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On surfaces with $p_g = 2$, $q = 1$ and $K^2 = 5$

Tommaso Gentile · Paolo A. Oliverio · Francesco Polizzi

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Abstract We consider minimal surfaces of general type with $p_g = 2$, $q = 1$ and $K^2 = 5$. We provide a stratification of the corresponding moduli space $\mathcal{M}$ and we give some bounds for the number and the dimensions of its irreducible components.

Keywords Surfaces of general type · Albanese map · Genus 2 fibration

Mathematics Subject Classification (2000) 14J29 · 14J10

1 Introduction

Recently there has been considerable interest in understanding the geometry of irregular surfaces of general type. Although the classification of such surfaces is still far from being achieved, their study has produced in the last years a considerable amount of results, see for instance the survey papers [2] and [14].

Minimal surfaces of general type satisfy the classical inequalities:

- $\chi(\mathcal{O}_S) := p_g - q + 1 \geq 1$,
- $K_S^2 \geq 2p_g$ if S is irregular (Debarre's inequality),
- $K_S^2 \leq 9\chi(\mathcal{O}_S)$ (Miyaoka–Yau inequality).

If S is irregular and $K_S^2 = 2\chi$, then it follows $q = 1$. In this case the Albanese map $f : S \longrightarrow \text{Alb}(S)$ is a genus 2 fibration whose fibres are all 2-connected. The corresponding

T. Gentile · P.A. Oliverio · F. Polizzi (✉)
Dipartimento di Matematica, Università della Calabria, Cubo 30B, 87036 Arcavacata di Rende, Cosenza, Italy
e-mail: polizzi@mat.unical.it

T. Gentile
e-mail: gentile@mat.unical.it

P.A. Oliverio
e-mail: oliverio@unical.it

classification was given by Catanese [7] for $K_S^2 = 2$, and by Horikawa [12] in the general case.

The study of irregular surfaces with $K_S^2 = 2\chi + 1$ was started by Catanese and Ciliberto in [4] and [5]. They investigated the case $\chi = 1$, i.e., $p_g = q = 1$ and $K_S^2 = 3$, proving that for this class of surfaces the genus g of the fibre of the Albanese map can be either 2 or 3. They also described all surfaces with $g = 3$ and started the classification of surfaces with $g = 2$, which was later completed by Catanese and Pignatelli in [6], by using a structure theorem for genus 2 fibrations which is proven in the same work.

For $\chi \geq 2$ the situation is far more complicated and not yet thoroughly studied. In this paper we consider the case $\chi = 2$, and we investigate the surfaces whose numerical invariants are

$$K_S^2 = 5, \qquad p_g = 2, \qquad q = 1.$$

By a result of Horikawa, given any irregular minimal surface of general type with $2\chi \leq K^2 < \frac{8}{3}\chi$, its Albanese map $f\colon S \longrightarrow \mathrm{Alb}(S)$ is a genus 2 fibration over a smooth curve of genus q. Then in our case we have a genus 2 fibration $f\colon S \longrightarrow B$ over an elliptic curve B.

We can therefore use the results of Horikawa-Xiao and those of Catanese-Pignatelli in order to construct our surfaces and describe their moduli space. In fact, we first study the rank 2 vector bundle $V_1 := f_*\omega_S$, distinguishing the two cases where V_1 is either decomposable or indecomposable. Then we divide the problem in various subcases, according to the behaviour of $V_2 := f_*\omega_S^2$, and for each subcase we study the corresponding stratum of the moduli space $\mathcal{M}$. By Riemann-Roch and [9], at a point $[S] \in \mathcal{M}$ we have

$$\dim_{[S]} \mathcal{M} \geq 10\chi(\mathcal{O}_S) - 2K_S^2 + p_g = 12,$$

hence, to understand the irreducible components of $\mathcal{M}$, we have to consider only those strata whose dimension is greater than or equal to 12.

Our main results are

Theorem 1.1 *Let $\mathcal{M}'$ be the subspace of $\mathcal{M}$ corresponding to surfaces such that V_1 is decomposable. There is a stratification into unirational algebraic subsets*:

$$\mathcal{M}' = \mathcal{M}_{\mathrm{I}} \cup \mathcal{M}_{\mathrm{IIa}} \cup \mathcal{M}_{\mathrm{IIb}} \cup \mathcal{M}_{\mathrm{IIc}} \cup \mathcal{M}_{\mathrm{IIIa}} \cup \mathcal{M}_{\mathrm{IIIc}} \cup \mathcal{M}_{\mathrm{IVa}} \cup \mathcal{M}_{\mathrm{IVb}} \cup \mathcal{M}_{\mathrm{IVc}} \cup \mathcal{M}_{\mathrm{V,gen}} \cup \mathcal{M}_{\mathrm{V,2}},$$

where $\mathcal{M}_{\mathrm{IIc}}$, $\mathcal{M}_{\mathrm{IVa}}$, $\mathcal{M}_{\mathrm{IVb}}$ and $\mathcal{M}_{\mathrm{IVc}}$ have dimension ≤ 11, *so they can be disregarded in the determination of the irreducible components*, *while*:

- $\mathcal{M}_{\mathrm{I}}$ *is nonempty*, *irreducible*, *of dimension at most* 13;
- $\mathcal{M}_{\mathrm{IIa}}$, $\mathcal{M}_{\mathrm{IIb}}$, $\mathcal{M}_{\mathrm{IIIa}}$, $\mathcal{M}_{\mathrm{IIIc}}$ *have dimension at most* 12;
- $\mathcal{M}_{\mathrm{V,\,gen}}$ *is non-empty*, *of dimension* 11;
- $\mathcal{M}_{\mathrm{V,2}}$ *is a generically smooth*, *irreducible component of dimension* 12.

Theorem 1.2 *Let $\mathcal{M}''$ be the subspace of $\mathcal{M}$ corresponding to surfaces such that V_1 is indecomposable. There is a stratification*

$$\mathcal{M}'' = \mathcal{M}_{\mathrm{VI}} \cup \mathcal{M}_{\mathrm{VIIa}} \cup \mathcal{M}_{\mathrm{VIIb}},$$

where the strata $\mathcal{M}_{\mathrm{VIIa}}$ and $\mathcal{M}_{\mathrm{VIIb}}$ have dimension ≤ 11, *while $\mathcal{M}_{\mathrm{VI}}$ has dimension at most* 12.

Using Theorems 1.1 and 1.2 and some easy additional arguments, one can prove the following

Corollary 1.3 *The moduli space $\mathcal{M}$ of minimal surfaces of general type with $p_g = 2$, $q = 1$ and $K^2 = 5$ is unirational and contains at least 2 irreducible components. Moreover, the dimension of each irreducible component is either 12 or 13, and there is at most one component of dimension 13.*

Of course, it would be interesting to exactly describe all irreducible components of $\mathcal{M}$ and also to understand how their closures intersect, but we will not try to develop this point here.

Now let us explain how this paper is organized.

In Sect. 2 we present some preliminaries, and we set up notation and terminology. In particular we recall Atiyah's classification of vector bundles over an elliptic curve and Horikawa's and Catanese–Pignatelli's approaches to the study of genus 2 fibrations.

In Sect. 3 we investigate the structure and the possible splitting types of the vector bundles $V_1 = f_*\omega_S$ and $V_2 = f_*\omega_S^2$.

Finally, Sect. 4 deals with the study of the moduli space $\mathcal{M}$.

2 Preliminaries

2.1 Vector bundles over an elliptic curve

The classification of vector bundles of an elliptic curve was given in [1]. Here we just recall the results needed in order to make this paper self-contained, and we refer the reader to Atiyah's paper for further details. Let B be an elliptic curve and let o be the identity element in the group law of B. If $\tau \in B$, we set $E_\tau(1, 1) := \mathcal{O}_B(\tau)$ and for all $r \geq 2$ we denote by $E_\tau(r, 1)$ the unique *indecomposable* rank r vector bundle on B defined recursively by the short exact sequence

$$0 \longrightarrow \mathcal{O}_B \longrightarrow E_\tau(r, 1) \longrightarrow E_\tau(r-1, 1) \longrightarrow 0.$$

Moreover, we set $F_1 := \mathcal{O}_B$ and for all $r \geq 2$ we denote by F_r the unique *indecomposable* rank r vector bundle on B defined recursively by the short exact sequence

$$0 \longrightarrow \mathcal{O}_B \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow 0.$$

Proposition 2.1 [1] (i) *For all $L \in \mathrm{Pic}^0(B)$ and for all $r \geq 2$ we have*

$$h^0(E_\tau(r,1) \otimes L) = 1, \qquad h^1(E_\tau(r,1) \otimes L) = 0.$$

Moreover every indecomposable rank r vector bundle V on B such that $\deg V = 1$ *is isomorphic to $E_\tau(r,1) \otimes L$ for some $L \in \mathrm{Pic}^0(B)$.*

(ii) *For all $L \in \mathrm{Pic}^0(B) \setminus \{\mathcal{O}_B\}$ we have*

$$h^0(F_r \otimes L) = h^1(F_r \otimes L) = 0,$$

whereas $h^0(F_r) = h^1(F_r) = 1$. Moreover every indecomposable rank r vector bundle V on B such that $\deg V = 0$ *is isomorphic to $F_r \otimes L$ for a unique $L \in \mathrm{Pic}^0(B)$.*

By using Proposition 2.1, we can prove

Proposition 2.2 *Let V be a rank 3 vector bundle on B, such that* $\det V = \mathcal{O}_B(\tau)$ *for some $\tau \in B$. Then the following holds.*

(i) *If $h^1(V \otimes L) = 0$ for all $L \in \mathrm{Pic}^0(B)$, then $V = E_\tau(3, 1)$.*
(ii) *If $h^1(V \otimes L) = 0$ for all $L \in \mathrm{Pic}^0(B) \setminus \{\mathcal{O}_B\}$ and $h^1(V) = 1$, then either $V = E_\tau(2, 1) \oplus \mathcal{O}_B$ or $V = F_2 \oplus \mathcal{O}_B(\tau)$.*
(iii) *If $h^1(V \otimes L) = 0$ for all $L \in \mathrm{Pic}^0(B) \setminus \{\mathcal{O}_B\}$ and $h^1(V) = 2$, then $V = \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau)$.*

Proof (i) Assume $h^1(V \otimes L) = 0$ for all $L \in \mathrm{Pic}^0(B)$. If V is indecomposable, then $V = E_\tau(3, 1)$ by Atiyah's classification. Suppose now that $V = W \oplus M$, where W is indecomposable of rank 2 and M is a line bundle. By our assumptions on the cohomology of V, it follows $0 \le \deg M \le 1$. If $\deg M = 0$, then $h^1(V \otimes M^{-1}) = 1$ yields a contradiction. If $\deg M = 1$, then $\deg W = 0$, hence $W = F_2 \otimes L$ for some $L \in \mathrm{Pic}^0(B)$. It follows $h^1(V \otimes L^{-1}) = 1$, again a contradiction. Finally, suppose that $V = L_1 \oplus L_2 \oplus L_3$, where the L_i are line bundles. We must have $\deg L_i \ge 0$, hence we may assume $\deg L_1 = 0$, $\deg L_2 = 0$, $\deg L_3 = 1$; therefore we have $h^1(V \otimes L_1^{-1}) \ge 1$, a contradiction. This concludes the proof of part (i).

(ii) Since $h^1(V) = 1$, the vector bundle V cannot be indecomposable. Suppose that $V = W \oplus M$, where W is indecomposable of rank 2 and M is a line bundle; as before, we have $0 \le \deg M \le 1$. If $\deg M = 0$ we have $\deg W = 1$, hence $h^1(M) = h^1(V) = 1$. It follows $M = \mathcal{O}_B$ and $V = E_\tau(2, 1) \oplus \mathcal{O}_B$. If $\deg M = 1$ we have $\deg W = 0$; since $h^1(V) = 1$, the only possibility is $V = F_2 \oplus \mathcal{O}_B(\tau)$. Finally, suppose that $V = L_1 \oplus L_2 \oplus L_3$, where the L_i are line bundles. Taking $L \in \mathrm{Pic}^0(B) \setminus \{\mathcal{O}_B\}$, we have $h^1((L_1 \oplus L_2 \oplus L_3) \otimes L) = 0$, hence $\deg L_i \ge 0$; on the other hand $\deg V = 1$, hence, as before, we may assume $\deg L_1 = 0$, $\deg L_2 = 0$, $\deg L_3 = 1$; moreover $L_1 \otimes L \neq \mathcal{O}_B$ and $L_2 \otimes L \neq \mathcal{O}_B$ for all $L \in \mathrm{Pic}^0(B) \setminus \{\mathcal{O}_B\}$. Hence we obtain $V = \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau)$, so $h^1(V) = 2$, a contradiction. This concludes the proof of part (ii).

(iii) Since $h^1(V) = 2$, arguing as before we see that $V = L_1 \oplus L_2 \oplus L_3$, where the L_i are line bundles. Moreover $h^1(V \otimes L) = 0$ for all $L \in \mathrm{Pic}^0(B)$ implies $\deg L_i \ge 0$. So we may assume $\deg L_1 = 0$, $\deg L_2 = 0$, $\deg L_3 = 1$, which implies $V = \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau)$. This concludes the proof of part (iii). □

Remark 2.3 A similar result holds if one replaces $\mathrm{Pic}^0(B) \setminus \{\mathcal{O}_B\}$ with $\mathrm{Pic}^0(B) \setminus \{M\}$, for any $M \in \mathrm{Pic}^0(B)$.

Proposition 2.4

(i) *Set $W := E_\tau(2, 1)$. Then we have*

$$\mathrm{S}^2 W = \bigoplus_{i=1}^{3} L_i(\tau), \qquad \mathrm{S}^3 W = W(\tau) \oplus W(\tau),$$

where the L_i are the three non-trivial 2-torsion line bundles on B.
(ii) *$S^{r-1} F_2 = F_r$, for all $r \ge 2$.*

Proof (*i*) If $\tau = o$, see [1, pp. 438–439]. The general case follows since, by Proposition 2.1, we have $E_\tau(2, 1) = E_o(2, 1) \otimes L$, where L is any line bundle on B such that $L^{\otimes 2} = \mathcal{O}_B(\tau - o)$.

