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# Generalized Nijenhuis Torsions and Block-Diagonalization of Operator Fields

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**Abstract**
The theory of generalized Nijenhuis torsions, which extends the classical notions due to Nijenhuis and Haantjes, offers new tools for the study of normal forms of operator fields. We prove a general result ensuring that, given a family of commuting operator fields whose generalized Nijenhuis torsion of level $m$ vanishes, there exists a local chart where all operators can be simultaneously block-diagonalized. We also introduce the notion of generalized Haantjes algebra, consisting of operators with a vanishing higher-level torsion, as a new algebraic structure naturally generalizing standard Haantjes algebras.

**Keywords** Nijenhuis geometry · Haantjes algebras · Block-diagonalization

**Mathematics Subject Classification** 53A45 · 58C40 · 58A30

## Contents



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## 1 Introduction

The purpose of this article is to establish a new geometric setting for the block-diagonalization of operator fields on a differentiable manifold.

The problem of finding normal forms of (1,1)-tensor fields on a differentiable manifold $M$ is relevant in many theoretical and applicative contexts. Indeed, it is useful in the determination of a reduced form for a system of partial differential equations Bogoyavlenskij (1996, 2006, 2007) and in the theory of hydrodynamic-type systems (Ferapontov and Marshall 2007; Ferapontov and Khusnutdinova 2006). Besides, it finds applications in classical mechanics, since it may allow to construct separation variables for completely integrable Hamiltonian systems (Chanu and Rastelli 2019).

In this article, we address this problem within a tensorial approach based on the notions of Nijenhuis and Haantjes torsions and their generalizations, recently introduced in Tempesta and Tondo (2022b).

In the last decades, the interest in the geometry of Nijenhuis and Haantjes tensors has considerably increased. Indeed, new applications have been found in many different contexts: for instance, the characterization of integrable chains of partial differential equations of hydrodynamic type (see, e.g., Ferapontov and Marshall 2007; Ferapontov and Khusnutdinova 2006) and their integrable reductions (Ferapontov and Pavlov 2022), as well as the study of topological field theories (Magri 2018). Also, Nijenhuis geometry, which plays a crucial role in the theory of almost complex structures (Newlander and Nirenberg 1957), has been extended to contexts as the study of analytic matrix functions, linearization theory, operator algebras (Bolsinov et al. 2022), etc.

In Tempesta and Tondo (2021), two of the present authors introduced the concept of Haantjes algebra. It has been shown that for an Abelian algebra of semisimple operator fields with vanishing Haantjes torsion there exist local coordinate charts where all the operators of the algebra can be simultaneously diagonalized. In Tempesta and Tondo (2022a); Reyes et al. (2022), the geometry of $\omega\mathscr{H}$ manifolds (namely symplectic manifolds endowed with an algebra of Haantjes operators) has been proposed as a natural setting for the formulation of the theory of finite-dimensional integrable Hamiltonian systems. The related class of $P\mathscr{H}$ manifolds has been discussed in Tondo (2018).

A new, infinite family of higher-order torsions of level $m$, generalizing both the Nijenhuis and the Haantjes ones, have been defined and discussed in Tempesta and

Tondo (2022b). These torsions can also be derived from a family of higher-order Haantjes brackets, which generalize the Frölicher–Nijenhuis (FN) one. As the FN bracket is the polarization of the Nijenhuis torsion, so the higher Haantjes brackets are related to the polarization of our higher-order torsions (Tempesta and Tondo 2023). Operator fields with a vanishing generalized torsion are called generalized Nijenhuis operators.

We remind that a different formulation of the theory of generalized Nijenhuis torsions has been proposed in Kosmann-Schwarzbach (2019).

Besides, a generalization of the classical Haantjes theorem was proved in Tempesta and Tondo (2022b). Precisely, it has been proved that, given an operator field $\boldsymbol{A} : \mathfrak{X}(M) \to \mathfrak{X}(M)$, the vanishing of any generalized Nijenhuis torsion is sufficient to guarantee the *mutual integrability* of the eigen-distributions of $\boldsymbol{A}$, namely the integrability of each of them and of all of their direct sums.

From a theoretical point of view, a first result of this work is Theorem 4. It states that for a set of commuting operators on a differentiable manifold, if at least one of them is a generalized Nijenhuis operator, then there exists a local coordinate chart where these operators take simultaneously a block-diagonal form.

This statement is further refined by requiring that all of the commuting operators are generalized Nijenhuis operators of a (common) level $m$ and their joint eigen-distributions are *pairwise integrable* (see Definition 11). Thus, our main result is Theorem 5 ensuring that, under these hypotheses, on a local coordinate chart all of them take a block-diagonal form with finer blocks.

These results naturally suggest the notion of *generalized Haantjes algebra* of level $m$. It consists of a set of generalized Nijenhuis operators of the same level $m$, forming a $C^\infty(M)$-module; besides, this set is closed under the product of operators.

We will discuss in detail the properties of the class of unigenerated Haantjes algebras of level $m$, which are generated by the independent powers of a single generalized Nijenhuis operator.

Another crucial result is that given an operator $\boldsymbol{A}$ whose generalized Nijenhuis torsion of level $m$ vanishes, any polynomial $P(\boldsymbol{A})$ whose coefficients are functions in $C^\infty(M)$ also has the same level-$m$ torsion vanishing. This property allows us to construct generalized Haantjes algebras in a very natural and direct way. We shall present some nontrivial examples of unigenerated generalized Haantjes algebras of level three and four. In particular, we will show how to construct local coordinate charts allowing the simultaneous block-diagonalization of all the operators forming these algebras.

The approach proposed in this work offers a new perspective on the classical problem of the normal form of operator fields. We remind that a relevant contribution to this problem, in the spirit of tensor analysis, was given in Bogoyavlenskij (2006, 2007), where necessary and sufficient conditions have been proposed for the existence of local charts were a given operator acquires a block-diagonal form. The main difference with respect to those results is that we solve the problem for a *complete family* of commuting operators. At the same time, no knowledge *a priori* of the eigen-distributions of the given operators is required. This aspect is crucial, since the spectral analysis of operator fields computationally becomes rapidly intractable by increasing the dimension of the underlying differentiable manifold. Thus, we offer sufficient tensorial conditions for the simultaneous block-diagonalization of a family of operators which can be easily checked with the aid of computer algebra, without entering the study of their eigen-distributions. Of course, for the explicit construction of local charts where the simultaneous block-diagonalization of the given family of operators takes place, it is necessary to determine their spectral properties. However, our result can be interpreted as a tensorial test, ensuring *a priori* the block-diagonalizability of the whole family. Once this property is ascertained for a concrete family, then one can proceed to the construction of the required local charts.

A thorough study of the rich algebraic properties of the generalized Haantjes algebras represents an interesting research perspective, that will be developed in future works.

The paper is organized as follows. In Sect. 2, the theory of Nijenhuis and Haantjes torsions is briefly reviewed. In Sect. 3, the concept of generalized Nijenhuis operator is discussed. In Sect. 4, the new notion of generalized Haantjes algebra is presented. In Sect. 5, the main theorems of this work, concerning simultaneous block-diagonalization of non-semisimple operator fields, are proved. In Sects. 6 and 7, examples of generalized Haantjes algebras are discussed and local charts ensuring simultaneous block-diagonalization are explicitly determined.

## 2 Preliminaries on the Nijenhuis and Haantjes Geometry

In this section, we shall review some basic notions concerning the geometry of Nijenhuis and Haantjes torsions, following the original papers (Haantjes 1955; Nijenhuis 1951) and the related ones (Nijenhuis 1955a, b; Frölicher and Nijenhuis 1956). Here, we shall focus only on the aspects of the theory which are relevant for the subsequent discussion.

Let $M$ be a differentiable manifold of dimension $n$, $\mathfrak{X}(M)$ the Lie algebra of vector fields on $M$ and $\boldsymbol{A} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a smooth $(1, 1)$-tensor field. In the following, all tensors will be considered to be smooth.

**Definition 1** The *Nijenhuis torsion* of $\boldsymbol{A}$ is the the vector-valued 2-form defined by

$$\tau_{\boldsymbol{A}}(X, Y) := \boldsymbol{A}^2[X, Y] + [\boldsymbol{A}X, \boldsymbol{A}Y] - \boldsymbol{A}\Big([X, \boldsymbol{A}Y] + [\boldsymbol{A}X, Y]\Big), \tag{2.1}$$

where $X, Y \in \mathfrak{X}(M)$ and $[\ ,\ ]$ denotes the commutator of two vector fields.

**Definition 2** The *Haantjes torsion* associated with $\boldsymbol{A}$ is the vector-valued 2-form defined by

$$\mathcal{H}_{\boldsymbol{A}}(X, Y) := \boldsymbol{A}^2\tau_{\boldsymbol{A}}(X, Y) + \tau_{\boldsymbol{A}}(\boldsymbol{A}X, \boldsymbol{A}Y) - \boldsymbol{A}\Big(\tau_{\boldsymbol{A}}(X, \boldsymbol{A}Y) + \tau_{\boldsymbol{A}}(\boldsymbol{A}X, Y)\Big) \tag{2.2}$$

The Haantjes (Nijenhuis) vanishing condition inspires the following definition.

