = \sigma^a_c \otimes \sigma^c_b, \quad S(X_a) = -X_b S(\sigma^b_a).$$

With compatible $*$-structures, using the second expression and requiring (2.2) one computes:

$$X_a \triangleleft f^* = -\sigma^b_a \triangleleft (X_b^\dagger \triangleleft f)^*. \tag{6.2}$$

By way of illustration let us consider the trivial right module $M = H$ with Hermitian form $h(m_1, m_2) = m_1^* m_2$. The analogue of the condition (3.2) in Definition 3.2 is read from (6.1) as

$$\nabla_{X_a} \triangleleft (mf) = m\, X_a \triangleleft (f) + \big(\nabla_{X_b} \triangleleft (m)\big)\, \sigma^b_a \triangleleft (f). \tag{6.3}$$

In turn, the compatibility with the Hermitian form reads:

$$X_a\big(h(m_1, m_2)\big) = h\big(m_1, \nabla_{X_a} m_2\big) - \sigma^b_a \triangleleft \big(h(\nabla_{X_b^\dagger} m_1, m_2)\big). \tag{6.4}$$

Indeed, using (6.1) and (6.2), we compute

$$\begin{aligned} X_a\big(h(m_1, m_2)\big) = X_a \triangleleft (m_1^* m_2) &= m_1^* X_a \triangleleft (m_2) + X_b \triangleleft (m_1^*)\, \sigma^b_a \triangleleft (m_2) \\ &= m_1^* X_a \triangleleft (m_2) - \sigma^c_b \triangleleft (X_c^\dagger \triangleleft m_1)^*\, \sigma^b_a \triangleleft (m_2) \\ &= m_1^* X_a \triangleleft (m_2) - \sigma^c_a \triangleleft \big(X_c^\dagger \triangleleft m_1)^*\, (m_2)\big) \end{aligned}$$

from which (6.4) follows.

Equations (6.3) and (6.4) can be the starting point for a theory of affine connections on a quantum group with a quantum tangent space. For a torsion freeness condition one would need (twisted) commutation relations among the elements of $T_\Gamma$. In general these commutation relations could be involved; in particular they do not need to be quadratic as in the classical case or in the example in (2.6)–(2.4). Details should await a different time.

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## Declarations

**Conflict of interest** The authors have no conflict of interest to declare that are relevant to the content of this article.



## Appendix A: The Calculus on the Sphere $S_q^2$ via the Right Action

As we have seen in Sect. 5 the left action of the $X_a$ does not preserve the algebra $S_q^2$, since their left action does not commute with that of $K$ defining the fibration. On the other hand, the right action of $X_a$ does preserve the algebra $S_q^2$ since the action does commute with the left one of $K$. Let us denote $Y_a = X_a$ for the right action. Then, it is easy to check that

$$\begin{aligned}
&B_0 \triangleleft Y_+ = q^{-1} B_- \quad B_0 \triangleleft Y_- = -q^{-1} B_+ \quad B_0 \triangleleft Y_z = 0, \\
&B_+ \triangleleft Y_+ = q\mathbb{1} - q(1+q^2)B_0 \quad B_+ \triangleleft Y_- = 0 \quad B_+ \triangleleft Y_z = -q^2(1+q^2)B_+, \\
&B_- \triangleleft Y_+ = 0 \quad B_- \triangleleft Y_- = -q^{-1}\mathbb{1} + q^{-1}(1+q^2)B_0 \quad B_- \triangleleft Y_z = (1+q^{-2})B_-.
\end{aligned} \tag{A.1}$$

Note that when restricted to $S_q^2$ the $Y_a$ are not independent. A long but straightforward computation shows that they are indeed related as

$$\begin{aligned}
&\big((f \triangleleft Y_+)B_+ q + (f \triangleleft Y_-)B_- q^{-1}\big)(1+q^2) + (f \triangleleft Y_z)\left(1 - 2\frac{1+q^2}{1+q^4}B_0\right) \\
&= (f \triangleleft Y_z^2)\, q^{-2}\left(\frac{1-q^2}{1+q^4}(2q^4+q^2+1)B_0 - (1-q^6)B_0^2\right) \\
&\quad + (f \triangleleft K^4)\, q^{-2}(1+q^2)\big((q^4-1)B_0 + (1-q^6)B_0^2\big),
\end{aligned} \tag{A.2}$$

for $f \in S_q^2$. This is checked on a vector space basis for the algebra $S_q^2$, a basis which can be taken as $X(m)(B_0)^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{N}$ with $X(m) = (B_+)^m$ for $m \geq 0$ and