(ii) See [1, Theorem 9]. □

2.2 Structure theorems for genus 2 fibrations

2.2.1 Horikawa's method

The following approach to genus 2 fibrations was introduced by Horikawa in [11]; see also [18, §1] for further details. Let $f: S \longrightarrow B$ be a relatively minimal genus 2 fibration over a smooth curve B of genus b, set $V_1 := f_* \omega_{S|B}$ and let $\pi_1: \mathbb{P}(V_1) \longrightarrow B$ be the associated $\mathbb{P}^1$-bundle. Let us consider the relative canonical map $\phi: S \dashrightarrow \mathbb{P}(V_1)$, whose indeterminacy locus is contained in the fibres of f which are not 2-connected. After composing with a finite number of blow-ups, we can extend ϕ to a generically finite, degree 2 morphism $\tilde{\phi}: \widetilde{S} \longrightarrow \mathbb{P}(V_1)$; let $\mathcal{B}$ be the branch divisor of $\tilde{\phi}$. There exists a divisor $\mathcal{F} \in \mathrm{Pic}(\mathbb{P}(V_1))$ such that $2\mathcal{F} = \mathcal{B}$, so we can consider the double cover $S' \longrightarrow \mathbb{P}(V_1)$ branched at $\mathcal{B}$, and it is no difficult to see that there exists a birational morphism $\widetilde{S} \longrightarrow S'$. The Néron Severi group of $\mathbb{P}(V_1)$ is generated by C_0 and Γ, that are the classes of $\mathcal{O}_{\mathbb{P}(V_1)}(1)$ and of a fiber, respectively; since $\mathcal{B}\Gamma = 6$, it follows that $\mathcal{B} = 6C_0 + \pi_1^* \alpha$, for some $\alpha \in \mathrm{Pic}(B)$. After applying a finite number of elementary transformations to the pair $(\mathbb{P}(V_1), \mathcal{B})$, we obtain that $\mathcal{B}$ has only the following types of singularities, defined when $k \geq 1$:

- (0) a double point or a simple triple point;
- ($\mathcal{I}_k$) a fibre Γ plus two triple points on it (hence these are quadruple points of B); each of these triple points is $(2k-1)$-fold or $2k$-fold;
- ($\mathcal{II}_k$) two triple points on a fibre, each of these is $2k$-fold or $(2k+1)$-fold;
- ($\mathcal{III}_k$) a fibre Γ plus a $(4k-2)$ or a $(4k-1)$-fold triple point on it which has a contact of order 6 with Γ;
- ($\mathcal{IV}_k$) a $4k$ or $(4k+1)$-fold triple point x which has a contact of order 6 with the fibre through x;
- ($\mathcal{V}$) a fibre Γ plus a quadruple point x on Γ, which after a blow-up in x results in a double point in the proper transform of Γ.

We recall that a k-fold triple point is a triple point that results in a simple triple point after $k-1$ blow-ups. Let us denote by $\nu(*)$ the number of fibres of type $*$.

Theorem 2.5 [11] *The following equality holds*:

$$K_S^2 = 2p_a(S) - 4 + 6b + \sum_k \{(2k-1)(\nu(\mathcal{I}_k) + \nu(\mathcal{III}_k)) + 2k(\nu(\mathcal{II}_k) + \nu(\mathcal{IV}_k))\} + \nu(\mathcal{V}).$$

2.2.2 Catanese-Pignatelli's method

Now we recall Catanese-Pignatelli approach to genus 2 fibrations, which roughly speaking consists in considering the relative *bicanonical* map instead of the canonical one. We closely follow the treatment given in [6] and [15], referring the reader to those papers for further details. For any relatively minimal genus 2 fibration $f: S \longrightarrow B$, we can consider the rank 3 vector bundle $V_2 := f_* \omega_{S|B}^2$ and the corresponding $\mathbb{P}^2$-bundle $\pi_2: \mathbb{P}(V_2) \longrightarrow B$. Therefore we can associate to the fibration f the 5-tuple (B, V_1, τ, ξ, w), where

- B is the base curve;
- $V_1 = f_*\omega_{S|B}$;
- τ is an effective divisor on B of degree $K_S^2 - 6(b-1) - 2\chi(\mathcal{O}_S)$, corresponding to the fibres of f which are not 2-connected;
- ξ is an element of $\mathrm{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_\tau, \mathrm{S}^2 V_1)/\mathrm{Aut}_{\mathcal{O}_B}(\mathcal{O}_\tau)$ giving the short exact sequence

$$0 \longrightarrow \mathrm{S}^2 V_1 \xrightarrow{\sigma_2} V_2 \longrightarrow \mathcal{O}_\tau \longrightarrow 0, \tag{1}$$

where σ_2 is the natural map induced by the tensor product of canonical sections of the fibres of f; then σ_2 yields a rational map $\mathbb{P}(V_1) \dashrightarrow \mathbb{P}(V_2)$ (the relative version of the 2-Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$) birational onto a conic bundle $\mathcal{C} \in |\mathcal{O}_{\mathbb{P}(V_2)}(2) \otimes \pi_2^*(\det V_1)^{-2}|$. More precisely, if x_0, x_1 are generators for the stalk of V_1, then the equation of $\mathcal{C}$ is locally given by

$$\sigma_2(x_0^2)\sigma_2(x_1^2) - (\sigma_2(x_0 x_1))^2 = 0. \tag{2}$$

- $w \in \mathbb{P}H^0(B, \widetilde{A}_6)$, where $\widetilde{A}_6 := A_6 \otimes (\det V_1 \otimes \mathcal{O}_B(\tau))^{-2}$ and A_6 is given by the following short exact sequence:

$$0 \longrightarrow (\det V_1)^2 \otimes V_2 \xrightarrow{i_3} \mathrm{S}^3 V_2 \longrightarrow A_6 \longrightarrow 0. \tag{3}$$

Here the map i_3 is locally defined as follows: if x_0, x_1 are generators for the stalk of V_1 and y_0, y_1, y_2 are generators for the stalk of V_2, then

$$i_3((x_0 \wedge x_1)^{\otimes 2} \otimes y_i) := \sigma_2(x_0^2)\sigma_2(x_1^2) y_i - \sigma_2(x_0 x_1)^2 y_i.$$

The relative bicanonical map, which is always a morphism, induces a factorization of the fibration f as

$$S \xrightarrow{r} X \xrightarrow{\psi} \mathcal{C} \xrightarrow{\pi_2|_{\mathcal{C}}} B,$$

where r is a contraction of (-2)-curves to Rational Double Points, and ψ is a finite double cover. The element $w \in \mathbb{P}H^0(\widetilde{A}_6) = |\mathcal{O}_{\mathcal{C}}(6) \otimes (\det V_1 \otimes \mathcal{O}_B(\tau))^{-2}|$ corresponds to the divisorial part Δ of the branch locus of ψ. In fact, the branch locus of ψ consists of a disjoint union $\Delta \cup \mathcal{P}$, where $\mathcal{P} \subset \mathrm{Sing}(\mathcal{C})$ is a finite set of points in natural bijection with $\mathrm{supp}(\tau)$. Notice that A_6 is the quotient of $\mathrm{S}^3 V_2$ by the subbundle of the relative cubics vanishing on $\mathcal{C}$; geometrically, this reflects the fact that, in general, not all the divisors in $|\mathcal{O}_{\mathcal{C}}(6) \otimes (\det V_1 \otimes \mathcal{O}_B(\tau))^{-2}|$ can be written as the complete intersection of $\mathcal{C}$ with a relative cubic $\mathcal{G} \in |\mathcal{O}_{\mathbb{P}(V_2)}(3) \otimes (\det V_1 \otimes \mathcal{O}_B(\tau))^{-2}|$. Finally, observe that if

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \widetilde{A}_6 \longrightarrow 0 \tag{4}$$

is the short exact sequence obtained by tensoring (3) with $(\det V_1 \otimes \mathcal{O}_B(\tau))^{-2}$, we obtain

$$h^0(\widetilde{A}_6) \le h^0(G_2) - h^0(G_1) + h^1(G_1). \tag{5}$$

We call (B, V_1, τ, ξ, w) the *associate* 5*-ple* of the fibration $f : S \longrightarrow B$.

Theorem 2.6 [6] *Assume that we have a* 5*-ple* (B, V_1, τ, ξ, w) *as before, such that the following (open) conditions are satisfied*:

($\mathcal{P}_1$) *the conic bundle* $\mathcal{C}$ *has only Rational Double Points as singularities*;
($\mathcal{P}_2$) *the curve* Δ *has only simple singularities, where "simple" means that the germ of the double cover of* $\mathcal{C}$ *branched on it has at most a Rational Double Point.*

Then there exists a unique relatively minimal genus 2 *fibration* $f\colon S \longrightarrow B$ *having the above as associate* 5*-ple. Moreover, the surface* S *has the following invariants*:

$$\chi(\mathcal{O}_S) = \deg V_1 + (b-1),$$
$$K_S^2 = 2\deg V_1 + \deg\tau + 8(b-1).$$

3 Surfaces of general type with $p_g = 2, q = 1$ and $K^2 = 5$

3.1 The sheaf V_1

Let S be a minimal surface of general type with $p_g = 2$, $q = 1$ and $K_S^2 = 5$. Its Albanese variety $B := \mathrm{Alb}(S)$ is an elliptic curve, and its Albanese map $f\colon S \longrightarrow B$ is a genus 2 fibration [12, Theorem 3.1]. Notice that since B is elliptic then $\omega_{S|B} = \omega_S$. By Theorem 2.6 we have $\deg(\tau) = 1$, i.e. τ is a point of B. The genus 2 fibration contains exactly one singular fibre, which comes from a singularity of $(\mathbb{P}(V_1), \mathcal{B})$ of type $\mathcal{I}_1$, $\mathcal{III}_1$ or $\mathcal{V}$, see Theorem 2.5. In particular, the curve $\mathcal{B}$ contains the fibre $\Gamma_\tau = \pi_1^*(\tau)$ of $\pi_1\colon \mathbb{P}(V_1) \longrightarrow B$. Standard calculations, see [3, Chap. V], show that $\mathcal{B}$ is algebraically equivalent to $6C_0 - 2\Gamma$, so we can write $\mathcal{B} = \mathcal{B}' + \Gamma$, where $\mathcal{B}'$ is an effective divisor algebraically equivalent to $6C_0 - 3\Gamma$.
Let now E_1 be a rank 1 subsheaf of maximal degree of $V_1 = f_*\omega_S$; then there is a short exact sequence

$$0 \longrightarrow E \longrightarrow V_1 \longrightarrow F \longrightarrow 0$$

such that F is locally free and $\deg F \geq 0$, see [10]; moreover one clearly has $1 \leq h^0(E) \leq h^0(V_1) = 2$. Setting $e := \deg E - \deg F$, by [18, Théorème 2.1, p. 16] there are exactly two possibilities:

- $\deg E = 1, \quad \deg F = 1, \quad e = 0$
- $\deg E = 2, \quad \deg F = 0, \quad e = 2.$

Proposition 3.1 (i) *If* $e = 0$ *then* (*up to translations*) *either* $V_1 = \mathcal{O}_B(p) \oplus \mathcal{O}_B(2o - p)$ *for some* $p \in B$ *or* $V_1 = F_2(\eta)$, *where* $\eta \in E$ *is a* 2*-torsion point.*

(ii) *If* $e = 2$ *then* $V_1 = \mathcal{O}_B(D) \oplus L$, *where* D *is an effective divisor of degree* 2 *on* B *and* $L \in \mathrm{Pic}^0(B)$ *is a non-trivial, torsion line bundle. This case occurs if and only if the canonical map* $\phi_{|K|}$ *of* S *factors through* f.

Proof (i) If $e = 0$, up to a translation we may assume $E = \mathcal{O}_B(p)$, $F = \mathcal{O}_B(2o - p)$, for some $p \in B$. If $F \neq E$, then $\mathrm{Ext}^1(F, E) = 0$ and we obtain $V_1 = \mathcal{O}_B(p) \oplus \mathcal{O}_B(2o - p)$. If $F = E$, then $\mathrm{Ext}^1(F, E) = \mathbb{C}$. In that case $2o = 2p$, so any non-trivial extension class corresponds to $V_1 = F_2(\eta)$, where $2\eta \in |2o|$.

(ii) If $e = 2$ then $\deg E = 2$, hence $E = \mathcal{O}_B(D)$ for some effective divisor D on B. We have $h^0(E) = 2$ and $h^1(E) = 0$, so $h^0(V_1) = h^0(E) + h^0(F)$, which implies $h^0(F) = 0$. Then F is a non-trivial, degree zero line bundle. Since $\mathrm{Ext}^1(F, E) = 0$, it follows $V_1 = \mathcal{O}_B(D) \oplus F$, and Simpson's results ([17]) imply that F is a non-trivial torsion line bundle on B. The last assertion follows from [18, Théorème 5.1, p. 71]. □

Proposition 3.2 *The case $e = 2$ does not occur.*

Proof If $e = 2$, then S would be the canonical resolution of the singularities of a degree 2 cover of $\mathbb{P}(V_1) = \mathbb{P}(\mathcal{O}_B(D) \oplus L)$. Since V_1 is decomposable, we can take global coordinates on the fibres of $\pi_1 : \mathbb{P}(V_1) \longrightarrow B$, namely

$$x_0 \in H^0(\mathcal{O}_{\mathbb{P}(V_1)}(1) \otimes \pi_1^* \mathcal{O}_B(-D)), \qquad x_1 \in H^0(\mathcal{O}_{\mathbb{P}(V_1)}(1) \otimes \pi_1^* L^{-1}).$$

Putting $M = \mathcal{O}_B(D)$, we obtain $x_0^i x_1^j \in H^0(\mathcal{O}_{\mathbb{P}(V_1)}(i+j) \otimes \pi_1^* M^{-i} \otimes \pi_1^* L^{-j})$. Since $\mathcal{B}'$ is algebraically equivalent to $6C_0 - 3\Gamma$, we have $\mathcal{B}' \in |H^0(\mathcal{O}_{\mathbb{P}(V_1)}(6) \otimes \pi_1^* T^{-1})|$ for a suitable degree 3 line bundle T on B, so the equation of $\mathcal{B}'$ can be written as

$$\sum_{i+j=6} a_{ij} x_0^i x_1^j = 0, \tag{6}$$

where $a_{ij} \in H^0(\mathbb{P}(V_1), \pi_1^*(T^{-1} \otimes M^i \otimes L^j))$. In particular $a_{06} = a_{15} = 0$, so x_0^2 divides the left-hand side of (6). Hence $\mathcal{B}'$ is non-reduced, a contradiction. □

Propositions 3.1 and 3.2 imply the following

Corollary 3.3 *Let S be a minimal surface of general type with $p_g = 2$, $q = 1$, $K_S^2 = 5$. Then the canonical map of S does not factor through the Albanese fibration.*