**Definition 3** A Haantjes (Nijenhuis) operator is a (1,1)-tensor field whose Haantjes (Nijenhuis) torsion identically vanishes.

A simple, relevant case of Haantjes operator is that of a tensor field $\boldsymbol{A}$ which takes a diagonal form in a local chart $\boldsymbol{x} = (x^1, \dots, x^n)$:

$$\boldsymbol{A}(\boldsymbol{x}) = \sum_{i=1}^{n} \lambda_i(\boldsymbol{x}) \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^i \ , \tag{2.3}$$

where $\lambda_i(\boldsymbol{x}) := \lambda_i^i(\boldsymbol{x})$ are the eigenvalue fields of $\boldsymbol{A}$ and $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$ are the fields forming the so called *natural frame* associated with the local chart $(x^1, \dots, x^n)$. As is well known, the Haantjes torsion of the diagonal operator (2.3) vanishes.

We also recall that two frames $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_n\}$ are said to be equivalent if $n$ nowhere vanishing smooth functions $f_i$ exist, such that

$$X_i = f_i(\boldsymbol{x}) Y_i \ , \qquad i = 1, \dots, n \ .$$

**Definition 4** An *integrable* frame is a reference frame equivalent to a natural frame.

It is interesting to observe that the algebraic properties of Haantjes operators are richer than those of Nijenhuis operators. Hereafter, $\boldsymbol{I} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ will denote the identity operator. A useful result is the following.

**Proposition 1** (Bogoyavlenskij 1996). *Let* $\boldsymbol{A}$ *be a (1,1)-tensor field. The following identity holds:*

$$\mathcal{H}_{f\boldsymbol{I}+g\boldsymbol{A}}(X, Y) = g^4 \mathcal{H}_{\boldsymbol{A}}(X, Y), \tag{2.4}$$

*where* $f, g : M \to \mathbb{R}$ *are* $C^\infty(M)$ *functions.*

***Proof*** See Proposition 1, p. 255 of Bogoyavlenskij (1996). □

Interestingly enough, such a property does not hold in the case of a Nijenhuis operator. Many more examples of Haantjes operators, relevant in classical mechanics and in Riemannian geometry, can be found for instance in the works (Reyes et al.2022; Tempesta and Tondo 2016; 2021; 2022a; 2022b; 2023).

## 3 The Generalized Nijenhuis Operators and Block-Diagonalization

In this section, for the sake of clarity, we shall briefly review some of the main algebraic and geometric properties of the new class of generalized Nijenhuis operators introduced and studied in Tempesta and Tondo (2021, 2022b).

**Definition 5** Let $\boldsymbol{A} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a (1,1)-tensor field. The generalized Nijenhuis torsion of $\boldsymbol{A}$ of level $m$, for each integer $m \geq 1$, is the skew-symmetric (1,2)-tensor field defined by

$$\begin{aligned}\tau_{\boldsymbol{A}}^{(m)}(X,Y)=\boldsymbol{A}^2\tau_{\boldsymbol{A}}^{(m-1)}(X,Y)+\tau_{\boldsymbol{A}}^{(m-1)}(\boldsymbol{A}X,\boldsymbol{A}Y)\\ -\boldsymbol{A}\Big(\tau_{\boldsymbol{A}}^{(m-1)}(X,\boldsymbol{A}Y)+\tau_{\boldsymbol{A}}^{(m-1)}(\boldsymbol{A}X,Y)\Big),\quad X,Y\in\mathfrak{X}(M)\,.\end{aligned}\tag{3.1}$$

Here, the notation $\tau_{\boldsymbol{A}}^{(0)}(X,Y):=[X,Y]$, $\tau_{\boldsymbol{A}}^{(1)}(X,Y):=\tau_{\boldsymbol{A}}(X,Y)$ and $\tau_{\boldsymbol{A}}^{(2)}(X,Y):=\mathcal{H}_{\boldsymbol{A}}(X,Y)$ is adopted.

**Definition 6** Let $\boldsymbol{A}:\mathfrak{X}(M)\to\mathfrak{X}(M)$ be a (1,1)-tensor field. If $\tau_{\boldsymbol{A}}^{(m)}(X,Y)=\mathbf{0}$ for some $m\in\mathbb{N}\backslash\{0\}$, we shall say that $\boldsymbol{A}$ is a generalized Nijenhuis operator of level $m$.

We recall a result which is crucial in the following analysis.

**Theorem 1** *Let $\boldsymbol{A}:\mathfrak{X}(M)\to\mathfrak{X}(M)$ be an operator. Assume that*

$$\tau_{\boldsymbol{A}}^{(m)}(X,Y)=\mathbf{0},\qquad X,Y\in\mathfrak{X}(M)\tag{3.2}$$

*for some $m\geq 1$. Then, each eigen-distribution of $\boldsymbol{A}$, as well as each direct sum of its eigen-distributions, is integrable.*

### 3.1 Eigen-Distributions and Spectral Properties of Generalized Nijenhuis Operators

We shall recall now some of the spectral properties of generalized Nijenhuis operators. Let us denote by $Spec(\boldsymbol{A}):=\{\lambda_1(\boldsymbol{x}),\lambda_2(\boldsymbol{x}),\ldots,\lambda_s(\boldsymbol{x})\}$, $\boldsymbol{x}\in M$, the set of the eigenvalues of an operator $\boldsymbol{A}:\mathfrak{X}(M)\to\mathfrak{X}(M)$, which will always be assumed to be real and pointwise distinct. We denote by

$$\mathcal{D}_i(\boldsymbol{x})=\ker\Big(\boldsymbol{A}(\boldsymbol{x})-\lambda_i(\boldsymbol{x})\boldsymbol{I}\Big)^{\rho_i},\qquad i=1,\ldots,s\tag{3.3}$$

the $i$-th eigen-distribution of index $\rho_i\geq 1$, which includes all the (generalized) eigenvectors corresponding to the eigenvalue $\lambda_i$. In Eq. (3.3), $\rho_i$ stands for the Riesz index of $\lambda_i$, which is the minimum integer such that

$$\ker\Big(\boldsymbol{A}(\boldsymbol{x})-\lambda_i(\boldsymbol{x})\boldsymbol{I}\Big)^{\rho_i}\equiv\ker\Big(\boldsymbol{A}(\boldsymbol{x})-\lambda_i(\boldsymbol{x})\boldsymbol{I}\Big)^{\rho_i+1}\,.\tag{3.4}$$

In the forthcoming considerations, we shall always suppose a *regularity condition*, namely that the rank of the distributions and $\rho_i$ are (locally) independent of $\boldsymbol{x}$. When $\rho_i=1$, $\mathcal{D}_i$ is a proper eigen-distribution.

It is also useful, from an applicative point of view, to consider the explicit action of a generalized Nijenhuis torsion on a pair of generalized eigenvectors of $\boldsymbol{A}$, as in the following result:

**Proposition 2** *Let $\boldsymbol{A}$ be a (1,1)-tensor and $X_\alpha\in\mathcal{D}_\mu$, $Y_\beta\in\mathcal{D}_\nu$ be two of its generalized eigenvectors corresponding to the eigenvalues $\mu$, $\nu$ respectively. Then, for any integer*

$m \geq 2$ *the following formula holds:*

$$\tau_{\boldsymbol{A}}^{(m)}(X_\alpha, Y_\beta) = \sum_{i,j=0}^{m} (-1)^{i+j} \binom{m}{i} \binom{m}{j} \Big(\boldsymbol{A} - \mu \mathbf{I}\Big)^{m-i} \Big(\boldsymbol{A} - \nu \mathbf{I}\Big)^{m-j} [X_{\alpha-i}, Y_{\beta-j}] \,, \tag{3.5}$$

where

$$\boldsymbol{A} X_\alpha = \mu X_\alpha + X_{\alpha-1}, \qquad \boldsymbol{A} Y_\beta = \nu Y_\beta + Y_{\beta-1} \,, \qquad 1 \leq \alpha \leq \rho_\mu, \quad 1 \leq \beta \leq \rho_\nu \,, \tag{3.6}$$

and $X_0$ and $Y_0$ are, by definition, null vector fields.

### 3.2 Block-Diagonalization

As a nontrivial application of Theorem 1, one can also prove that, given an operator $\boldsymbol{A}$, condition (3.2) is also sufficient to ensure the existence of a local chart where the operator $\boldsymbol{A}$ can be *block-diagonalized*. We envisage relevant applications, for instance, in the theory of hydrodynamic-type systems (Bogoyavlenskij 2006), in the study of partial separability of Hamiltonian systems (Chanu and Rastelli 2019) and, more generally, in the context of Courant's problems for first-order hyperbolic systems of partial differential equations (Courant and Hilbert 1962).