$X(m) = (B_-)^{-m}$ for $m < 0$ (cf. [19]). From the expression in (5.3) one writes the differential $d$ on $S_q^2$ in terms of the right acting operators $Y_a$.

**Lemma A.1** *For $f \in S_q^2$, the differential in (5.2) can be written as*

$$df = (f \triangleleft V_+)\, dB_+ + (f \triangleleft V_-)\; dB_- + (f \triangleleft V_0)\, dB_0, \tag{A.3}$$

*where*

$$V_+ = Y_+\big(1 - q^{-2}(1+q^2)B_0\big)q^{-1} - Y_z\; B_- \frac{q^{-2}(1+q^6)}{1+q^4} + Y_z^2\; B_- \frac{1-q^2}{(1+q^2)(1+q^4)},$$

$$V_- = -Y_-\big(1 - q^2(1+q^2)B_0\big)q + Y_z\; B_+ \frac{q^{-2}(1+q^6)}{1+q^4} - Y_z^2\; B_+ \frac{1-q^2}{(1+q^2)(1+q^4)},$$

$$V_0 = \big(Y_+\; B_+ q^{-1} - Y_-\; B_- q\big)(1+q^2) + Y_z\; B_0 \frac{(1-q^4)(1+q^6)}{1+q^4} - Y_z^2\; B_0 \frac{1-q^2}{1+q^4}.$$

***Proof*** By acting on the vector space basis $X(m)(B_0)^n$ (as introduced previously), one explicitly checks the equality of (5.2) and (A.3) via a tedious but straightforward computation. □

***Remark A.2*** When $q = 1$ the derivative (A.3) reduces to

$$\begin{aligned} df &= 2\big((f \triangleleft Y_+)\; B_+ - (f \triangleleft Y_-)\; B_-\big)dB_0 \\ &\quad + \big((f \triangleleft Y_+)\; (1 - 2B_0) - (f \triangleleft Y_z)\; B_-\big)dB_+ \\ &\quad + \big(-(f \triangleleft Y_-)\; (1 - 2B_0) + (f \triangleleft Y_z)\; B_+\big)dB_-. \end{aligned} \tag{A.4}$$

Classically, the vector field $X_a$ are the left invariant vector fields on $S^3 = SU(2)$ with dual left invariant forms $\omega_a$. Thus they do not project to vector fields on the base space $S^2$ with commuting coordinates $(B_+, B_-, B_0)$ and relation $B_+B_- = B_0(1 - B_0)$: $X_a \triangleright f$ is not a function on $S^2$ even when $f$ is. On the other hand, the vector fields $Y_a$ are the right invariant vector fields on $SU(2)$ and thus they project to vector fields on $S^2$, where they are not independent any longer and are related by

$$2(B_+Y_+ + B_-Y_-) + (1 - 2B_0)Y_z = 0,$$

which is just the relation to which (A.2) reduces when $q = 1$.

By changing coordinates $B_0 = \frac{1}{2}(1 - x)$ so that the radius condition for $S^2$ is written as $r^2 = 4B_+B_- + x^2$, the exterior derivative operator in (A.4) becomes

$$df = \partial_x f\; dx + \partial_+ f\; dB_+ + \partial_- f\; dB_- - (\Delta f)\,(x\, dx + 2B_-\, dB_+ + 2B_+\, dB_-)$$

where $\Delta = x\,\partial_x + B_+\,\partial_+ + B_-\,\partial_-$ is the Euler (dilatation) vector field. One then computes $dr^2 = 2(1-r^2)(x\,dx + 2B_-\,dB_+ + 2B_+\,dB_-)$, which vanishes when restricting to $S^2$: $r^2 - 1 = 0$.

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