3.2 The sheaf V_2

3.2.1 The case where V_1 is decomposable

If V_1 is decomposable then Propositions 3.1 and 3.2 yield $V_1 = \mathcal{O}_B(p) \oplus \mathcal{O}_B(2o - p)$, so we have $S^2 V_1 = \bigoplus_{i=1}^3 P_i$, where $P_1 = \mathcal{O}_B(2p)$, $P_2 = \mathcal{O}_B(2o)$, $P_3 = \mathcal{O}_B(4o - 2p)$. Fix a section $f_0 \in H^0(\mathcal{O}_B(\tau)) \setminus \{0\}$; applying the functor $\mathrm{Hom}(-, S^2 V_1)$ to the exact sequence

$$0 \longrightarrow \mathcal{O}_B(o - \tau) \xrightarrow{(-f_0)} \mathcal{O}_B(o) \longrightarrow \mathcal{O}_\tau \longrightarrow 0$$

we obtain

$$\mathrm{Ext}^1(\mathcal{O}_\tau, S^2 V_1) = \bigoplus_{i=1}^3 \frac{H^0(P_i(\tau - o))}{H^0(P_i(-o))} \cong \mathbb{C}^3, \tag{7}$$

that is $\mathrm{Ext}^1(\mathcal{O}_\tau, S^2 V_1)$ can be identified with the space of global sections of $\bigoplus H^0(P_i(\tau - o))$, modulo the subspace of sections vanishing in τ. For any $(f_1, f_2, f_3) \in \bigoplus H^0(P_i(\tau - o))$, we denote by $(\bar{f_1}, \bar{f_2}, \bar{f_3})$ its image in $\mathrm{Ext}^1(\mathcal{O}_\tau, S^2 V_1)$. Arguing as in [6, p.1032], this implies that $V_2 = f_* \omega_S^2$ is the cokernel of a short exact sequence

$$0 \longrightarrow \mathcal{O}_B(o - \tau) \xrightarrow{i} \mathcal{O}_B(o) \oplus \bigoplus_{i=1}^3 P_i \longrightarrow V_2 \longrightarrow 0, \tag{8}$$

where the injective map i is given by ${}^t(f_0, f_1, f_2, f_3)$.

Remark 3.4 If we choose the map i' given by ${}^t(f_0, f_1 + f_0 g_1, f_2 + f_0 g_2, f_3 + f_0 g_3)$, with $g_i \in H^0(P_i(-o))$, we obtain a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_B(o-\tau) & \xrightarrow{i'} & \mathcal{O}(o) \oplus \bigoplus_{i=1}^3 P_i & \longrightarrow & V_2' & \longrightarrow & 0 \\
 & & \| & & \uparrow M & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_B(o-\tau) & \xrightarrow{i} & \mathcal{O}(o) \oplus \bigoplus_{i=1}^3 P_i & \longrightarrow & V_2 & \longrightarrow & 0
\end{array}$$

where the matrix M is given by

$$\begin{pmatrix} 1 & g_1 & g_2 & g_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence $V_2' \cong V_2$, so the isomorphism class of V_2 only depends on $(\bar{f}_1, \bar{f}_2, \bar{f}_3)$.

Notice that V_2 is a vector bundle if and only if f_1, f_2, f_3 do not vanish simultaneously in τ, that is if and only if $\xi = (\bar{f}_1, \bar{f}_2, \bar{f}_3)$ is not the trivial extension class. Let m be the cardinality of the set $\{i \mid \bar{f}_i = 0\}$; hence $0 \le m \le 2$. Now we give the description of V_2 in the different cases.

Proposition 3.5 *Assume $V_1 = \mathcal{O}_B(p) \oplus \mathcal{O}_B(2o-p)$. Then there are precisely the following possibilities*:

(I) $m = 0$, $\mathcal{O}_B(4o-4p) \neq \mathcal{O}_B$, $V_2(-2o) = E_\tau(3,1)$
(IIa) $m = 0$, $\mathcal{O}_B(4o-4p) = \mathcal{O}_B, \mathcal{O}_B(2o-2p) \neq \mathcal{O}_B$, $V_2(-2o) = F_2(2o-2p) \oplus \mathcal{O}_B(\tau)$
(IIb) $m = 0$, $\mathcal{O}_B(4o-4p) = \mathcal{O}_B, \mathcal{O}_B(2o-2p) \neq \mathcal{O}_B$, $V_2(-2o) = E_\tau(2,1) \oplus \mathcal{O}_B$
(IIc) $m = 1$, $\mathcal{O}_B(4o-4p) = \mathcal{O}_B, \mathcal{O}_B(2o-2p) \neq \mathcal{O}_B$, $V_2(-2o) = \mathcal{O}_B(2o-2p) \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau+2p-2o)$
(IIIa) $m = 1$, $\mathcal{O}_B(2o-2p) \neq \mathcal{O}_B$, $V_2(-2o) = E_{\tau+2o-2p}(2,1) \oplus \mathcal{O}_B(2p-2o)$
(IIIb) $m = 1$, $\mathcal{O}_B(2o-2p) \neq \mathcal{O}_B$, $V_2(-2o) = E_{\tau+2p-2o}(2,1) \oplus \mathcal{O}_B(2o-2p)$
(IIIc) $m = 1$, $\mathcal{O}_B(2o-2p) \neq \mathcal{O}_B$, $V_2(-2o) = E_\tau(2,1) \oplus \mathcal{O}_B$
(IVa) $m = 2$, $\mathcal{O}_B(2o-2p) \neq \mathcal{O}_B$, $V_2(-2o) = \mathcal{O}_B(2p-2o) \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau+2o-2p)$
(IVb) $m = 2$, $\mathcal{O}_B(2o-2p) \neq \mathcal{O}_B$, $V_2(-2o) = \mathcal{O}_B(2o-2p) \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau+2p-2o)$
(IVc) $m = 2$, $\mathcal{O}_B(2o-2p) \neq \mathcal{O}_B$, $V_2(-2o) = \mathcal{O}_B(2p-2o) \oplus \mathcal{O}_B(2o-2p) \oplus \mathcal{O}_B(\tau)$
(V) $0 \le m \le 2$, $\mathcal{O}_B(2o-2p) = \mathcal{O}_B$, $V_2(-2o) = \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{O}_B(\tau)$

Proof The proof is not difficult, but one needs to consider several cases; for the reader's convenience, we will write it in detail. Let $L \in \mathrm{Pic}^0(B)$; tensoring the exact sequence (8) with $L(-2o)$ we obtain

$$\begin{aligned} 0 &\longrightarrow L(-o-\tau) \longrightarrow L(-o) \oplus L(2p-2o) \oplus L \oplus L(2o-2p) \\ &\longrightarrow V_2(-2o) \otimes L \longrightarrow 0, \end{aligned} \tag{9}$$

which in turn induces a linear map in cohomology

$$\alpha\colon H^1(L(-o-\tau)) \longrightarrow H^1(L(-o) \oplus L(2p-2o) \oplus L \oplus L(2o-2p))$$

such that $H^1(V_2(-2o) \otimes L)$ is isomorphic to the cokernel of α. Notice that $\det V_2(-2o) = \mathcal{O}_B(\tau)$. The first component of α is always surjective, since it is induced by the short exact sequence

$$0 \longrightarrow L(-o-\tau) \longrightarrow L(-o) \longrightarrow \mathcal{O}_\tau \longrightarrow 0,$$

therefore if $L \notin \{\mathcal{O}_B(2o-2p), \mathcal{O}_B, \mathcal{O}_B(2p-2o)\}$ the map α is surjective and $H^1(V_2(-2o)\otimes L)=0$. Taking the dual of α, we obtain the map

$$\alpha^*\colon H^0\big(L^*(o)\oplus L^*(2o-2p)\oplus L^*\oplus L^*(2p-2o)\big)\longrightarrow H^0(L^*(o+\tau)),$$

which is given by (f_0, f_1, f_2, f_3); moreover $H^1(V_2(-2o)\otimes L)^*$ is isomorphic to $\ker\alpha^*$.

If $\mathcal{O}_B(2o-2p)=\mathcal{O}_B$, then α^* is injective for all $L\in \mathrm{Pic}^0(B)\setminus\{\mathcal{O}_B\}$, whereas for $L=\mathcal{O}_B$ it has a 2-dimensional kernel; by using Proposition 2.2 we conclude that $V_2(-2o)=\mathcal{O}_B\oplus\mathcal{O}_B\oplus\mathcal{O}_B(\tau)$, so we are in case (V). Therefore we may assume $\mathcal{O}_B(2o-2p)\neq\mathcal{O}_B$. Since α^* is injective unless $L\in\{\mathcal{O}_B(2o-2p), \mathcal{O}_B, \mathcal{O}_B(2p-2o)\}$, we have just to consider these three cases.

If $L=\mathcal{O}_B(2o-2p)$ we obtain

$$h^1(V_2(-2o)\otimes L)=\begin{cases}0 & \text{if } \mathcal{O}_B(4o-4p)\neq\mathcal{O}_B \text{ and } \bar{f}_1\neq 0;\\ 1 & \text{if } \mathcal{O}_B(4o-4p)\neq\mathcal{O}_B \text{ and } \bar{f}_1=0;\\ 1 & \text{if } \mathcal{O}_B(4o-4p)=\mathcal{O}_B \text{ and } \bar{f}_1\neq 0 \text{ or } \bar{f}_3\neq 0;\\ 2 & \text{if } \mathcal{O}_B(4o-4p)=\mathcal{O}_B \text{ and } \bar{f}_1=\bar{f}_3=0.\end{cases}$$

Analogously, if $L=\mathcal{O}_B(2p-2o)$ we obtain

$$h^1(V_2(-2o)\otimes L)=\begin{cases}0 & \text{if } \mathcal{O}_B(4o-4p)\neq\mathcal{O}_B \text{ and } \bar{f}_3\neq 0;\\ 1 & \text{if } \mathcal{O}_B(4o-4p)\neq\mathcal{O}_B \text{ and } \bar{f}_3=0;\\ 1 & \text{if } \mathcal{O}_B(4o-4p)=\mathcal{O}_B \text{ and } \bar{f}_1\neq 0 \text{ or } \bar{f}_3\neq 0;\\ 2 & \text{if } \mathcal{O}_B(4o-4p)=\mathcal{O}_B \text{ and } \bar{f}_1=\bar{f}_3=0.\end{cases}$$

Finally, if $L=\mathcal{O}_B$ we obtain

$$h^1(V_2(-2o)\otimes L)=\begin{cases}0 & \text{if } \bar{f}_2\neq 0;\\ 1 & \text{if } \bar{f}_2=0.\end{cases}$$

Now we observe that if $\bar{f}_i=0$ then $P_i(-2o)$ is a direct summand of $V_2(-2o)$, and we analyze the different possibilities.

Assume first $\mathcal{O}_B(4o-4p)\neq\mathcal{O}_B$. In this case there exist exactly m line bundles L such that $H^1(V_2(-2o)\otimes L)\neq 0$. By a straightforward application of Proposition 2.2 and Remark 2.3 we obtain cases (I), (IIIa), (IIIb), (IIIc), (IVa), (IVb), (IVc).

Now assume $\mathcal{O}_B(4o-4p)=\mathcal{O}_B$. Then the only new possibilities are:

- $\bar{f}_i\neq 0$ for all i, that is $m=0$; then $H^1(V_2(-2o)\otimes L)$ is trivial for all $L\in\mathrm{Pic}^0(B)$, except in the case $L=\mathcal{O}_B(2o-2p)=\mathcal{O}_B(2p-2o)$ where it is 1-dimensional. By Proposition 2.2 and Remark 2.3 this is either (IIa) or (IIb).
- $\bar{f}_1\neq 0$, $\bar{f}_2=0$, $\bar{f}_3\neq 0$; then $H^1(V_2(-2o)\otimes L)$ is trivial for all $L\in\mathrm{Pic}^0(B)$, except in the cases $L=\mathcal{O}_B(2o-2p)$ and $L=\mathcal{O}_B$ where it is 1-dimensional; this is (IIc).

The proof is now complete. □

3.2.2 The case where V_1 is indecomposable

If V_1 is indecomposable, then $V_1=F_2(\eta)$, where η is a 2-torsion point, so Proposition 2.4 yields $\mathrm{S}^2V_1=F_3(2o)$. Arguing as in Sect. 3.2.1, we obtain

$$\mathrm{Ext}^1(\mathcal{O}_\tau, \mathrm{S}^2V_1)=\frac{H^0(F_3(o+\tau))}{H^0(F_3(o))}\cong\mathbb{C}^3, \tag{10}$$

that is $\mathrm{Ext}^1(\mathcal{O}_\tau, \mathrm{S}^2 V_1)$ can be identified with the space of global sections of $F_3(o + \tau)$, modulo the subspace of sections vanishing in τ. For any $v \in H^0(F_3(o+\tau))$, we will denote by $\bar{v}$ its image in $\mathrm{Ext}^1(\mathcal{O}_\tau, \mathrm{S}^2 V_1)$. Now let us fix a section $f_0 \in H^0(\mathcal{O}_B(\tau)) \setminus \{0\}$. Then V_2 is the cokernel of a short exact sequence

$$0 \longrightarrow \mathcal{O}_B(o-\tau) \stackrel{i}{\longrightarrow} \mathcal{O}_B(o) \oplus F_3(2o) \longrightarrow V_2 \longrightarrow 0, \tag{11}$$

where the injective map i is given by ${}^t(f_0, v)$. Notice that V_2 is a vector bundle if and only if v does not vanish in τ, that is if and only if $\xi := \bar{v}$ is not the trivial extension class. We can now give a more precise description of V_2.

Proposition 3.6 *Assume $V_1 = F_2(\eta)$, where $\eta \in E$ is a 2–torsion point. Then we have the following possibilities*:

(VI) $V_2(-2o) = E_\tau(3, 1)$
(VIIa) $V_2(-2o) = F_2 \oplus \mathcal{O}_B(\tau)$
(VIIb) $V_2(-2o) = E_\tau(2, 1) \oplus \mathcal{O}_B$.