Let $\boldsymbol{A}$ be an operator satisfying condition (3.2); we denote by $r_i$ the rank of the distribution $\mathcal{D}_i$ of $\boldsymbol{A}$. We also introduce the distribution (of corank $r_i$)

$$\mathcal{E}_i := Im\Big(\boldsymbol{A} - \lambda_i \mathbf{I}\Big)^{\rho_i} = \bigoplus_{j=1,\, j\neq i}^{s} \mathcal{D}_j, \qquad i = 1, \ldots, s \tag{3.7}$$

which is spanned by all the generalized eigenvectors of $\boldsymbol{A}$, except those associated with the eigenvalue $\lambda_i$ (we remind that by hypothesis $\boldsymbol{A}$ has real eigenvalues). We shall say that $\mathcal{E}_i$ is a *characteristic distribution* of $\boldsymbol{A}$. Let $\mathcal{E}_i^\circ$ denote the annihilator of the distribution $\mathcal{E}_i$. The cotangent spaces of $M$ can be decomposed as

$$T_{\boldsymbol{x}}^* M = \bigoplus_{i=1}^{s} \mathcal{E}_i^\circ(\boldsymbol{x}). \tag{3.8}$$

As a consequence of Theorem 1, each characteristic distribution $\mathcal{E}_i$ is integrable. By $\mathrm{E}_i$, we denote the foliation associated with $\mathcal{E}_i$ and by $E_i(\boldsymbol{x})$ the connected leave through $\boldsymbol{x}$, belonging to $\mathrm{E}_i$. Given the set of distributions $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_s\}$, we have associated an equal number of foliations $\{\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_s\}$. This set of foliations is said to be the *characteristic web* of $\boldsymbol{A}$. The leaves $E_i(\boldsymbol{x})$ of each foliation $\mathrm{E}_i$ are usually referred to as the *characteristic fibers* of the web.

**Definition 7** Let $\boldsymbol{A} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a generalized Nijenhuis operator of level $m$. A collection of $r_i$ smooth functions will be said to be adapted to the foliation $\mathrm{E}_i$ of the characteristic *web* of $\boldsymbol{A}$ if the level sets of such functions coincide with the characteristic fibers of $\mathrm{E}_i$.

**Definition 8** Let $\boldsymbol{A} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a generalized Nijenhuis operator of level $m$. A parametrization of the characteristic web of $\boldsymbol{A}$ is an ordered set of $n$ independent smooth functions listed as $(\boldsymbol{f}^1, \ldots, \boldsymbol{f}^i, \ldots, \boldsymbol{f}^s)$, such that for any $i = 1, \ldots, s$, the ordered subset $\boldsymbol{f}^i = (f^{i,1}, \ldots, f^{i,r_i})$ is adapted to the $i$-th characteristic foliation of the web:

$$f^{i,k}_{|E_i(\mathbf{x})} = c^{i,k} \qquad \forall E_i(\mathbf{x}) \in \mathrm{E}_i \ , \quad k = 1, \ldots, r_i \ . \tag{3.9}$$

Here, $c^{i,k}$ are real constants depending on the indices $i$ and $k$ only. In this case, we shall say that the collection of these functions is adapted to the web and that each of them is a *characteristic function*.

In the case of a single operator with a vanishing higher-order torsion, the following result gives a simple and very general tensorial criterion for the existence of local coordinates ensuring block-diagonalizability.

**Theorem 2** *(Tempesta and Tondo 2022b) Let $\boldsymbol{A} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be an operator. If*

$$\tau^{(m)}_{\boldsymbol{A}}(X, Y) = \mathbf{0} \ , \qquad X, Y \in \mathfrak{X}(M) \tag{3.10}$$

*for some $m \geq 1$, then $\boldsymbol{A}$ admits local charts adapted to the spectral decomposition of the tangent spaces into generalized eigenspaces, where it takes a block-diagonal form.*

One of the main achievements of this work is to generalize this result to the case of families of commuting operators. To this aim, we shall introduce a new class of operator algebras.

## 4 Generalized Haantjes Algebras

### 4.1 Definitions

The notion of Haantjes algebra has been introduced and discussed in Tempesta and Tondo (2021). In this section, we define a class of new, generalized Haantjes algebras.

**Definition 9** A generalized Haantjes algebra of level $m$ is a pair $(M, \mathscr{H}^{(m)})$ with the following properties:

- $M$ is a differentiable manifold of dimension n;
- $\mathscr{H}^{(m)}$ is a set of operators $\boldsymbol{K} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ whose Nijenhuis torsion of level $m$ vanishes: $\tau^{(m)}_{\boldsymbol{K}} = \mathbf{0}$. Also, they generate:

- a free module over the ring of smooth functions on $M$:

$$\tau^{(m)}_{\left(f\boldsymbol{K}_1+g\boldsymbol{K}_2\right)}(X,Y)=\mathbf{0}\,,\qquad \forall\, X,Y\in\mathfrak{X}(M)\,,\quad f,g\in C^\infty(M)\,,\\ \forall\,\boldsymbol{K}_1,\boldsymbol{K}_2\in\mathscr{H}^{(m)}; \tag{4.1}$$

- a ring w.r.t. the composition operation

$$\tau^{(m)}_{\left(\boldsymbol{K}_1\,\boldsymbol{K}_2\right)}(X,Y)=\mathbf{0}\,,\qquad \forall\,\boldsymbol{K}_1,\boldsymbol{K}_2\in\mathscr{H}^{(m)},\quad \forall\, X,Y\in\mathfrak{X}(M)\,. \tag{4.2}$$

If

$$\boldsymbol{K}_1\,\boldsymbol{K}_2=\boldsymbol{K}_2\,\boldsymbol{K}_1\,,\qquad \boldsymbol{K}_1,\boldsymbol{K}_2\in\mathscr{H}^{(m)}\,, \tag{4.3}$$

the algebra $(M,\mathscr{H}^{(m)})$ will be said to be an Abelian generalized Haantjes algebra. Moreover, if the identity operator $\boldsymbol{I}\in\mathscr{H}^{(m)}$, then $(M,\mathscr{H}^{(m)})$ will be said to be a generalized Haantjes algebra with identity.

In other words, the set $\mathscr{H}^{(m)}$ can be regarded as an associative algebra of generalized Haantjes operators.

The case $m=2$, namely that of standard Haantjes algebras, possesses several important properties. Among them, we recall that for a given Abelian Haantjes algebra $\mathscr{H}^{(2)}\equiv\mathscr{H}$ there exists an associated set of coordinates, called *Haantjes coordinates*, by means of which all $\boldsymbol{K}\in\mathscr{H}$ can be written simultaneously in a block-diagonal form. In particular, if $\mathscr{H}$ is also semisimple, on each set of Haantjes coordinates all $\boldsymbol{K}\in\mathscr{H}$ can be written simultaneously in a purely diagonal form. We mention that Haantjes algebras play a relevant role in the theory of classical separable and multiseparable Hamiltonian systems (Reyes et al. 2022).

The following result simplifies the study of the integrability of the eigen-distributions of a family of operators forming an Abelian, generalized Haantjes algebra.

**Lemma 1** *Let $(M,\mathscr{H}^{(m)})$ be an Abelian generalized Haantjes algebra of level $m$. We shall assume that the rank of the eigen-distributions of the operators belonging to $\mathscr{H}^{(m)}$ is independent of $\boldsymbol{x}\in M$. Then, each nontrivial intersection of these eigen-distributions is integrable.*

***Proof*** Let $\{\boldsymbol{K}_1,\ldots,\boldsymbol{K}_w\}$ be a basis of $(M,\mathscr{H}^{(m)})$, and $(\mathcal{D}^{(1)}_{i_1},\ldots,\mathcal{D}^{(w)}_{i_w})$, $i_1=1,\ldots,s_1,\ldots,i_w=1,\ldots,s_w$, where $s_j=\text{card Spec}\,(\boldsymbol{K}_j)$, be the set of their eigen-distributions. Let

$$\mathcal{V}_a=\mathcal{D}^{(1)}_{i_1}(\boldsymbol{x})\bigcap\ldots\bigcap\mathcal{D}^{(w)}_{i_w}(\boldsymbol{x})\qquad a=1,\ldots,v,\qquad v\le n \tag{4.4}$$

denote a nontrivial intersection of eigen-distributions of the operators $\{\boldsymbol{K}_1,\ldots,\boldsymbol{K}_w\}$, which we call a *joint eigen-distribution*. Consequently, the distribution $\mathcal{V}_a$, being the intersection of distributions which are involutive due to Theorem 1, is also involutive. $\square$

### 4.2 Generalized and Unigenerated Haantjes Algebras

A special class of commutative generalized Haantjes algebras are those generated by a single operator $\boldsymbol{L}$ and its independent powers. We denote by $h$ the degree of the minimal polynomial of $\boldsymbol{L}$.

**Definition 10** A generalized Haantjes algebra $(M, \mathscr{H}^{(m)})$ will be said to be unigenerated (or cyclic) [1] if there exists an operator $\boldsymbol{L} : \mathfrak{X}(M) \to \mathfrak{X}(M)$, with $\tau_{\boldsymbol{L}}^{(m)}(X, Y) = 0$, such that $\mathscr{H}^{(m)} \subseteq \mathcal{L}(\boldsymbol{L})$, where $\mathcal{L}(\boldsymbol{L})$ is the set of all polynomials $\sum_{k=0}^{h-1} c_k(\boldsymbol{x})\boldsymbol{L}^k$ in the powers of $\boldsymbol{L}$ with coefficients $c_k(\boldsymbol{x}) \in C^\infty(M)$.

Given a unigenerated algebra $(M, \mathscr{H}^{(m)})$ generated by an operator $\boldsymbol{L}$, its rank is less than or equal to $h$.

***Remark 1*** Unigenerated Haantjes algebras of level $m$ as algebraic structures are not rare. Indeed, as we will prove in Theorem 3, given an operator $\boldsymbol{L}$ with $\tau_{\boldsymbol{L}}^{(m)}(X, Y) = 0$, any polynomial in $\boldsymbol{L}$ having coefficients in $C^\infty(M)$ is still an operator having a vanishing torsion of the same level. Thus, the independent powers of any operator with a given vanishing generalized torsion provide naturally a basis of a unigenerated generalized algebra.

In the following, we shall prove this fundamental fact. To this aim, we shall use and adapt to our general case several algebraic techniques and results introduced and proved in Bogoyavlenskij (2004) for the case of Nijenhuis and Haantjes operators. In particular, we shall use a representation of (1,2)-tensors in the ring $S_3$ of polynomials of three independent variables $z, \lambda, \mu$ with coefficients depending on $\boldsymbol{x} \in M$. A generic polynomial of this ring has the form

$$S(z, \lambda, \mu) := \sum_{i,j,k}^{N} s_{ijk}(\boldsymbol{x}) z^i \lambda^j \mu^k, \qquad \boldsymbol{x} \in M$$

with $N \in \mathbb{N}\backslash\{0\}$, $s_{ijk}(\boldsymbol{x}) \in C^\infty(M)$. Given an operator $\boldsymbol{A}$ and a (1,2)-tensor $T(X, Y)$, we introduce the representation defined by Bogoyavlenskij (2004)

$$R_S(T)(X, Y) = \sum_{i,j,k}^{N} s_{ijk}(\boldsymbol{x}) \boldsymbol{A}^i T(\boldsymbol{A}^j X, \boldsymbol{A}^k Y) \,. \tag{4.5}$$

Thus, the action of $\lambda$ and $\mu$ is associated with the first and second arguments of $T(X, Y)$, whereas $z$ is associated with the powers of $\boldsymbol{A}$ acting on the values of $T(X, Y)$. Representation (4.5) satisfies the basic properties

$$R_{S_1+S_2} = R_{S_1} + R_{S_2}, \tag{4.6}$$

[1] We borrow the terminology of unigenerated polynomial algebras from Kreuzer and Robbiano (1993), although in our context it is defined for the more general case of polynomial algebras with variable coefficients. In our previous works, the same notion was equivalently denominated "cyclic algebras."

$$R_{S_1 \cdot S_2} = R_{S_2 \cdot S_1} = R_{S_1} \cdot R_{S_2} \, . \tag{4.7}$$

This framework allows us to represent the "tower" of generalized Nijenhuis torsions in a direct way.

**Lemma 2** *Let $\boldsymbol{A} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be an operator. We have:*

$$\tau_{\boldsymbol{A}}^{(m+1)}(X, Y) = R_\sigma \tau_{\boldsymbol{A}}^{(m)}(X, Y) \tag{4.8}$$

*where $\sigma$ is the polynomial $\sigma(z, \lambda, \mu) = (z - \lambda)(z - \mu)$.*

***Proof*** It is an immediate consequence of formula (3.1), which indeed can be written as

$$\begin{aligned} R_{(z^2 - z\lambda - z\mu + \lambda\mu)} \tau_{\boldsymbol{A}}^{(m)}(X, Y) &= \boldsymbol{A}^2 \tau_{\boldsymbol{A}}^{(m)}(X, Y) + \tau_{\boldsymbol{A}}^{(m)}(\boldsymbol{A}X, \boldsymbol{A}Y) \\ &\quad - \boldsymbol{A}\Big(\tau_{\boldsymbol{A}}^{(m)}(X, \boldsymbol{A}Y) + \tau_{\boldsymbol{A}}^{(m)}(\boldsymbol{A}X, Y)\Big) \\ &= \tau_{\boldsymbol{A}}^{(m+1)}(X, Y) \, . \end{aligned} \tag{4.9}$$

□

When $m = 2$, we recover the formula

$$\mathcal{H}_{\boldsymbol{A}}(X, Y) = R_\sigma \tau_{\boldsymbol{A}}(X, Y)$$

first stated in Bogoyavlenskij (2004). We also remind that, given a polynomial $P(z) = \sum_{k=0}^{N} c_k(\boldsymbol{x}) z^k$, the *Bézout identity* holds:

$$P(z) - P(\lambda) = (z - \lambda) Q_P(z, \lambda) \tag{4.10}$$

where

$$Q_P(z, \lambda) = \sum_{k=1}^{N} c_k(\boldsymbol{x}) \sum_{p+q=k-1} z^p \lambda^q \, .$$

We prove now a useful technical result.

**Lemma 3** *Let $\boldsymbol{A} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be an operator. Let $\boldsymbol{P} := P(\boldsymbol{A}) = \sum_{k=0}^{N} c_k(\boldsymbol{x}) \boldsymbol{A}^k$ be a polynomial in $\boldsymbol{A}$ with variable coefficients. We have*

$$\tau_{\boldsymbol{P}}^{(m)}(X, Y) = R_{(Q_P(z,\lambda))^m (Q_P(z,\mu))^m} \tau_{\boldsymbol{A}}^{(m)}(X, Y), \qquad m \geq 2 \, . \tag{4.11}$$

***Proof*** We shall proceed by induction over $m \geq 2$. The case $m = 2$, corresponding to the Haantjes torsion, has been proved in Bogoyavlenskij (2004). Thus, we assume that the property is true for the case of a torsion of level $m-1$ and we prove that it holds true for the case of a torsion of level $m$. According to Lemma 2, we have

$$\tau_{\boldsymbol{A}}^{(m)}(X,Y) = R_\sigma \tau_{\boldsymbol{A}}^{(m-1)}(X,Y) \tag{4.12}$$

where $\sigma = (z-\lambda)(z-\mu)$. Also, in terms of the operator $\boldsymbol{P}$ expression (4.12) can be written as

$$\tau_{\boldsymbol{P}}^{(m)}(X,Y) = R_{(P(z)-P(\lambda))(P(z)-P(\mu))} \tau_{\boldsymbol{P}}^{(m-1)}(X,Y)\ . \tag{4.13}$$

Thus, applying twice the Bézout identity (4.10), we get

$$(P(z)-P(\lambda))(P(z)-P(\mu)) = (z-\lambda)(z-\mu) Q_P(z,\lambda) Q_P(z,\mu)\ .$$

Exploiting property (4.7), we obtain

$$\tau_{\boldsymbol{P}}^{(m)}(X,Y) = R_{Q_P(z,\lambda)Q_P(z,\mu)} R_\sigma \tau_{\boldsymbol{P}}^{(m-1)}(X,Y)\ . \tag{4.14}$$

By the induction hypothesis, we have

$$\tau_{\boldsymbol{P}}^{(m-1)}(X,Y) = R_{(Q_P(z,\lambda))^{m-1}(Q_P(z,\mu))^{m-1}} \tau_{\boldsymbol{A}}^{(m-1)}(X,Y)\ .$$

Consequently, due to Eq. (4.14) and Lemma 2, we deduce that

$$\begin{aligned}\tau_{\boldsymbol{P}}^{(m)}(X,Y) &= R_{(Q_P(z,\lambda))^m(Q_P(z,\mu))^m} R_\sigma \tau_{\boldsymbol{A}}^{(m-1)}(X,Y) \\ &= R_{(Q_P(z,\lambda))^m(Q_P(z,\mu))^m} \tau_{\boldsymbol{A}}^{(m)}(X,Y)\ ,\end{aligned} \tag{4.15}$$

which completes the proof. □

We can now state the main result of this section.

**Theorem 3** *Let $\boldsymbol{A} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be an operator with $\tau_{\boldsymbol{A}}^{(m)}(X,Y) = \mathbf{0}$, $m \geq 2$. Then*

$$\tau^{(m)}_{\left(\sum_{k=0}^N c_k(\boldsymbol{x})\boldsymbol{A}^k\right)}(X,Y) = \mathbf{0}, \qquad X,Y \in \mathfrak{X}(M)\ , \tag{4.16}$$

*where $c_k(\boldsymbol{x}) \in C^\infty(M)$.*

***Proof*** Let $\boldsymbol{P} := P(\boldsymbol{A}) = \sum_{k=0}^N c_k(\boldsymbol{x})\boldsymbol{A}^k$. It is sufficient to apply Lemma 3, taking into account the assumption $\tau_{\boldsymbol{A}}^{(m)}(X,Y) = \mathbf{0}$. □

Interestingly enough, Theorem 3 does not hold in general for the case of the Nijenhuis torsion (i.e., $m = 1$). However, as is well known, relation (4.16) is valid for the Nijenhuis torsion if the coefficients $c_k(\boldsymbol{x})$ are all constant.

## 5 Generalized Nijenhuis Torsions and Simultaneous Block-Diagonalization

In this section, we state our main theoretical results. We first recall the following

**Definition 11** We shall say that a set of distributions $\{\mathcal{D}_i, \mathcal{D}_j, \ldots, \mathcal{D}_k\}$ is pairwise integrable if each pair $\mathcal{D}_i + \mathcal{D}_j$, $i \neq j$ is integrable.

**Definition 12** Let us consider a set of distributions $\{\mathcal{D}_i, \mathcal{D}_j, \ldots, \mathcal{D}_k\}$. We shall say that these distributions are *mutually integrable* if

(i) each of them is integrable
(ii) they are pairwise integrable.