Moreover, for a general choice of $\xi \in \mathrm{Ext}^1(\mathrm{S}^2 V_1, \mathcal{O}_\tau)$ only (VI) *occurs.*

Proof Let $L \in \mathrm{Pic}^0(B)$; tensoring the exact sequence (11) with $L(-2o)$ we obtain

$$0 \longrightarrow L(-o-\tau) \longrightarrow L(-o) \oplus (F_3 \otimes L) \longrightarrow V_2(-2o) \otimes L \longrightarrow 0, \tag{12}$$

which in turn induces a linear map in cohomology

$$\alpha\colon H^1(L(-o-\tau)) \longrightarrow H^1(L(-o)) \oplus H^1(F_3 \otimes L)$$

such that $H^1(V_2(-2o) \otimes L)$ is isomorphic to the cokernel of α. As in the proof of Proposition 3.5, the first component of α is always surjective. If $L \neq \mathcal{O}_B$ then $H^1(F_3 \otimes L) = 0$ (see Proposition 2.1); consequently, α is surjective and $H^1(V_2(-2o) \otimes L) = 0$. We must now investigate what happens for $L = \mathcal{O}_B$. Let $v \in \mathrm{Hom}(\mathcal{O}_B(-o-\tau), F_3) \cong H^0(F_3(o+\tau))$, and let Q be the cokernel of the corresponding map $v\colon \mathcal{O}_B(-o-\tau) \longrightarrow F_3$.

Claim 3.7 *For a general choice of v, we have*

$$Q = \mathcal{O}_B(q) \oplus \mathcal{O}_B(o+\tau-q)$$

for some $q \in B$. Moreover, $Q = \mathcal{O}_B \oplus \mathcal{O}_B(o+\tau)$ if and only if $\mathrm{im}\, v \subset W$, *where W is the unique subbundle of F_3 isomorphic to F_2, see* [1, p. 433].

Proof Since $F_3(o+\tau)$ is globally generated, for a general choice of v the sheaf Q is locally free. If Q were indecomposable then $Q = F_2(u)$, where $u \in B$ is such that $\mathcal{O}_B(2u) = \mathcal{O}_B(o+\tau)$. Since F_r is self-dual, by taking duals we obtain the exact sequence

$$0 \longrightarrow F_2(-u) \longrightarrow F_3 \longrightarrow \mathcal{O}_B(o+\tau) \longrightarrow 0.$$

By composing it with the injective morphism $\mathcal{O}_B(-u) \to F_2(-u)$ induced by the section of F_2, we conclude that $\mathcal{O}_B$ is a sub-vector bundle of $F_3(u)$, but this is a contradiction, since every section of $F_3(u)$ vanishes in u (see [8, Sect. 5, p. 108]); thus Q must be decomposable.

Moreover, we have $Q \cong \mathcal{O}_B \oplus \mathcal{O}_B(o+\tau)$ if and only if there exists a surjective map $F_3 \longrightarrow \mathcal{O}_B$ whose kernel contains im v. But such a kernel is exactly W, so we are done. □

In order to complete the proof of Proposition 3.6, let us take a general $v \in H^0(F_3(o+\tau))$. We must then study the exact sequence

$$0 \longrightarrow \mathcal{O}_B(-o-\tau) \xrightarrow{v} F_3 \xrightarrow{j} \mathcal{O}_B(q) \oplus \mathcal{O}_B(o+\tau-q) \longrightarrow 0,$$

and in particular the map β induced in cohomology as follows:

$$H^0(\mathcal{O}_B(q) \oplus \mathcal{O}_B(o+\tau-q)) \longrightarrow H^1(\mathcal{O}_B(-o-\tau)) \xrightarrow{\beta} H^1(F_3) \longrightarrow 0. \tag{13}$$

Dualizing (13), using Serre duality and exploiting the isomorphism $F_3^* \cong F_3$ we obtain

$$0 \longrightarrow H^0(F_3) \xrightarrow{\beta^*} H^0(\mathcal{O}_B(o+\tau)) \longrightarrow H^1(\mathcal{O}_B(-q) \oplus \mathcal{O}_B(-o-\tau+q)),$$

hence im β^* can be identified with $\langle s_q \rangle$, the line generated by the unique non-zero section $s_q \in H^0(\mathcal{O}_B(o+\tau))$ such that $s_q(q) = 0$. Now, looking at sequence (12) for $L = \mathcal{O}_B$, we see that α is dual to

$$\alpha^* \colon H^0(\mathcal{O}_B(o)) \oplus H^0(F_3) \xrightarrow{(f_0, \beta^*)} H^0(\mathcal{O}_B(o+\tau)),$$

so the image of α^* is the subspace spanned by s_o and s_q. Since v is general we have $o \neq q$, hence s_o and s_q are linearly independent sections in $H^0(\mathcal{O}_B(o+\tau))$ and this implies that α^* is an isomorphism. Consequently, α is also an isomorphism and for a general choice of $\xi = \bar{v}$ we obtain $h^1(V_2(-2o)) = 0$. For some special choice of $v \in H^0(F_3(o+\tau))$ it may happen that α^* has a 1-dimensional kernel; consequently, α has a 1-dimensional cokernel and $h^1(V_2(-2o)) = 1$. Therefore we can apply Proposition 2.2, concluding the proof of Proposition 3.6. □

4 The moduli space

Let $\mathcal{M}$ be the moduli space of minimal surfaces of general type S with $p_g(S) = 2$, $q(S) = 1$ and $K_S^2 = 5$. We write $\mathcal{M} = \mathcal{M}' \cup \mathcal{M}''$, where $\mathcal{M}'$ corresponds to surfaces such that V_1 is decomposable and $\mathcal{M}''$ corresponds to surfaces such that V_1 is indecomposable.

Definition 4.1 *We stratify $\mathcal{M}'$ and $\mathcal{M}''$ as*

$$\mathcal{M}' = \mathcal{M}_{\mathrm{I}} \cup \mathcal{M}_{\mathrm{IIa}} \cup \cdots \cup \mathcal{M}_{\mathrm{V}}$$

$$\mathcal{M}'' = \mathcal{M}_{\mathrm{VI}} \cup \mathcal{M}_{\mathrm{VIIa}} \cup \mathcal{M}_{\mathrm{VIIb}},$$

according to the decomposition type for V_2, as in Propositions 3.5 and 3.6.

Now we want to estimate the dimensions of these strata. By Catanese-Pignatelli's structure theorem for genus 2 fibrations, we can consider a surjective map $\Phi \colon \mathcal{D} \longrightarrow \mathcal{M}$, where $\mathcal{D}$ is the set of admissible 5-tuples (B, V_1, τ, ξ, w) which give surfaces with our numerical invariants and belonging to a given stratum. Therefore in each case the dimension of the stratum is less than or equal to the dimension of $\mathcal{D}$.

Moreover, we will see that each strata can be parametrized via a unirational family; therefore $\mathcal{M}$ itself is unirational.

Remark 4.2 In order to compute the exact dimension of each strata of the moduli space, we must compute the dimension of the corresponding parameter space $\mathcal{D}$, and then subtract from the result the dimension of the general fibre of Φ. Such a fibre will correspond to the orbit of the action of certain automorphism groups over our construction data.

Locally around the point $[S] \in \mathcal{M}$, the coarse moduli space $\mathcal{M}$ is analytically isomorphic to the quotient of the base T of the Kuranishi family by the finite group $\mathrm{Aut}(S)$. Hence

$$h^1(S, T_S) \geq \dim_{[S]} \mathcal{M} \geq h^1(S, T_S) - h^2(S, T_S) = 10\chi(\mathcal{O}_S) - 2K_S^2 = 10.$$

When $q = 1$ one obtains the better lower bound $10\chi(\mathcal{O}_S) - 2K_S^2 + p_g = 12$, see [16] and [9]. So in our case we have

$$h^1(S, T_S) \geq \dim_{[S]} \mathcal{M} \geq 12.$$

This implies that those strata whose dimension is less than 12 can be disregarded for the determination of the irreducible components of $\mathcal{M}$.

For further application, let us describe a method that can be used in order to estimate $h^1(S, T_S)$, see [15]. There is an exact sequence

$$0 \longrightarrow \omega_S \longrightarrow \Omega_S^1 \otimes \omega_S \longrightarrow \omega_S^{\otimes 2} \longrightarrow \mathcal{O}_{\mathrm{Crit}(f)}(\omega_S^{\otimes 2}) \longrightarrow 0,$$

where $f \colon S \longrightarrow B := \mathrm{Alb}(S)$ is the Albanese map of S. Setting $\mathcal{F} := (\Omega_S^1 \otimes \omega_S)/\omega_S$, we get

$$0 \longrightarrow \mathcal{F} \longrightarrow \omega_S^{\otimes 2} \longrightarrow \mathcal{O}_{\mathrm{Crit}(f)}(\omega_S^{\otimes 2}) \longrightarrow 0.$$

Therefore

$$2 = h^0(S, \omega_S) \leq h^0(S, \Omega_S^1 \otimes \omega_S) \leq h^0(S, \omega_S) + h^0(S, \mathcal{F}) = 2 + h^0(S, \mathcal{F}), \qquad (14)$$

and by the Serre duality $h^2(S, T_S) = h^0(S, \Omega_S^1 \otimes \omega_S)$. Finally,

$$0 \longrightarrow H^0(S, \mathcal{F}) \longrightarrow H^0(S, \omega_S^{\otimes 2}) \longrightarrow H^0(S, \omega_S^{\otimes 2} \otimes \mathcal{O}_{\mathrm{Crit}(f)}) \longrightarrow 0$$

implies that $H^0(S, \mathcal{F})$ is the vector space given by the bicanonical curves of S passing through $\mathrm{Crit}(f)$.

Let us start by studying $\mathcal{M}'$. We have $\mathcal{O}_B(p) \oplus \mathcal{O}_B(2o - p) \cong \mathcal{O}_B(q) \oplus \mathcal{O}_B(2o - q)$ if and only if either $p = q$ or $p + q \in |2o|$; therefore, when p varies in B, the vector bundle V_1 varies into a 1-dimensional family isomorphic to $\mathbb{P}^1$.

Proposition 4.3 *The stratum* $\mathcal{M}_I$ *is nonempty, irreducible, of dimension at most* 13.

Proof Set $W := E_\tau(3, 1)$; then $V_2 = W(2o)$ and we have a short exact sequence

$$0 \longrightarrow W(2o - 2\tau) \longrightarrow \mathrm{S}^3 W(2o - 2\tau) \longrightarrow \widetilde{A}_6 \longrightarrow 0,$$

see (3) and (4). By [5, Sect. 1] we obtain

$$h^0(W(2o - 2\tau)) = 1, \qquad h^1(W(2o - 2\tau)) = 0, \qquad h^0(\mathrm{S}^3 W(2o - 2\tau)) = 10,$$

hence $h^0(\widetilde{A}_6) = 9$. We have 1 parameter for B, 1 parameter for V_1, 2 parameters for ξ, 1 parameter for τ and 8 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$. Therefore $\mathcal{M}_{\mathrm{I}}$ has dimension at most 13, and it is irreducible since it can be parametrized via an irreducible family.

Now let us show that it is non-empty. For the sake of simplicity, we assume $\tau = o$ and we write $\pi : \mathbb{P}(W) \longrightarrow B$ and $\pi_2 : \mathbb{P}(V_2) \longrightarrow B$ for the projective bundles associated to W and V_2, respectively. There is an isomorphism of projective bundles $\psi : \mathbb{P}(W) \longrightarrow \mathbb{P}(V_2)$ such that

$$\psi^* \mathcal{O}_{\mathbb{P}(V_2)}(1) \cong \mathcal{O}_{\mathbb{P}(W)}(1) \otimes \pi^* \mathcal{O}_B(2o). \tag{15}$$

The projective bundle $\mathbb{P}(W)$ can be identified with $\mathrm{Sym}^3 B$, see for instance [5]. For all $x \in B$, set:

$$D_x = \{x + x_2 + x_3 \mid x_2, x_3 \in B\},$$

$$F_x = \{x_1 + x_2 + x_3 \mid \text{the sum of } x_1, x_2, x_3 \text{ in the group law of } B \text{ equals } x\}$$

Then D_o is the divisor class of $\mathcal{O}_{\mathbb{P}(W)}(1)$, and (15) implies that

$$\mathcal{O}_{\mathbb{P}(V_2)}(1) = \mathcal{O}_{\mathbb{P}(V_2)}(D_o + 2F_o). \tag{16}$$

Thus $\mathcal{C} \in |\mathcal{O}_{\mathbb{P}(V_2)}(2) \otimes \pi_2^*(\det(V_1))^{-2}| = |2D_o + 4F_o - 4F_o| = |2D_o|$.

Let now $\varphi : \widetilde{B} \longrightarrow B$ be an isogeny of degree 3, and set $G := \ker(\varphi) \cong \mathbb{Z}_3$. If we write

$$\varphi^{-1}(o) = \{\tilde{o}, \tilde{a}, \tilde{b}\},$$

we have $G = \langle t_{\tilde{a}}^* \rangle$, where $t_{\tilde{a}}^*$ is the translation by ã.