We consider first the case of a single generalized Nijenhuis operator belonging to an arbitrary family of commuting operators.

**Theorem 4** *Let $\mathcal{S} = \{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_w\}$, $\boldsymbol{K}_\alpha : \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\alpha = 1, \ldots, w$ be a set of commuting operator fields; we assume that one of them, say $\boldsymbol{K}_1$, satisfies the condition*

$$\tau^{(m)}_{\boldsymbol{K}_1}(X, Y) = \mathbf{0}, \qquad X, Y \in \mathfrak{X}(M)\ , \tag{5.1}$$

*for some $m \geq 1$. Then, there exist local charts in which all of the operators $\boldsymbol{K}_\alpha$ can be written simultaneously in a block-diagonal form.*

***Proof*** Assuming that condition (5.1) is satisfied, Proposition 2 ensures the existence of an equivalence class of integrable frames and local charts where the operator $\boldsymbol{K}_1$ takes a block-diagonal form. Such coordinates are adapted to the characteristic web associated with the spectral decomposition of $\boldsymbol{K}_1$:

$$T_{\boldsymbol{x}} M = \mathcal{D}_i(\boldsymbol{x}) \bigoplus \mathcal{E}_i(\boldsymbol{x}) = \bigoplus_{i=1}^{s} \mathcal{D}_i(\boldsymbol{x})\ . \tag{5.2}$$

with

$$\boldsymbol{x} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^i, \ldots, \boldsymbol{x}^s)\ . \tag{5.3}$$

The variables $\boldsymbol{x}^i = (x^{i,1}, \ldots, x^{i,r_i})$ are defined over the integral leaves of the eigen-distribution $\mathcal{D}_i$, whereas the remaining ones, namely

$$(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{i-1}, \boldsymbol{x}^{i+1}, \ldots, \boldsymbol{x}^s)$$

are coordinates of the leaves, i.e., are constant ($\boldsymbol{x}^1 = \boldsymbol{c}^1, \ldots, \boldsymbol{x}^{i-1} = \boldsymbol{c}^{i-1}, \boldsymbol{x}^{i+1} = \boldsymbol{c}^{i+1}, \ldots, \boldsymbol{x}^s = \boldsymbol{c}^s$) on each leaf $D_i(\boldsymbol{c})$ of the foliation. Here $\boldsymbol{c} := (\boldsymbol{c}^1, \ldots, \boldsymbol{c}^{i-1}, \boldsymbol{c}^{i+1}, \ldots, \boldsymbol{c}^s)$. Since all operators of the set $\mathcal{S}$ commute, every distribution $\mathcal{D}_i$ is invariant under the action of the operators $\{\boldsymbol{K}_2, \ldots, \boldsymbol{K}_w\}$. As a direct consequence of this property, all the operators $\boldsymbol{K}_\alpha \in \mathcal{S}$ in the local chart (5.3) take a block-diagonal form, where the $i$-th ($r_i \times r_i$) block matrix $[\boldsymbol{K}_\alpha^{(i)}]_{|D_i(\boldsymbol{c})}$ coincides with the matrix $[\boldsymbol{K}_{\alpha|D_i(\boldsymbol{c})}]$, $i = 1, \ldots, s$, which represents the restricted operator $\boldsymbol{K}_\alpha$ to the leaf $D_i(\boldsymbol{c})$. □

**Theorem 5** *Let $\mathcal{S} = \{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_w\}$, $\boldsymbol{K}_\alpha : \mathfrak{X}(M) \to \mathfrak{X}(M)$, be a set of commuting operator fields. If*

*(i) all the operators of the family have vanishing generalized torsion of level $m$:*

$$\tau_{\boldsymbol{K}_\alpha}^{(m)}(X, Y) = \mathbf{0}\,, \qquad \alpha = 1, \ldots, w, \qquad X, Y \in \mathfrak{X}(M) \tag{5.4}$$

*(ii) their joint eigen-distributions*

$$\mathcal{V}_a(\boldsymbol{x}) := \bigoplus_{i_1,\ldots,i_w}^{s_1,\ldots,s_w} \mathcal{D}_{i_1}^{(1)}(\boldsymbol{x}) \bigcap \ldots \bigcap \mathcal{D}_{i_w}^{(w)}(\boldsymbol{x})\,, \qquad a = 1, \ldots, v \leq n \tag{5.5}$$

*are pairwise integrable, then there exist sets of local coordinates, adapted to the decomposition*

$$T_{\boldsymbol{x}}M = \bigoplus_{a=1}^{v} \mathcal{V}_a(\boldsymbol{x}) \qquad \boldsymbol{x} \in M, \tag{5.6}$$

*in which all operators $\boldsymbol{K}_\alpha$ admit simultaneously a block-diagonal form with possibly finer blocks with respect to those considered in Theorem 4.*

***Proof*** According to Theorem 4, since by hypothesis $\tau_{\boldsymbol{K}_1}^{(m)}(X, Y) = \mathbf{0}$, $X, Y \in \mathfrak{X}(M)$, all the operators take simultaneously a block-diagonal form in a local chart adapted to the spectral decomposition of $\boldsymbol{K}_1$. Assume also that $\tau_{\boldsymbol{K}_2}^{(m)}(X, Y) = \mathbf{0}$. Then, the tangent space at any point $\boldsymbol{x}$ admits the finer decomposition

$$T_{\boldsymbol{x}}M = \bigoplus_{i_1,i_2}^{s_1,s_2} \mathcal{D}_{i_1}^{(1)}(\boldsymbol{x}) \cap \mathcal{D}_{i_2}^{(2)}(\boldsymbol{x})\,, \tag{5.7}$$

where $\mathcal{D}_{i_2}^{(2)}$ are the eigen-distributions of $\boldsymbol{K}_2$. These eigen-distributions are integrable by virtue of Theorem 1. Consequently, the generalized Haantjes Theorem 1 can also be applied to the restriction of $\boldsymbol{K}_2$ to $D_{i_1}^{(1)}(\boldsymbol{c}_1)$. Therefore, there exists a transformation of coordinates, acting only on the coordinates over the leaves of the foliation $D_{i_1}^{(1)}$

$$\Phi : M \to M, \qquad (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{i_1}, \ldots, \boldsymbol{x}^{s_1}) \mapsto (\boldsymbol{x}^1, \ldots, \boldsymbol{y}^{i_1}, \ldots, \boldsymbol{x}^{s_1})\,, \tag{5.8}$$

such that the new coordinates $\boldsymbol{y}^{i_1} = (y^{i_1,1}, \ldots, y^{i_1,r_{i_1}}) = \boldsymbol{f}^{i_1}(\boldsymbol{x}^{i_1})$ are adapted to the decomposition

$$T_{\boldsymbol{x}} D_{i_1}^{(1)}(\boldsymbol{c}_1) = \bigoplus_{i_2}^{s_2} \mathcal{D}_{i_1}^{(1)}(\boldsymbol{x}) \cap \mathcal{D}_{i_2}^{(2)}(\boldsymbol{x}), \qquad \boldsymbol{x} \in D_{i_1}^{(1)}(\boldsymbol{c}_1)\,. \tag{5.9}$$

Thus, we have

$$[\boldsymbol{K}_\alpha^{(i_1)}] = \left[\begin{array}{c|c|c} \boldsymbol{K}_\alpha^{(i_1,1)} & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \boldsymbol{K}_\alpha^{(i_1,s_2)} \end{array}\right], \qquad \alpha = 1, \ldots, w \tag{5.10}$$

where $[\boldsymbol{K}_\alpha^{(i_1,j)}]|_{D_{i_1}^{(1)}(\boldsymbol{c}_1)\cap D_j^{(2)}(\boldsymbol{c}_2)} = \left[\boldsymbol{K}_\alpha|_{D_{i_1}^{(1)}(\boldsymbol{c}_1)\cap D_j^{(2)}(\boldsymbol{c}_2)}\right]$, $j = 1, \ldots, s_2$. Let us consider the decomposition

$$T_{\boldsymbol{x}} M = \bigoplus_{i_1,i_2}^{s_1,s_2} \mathcal{D}_{i_1}^{(1)}(\boldsymbol{x}) \cap \mathcal{D}_{i_2}^{(2)}(\boldsymbol{x}) = \bigoplus_{\gamma=1}^{u} \mathcal{U}_\gamma(\boldsymbol{x}) \qquad \boldsymbol{x} \in M\,, \tag{5.11}$$

where in the direct sum (5.7) $\mathcal{U}_\gamma \neq \{\boldsymbol{0}\}$, $u \leq n$ and $r_\gamma$ denotes the rank of $\mathcal{U}_\gamma$ ($\sum_{\gamma=1}^{u} r_\gamma = n$). Clearly, the distributions $\mathcal{U}_\gamma$ are invariant under the action of each operator $\boldsymbol{K}_\alpha \in \mathcal{S}$. Besides, these distributions are involutive as are realized as the intersection of involutive distributions. By assumption, their direct sums are also integrable. Therefore, there exist local charts in $M$ of the form

$$\{U, (y^{1,1}, \ldots, y^{1,r_1}; \ldots; y^{i_1,1}, \ldots, y^{i_1,r_{i_1}}; \ldots; y^{s_1,1}, \ldots, y^{s_1,r_{s_1}})\} \tag{5.12}$$

adapted to the decomposition (5.7), where all the operators $\boldsymbol{K}_\alpha \in \mathcal{S}$ admit simultaneously a (possibly) finer block-diagonal form. By extending the previous procedure to the Haantjes operators $\boldsymbol{K}_3, \ldots, \boldsymbol{K}_w$, we obtain the decomposition

$$T_{\boldsymbol{x}} M = \bigoplus_{i_1,\ldots,i_w}^{s_1,\ldots,s_w} \mathcal{D}_{i_1}^{(1)}(\boldsymbol{x}) \bigcap \cdots \bigcap \mathcal{D}_{i_w}^{(w)}(\boldsymbol{x}) = \bigoplus_{a=1}^{v} \mathcal{V}_a(\boldsymbol{x}), \tag{5.13}$$

where in the direct sum (5.13) $\mathcal{V}_a \neq \{\boldsymbol{0}\}$, $v \leq n$ and $r_a$ denotes the rank of $\mathcal{V}_a$ ($\sum_{a=1}^{v} r_a = n$). Since the involutive distributions $\mathcal{V}_a$ are mutually integrable as a consequence of the previous reasoning, then there exist local charts

$$\{U, (\boldsymbol{y}^1, \ldots, \boldsymbol{y}^a, \ldots, \boldsymbol{y}^v)\} \tag{5.14}$$

adapted to the decomposition (5.13), such that

$$\mathcal{V}_a = \left\langle \frac{\partial}{\partial y^{a,1}}, \ldots, \frac{\partial}{\partial y^{a,r_a}} \right\rangle \qquad a = 1, \ldots, v, \tag{5.15}$$

where the natural frame $\left\{\frac{\partial}{\partial y^{a,1}}, \ldots, \frac{\partial}{\partial y^{a,r_a}}\right\}$ over the leaves of $\mathcal{V}_a$ is formed by joint *generalized* eigenvector fields of the operators $\{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_w\}$. $\square$

***Remark 2*** Given a generalized Haantjes algebra $(M, \mathscr{H}^{(m)})$ with generator $\boldsymbol{L}$, the hypothesis $(ii)$ of Theorem 5 is automatically satisfied. Indeed, the joint eigen-distributions $\mathcal{V}_a$ either coincide with the eigen-distributions of the generator $\boldsymbol{L}$ or are direct sums of them. In both cases, since (due to Theorem 1) the eigen-distributions of $\boldsymbol{L}$ are mutually integrable, it turns out that the direct sums of the joint eigen-distributions $\mathcal{V}_a$ are integrable as well.

In the following sections, we shall illustrate the results proposed. First, we will show some examples of generalized Haantjes algebras of operator fields. Then, we shall present the explicit construction of local charts of coordinates allowing their simultaneous block-diagonalization.

## 6 Block-Diagonalization of Generalized Haantjes Algebras

As stated in Theorems 4 and 5, we can block-diagonalize families of commuting generalized Nijenhuis operators by constructing a suitable coordinate chart. To this aim, we propose the following procedure.

(1) Determine the joint eigen-distributions $\mathcal{V}_a$ of the given family of operator fields
(2) Determine a basis of closed one-forms for each of the corresponding annihilators
(3) Integrate them to find the characteristic coordinates
(4) Compute the expression of the operators of the given family in these coordinates

To illustrate this procedure, we shall first construct families of operator fields, depending on arbitrary functions and having a prescribed vanishing higher-level torsion. A direct approach consists in considering an operator field whose entries are all arbitrary functions, and imposing both that a certain higher-level torsion is vanishing, and simultaneously that the lower-level ones are not. In this way, one can obtain a set of differential constraints that can be solved (possibly with suitable ansätze) to obtain families of operator fields with a vanishing higher-level torsion, still depending on arbitrary functions. Generally speaking, there is a lot of freedom in this approach.

From this analysis, it emerges that generalized Haantjes algebras are not rare. Thus, by specializing the arbitrary functions contained in the families of operators determined according to the procedure proposed, we can obtain infinitely many examples of new, higher-level Haantjes algebras.

### 6.1 A Third-Level Generalized Haantjes Algebra

Let $M$ be a five-dimensional differentiable manifold. In local coordinates $\boldsymbol{x} = (x^1, x^2, x^3, x^4, x^5)$, a first example is given by the family of operators

$$\boldsymbol{L}(\boldsymbol{x}) = \begin{bmatrix} f_1 & 1 & 0 & 1 & 0 \\ f_1 - f_2 + 1 & f_1 + 1 & -f_3 & f_1 - f_2 + 1 & -f_3 \\ 1 & 0 & f_2 + f_3 & 1 & f_3 \\ f_2 - f_1 & -1 & f_3 & f_2 - 1 & f_3 \\ -1 & 0 & -f_3 - 1 & -1 & f_2 - f_3 - 1 \end{bmatrix}, \tag{6.1}$$

where $f_1 = f_1(x^1, x^2)$, $f_2 = f_2(x^3, x^4, x^5)$ and $f_3 = f_3(x^3)$ are arbitrary functions.

A direct computation allows us to prove the following result.

**Proposition 3** *The family of operators* $\boldsymbol{L}$ *satisfies the following properties:*

- $\tau_{\boldsymbol{L}}^{(3)}(X, Y) = \mathbf{0}$
- $\mathcal{H}_{\boldsymbol{L}}(X, Y) \neq \mathbf{0}$ *if and only if* $f_3' \neq 0$.

The minimal polynomial of $\boldsymbol{L}$ is generically of fifth degree and reads

$$m_{\boldsymbol{L}}(\boldsymbol{x}, \lambda) = (\lambda - f_1 - 1)(\lambda - f_1 + 1)\,(\lambda - f_2 + 1)\,(\lambda - f_2)^2\,. \tag{6.2}$$

Thus, it coincides with the characteristic polynomial; therefore $\boldsymbol{L}$ is a cyclic operator and the Riesz indices $\rho_i$ coincide with the rank of the eigen-distributions $\mathcal{D}_i$. Also, notice that $\boldsymbol{L}$ is generically non-semisimple. The eigenvalues and generalized eigenvectors of $\boldsymbol{L}$ are

$$\begin{aligned}
\lambda_1 = f_1 + 1, \quad \rho_1 = 1, \quad \mathcal{D}_1 &= \left\langle \frac{\partial}{\partial x^1} + 2\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4} \right\rangle, \\
\lambda_2 = f_1 - 1, \quad \rho_2 = 1, \quad \mathcal{D}_2 &= \left\langle \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^4} \right\rangle, \\
\lambda_3 = f_2 - 1, \quad \rho_3 = 1, \quad \mathcal{D}_3 &= \left\langle f_3\frac{\partial}{\partial x^2} - f_3\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5} \right\rangle, \\
\lambda_4 = f_2, \quad \rho_4 = 2, \quad \mathcal{D}_4 &= \left\langle \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4} \right\rangle.
\end{aligned} \tag{6.3}$$

From these eigen-distributions, we can compute their characteristic distributions and consequently their annihilators:

$$\begin{aligned}
\mathcal{E}_1 = \oplus_{i\neq 1}\mathcal{D}_i &= \left\langle \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^4}, f_3\frac{\partial}{\partial x^2} - f_3\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^5} \right\rangle \\
&\Longrightarrow \mathcal{E}_1^\circ = \langle dx^1 + dx^2 + dx^4 \rangle, \\
\mathcal{E}_2 = \oplus_{i\neq 2}\mathcal{D}_i &= \left\langle \frac{\partial}{\partial x^1} + 2\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4}, f_3\frac{\partial}{\partial x^2} - f_3\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^5} \right\rangle \\
&\Longrightarrow \mathcal{E}_2^\circ = \langle dx^1 - dx^2 - dx^4 \rangle, \\
\mathcal{E}_3 = \oplus_{i\neq 3}\mathcal{D}_i &= \left\langle \frac{\partial}{\partial x^1} + 2\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^5} \right\rangle \\
&\Longrightarrow \mathcal{E}_3^\circ = \langle dx^3 + dx^5 \rangle, \\
\mathcal{E}_4 = \oplus_{i\neq 4}\mathcal{D}_i &= \left\langle \frac{\partial}{\partial x^1} + 2\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^4}, f_3\frac{\partial}{\partial x^2} - f_3\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5} \right\rangle \\
&\Longrightarrow \mathcal{E}_4^\circ = \langle dx^3, dx^1 + dx^4 + f_3 dx^5 \rangle.
\end{aligned} \tag{6.4}$$

***Remark 3*** We can easily extract an Abelian unigenerated algebra $\mathscr{H}^{(3)}$ from the class (6.1). In order to achieve that, we set $f_3 = x^3$ for all the operators of the algebra and vary the other arbitrary functions. Observe that all the operators of the family obtained in this way will share both the same eigen-distributions and the degree of the minimal polynomial. Thus, any of them generate the whole algebra $\mathscr{H}^{(3)}$ with its independent powers. Notice that for this algebra the hypotheses of Theorem 5 are all satisfied, as discussed in Remark 2.