By [1] there exists a line bundle $L \in \mathrm{Pic}(\widetilde{B})$ of degree 1 such that

$$\varphi_* L = W$$

and moreover

$$\begin{aligned} \varphi^* \varphi_* L = \varphi^* E_\tau(3, 1) &= \mathcal{O}_{\widetilde{B}}(\tilde{o}) \oplus t_{\tilde{a}}^* \mathcal{O}_{\widetilde{B}}(\tilde{o}) \oplus (t_{\tilde{a}}^*)^2 \mathcal{O}_{\widetilde{B}}(\tilde{o}) \\ &= \mathcal{O}_{\widetilde{B}}(\tilde{o}) \oplus \mathcal{O}_{\widetilde{B}}(\tilde{a}) \oplus \mathcal{O}_{\widetilde{B}}(\tilde{b}), \end{aligned} \tag{17}$$

see [13, Theorem 2.2]. Let us define $\widetilde{E} := \varphi^*(W \otimes \mathcal{O}_B(2o))$; since the divisor $2\tilde{a} + 2\tilde{b}$ is linearly equivalent to $4\tilde{o}$, (17) yields

$$\begin{aligned} \widetilde{E} &= \varphi^* W \otimes \mathcal{O}_{\widetilde{B}}(2\tilde{o} + 2\tilde{a} + 2\tilde{b}) \\ &= \mathcal{O}_{\widetilde{B}}(3\tilde{o} + 2\tilde{a} + 2\tilde{b}) \oplus \mathcal{O}_{\widetilde{B}}(2\tilde{o} + 3\tilde{a} + 2\tilde{b}) \oplus \mathcal{O}_{\widetilde{B}}(2\tilde{o} + 2\tilde{a} + 3\tilde{b}) \\ &= \mathcal{O}_{\widetilde{B}}(7\tilde{o}) \oplus \mathcal{O}_{\widetilde{B}}(6\tilde{o} + \tilde{a}) \oplus \mathcal{O}_{\widetilde{B}}(6\tilde{o} + \tilde{b}). \end{aligned}$$

From the commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\widetilde{E}) & \xrightarrow{\Phi} & \mathbb{P}(V_2) \\ {\scriptstyle \tilde{\pi}_2} \downarrow & & \downarrow {\scriptstyle \pi_2} \\ \widetilde{B} & \xrightarrow{\varphi} & B \end{array}$$

it follows

$$\begin{aligned}\Phi_*\Phi^*\mathcal{O}_{\mathbb{P}(V_2)}(D_o) &= \mathcal{O}_{\mathbb{P}(V_2)}(D_o) \otimes \Phi_*\mathcal{O}_{\mathbb{P}(\widetilde{E})}\\ &= \mathcal{O}_{\mathbb{P}(V_2)}(D_o) \otimes (\mathcal{O}_{\mathbb{P}(V_2)} \oplus \mathcal{L} \oplus \mathcal{L}^2)\\ &= \mathcal{O}_{\mathbb{P}(V_2)}(D_o) \oplus (\mathcal{O}_{\mathbb{P}(V_2)}(D_o) \otimes \mathcal{L}) \oplus (\mathcal{O}_{\mathbb{P}(V_2)}(D_o) \otimes \mathcal{L}^2),\end{aligned}$$

where $\mathcal{L}$ is the 3-torsion line bundle inducing the étale $\mathbb{Z}_3$-cover $\Phi\colon \mathbb{P}(\widetilde{E}) \longrightarrow \mathbb{P}(V_2)$. By (16) we see that

$$\begin{aligned}\Phi^*\mathcal{O}_{\mathbb{P}(V_2)}(D_o) &= \Phi^*(\mathcal{O}_{\mathbb{P}(V_2)}(1) \otimes \pi_2^*\mathcal{O}_B(-2o))\\ &= \mathcal{O}_{\mathbb{P}(\widetilde{E})}(1) \otimes \tilde{\pi}_2^*\mathcal{O}_{\widetilde{B}}(-6\tilde{o}).\end{aligned}$$

Let y_0, y_1 and y_2 be global coordinates on the fibers of $\tilde{\pi}_2$, namely

$$\begin{aligned}y_0 &\in H^0(\mathcal{O}_{\mathbb{P}(\widetilde{E})}(1) \otimes \tilde{\pi}_2^*\mathcal{O}_{\widetilde{B}}(-7\tilde{o})),\\ y_1 &\in H^0(\mathcal{O}_{\mathbb{P}(\widetilde{E})}(1) \otimes \tilde{\pi}_2^*\mathcal{O}_{\widetilde{B}}(-6\tilde{o} - \tilde{a})),\\ y_2 &\in H^0(\mathcal{O}_{\mathbb{P}(\widetilde{E})}(1) \otimes \tilde{\pi}_2^*\mathcal{O}_{\widetilde{B}}(-6\tilde{o} - \tilde{b})).\end{aligned}$$

We have $h^0(\Phi^*\mathcal{O}_{\mathbb{P}(V_2)}(D_o)) = 3$ and a general section of $\Phi^*\mathcal{O}_{\mathbb{P}(V_2)}(D_o)$ can be written as

$$\sigma = \lambda_0 y_0 + \lambda_1 y_1 + \lambda_2 y_2,$$

where $\lambda_0 \in H^0(\tilde{\pi}_2^*\mathcal{O}_{\widetilde{B}}(\tilde{o}))$, $\lambda_1 \in H^0(\tilde{\pi}_2^*\mathcal{O}_{\widetilde{B}}(\tilde{a}))$ and $\lambda_2 \in H^0(\tilde{\pi}_2^*\mathcal{O}_{\widetilde{B}}(\tilde{b}))$.

Then a straightforward computation shows that we can choose the y_i so that the action of $t_{\tilde{a}}^* \in G$ on the y_i is given by

$$t_{\tilde{a}}^*\colon \begin{cases} y_0 \mapsto y_1,\\ y_1 \mapsto y_2,\\ y_2 \mapsto y_0.\end{cases} \tag{18}$$

Therefore $t_{\tilde{a}}^*\sigma = (t_{\tilde{a}}^*\lambda_0)y_1 + (t_{\tilde{a}}^*\lambda_1)y_2 + (t_{\tilde{a}}^*\lambda_2)y_0$, so σ is G-invariant if and only if $t_{\tilde{a}}^*\lambda_0 = \lambda_1$, $t_{\tilde{a}}^*\lambda_1 = \lambda_2$ and $t_{\tilde{a}}^*\lambda_2 = \lambda_0$. Since $(t_{\tilde{a}}^*)^2 = t_{\tilde{b}}^*$, this is equivalent to require $\lambda_1 = t_{\tilde{a}}^*\lambda_0$ and $\lambda_2 = t_{\tilde{b}}^*\lambda_0$. So a general invariant section of $\Phi^*\mathcal{O}_{\mathbb{P}(V_2)}(D_o)$ is given by

$$\lambda y_0 + (t_{\tilde{a}}^*\lambda)y_1 + (t_{\tilde{b}}^*\lambda)y_2,$$

where $\lambda \in H^0(\mathcal{O}_{\widetilde{B}}(\tilde{o}))$.

Now a general section of $\Phi^*\mathcal{O}_{\mathbb{P}(V_2)}(2D_o)$ is of the form:

$$\begin{aligned}\sigma &= \sum_{i+j+k=2} \lambda_{ijk} y_0^i y_1^j y_2^k\\ &= \lambda_{200}y_0^2 + \lambda_{020}y_1^2 + \lambda_{002}y_2^2 + \lambda_{110}y_0y_1 + \lambda_{101}y_0y_2 + \lambda_{011}y_1y_2,\end{aligned}$$

where the λ_{ijk} are sections of pullbacks of suitable line bundles on $\widetilde{B}$.

By (18), $t_{\tilde{a}}^*$ acts on σ as

$$\begin{aligned}t_{\tilde{a}}^*\sigma = (t_{\tilde{a}}^*\lambda_{200})y_1^2 + (t_{\tilde{a}}^*\lambda_{020})y_2^2 + (t_{\tilde{a}}^*\lambda_{002})y_0^2 + (t_{\tilde{a}}^*\lambda_{110})y_1y_2 + (t_{\tilde{a}}^*\lambda_1 01)y_0y_1\\ + (t_{\tilde{a}}^*\lambda_{011})y_0y_2,\end{aligned}$$

so σ is G-invariant if and only if

$$\lambda_{020} = t_{\tilde{a}}^* \lambda_{200}, \qquad \lambda_{002} = t_{\tilde{a}}^* \lambda_{020} = t_{\tilde{b}}^* \lambda_{200},$$
$$\lambda_{011} = t_{\tilde{a}}^* \lambda_{110}, \qquad \lambda_{101} = t_{\tilde{a}}^* \lambda_{011} = t_{\tilde{b}}^* \lambda_{110}.$$

Hence a general invariant section of $\Phi^* \mathcal{O}_{\mathbb{P}(V_2)}(2D_o)$ can be written as

$$\lambda y_0^2 + (t_{\tilde{a}}^* \lambda) y_1^2 + (t_{\tilde{b}}^* \lambda) y_2^2 + \mu y_0 y_1 + (t_{\tilde{b}}^* \mu) y_0 y_2 + (t_{\tilde{a}}^* \mu) y_1 y_2, \tag{19}$$

with $\lambda \in H^0(\mathcal{O}_{\widetilde{B}}(2\tilde{o}))$, $\mu \in H^0(\mathcal{O}_{\widetilde{B}}(\tilde{o} + \tilde{a}))$.

Denoting by $\tilde{p} \in \widetilde{B}$ any of the points in $\varphi^{-1}(p)$, the short exact sequence (1) lifts to

$$0 \longrightarrow \mathcal{O}_{\widetilde{B}}(6\tilde{p}) \oplus \mathcal{O}_{\widetilde{B}}(6\tilde{o}) \oplus \mathcal{O}_{\widetilde{B}}(12\tilde{o} - 6\tilde{p}) \xrightarrow{\tilde{\sigma}_2} \widetilde{E} \longrightarrow \mathcal{O}_{\tilde{o}+\tilde{a}+\tilde{b}} \longrightarrow 0. \tag{20}$$

Taking global coordinates $\tilde{x}_0$, $\tilde{x}_1$ on the fibres of $\varphi^* V_1 = \mathcal{O}_{\widetilde{B}}(3\tilde{p}) \oplus \mathcal{O}_B(6\tilde{o} - 3\tilde{p})$, the map $\tilde{\sigma}_2$ is given by

$$\begin{cases} \tilde{\sigma}_2(\tilde{x}_0^2) = a_{00} y_0 + a_{01} y_1 + a_{02} y_2, \\ \tilde{\sigma}_2(\tilde{x}_0 \tilde{x}_1) = a_{10} y_0 + a_{11} y_1 + a_{12} y_2, \\ \tilde{\sigma}_2(\tilde{x}_1^2) = a_{20} y_0 + a_{21} y_1 + a_{22} y_2, \end{cases}$$

where

$$a_{00} \in H^0(\tilde{\pi}_2^* \mathcal{O}_{\widetilde{B}}(7\tilde{o} - 6\tilde{p})), \qquad a_{01} \in H^0(\tilde{\pi}_2^* \mathcal{O}_{\widetilde{B}}(6\tilde{o} - 6\tilde{p} + \tilde{a})),$$
$$a_{02} \in H^0(\tilde{\pi}_2^* \mathcal{O}_{\widetilde{B}}(6\tilde{o} - 6\tilde{p} + \tilde{b})), \qquad a_{10} \in H^0(\tilde{\pi}_2^* \mathcal{O}_{\widetilde{B}}(\tilde{o})), \qquad a_{11} \in H^0(\tilde{\pi}_2^* \mathcal{O}_{\widetilde{B}}(\tilde{a})),$$
$$a_{12} \in H^0(\tilde{\pi}_2^* \mathcal{O}_{\widetilde{B}}(\tilde{b})), \qquad a_{20} \in H^0(\tilde{\pi}_2^* \mathcal{O}_{\widetilde{B}}(6\tilde{p} - 5\tilde{o})),$$
$$a_{21} \in H^0(\tilde{\pi}_2^* \mathcal{O}_{\widetilde{B}}(6\tilde{p} - 6\tilde{o} + \tilde{a})), \qquad a_{22} \in H^0(\tilde{\pi}_2^* \mathcal{O}_{\widetilde{B}}(6\tilde{p} - 6\tilde{o} + \tilde{b})).$$

Let us consider now the conic bundle $\widetilde{\mathcal{C}} \subset \mathbb{P}(\widetilde{E})$ given by

$$(a_{00} y_0 + a_{01} y_1 + a_{02} y_2)(a_{20} y_0 + a_{21} y_1 + a_{22}) - (a_{10} y_0 + a_{11} y_1 + a_{12} y_2)^2 = 0.$$

If we choose

$$a_{01} = t_{\tilde{a}}^* a_{00}, \qquad a_{02} = t_{\tilde{b}}^* a_{00}, \qquad a_{11} = t_{\tilde{a}}^* a_{10},$$
$$a_{12} = t_{\tilde{b}}^* a_{10}, \qquad a_{21} = t_{\tilde{a}}^* a_{20}, \qquad a_{22} = t_{\tilde{b}}^* a_{20}$$

the equation of $\widetilde{\mathcal{C}}$ is G-invariant, hence of the form (19); in fact, we have

$$\lambda = a_{00} a_{20} - a_{10}^2, \qquad \mu = a_{00}(t_{\tilde{a}}^* a_{20}) + (t_{\tilde{a}}^* a_{00}) a_{20} - 2a_{10}(t_{\tilde{a}}^* a_{10}).$$

We claim that, for a general choice of a_{00}, a_{10}, a_{20}, the only singularities of $\widetilde{\mathcal{C}}$ are three rational double points of type A_1, lying over the three points $\tilde{o}$, $\tilde{a}$, $\tilde{b}$. Since $\tilde{\sigma}_2$ is of maximal rank outside these points, and since they form an orbit for the G-action, it is sufficient to check that the fibre over $\tilde{o}$ has a node (which will be automatically a point of type A_1 for $\widetilde{\mathcal{C}}$). In a neighborhood of this fibre, set

$$u_0 := a_{00}(\tilde{o}) y_0 + a_{01}(\tilde{o}) y_1 + a_{02}(\tilde{o}) y_2,$$
$$u_1 := a_{10}(\tilde{o}) y_0 + a_{11}(\tilde{o}) y_1 + a_{12}(\tilde{o}) y_2,$$
$$u_2 := a_{20}(\tilde{o}) y_0 + a_{21}(\tilde{o}) y_1 + a_{22}(\tilde{o}) y_2.$$

Since $\tilde{\sigma}_2$ drops rank in $\tilde{o}$, we can find $c_0, c_2 \in \mathbb{C}$ such that $u_1 = c_0 u_0 + c_2 u_2$; then a local equation of the fibre of $\widetilde{\mathcal{C}}$ over $\tilde{o}$ is given by

$$u_0 u_2 - (c_0 u_0 + c_2 u_2)^2 = 0. \tag{21}$$

Since for a general choice of a_{00}, a_{10}, a_{20} (i.e. for a general choice of c_0, c_2) the quadratic form (21) splits into two *distinct* linear forms, our claim is proven.

Therefore the image of $\widetilde{\mathcal{C}}$ in $\mathbb{P}(V_2)$ is a conic bundle $\mathcal{C}$ with a unique singular point of type A_1, lying over the point $o \in B$. Moreover, by construction, $\mathcal{C}$ is the conic bundle associated with the map $\sigma_2 \colon S^2 V_1 \to V_2$, so condition $(\mathcal{P}_1)$ of Theorem 2.6 is satisfied.

The relative cubic $\mathcal{G}$ belongs to the linear system $|\mathcal{O}_{\mathbb{P}(V_2)}(3) \otimes \pi_2^* \mathcal{O}_B(-4o - 2\tau)| = |3D_o + 6F_o - 6F_o| = |3D_o|$. By [5], the linear system $|3D_o|$ is base point free, hence its restriction to $\mathcal{C}$ is base point free too. This implies that a general complete intersection of the form $\mathcal{G} \cap \mathcal{C}$ is smooth and does not contain the unique singular point of $\mathcal{C}$. Thus condition $(\mathcal{P}_2)$ is also satisfied, and consequently $\mathcal{M}_{\mathrm{I}}$ is not empty. □

Proposition 4.4 *The stratum $\mathcal{M}_{\mathrm{IIa}}$ has dimension at most* 12.