By way of an example of simultaneous block-diagonalization, we can consider the three commuting operators

$$\boldsymbol{L}_1(\boldsymbol{x}) = \begin{bmatrix} x^1 & 1 & 0 & 1 & 0 \\ x^1+1 & x^1+1 & -x^3 & x^1+1 & -x^3 \\ 1 & 0 & x^3 & 1 & x^3 \\ -x^1 & -1 & x^3 & -1 & x^3 \\ -1 & 0 & -x^3-1 & -1 & -x^3-1 \end{bmatrix}, \tag{6.5}$$

$$\boldsymbol{L}_2(\boldsymbol{x}) = \begin{bmatrix} x^2 & 1 & 0 & 1 & 0 \\ x^2-x^5+1 & x^2+1 & -x^3 & x^2-x^5+1 & -x^3 \\ 1 & 0 & x^3+x^5 & 1 & x^3 \\ -x^2+x^5 & -1 & x^3 & x^5-1 & x^3 \\ -1 & 0 & -x^3-1 & -1 & -x^3+x^5-1 \end{bmatrix}, \tag{6.6}$$

$$\boldsymbol{L}_3(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ -x^4-x^3x^5+1 & 1 & -x^3 & -x^4-x^3x^5+1 & -x^3 \\ 1 & 0 & x^4+x^3\left(x^5+1\right) & 1 & x^3 \\ x^4+x^3x^5 & -1 & x^3 & x^4+x^3x^5-1 & x^3 \\ -1 & 0 & -x^3-1 & -1 & x^4+x^3\left(x^5-1\right)-1 \end{bmatrix}. \tag{6.7}$$

These operators are $C^\infty(\mathbb{R})$-linearly independent. Any of these three operators can be chosen to be the generator of a level-three Haantjes unigenerated algebra $\mathscr{H}^{(3)}$ of rank 5. A basis of this algebra is, for instance, $\mathscr{B} = \{\boldsymbol{I}, \boldsymbol{L}_1, \boldsymbol{L}_1^2, \boldsymbol{L}_1^3, \boldsymbol{L}_1^4\}$. In the following subsection, we shall determine a set of coordinates which block-diagonalize the algebra $\mathscr{H}^{(3)}$.

### 6.2 Block-Diagonalization

By integrating the annihilators (6.4) of the eigen-distributions of operators (6.5)-(6.7) and leaving $f_3 = f_3(x^3)$ arbitrary, we obtain a set of coordinates which block-diagonalize the algebra $\mathscr{H}^{(3)}$. We have explicitly:

$$\begin{aligned} y^1 &= x^1 + x^2 + x^4, \\ y^2 &= x^1 - x^2 - x^4, \\ y^3 &= x^3 + x^5, \\ y^4 &= x^3, \\ y^5 &= x^1 + x^4 + f_3(x^3)x^5. \end{aligned} \tag{6.8}$$

Precisely, in these block-separation coordinates, the operators $\boldsymbol{L}_1, \boldsymbol{L}_2, \boldsymbol{L}_3$ read:

$$
\boldsymbol{L}_1(\boldsymbol{y}) = \begin{bmatrix}
\frac{1}{2}\left(y^1+y^2\right)+1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}\left(y^1+y^2\right)-1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -y^3+2y^4 & 1 \\
0 & 0 & 0 & -\left(y^3-2y^4\right)^2 & y^3-2y^4
\end{bmatrix}, \tag{6.9}
$$

$$
\boldsymbol{L}_2(\boldsymbol{y}) = \begin{bmatrix}
y^1+(y^3-y^4)y^4-y^5+1 & 0 & 0 & 0 & 0 \\
0 & y^1+(y^3-y^4)y^4-y^5-1 & 0 & 0 & 0 \\
0 & 0 & y^3-y^4-1 & 0 & 0 \\
0 & 0 & 0 & y^4 & 1 \\
0 & 0 & 0 & -\left(y^3-2y^4\right)^2 & 2y^3-3y^4
\end{bmatrix}, \tag{6.10}
$$

$$
\boldsymbol{L}_3(\boldsymbol{y}) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2}(y^1+y^2)+y^5-1 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2}(y^1+y^2)-y^3+2y^4+y^5 & 1 \\
0 & 0 & 0 & -\left(y^3-2y^4\right)^2 & -\frac{1}{2}(y^1+y^2)+y^3-2y^4+y^5
\end{bmatrix} \tag{6.11}
$$

Observe that more generally, in coordinates (6.8) the whole class (6.1) takes the block-diagonal form

$$
\boldsymbol{L}(\boldsymbol{y}) = \begin{bmatrix}
f_1+1 & 0 & 0 & 0 & 0 \\
0 & f_1-1 & 0 & 0 & 0 \\
0 & 0 & f_2-1 & 0 & 0 \\
0 & 0 & 0 & f_2+f_3-(y^3-y^4)f_3' & 1 \\
0 & 0 & 0 & -\left(f_3-(y^3-y^4)f_3'\right)^2 & f_2-f_3+(y^3-y^4)f_3'
\end{bmatrix}, \tag{6.12}
$$

where $f_1 = f_1\left(\frac{1}{2}(y^1+y^2),\, y^1+(y^3-y^4)f_3-y^5\right)$, $f_2 = f_2\left(y^4, -\frac{1}{2}(y^1+y^2) -(y^3-y^4)f_3+y^5,\, y^3-y^4\right)$ and $f_3 = f_3(y^4)$ are arbitrary functions (with $f_3' \neq 0$).

## 7 A Fourth-Level Generalized Haantjes Algebra

Let us consider the following fourth-level generalized Nijenhuis family of operator fields defined over a seven-dimensional differentiable manifold $M$, which is obtained with the same procedure described in the previous section:

$$
\boldsymbol{K}(\boldsymbol{x}) = \begin{bmatrix}
g_1 & x^1 & x^1+g_2 & -x^1 & x^1 & -x^1 & x^1 \\
1 & g_5 & -g_1+g_5+x^1 & g_4-g_5 & 0 & g_1-g_5-x^1 & -1-g_4+g_5 \\
0 & 0 & g_1 & 0 & 0 & 0 & 0 \\
1+g_1-g_4 & x^1 & x^1+g_2 & -1-x^1+g_4 & x^1 & -x^1 & x^1 \\
0 & g_1-g_5 & g_1+g_3-g_5+\frac{1}{x^1} & -g_1+g_5 & g_1 & -g_1+g_5-\frac{1}{x^1} & g_1-g_5 \\
0 & 0 & x^1 & 0 & 0 & g_1-x^1 & 0 \\
g_1-g_4 & x^1 & x^1+g_2 & -1-x^1 & x^1 & -x^1 & 1+x^1+g_4
\end{bmatrix}. \tag{7.1}
$$

Here, $g_1 = g_1(x^1,x^2,x^3,x^4)$, $g_2 = g_2(x^1,x^2,x^3,x^4)$, $g_3 = g_3(x^1,x^2,x^3,x^4)$, $g_4 = g_4(x^5,x^6)$, $g_5 = g_5(x^7)$.

As a result of a direct computation, one can prove the following

**Proposition 4** *The operator family (7.1) satisfies the property:*

$$\tau_{\boldsymbol{K}}^{(4)}(X, Y) = 0\,.$$

*In addition, its third-level torsion does not vanish whenever $g_3 \neq 0$.*

### 7.1 Spectral Analysis

Its minimal polynomial is generically of seventh degree,

$$m_{\boldsymbol{K}}(\boldsymbol{x}, \lambda) = (\lambda - g_5)(\lambda - g_4 - 1)(\lambda - g_4 + 1)(\lambda - g_1 + x^1)(\lambda - g_1)^3, \quad (7.2)$$

and its eigenvalues and generalized eigenvectors are

$$\begin{aligned}
&\lambda_1 = g_5, \quad \rho_1 = 1, \qquad \mathcal{D}_1 = \left\langle \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^5} \right\rangle, \\
&\lambda_2 = g_4 + 1, \quad \rho_2 = 1, \quad \mathcal{D}_2 = \left\langle \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^7} \right\rangle, \\
&\lambda_3 = g_4 - 1, \quad \rho_3 = 1, \quad \mathcal{D}_3 = \left\langle \frac{\partial}{\partial x^2} + 2\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7} \right\rangle, \\
&\lambda_4 = g_1 - x^1, \rho_4 = 1, \quad \mathcal{D}_4 = \left\langle \frac{\partial}{\partial x^1} - (x^1)^2\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^5} - (x^1)^2\frac{\partial}{\partial x^6} + \frac{\partial}{\partial x^7} \right\rangle, \\
&\lambda_5 = g_1, \quad \rho_5 = 3, \qquad \mathcal{D}_5 = \left\langle \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^6} \right\rangle.
\end{aligned} \quad (7.3)$$