Proof In case (IIa) we have $\mathcal{O}_B(4o - 4p) = \mathcal{O}_B$, so there are no parameters for V_1. The vector bundle $\widetilde{A}_6$ fits into the short exact sequence

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \widetilde{A}_6 \longrightarrow 0,$$

where

$$G_1 = F_2(2p - 2\tau) \oplus \mathcal{O}_B(2o - \tau),$$
$$G_2 = \mathrm{S}^3 F_2(2p - 2\tau) \oplus \mathrm{S}^2 F_2(2o - \tau) \oplus F_2(2p) \oplus \mathcal{O}_B(2o + \tau).$$

By Proposition 2.4 we have $\mathrm{S}^2 F_2 = F_3$, $\mathrm{S}^3 F_2 = F_4$. Now there are two possibilities.

- $\mathcal{O}_B(2p - 2\tau) \neq \mathcal{O}_B$. In this case

$$h^0(G_1) = 1, \qquad h^1(G_1) = 0, \qquad h^0(G_2) = 10,$$

hence $h^0(\widetilde{A}_6) = h^0(G_2) - h^0(G_1) = 9$. We have 1 parameter for B, 2 parameters for ξ, 1 parameter for τ and 8 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$.

- $\mathcal{O}_B(2p - 2\tau) = \mathcal{O}_B$. In this case

$$h^0(G_1) = 2, \qquad h^1(G_1) = 1, \qquad h^0(G_2) = 11,$$

hence $h^0(\widetilde{A}_6) \leq 10$ by (5). We have 1 parameter for B, 2 parameters for ξ, no parameters for τ and V_1 and at most 9 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$.

Summing up, we conclude that $\mathcal{M}_{\mathrm{IIa}}$ has dimension at most 12. □

Proposition 4.5 *The stratum $\mathcal{M}_{\mathrm{IIb}}$ has dimension at most* 12.

Proof Set $W = E_\tau(2, 1)$; then $V_2(-2o) = W \oplus \mathcal{O}_B$ and tensoring the exact sequence (3) with $\mathcal{O}_B(-6o)$ we obtain

$$0 \longrightarrow W \oplus \mathcal{O}_B \xrightarrow{i_3} \left(\mathrm{S}^3 W \oplus \mathrm{S}^2 W\right) \oplus (W \oplus \mathcal{O}_B) \longrightarrow A_6(-6o) \longrightarrow 0. \tag{22}$$

Arguing as in [6, Lemma 6.14], we see that the second component of the map i_3 is actually the identity, hence the exact sequence (22) splits, giving

$$\widetilde{A}_6 = A_6(-4o - 2\tau) = (\mathrm{S}^3 W \oplus \mathrm{S}^2 W)(2o - 2\tau).$$

By Proposition 2.4 this in turn implies

$$\widetilde{A}_6 = \left(W \oplus W \oplus \bigoplus_{i=1}^{3} L_i \right)(2o - \tau),$$

hence $h^0(\widetilde{A}_6) = 9$. We have 1 parameter for B, no parameters for V_1, 2 parameters for ξ, 1 parameter for τ and 8 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$. Therefore $\mathcal{M}_{\mathrm{IIb}}$ has dimension at most 12.

The fact that it is nonempty can be proven as in case $\mathcal{M}_{\mathrm{I}}$ (using an isogeny of degree 2 instead of 3); the details are left to the reader. □

Proposition 4.6 *The stratum* $\mathcal{M}_{\mathrm{IIc}}$ *has dimension at most* 11.

Proof In case (IIc) we have $\mathcal{O}_B(2o - 2p) = \mathcal{O}_B(2p - 2o)$, with $\mathcal{O}_B(2o - 2p) \neq \mathcal{O}_B$, and the map σ_2 has the form

$$\sigma_2 \colon \mathcal{O}_B(2p) \oplus \mathcal{O}_B(2o) \oplus \mathcal{O}_B(4o - 2p) \longrightarrow \mathcal{O}_B(2p) \oplus \mathcal{O}_B(2o) \oplus \mathcal{O}_B(4o - 2p + \tau).$$

Take global coordinates

$$x_0 \in H^0\big(\mathcal{O}_{\mathbb{P}(V_1)}(1) \otimes \pi_1^* \mathcal{O}_B(-p)\big), \qquad x_1 \in H^0\big(\mathcal{O}_{\mathbb{P}(V_1)}(1) \otimes \pi_1^* \mathcal{O}_B(-2o + p)\big)$$

on the fibres of $\mathbb{P}(V_1)$ and, similarly, global coordinates y_0, y_1, y_2 on the fibres of $\mathbb{P}(V_2)$. With respect to these coordinates, σ_2 is given by

$$\begin{cases} \sigma_2(x_0^2) = a_{00} y_0 + a_{02} y_2, \\ \sigma_2(x_0 x_1) = a_{11} y_1 + a_{12} y_2, \\ \sigma_2(x_1^2) = a_{20} y_0 + a_{22} y_2, \end{cases}$$

where $a_{00}, a_{11}, a_{20} \in \mathbb{C}$, $a_{02}, a_{22} \in H^0(\mathcal{O}_B(\tau))$, $a_{12} \in H^0(\mathcal{O}_B(\tau + 2o - 2p))$. Therefore the equation of the conic bundle $\mathcal{C} \subset \mathbb{P}(V_2)$ is

$$(a_{00} y_0 + a_{02} y_2)(a_{20} y_0 + a_{22} y_2) - (a_{11} y_1 + a_{12} y_2)^2 = 0.$$

Moreover, since the rank of σ_2 drops exactly at the point τ, it follows $a_{11} \neq 0$. This means that the coefficient of the term y_1^2 is a non-zero constant, hence the same argument of [15, Lemma 3.5] shows that exact sequence (3) splits. Therefore we obtain

$$\begin{aligned} \widetilde{A}_6 = {} & \mathcal{O}_B(2p - 2\tau) \oplus \mathcal{O}_B(4o - 2p + \tau) \oplus \mathcal{O}_B(4p - 2o - 2\tau) \\ & \oplus \mathcal{O}_B(2p - \tau) \oplus \mathcal{O}_B(4o - 2p) \oplus \mathcal{O}_B(6o - 4p) \oplus \mathcal{O}_B(2o - \tau), \end{aligned}$$

so

$$h^0(\widetilde{A}_6) = \begin{cases} 10 & \text{if either } \mathcal{O}(2p - 2\tau) = \mathcal{O}_B \text{ or } \mathcal{O}_B(4p - 2o - 2\tau) = \mathcal{O}_B; \\ 9 & \text{otherwise.} \end{cases}$$

So we have 1 parameter for B, 1 parameter for ξ, no parameters (resp. 1 parameter) for τ and 9 parameters (resp. 8 parameters) from $\mathbb{P}H^0(\widetilde{A}_6)$. It follows that $\mathcal{M}_{\mathrm{IIc}}$ has dimension at most 11. □

Proposition 4.7 *We have* $\mathcal{M}_{\mathrm{IIIa}} = \mathcal{M}_{\mathrm{IIIb}}$. *Moreover the dimension of this stratum is at most* 12.

Proof Case (IIIb) is obtained from case (IIIa) by considering $2o - p$ instead of p; this shows that the corresponding strata coincide. So it is sufficient to consider case (IIIa); set

$$W := E_{\tau+2o-2p}, \qquad L := \mathcal{O}_B(2p - 2o).$$

Then we have $V_2(-2o) = W \oplus L$ and tensoring the exact sequence (3) with $\mathcal{O}_B(-6o)$ we obtain

$$0 \longrightarrow W \oplus L \xrightarrow{i_3} \mathrm{S}^3 W \oplus (\mathrm{S}^2 W \otimes L) \oplus (W \otimes L^2) \oplus L^3 \longrightarrow A_6(-6o) \longrightarrow 0.$$

Hence $\widetilde{A}_6 = A_6(-4o - 2\tau)$ fits into the short exact sequence

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \widetilde{A}_6 \longrightarrow 0,$$

where

$$G_1 = (W \oplus L)(2o - 2\tau),$$
$$G_2 = \big(\mathrm{S}^3 W \oplus (\mathrm{S}^2 W \otimes L) \oplus (W \otimes L^2) \oplus L^3\big)(2o - 2\tau).$$

There are several possibilities.

- $L(2o - 2\tau) \neq \mathcal{O}_B$, $L^3(2o - 2\tau) \neq \mathcal{O}_B$. In this case

$$h^0(G_1) = 1, \qquad h^1(G_1) = 0, \qquad h^0(G_2) = 10,$$

hence $h^0(\widetilde{A}_6) = 9$. We have 1 parameter for B, 1 parameter for V_1, 1 parameter for ξ, 1 parameter for τ and 8 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$.

- $L(2o - 2\tau) \neq \mathcal{O}_B$, $L^3(2o - 2\tau) = \mathcal{O}_B$. In this case

$$h^0(G_1) = 1, \qquad h^1(G_1) = 0, \qquad h^0(G_2) = 11,$$

hence $h^0(\widetilde{A}_6) = 10$. We have 1 parameter for B, 1 parameter for V_1, 1 parameter for ξ, no parameters for τ and 9 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$.

- $L(2o - 2\tau) = \mathcal{O}_B$, $L^3(2o - 2\tau) \neq \mathcal{O}_B$. We have

$$h^0(G_1) = 2, \qquad h^1(G_1) = 1, \qquad h^0(G_2) = 10,$$

hence $h^0(\widetilde{A}_6) \leq 9$ by (5). We have 1 parameter for B, 1 parameter for V_1, 1 parameter for ξ, no parameters for τ and at most 8 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$.

- $L(2o - 2\tau) = \mathcal{O}_B$, $L^3(2o - 2\tau) = \mathcal{O}_B$. Notice that this implies $L^2 = \mathcal{O}_B$, so there are no parameters for V_1. We obtain

$$h^0(G_1) = 2, \qquad h^1(G_1) = 1, \qquad h^0(G_2) = 11,$$

hence $h^0(\widetilde{A}_6) \leq 10$ by (5). We have 1 parameter for B, 1 parameter for ξ, no parameters for τ and at most 9 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$.

Summing up, we conclude that the dimension of the stratum $\mathcal{M}_{\mathrm{IIIa}} = \mathcal{M}_{\mathrm{IIIb}}$ is at most 12. □

Proposition 4.8 *The stratum* $\mathcal{M}_{\mathrm{IIIc}}$ *has dimension at most* 12.

Proof As in the proof of Proposition 4.5, $h^0(\widetilde{A}_6) = 9$. We have 1 parameter for B, 1 parameter for V_1, 1 parameter for ξ, 1 parameter for τ and 8 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$. Therefore $\mathcal{M}_{\mathrm{IIIc}}$ has dimension at most 12. □

Proposition 4.9 *The strata* $\mathcal{M}_{\mathrm{IVa}}$, $\mathcal{M}_{\mathrm{IVb}}$ *have dimension at most* 11.

Proof The proof is the same as in case (IIc); the details are left to the reader. □

Proposition 4.10 *The stratum* $\mathcal{M}_{\mathrm{IVc}}$ *has dimension at most* 11.

Proof In case (IVc) the vector bundles G_1, G_2 in exact sequence (4) are as follows:

$$\begin{aligned} G_1 &= \mathcal{O}_B(2p-2\tau) \oplus \mathcal{O}_B(4o-2p-2\tau) \oplus \mathcal{O}_B(2o-\tau), \\ G_2 &= \mathcal{O}_B(6p-4o-2\tau) \oplus \mathcal{O}_B(8o-6p-2\tau) \oplus \mathcal{O}_B(2o+\tau) \oplus \mathcal{O}_B(2p-2\tau) \\ &\quad \oplus \mathcal{O}_B(4p-2o-\tau) \oplus \mathcal{O}_B(4o-2p-2\tau) \oplus \mathcal{O}_B(6o-4p-\tau) \oplus \mathcal{O}_B(2p) \\ &\quad \oplus \mathcal{O}_B(4o-2p) \oplus \mathcal{O}_B(2o-\tau). \end{aligned}$$

A tedious but elementary analysis of all possibilities, together with inequality (5), shows that the number of parameters involved in the construction never exceeds 11. Hence $\mathcal{M}_{\mathrm{IVc}}$ has dimension at most 11. □

Now let us write $\mathcal{M}_{\mathrm{V}} = \mathcal{M}_{\mathrm{V,\,gen}} \cup \mathcal{M}_{\mathrm{V},2}$, where $\mathcal{M}_{\mathrm{V},2}$ consists of surfaces with $\mathcal{O}_B(2o - 2\tau) = \mathcal{O}_B$ and $\mathcal{M}_{\mathrm{V,gen}}$ is the rest.

Proposition 4.11 $\mathcal{M}_{\mathrm{V,gen}}$ *and* $\mathcal{M}_{\mathrm{V},2}$ *are both non-empty.*

Proof In case (V) we have $\mathcal{O}_B(2o-2p) = \mathcal{O}_B$, hence the map $\sigma_2 \colon S^2V_1 \to V_2$ has the form

$$\sigma_2 \colon \mathcal{O}_B(2o)^3 \longrightarrow \mathcal{O}_B(2o)^2 \oplus \mathcal{O}_B(2o+\tau).$$

Recall that for a general choice of σ_2 we have $\bar{f}_i \neq 0$ for all $i \in \{1, 2, 3\}$. Take coordinates x_0, x_1 on the fibres of V_1 and y_0, y_1, y_2 on the fibres of V_2; with respect to these coordinates, σ_2 is given by

$$\begin{cases} \sigma_2(x_0^2) = a_{00}y_0 + a_{01}y_1 + a_{02}f_0y_2, \\ \sigma_2(x_0x_1) = a_{10}y_0 + a_{11}y_1 + a_{12}f_0y_2, \\ \sigma_2(x_1^2) = a_{20}y_0 + a_{21}y_1 + a_{22}f_0y_2, \end{cases}$$

where $a_{ij} \in \mathbb{C}$ and $f_0 \in H^0(\mathcal{O}_B(\tau))$. Moreover, since the rank of σ_2 drops precisely at the point τ, it follows $\det(a_{ij}) \neq 0$.

Therefore the global equation of the relative conic $\mathcal{C} \subset \mathbb{P}(V_2)$ is

$$(a_{00}y_0 + a_{01}y_1 + a_{02}f_0y_2)(a_{20}y_0 + a_{21}y_1 + a_{22}f_0y_2)$$
$$-(a_{10}y_0 + a_{11}y_1 + a_{12}f_0y_2)^2 = 0.$$

Notice that at least one of the coefficient of y_0^2, y_1^2 or y_0y_1 in the equation of $\mathcal{C}$ is not zero, otherwise y_2^2 divides the equation of $\mathcal{C}$. Since each of these coefficients is a non-zero constant, by the argument in [15, Lemma 3.5] one sees that in any case the exact sequence (3) splits. Therefore we obtain

$$\widetilde{A}_6 = \mathcal{O}_B(2o-2\tau)^2 \oplus \mathcal{O}_B(2o-\tau)^2 \oplus \mathcal{O}_B(2o)^2 \oplus \mathcal{O}_B(2o+\tau),$$

so

$$h^0(\widetilde{A}_6) = \begin{cases} 11 & \text{if } \mathcal{O}_B(2o-2\tau) = \mathcal{O}_B, \\ 9 & \text{otherwise.} \end{cases}$$

Choosing $a_{02} = a_{22} = a_{10} = a_{11} = 0$, $a_{00} = a_{01} = a_{20} = a_{12} = 1$, $a_{21} = -1$, the equation of $\mathcal{C}$ becomes

$$y_0^2 - y_1^2 - f_0^2 y_2^2 = 0.$$

Hence $\mathcal{C}$ has a unique singular point (of type A_1), namely the point P with homogeneous coordinates $[0:0:1]$ lying on the fibre over τ; in particular, condition $(\mathcal{P}_1)$ of Theorem 2.6 is satisfied. Since (3) splits, the curve Δ defined by the section $w \in H^0(\widetilde{A}_6)$ is cut by a relative cubic $\mathcal{G} \in |\mathcal{O}_{\mathbb{P}(V_2)}(3) \otimes \pi_2^*\mathcal{O}_B(-4o-2\tau)|$; let us write the equation of $\mathcal{G}$ as

$$\sum_{i+j+k=3} b_{ijk}\, y_0^i y_1^j y_2^k = 0, \tag{23}$$

where $b_{ijk} \in H^0(\mathbb{P}(V_2), \pi_2^*\mathcal{O}_B(2o+(k-2)\tau))$. If $\mathcal{O}_B(2o-2\tau) = \mathcal{O}_B$ then all the coefficients of $\mathcal{G}$ are generically non-zero; one checks that in this case the linear system $|\mathcal{G}|$ in $\mathbb{P}(V_2)$ is base-point free, hence the linear system $|\Delta|$ in $\mathcal{C}$ is base-point free too; by Bertini theorem, we conclude that for a general choice of Δ condition $(\mathcal{P}_2)$ is also satisfied, hence $\mathcal{M}_{\mathrm{V},2}$ is non-empty.

If $\mathcal{O}_B(2o-2\tau) \neq \mathcal{O}_B$, then $b_{300} = b_{210} = b_{120} = b_{030} = 0$. So the relative cubic $\mathcal{G}$ splits as $\mathcal{G} = \mathcal{H} \cup \mathcal{G}'$, where $\mathcal{H}$ is the relative hyperplane $\{y_2 = 0\}$ and $\mathcal{G}'$ is the relative conic

$$b_{201}y_0^2 + b_{111}y_0y_1 + b_{102}y_0y_2 + b_{021}y_1^2 + b_{012}y_1y_2 + b_{003}y_2^2 = 0.$$

Consequently, Δ splits as $\Delta = \mathcal{H}_{\mathcal{C}} \cup \Delta'$, where $\mathcal{H}_{\mathcal{C}} = \mathcal{H} \cap \mathcal{C}$ and $\Delta' = \mathcal{G}' \cap \mathcal{C}$. The sections $b_{201}, b_{021}, b_{111}$ all vanish at the same point, namely the unique point $q \in B$ such that $\mathcal{O}_B(2o-\tau) = \mathcal{O}_B(q)$; notice that $q \neq \tau$. Hence the base locus of $|\mathcal{G}'|$ is the line $y_2 = 0$ in the fibre $\pi^{-1}(q)$, and this in turn implies that the base locus of $|\Delta'|$ in $\mathcal{C}$ are the two points $P_1 = [1:1:0]$ and $P_2 = [1:-1:0]$ on the fibre of $\mathcal{C}$ over q. Now let us make a general choice of the coefficients in (23). Then Δ does not contain the unique singular point of $\mathcal{C}$; moreover, a standard local computation together with Bertini theorem show that

- Δ' is smooth;
- Δ' and $\mathcal{H}_{\mathcal{C}}$ intersect transversally at P_1 and P_2.

So condition $(\mathcal{P}_2)$ is satisfied and $\mathcal{M}_{\mathrm{V,gen}}$ is non-empty. □

Let us compute now the dimensions of $\mathcal{M}_{V,2}$ and $\mathcal{M}_{V,\mathrm{gen}}$.

Proposition 4.12 *$\mathcal{M}_{\mathrm{V},2}$ has dimension* 12, *whereas $\mathcal{M}_{\mathrm{V},\mathrm{gen}}$ has dimension* 11. *Moreover, $\mathcal{M}_{\mathrm{V},2}$ is a generically smooth, irreducible component of $\mathcal{M}$.*

Proof We first compute the dimension of the parameter space $\mathcal{D}$ in each case. If $\mathcal{O}_B(2o - 2\tau) = \mathcal{O}_B$ we have 1 parameter for B, 2 parameters for ξ and 10 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$; otherwise we have 1 parameter for B, 2 parameters for ξ, 1 parameter from τ and 8 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$. Therefore $\mathcal{M}_{V,2}$ has dimension at most 13, whereas $\mathcal{M}_{V,\mathrm{gen}}$ has dimension at most 12.

By Remark 4.2, we have now to find the dimension of the general fibre of $\Phi\colon \mathcal{D} \to \mathcal{M}$, and for this we have to consider the action of certain automorphism groups over our data.

Observe first that in both cases we can forget the action of $\mathrm{Aut}(B)$, since we have fixed a point of B by choosing $\det V_1 = \mathcal{O}_B(2o)$. So we have only to consider the action of $\mathrm{Aut}(V_1) \times \mathrm{Aut}(V_2)$.

We are therefore reduced to solve the following problem: given an admissible 5-tuple (B, V_1, t, ξ, w), corresponding to the genus 2 fibration $f : S \to B$, we must find the dimension of the subvariety $Z \subset \mathrm{Aut}(V_1) \times \mathrm{Aut}(V_2)$ given by the pairs (ϕ_1, ϕ_2) which make the following diagram commuting:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^2V_1 & \xrightarrow{\sigma_2} & V_2 & \longrightarrow & \mathcal{O}_\tau & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle S^2\phi_1} & & \downarrow{\scriptstyle \phi_2} & & \| & & \\ 0 & \longrightarrow & S^2V_1 & \xrightarrow{\sigma_2} & V_2 & \longrightarrow & \mathcal{O}_\tau & \longrightarrow & 0. \end{array} \tag{24}$$

In fact, the dimension of the fibre $\Phi^{-1}([S])$ is given by $\dim Z - 1$. Geometrically, this expresses the fact that the points in such a fibre are in 1-to-1 correspondence with the family of automorphisms of the projective bundle $\mathbb{P}(V_2)$ fixing the conic bundle $\mathcal{C}$.

Now we claim that, if S is *general* in either $\mathcal{M}_{V,2}$ or $\mathcal{M}_{V,\mathrm{gen}}$, by choosing suitable coordinates for V_1 and V_2 we can put the equation of the conic bundle $\mathcal{C}$ in the form

$$y_0(y_1 + f_0 y_2) - y_1^2 = 0. \tag{25}$$

In fact, in the general case $\mathcal{C}$ has a nodal fibre over the point τ; without loss of generality we can assume that such a fibre has equation $y_1(y_0 - y_1) = 0$, so that the conic bundle has the form $(y_0 + a_{02} f_0 y_2)(y_1 + a_{22} f_0 y_2) - (y_1 + a_{12} f_0 y_2)^2 = 0$. Now the claim follows by using the linear change of coordinates

$$y_0' := y_0 + a_{02} f_0 y_2, \qquad y_1' := y_1 + a_{12} f_0 y_2, \qquad y_2' := (a_{22} - a_{12}) y_2.$$

Therefore, in order to compute the dimension of the general fibre of Φ, we may assume that the matrix associated with $\sigma_2\colon S^2V_1 \to V_2$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & f_0 \end{pmatrix}.$$

Let now $\phi_1 \in \mathrm{Aut}(V_1)$, given by $\phi_1(x_0) = ax_0 + cx_1$ and $\phi_1(x_1) = bx_0 + dx_1$, $a, b, c, d \in \mathbb{C}$. Then the action of $S^2\phi_1$ on $S^2 V_1$ is expressed by the matrix

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

On the other hand, the general $\phi_2 \in \mathrm{Aut}(V_2)$ is given by the matrix

$$\begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} f_0 & b_{32} f_0 & b_{33} \end{pmatrix},$$

where $b_{ij} \in \mathbb{C}$. Hence, imposing that the diagram (24) commutes, by straightforward computations one finds that any pair $(\phi_1, \phi_2) \in Z$ is either of the form

$$\phi_1 = \begin{pmatrix} a & a \\ c & -a \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} a^2 & a^2 & 0 \\ 2ac + c^2 & -a^2 & 0 \\ c^2 f_0 & -acf_0 & a(a+c) \end{pmatrix}$$

or of the form

$$\phi_1 = \begin{pmatrix} a & 0 \\ c & a+c \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} a^2 & 0 & 0 \\ 2ac + c^2 & (a+c)^2 & 0 \\ c^2 f_0 & c(a+c) f_0 & a(a+c) \end{pmatrix}.$$

It follows that $Z \subset \mathrm{Aut}(V_1) \times \mathrm{Aut}(V_2)$ is a subvariety of dimension 2. Consequently, the general fibre of Φ has dimension 1; this means that the dimension of $\mathcal{M}_{V,2}$ equals 12, whereas the dimension of $\mathcal{M}_{V,\mathrm{gen}}$ equals 11.

Now we want to prove that $\mathcal{M}_{V,2}$ is an irreducible component of $\mathcal{M}$. In order to do this, we will show that $h^1(S, T_S) = 12$ for a general $S \in \mathcal{M}_{V,2}$. Since $\dim \mathcal{M}_{V,2} = 12$, this will also prove that this component is generically smooth.

The condition $h^1(S, T_S) \le 12$ is equivalent to $h^2(S, T_S) = h^0(S, \Omega^1_S \otimes \omega_S) \le 2$. By Remark 4.2, it is therefore enough to prove that $h^0(\mathcal{F}) = 0$, where $\mathcal{F} := (\Omega^1_S \otimes \omega_S)/\omega_S$ or, equivalently, that there are no bicanonical curves of S containing the 0–dimensional scheme $\mathrm{Crit}(f)$.

By the results in Sect. 2.2.2, the Albanese fibration $f: S \to B$ factors as the composition of the conic bundle $\mathcal{C} \longrightarrow B$ and a finite double cover $\psi : S \longrightarrow \mathcal{C}$ branched on the node of $\mathcal{C}$ and on a smooth curve Δ not passing through the node.

Let us study the 0-dimensional scheme $\mathrm{Crit}(f)$. Since all the fibres of $\mathcal{C}$ are reduced, the critical points of f must be fixed by the involution of S. The isolated fixed point is the preimage of the node of $\mathcal{C}$, and it is critical for f. The other critical points of f are the points of S whose images in $\mathcal{C}$ are the ramification points for the map $\Delta \longrightarrow B$.

As before, we can choose $\mathcal{C}$ of equation $y_0^2 - y_1^2 + f_0 y_2^2 = 0$, and the curve Δ is defined as the complete intersection of $\mathcal{C}$ with a relative cubic $\mathcal{G} \in |\mathcal{O}_{\mathcal{C}}(3) \otimes \mathcal{O}_B(-4o - 2\tau)|$. Since $\mathcal{O}_B(2o - 2\tau) = \mathcal{O}_B$, we can choose $\mathcal{G}$ of equation

$$ay_0^3 + by_1^3 + \lambda y_2^3 = 0,$$

where $a, b \in \mathbb{C}$ and $\lambda \in H^0(\mathbb{P}(V_2), \pi_2^* \mathcal{O}_B(3\tau))$, see (23). The node P of $\mathcal{C}$ is the point with homogeneous coordinates $[0:0:1]$ lying on the fibre over τ, and $\mathrm{Crit}(\Delta \longrightarrow B)$ is defined by

$$\operatorname{rank}\begin{pmatrix} y_0 & -y_1 & f_0^2 y_2 \\ a y_0^2 & b y_1^2 & \lambda y_2^2 \end{pmatrix} \le 1.$$

This is obviously equivalent to set equal 0 all the minors of order 2. So we must solve the system of equations

$$\begin{cases} b y_0 y_1^2 + a y_0^2 y_1 = 0 \\ \lambda y_0 y_2^2 - a f_0^2 y_0^2 y_2 = 0 \\ \lambda y_1 y_2^2 + b y_1^2 y_2 = 0, \end{cases} \quad \text{that is} \quad \begin{cases} y_0 y_1 (b y_1 + a y_0) = 0 \\ y_0 y_2 (\lambda y_2 - a f_0^2 y_0) = 0 \\ y_1 y_2 (\lambda y_2 + b f_0^2 y_1) = 0. \end{cases}$$

This yields

$$\begin{aligned} &\{y_0 = y_1 = 0\} \cup \{y_0 = y_2 = 0\} \cup \{y_0 = \lambda y_2 + b f_0^2 y_1 = 0\} \\ &\quad \cup \{y_1 = y_2 = 0\} \cup \{y_1 = \lambda y_2 - a f_0^2 y_0 = 0\} \\ &\quad \cup \{y_2 = b y_1 + a y_0 = 0\} \cup \{\lambda y_2 + b f_0^2 y_1 = \lambda y_2 - a f_0^2 y_0 = b y_1 + a y_0 = 0\}. \end{aligned}$$

Let us compute, in each case, the solutions in $\mathcal{C}$:

$\{y_0 = y_1 = 0\}$ In this case, because $f_0(\tau) = 0$, the unique solution in $\mathcal{C}$ is the point P.

$\{y_0 = y_2 = 0\}$ By looking at the equation of $\mathcal{C}$ we have also that $y_1 = 0$, and this is impossible. So in this case there are no solutions.