The characteristic distributions and the corresponding annihilators are

$$\begin{aligned}
\mathcal{E}_1 = \oplus_{i\neq 1}\, \mathcal{D}_i = \Big\langle & \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^2} + 2\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^1} - (x^1)^2\frac{\partial}{\partial x^2} \\
& + \frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^5} - (x^1)^2\frac{\partial}{\partial x^6} \\
& + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^6}, \frac{\partial}{\partial x^5} \Big\rangle \\
& \Longrightarrow \mathcal{E}_1^\circ = \langle dx^2 + dx^3 - dx^4 - dx^6 + dx^7 \rangle, \\
\mathcal{E}_2 = \oplus_{i\neq 2}\, \mathcal{D}_i = \Big\langle & \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^2} + 2\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^1} \\
& - (x^1)^2\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^5} - (x^1)^2\frac{\partial}{\partial x^6} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^1} \\
& + \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^6}, \frac{\partial}{\partial x^5} \Big\rangle \\
& \Longrightarrow \mathcal{E}_2^\circ = \langle dx^1 + dx^4 - 2dx^7 \rangle,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{E}_3 = \oplus_{i \neq 3} \mathcal{D}_i = \Big\langle & \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^1} - (x^1)^2 \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^5} \\
& - (x^1)^2 \frac{\partial}{\partial x^6} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^6}, \frac{\partial}{\partial x^5} \Big\rangle \\
& \Longrightarrow \mathcal{E}_3^\circ = \langle dx^1 - dx^4 \rangle, \\
\mathcal{E}_4 = \oplus_{i \neq 4} \mathcal{D}_i = \Big\langle & \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^2} + 2\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \\
& \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^6}, \frac{\partial}{\partial x^5} \Big\rangle \\
& \Longrightarrow \mathcal{E}_4^\circ = \langle dx^3 - dx^6 \rangle, \\
\mathcal{E}_5 = \oplus_{i \neq 5} \mathcal{D}_i = \Big\langle & \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^2} + 2\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \\
& \frac{\partial}{\partial x^1} - (x^1)^2 \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^5} - (x^1)^2 \frac{\partial}{\partial x^6} + \frac{\partial}{\partial x^7} \Big\rangle \\
& \Longrightarrow \mathcal{E}_5^\circ = \langle dx^3, (x^1)^2 dx^1 + dx^6, dx^1 + dx^2 - dx^4 + dx^5 - dx^6 + dx^7 \rangle \,.
\end{aligned}
\tag{7.4}
$$

By specializing the arbitrary functions $g_1$, $g_2$, $g_4$, $g_5$, but keeping the same function $g_3$ in all operators to have commutativity, we easily obtain an Abelian, unigenerated and fourth-level Haantjes algebra. As in the previous example, all the operators of the family obtained in this way will share both the same eigen-distributions and the degree of the minimal polynomial. Thus, any of them generates the whole algebra.

To illustrate the simultaneous block-diagonalization procedure, by way of an example we can consider the three commuting operators

$$
\boldsymbol{K}_1(\boldsymbol{x}) = \begin{bmatrix}
x^2 & x^1 & x^1 & -x^1 & x^1 & -x^1 & x^1 \\
1 & 0 & x^1 - x^2 & 0 & 0 & -x^1 + x^2 & -1 \\
0 & 0 & x^2 & 0 & 0 & 0 & 0 \\
1 + x^2 & x^1 & x^1 & -1 - x^1 & x^1 & -x^1 & x^1 \\
0 & x^2 & x^2 & -x^2 & x^2 & -\frac{1}{x^1} - x^2 & x^2 \\
0 & 0 & x^1 & 0 & 0 & -x^1 + x^2 & 0 \\
x^2 & x^1 & x^1 & -1 - x^1 & x^1 & -x^1 & 1 + x^1
\end{bmatrix},
\tag{7.5}
$$

$$
\boldsymbol{K}_2(\boldsymbol{x}) = \begin{bmatrix}
x^3 & x^1 & -\frac{1}{x^1} & -x^1 & x^1 & -x^1 & x^1 \\
1 & x^7 & x^1 - x^3 + x^7 & x^6 - x^7 & 0 & -x^1 + x^3 - x^7 & -1 - x^6 + x^7 \\
0 & 0 & x^3 & 0 & 0 & 0 & 0 \\
1 + x^3 - x^6 & x^1 & -\frac{1}{x^1} & -1 - x^1 + x^6 & x^1 & -x^1 & x^1 \\
0 & x^3 - x^7 & x^3 - x^7 & -x^3 + x^7 & x^3 & -\frac{1}{x^1} - x^3 + x^7 & x^3 - x^7 \\
0 & 0 & x^1 & 0 & 0 & -x^1 + x^3 & 0 \\
x^3 - x^6 & x^1 & -\frac{1}{x^1} & -1 - x^1 & x^1 & -x^1 & 1 + x^1 + x^6
\end{bmatrix},
\tag{7.6}
$$

$$
\boldsymbol{K}_3(\boldsymbol{x}) = \begin{bmatrix} x^4 & x^1 & 0 & -x^1 & x^1 & -x^1 & x^1 \\ 1 & 1 & 1+x^1-x^4 & -1+x^5 & 0 & -1-x^1+x^4 & -x^5 \\ 0 & 0 & x^4 & 0 & 0 & 0 & 0 \\ 1+x^4-x^5 & x^1 & 0 & -1-x^1+x^5 & x^1 & -x^1 & x^1 \\ 0 & -1+x^4 & -1+x^4 & 1-x^4 & x^4 & 1-\frac{1}{x^1}-x^4 & -1+x^4 \\ 0 & 0 & x^1 & 0 & 0 & -x^1+x^4 & 0 \\ x^4-x^5 & x^1 & 0 & -1-x^1 & x^1 & -x^1 & 1+x^1+x^5 \end{bmatrix}. \tag{7.7}
$$

These operators are $C^\infty(\mathbb{R})$-linearly independent. Each of them generates a fourth-level Haantjes algebra $\mathscr{H}^{(4)}$ of rank 7. A basis of this algebra is given, for instance, by $\mathscr{B} = \{\boldsymbol{I}, \boldsymbol{K}_1, \boldsymbol{K}_1^2, \boldsymbol{K}_1^3, \boldsymbol{K}_1^4, \boldsymbol{K}_1^5, \boldsymbol{K}_1^6\}$. In the following subsection, we shall determine a set of coordinates which block-diagonalize the whole algebra $\mathscr{H}^{(4)}$.

### 7.2 Block-Diagonalization

By integrating the annihilators (7.4) of the eigen-distributions of these operators, we get the block-separating coordinates:

$$
\begin{aligned}
y^1 &= x^2 + x^3 - x^4 - x^6 + x^7, \\
y^2 &= x^1 + x^4 - 2x^7, \\
y^3 &= x^1 - x^4, \\
y^4 &= x^3 - x^6, \\
y^5 &= x^3, \\
y^6 &= \frac{(x^1)^3}{3} + x^6, \\
y^7 &= x^1 + x^2 - x^4 + x^5 - x^6 + x^7.
\end{aligned} \tag{7.8}
$$

In these coordinates, all the operators of the algebra $\mathscr{H}^{(4)}$ block-diagonalize simultaneously. In particular, by defining $\chi = \sqrt[3]{3(y^4 - y^5 + y^6)}$, the operators $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, $\boldsymbol{K}_3$ take the form:

$$
\boldsymbol{K}_1(\boldsymbol{y}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{y^1 + \frac{1}{2}(y^2 - y^3) - y^4 - \chi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y^1 + \frac{1}{2}(y^2 - y^3) - y^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \chi + \chi^3 & y^1 + \frac{1}{2}(y^2 - y^3) - y^4 - \chi & \chi^3 \\ 0 & 0 & 0 & 0 & \chi & -\chi^{-1} & y^1 + \frac{1}{2}(y^2 - y^3) - y^4 + \chi \end{bmatrix}, \tag{7.9}
$$

$$
\boldsymbol{K}_2(\boldsymbol{y}) = \begin{bmatrix} -\frac{1}{2}(y^2+y^3)+\chi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-y^4+y^5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1-y^4+y^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^5-\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y^5-\chi & \chi^3 \\ 0 & 0 & 0 & 0 & -\chi^{-1} & -\chi^{-1} & y^5+\chi \end{bmatrix}, \tag{7.10}
$$

$$
\boldsymbol{K}_3(\boldsymbol{y}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-y^1+y^5-\chi+y^7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1-y^1+y^5-\chi+y^7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y^3+\chi & 0 & 0 \\ 0 & 0 & 0 & 0 & \chi & -y^3 & \chi^3 \\ 0 & 0 & 0 & 0 & 0 & -\chi^{-1} & -y^3+2\chi \end{bmatrix}. \tag{7.11}
$$

Also, in the block-separation coordinates, the family (7.1) takes a block-diagonal form:

$$
\boldsymbol{K}(\boldsymbol{y}) = \begin{bmatrix} g_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_4+1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_4-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_1-\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_2\chi^2+\chi\left(1+\chi^2\right) & g_1-\chi & \chi^3 \\ 0 & 0 & 0 & 0 & g_3+g_2+\chi^{-1}\left(1+\chi^2\right) & -\chi^{-1} & g_1+\chi \end{bmatrix}. \tag{7.12}
$$

Here, $g_1 = g_1(\chi, y^1+\frac{1}{2}(y^2-y^3)-y^4, y^5)$, $g_2 = g_2(\chi, y^1+\frac{1}{2}(y^2-y^3)-y^4, y^5)$, $g_3 = g_3(\chi, y^1+\frac{1}{2}(y^2-y^3)-y^4, y^5)$, $g_4 = g_4(-y^1-\chi+y^5+y^7, -y^4+y^5)$ and $g_5 = g_5(-\frac{1}{2}(y^2+y^3)+\chi)$ are arbitrary functions of their arguments.

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