$\{y_0 = \lambda y_2 + b f_0^2 y_1 = 0\}$ We must solve

$$\begin{cases} y_0 = \lambda y_2 + b f_0^2 y_1 = 0 \\ y_0^2 - y_1^2 + f_0^2 y_2^2 = 0, \end{cases} \quad \text{that is} \quad \begin{cases} y_0 = 0 \\ \lambda y_2 + b f_0^2 y_1 = 0 \\ (-y_1 + f_0 y_2)(y_1 + f_0 y_2) = 0, \end{cases}$$

which gives

$$\begin{cases} y_0 = 0 \\ y_1 = f_0 y_2 \\ y_2(\lambda + b f_0^3) = 0 \end{cases} \cup \begin{cases} y_0 = 0 \\ y_1 = -f_0 y_2 \\ y_2(\lambda - b f_0^3) = 0. \end{cases}$$

Since $y_2 \neq 0$ the solutions are the three points $[0 : f_0(\rho_i) : 1]$ lying on the fibres over ρ_i, where $\rho_1 + \rho_2 + \rho_3 = \operatorname{div}(\lambda + b f_0^3)$, and the three points $[0 : -f_0(\rho_i') : 1]$ lying on the fibres over ρ_i', where $\rho_1' + \rho_2' + \rho_3' = \operatorname{div}(\lambda - b f_0^3)$.

$\{y_1 = y_2 = 0\}$ The equation of $\mathcal{C}$ also gives $y_0 = 0$, which is impossible; so in this case there are no solutions.

$\{y_1 = \lambda y_2 - a f_0^2 y_0 = 0\}$ The computations are the same as in the case $\{y_0 = \lambda y_2 + b f_0^2 y_1 = 0\}$. The solutions are the three points $[-\sqrt{-1} f_0(\varepsilon_i) : 0 : 1]$ lying on the fibres over ε_i, where $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \operatorname{div}(\lambda + a\sqrt{-1} f_0^3)$, and the three points $[\sqrt{-1} f_0(\varepsilon_i) : 0 : 1]$ lying on the fibres over ε_i', where $\varepsilon_1' + \varepsilon_2' + \varepsilon_3' = \operatorname{div}(\lambda - a\sqrt{-1} f_0^3)$.

$\{y_2 = b y_1 + a y_0 = 0\}$ From the equation of $\mathcal{C}$ it follows that for a generic choice of a and b we must have $y_0 = y_1 = y_2 = 0$, which is impossible. So in this case there are no solutions.

$\{\lambda y_2 + b f_0^2 y_1 = \lambda y_2 - a f_0^2 y_0 = b y_1 + a y_0 = 0\}$ In this case we find six points, three on the curve

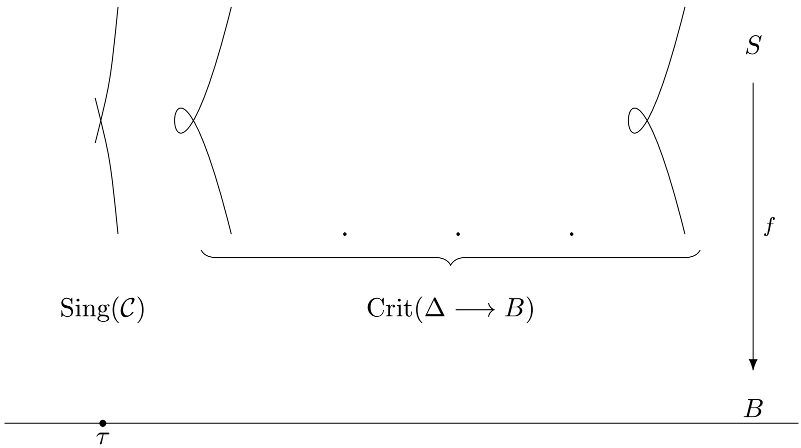
$$\begin{cases} b y_1 + a y_0 = 0 \\ c y_0 + b f_0 y_2 = 0 \end{cases}$$

and three on the curve

$$\begin{cases} by_1 + ay_0 = 0 \\ cy_0 - bf_0 y_2 = 0, \end{cases}$$

where $-c^2 = b^2 - a^2$. In general, a, b and c are nonzero and, in such a case, the solutions are the three points $[-bf_0(\sigma_i) : af_0(\sigma_i) : c]$ lying on the fibres over σ_i, where $\sigma_1 + \sigma_2 + \sigma_3 = \mathrm{div}(c\lambda + abf_0^3)$ and the three points $[bf_0(\sigma_i') : -af_0(\sigma_i') : c]$ lying on the fibres over σ_i', where $\sigma_1' + \sigma_2' + \sigma_3' = \mathrm{div}(c\lambda - abf_0^3)$.

Summing up, for a general $S \in \mathcal{M}_{\mathrm{V},2}$ the 0-dimensional scheme $\mathrm{Crit}(f)$ consists precisely of 19 distinct points. One is the preimage $Q := \psi^{-1}(P)$ of P in S, and the others correspond to the singularities of eighteen 2-connected nodal curves, as in the following picture:



Notice that this agrees with the Zeuthen–Segre formula

$$\begin{aligned} 19 = \chi_{\mathrm{top}}(S) &= \chi_{\mathrm{top}}(B)\chi_{\mathrm{top}}(F) + \sum \chi_{\mathrm{top}}(F_p) - \chi_{\mathrm{top}}(F) \\ &= \sum \chi_{\mathrm{top}}(F_p) - \chi_{\mathrm{top}}(F), \end{aligned}$$

where the sum runs over the singular fibres of f. Thus for a general $S \in \mathcal{M}_{V,2}$, the Albanese map has exactly 19 singular fibres.

Since the linear system $|2K_S|$ is the pullback via the relative bicanonical map of the linear system $|\mathcal{O}_{\mathbb{P}(V_2)}(1)|$, we must now compute the dimension of the vector space of elements in $H^0(\mathcal{O}_{\mathbb{P}(V_2)}(1))$ which contain $\mathrm{Crit}(f)$.

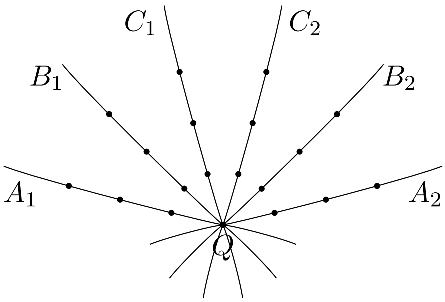
Let us consider the six curves

$$A_1 \colon \begin{cases} y_0 = 0 \\ y_1 - f_0 y_2 = 0, \end{cases} \qquad A_2 \colon \begin{cases} y_0 = 0 \\ y_1 + f_0 y_2 = 0, \end{cases}$$

$$B_1 \colon \begin{cases} y_1 = 0 \\ y_0 - \sqrt{-1} f_0 y_2 = 0, \end{cases} \qquad B_2 \colon \begin{cases} y_0 = 0 \\ y_0 + \sqrt{-1} f_0 y_2 = 0, \end{cases}$$

$$C_1 \colon \begin{cases} by_1 + ay_0 = 0 \\ cy_0 + bf_0 y_2 = 0, \end{cases} \qquad C_2 \colon \begin{cases} by_1 + ay_0 = 0 \\ cy_0 - bf_0 y_2 = 0. \end{cases}$$

Each curve contains Q and three other points of $\mathrm{Crit}(f)$ as in the following picture:



The Néron–Severi group $\mathrm{NS}(\mathbb{P}(V_2))$ is generated by H and Ψ, where H is the class of $\mathcal{O}_{\mathbb{P}(V_2)}(1)$ and Ψ is the class of a fibre.

Let Y be an element of $|\mathcal{O}_{\mathbb{P}(V_2)}(1)|$ containing $\mathrm{Crit}(f)$. Thus Y contains 4 points in each curve A_j, B_j, C_j, $j = 1, 2$. Since the numerical class of these curves is $(H - 2\Psi)^2$, we have

$$H(H - 2\Psi)^2 = H(H^2 - 2H\Psi) = H^3 - 4H\Psi = 7 - 4 = 3$$

and so, by Bézout theorem, Y contains all the curves A_j, B_j, C_j. Let us write the equation of Y as $\alpha y_0 + \beta y_1 + \gamma y_2 = 0$, where $\alpha, \beta \in H^0(\pi_2^* \mathcal{O}_B(2o))$ and $\gamma \in H^0(\pi_2^* \mathcal{O}_B(2o + \tau))$.

By imposing that Y contains A_1, we find

$$\beta f_0 y_2 + \gamma y_2 \equiv 0,$$

which implies $\gamma = -\beta f_0$. By imposing that Y contains A_2, we find

$$-\beta f_0 y_2 + \gamma y_2 \equiv 0,$$

which implies $\gamma = \beta f_0$. It follows $\gamma = \beta = 0$, hence Y has equation $\alpha y_0 = 0$. Similarly, by imposing that Y contains both B_1 and B_2, we obtain that Y is of the form $\beta y_1 = 0$. Thus $Y \equiv 0$, i.e.

$$\mathrm{Ker}[H^0(\omega_S^{\otimes 2}) \longrightarrow H^0(\mathcal{O}_{\mathrm{Crit}(f)}(\omega_S^{\otimes 2}))] = 0,$$

which implies $h^1(T_S) = 12$. This shows that $\mathcal{M}_{\mathrm{V},2}$ is a generically smooth, irreducible component of $\mathcal{M}$ of dimension 12. □

Finally, we consider the strata belonging to $\mathcal{M}''$. The surfaces in these strata satisfy $V_1 = F_2(\eta)$, where η is a 2–torsion point, hence V_1 will not play any role in the computation of parameters.

Proposition 4.13 *The stratum* $\mathcal{M}_{\mathrm{VI}}$ *has dimension at most* 12.

Proof Set $W := E_\tau(3, 1)$; then we have a short exact sequence

$$0 \longrightarrow W(2o - 2\tau) \longrightarrow \mathrm{S}^3 W(2o - 2\tau) \longrightarrow \widetilde{A}_6 \longrightarrow 0.$$

By [5, Sect. 1] we obtain

$$h^0(W(2o - 2\tau)) = 1, \qquad h^1(W(2o - 2\tau)) = 0, \qquad h^0(\mathrm{S}^3 W(2o - 2\tau)) = 10,$$

hence $h^0(\widetilde{A}_6) = 9$. We have 1 parameter for B, 2 parameters for ξ, 1 parameter for τ and 8 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$. Therefore $\mathcal{M}_{\mathrm{VI}}$ has dimension at most 12. □

Proposition 4.14 *The stratum* $\mathcal{M}_{\mathrm{VIIa}}$ *has dimension at most* 11.

Proof In this case $V_2(-2o) = F_2 \oplus \mathcal{O}_B(\tau)$, and ξ belongs to a family which is at most 1-dimensional, see Proposition 3.6. The vector bundle $\widetilde{A}_6$ fits into a short exact sequence

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \widetilde{A}_6 \longrightarrow 0,$$

where

$$G_1 = \big(F_2 \oplus \mathcal{O}_B(\tau)\big)(2o - 2\tau), \qquad G_2 = \big(F_4 \oplus F_3(\tau) \oplus F_2(2\tau) \oplus \mathcal{O}_B(3\tau)\big)(2o - 2\tau).$$

We distinguish two cases.

(i) $\mathcal{O}_B(2o - 2\tau) \neq \mathcal{O}_B$. We obtain

$$h^0(G_1) = 1, \qquad h^1(G_1) = 0, \qquad h^0(G_2) = 10,$$

therefore $h^0(\widetilde{A}_6) = 9$. We have 1 parameter for B, at most one parameter for ξ, one parameter for τ and 8 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$.

(ii) $\mathcal{O}_B(2o - 2\tau) = \mathcal{O}_B$. We obtain

$$h^0(G_1) = 2, \qquad h^1(G_1) = 1, \qquad h^0(G_2) = 11,$$

hence $h^0(\widetilde{A}_6) \leq 10$, see (5). We have 1 parameter for B, at most one parameter for ξ, no parameters for τ and at most 9 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$.

It follows that $\mathcal{M}_{\mathrm{VIIa}}$ has dimension at most 11. □

Proposition 4.15 *The stratum* $\mathcal{M}_{\mathrm{VIIb}}$ *has dimension at most* 11.

Proof In this case ξ belongs to a family which is at most 1-dimensional. Set $W = E_\tau(2, 1)$; then $V_2(-2o) = W \oplus \mathcal{O}_B$ and tensoring the exact sequence (3) with $\mathcal{O}_B(-4o - 2\tau)$ we obtain

$$\begin{aligned} 0 \longrightarrow (W \oplus \mathcal{O}_B)(2o - 2\tau) &\xrightarrow{i_3} \big[(\mathrm{S}^3 W \oplus \mathrm{S}^2 W) \oplus (W \oplus \mathcal{O}_B)\big](2o - 2\tau) \\ &\longrightarrow \widetilde{A}_6 \longrightarrow 0. \end{aligned} \tag{26}$$

Arguing as in [6, Lemma 6.14], we see that the second component of the map i_3 is the identity, hence the exact sequence (26) splits, giving

$$\widetilde{A}_6 = (\mathrm{S}^3 W \oplus \mathrm{S}^2 W)(2o - 2\tau).$$

By Proposition 2.4 this in turn implies

$$\widetilde{A}_6 = \left(W \oplus W \oplus \bigoplus_{i=1}^{3} L_i\right)(2o - \tau),$$

hence $h^0(\widetilde{A}_6) = 9$. We have 1 parameter for B, at most 1 parameter for ξ, 1 parameter for τ and 8 parameters from $\mathbb{P}H^0(\widetilde{A}_6)$. Therefore $\mathcal{M}_{\mathrm{VIIb}}$ has dimension at most 11. □

Summing up, we have the following

Corollary 4.16 *The moduli space $\mathcal{M}$ of minimal surfaces of general type with $p_g = 2$, $q = 1$ and $K^2 = 5$ is unirational and contains at least* 2 *irreducible components. Moreover, the dimension of each irreducible component is either* 12 *or* 13, *and there is at most one component of dimension* 13.

Proof Notice first that $\mathcal{M}_{\mathrm{V,gen}}$ is not contained in the closure of $\mathcal{M}_{\mathrm{V},2}$, since in the former case τ is a general point, whereas in the latter τ is a 2-torsion point. So $\mathcal{M}$ contains at least two irreducible components, namely $\mathcal{M}_{\mathrm{V},2}$ and the component containing $\mathcal{M}_{\mathrm{V,gen}}$. Moreover there is at most one component of dimension 13, namely $\mathcal{M}_{\mathrm{I}}$. □

It would be desirable to exactly describe all irreducible components of $\mathcal{M}$ and to understand how their closures intersect, but we will not try to develop this point here.

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