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# A new family of surfaces with $p_g = q = 2$ and $K^2 = 6$ whose Albanese map has degree 4

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Abstract

We construct a new family of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 6$, whose Albanese map is a quadruple cover of an abelian surface with polarization of type $(1, 3)$. We also show that this family provides an irreducible component of the moduli space of surfaces with $p_g = q = 2$ and $K^2 = 6$. Finally, we prove that such a component is generically smooth of dimension 4 and that it contains the two-dimensional family of product–quotient examples previously constructed by the first author. The main tools we use are the Fourier–Mukai transform and the Schrödinger representation of the finite Heisenberg group $\mathscr{H}_3$.

## *Introduction*

In recent years, both the geographical problem and the fine classification for *irregular* algebraic surfaces (that is, surfaces with irregularity $q > 0$) have attracted the attention of several authors; in particular, minimal surfaces of general type with $\chi(\mathscr{O}) = 1$, that is, $p_g = q$, were thoroughly investigated.

In this case, some well-known results imply $p_g \leqslant 4$. While surfaces with $p_g = q = 4$ and $p_g = q = 3$ have been completely described (see [**10**, **13**, **14**, **25**]), the classification of those with $p_g = q = 2$ is still missing (see [**22**, **23**] for a recent account on this topic). As the title suggests, in the present paper we consider some new surfaces $\hat{S}$ with $p_g = q = 2$ and $K^2 = 6$ whose Albanese map $\hat{\alpha}\colon \hat{S} \to \hat{A} := \mathrm{Alb}(\hat{S})$ is a quadruple cover of an abelian surface $\hat{A}$.

Our construction presents some analogies with the one presented in [**11**, **22**] for the case $p_g = q = 2$ and $K^2 = 5$. Indeed, in both situations the Tschirnhausen bundle $\mathscr{E}^\vee$ associated with the Albanese cover is of the form $\Phi^{\mathscr{P}}(\mathscr{L}^{-1})$, where $\mathscr{L}$ is a polarization on $A$ (the dual abelian variety of $\hat{A}$) and $\Phi^{\mathscr{P}}$ denotes the Fourier–Mukai transform. More precisely, in the case studied in [**11**] the surfaces are triple covers, $\mathscr{E}^\vee$ has rank 2 and $\mathscr{L}$ is a polarization of type $(1, 2)$; in our setting, instead, the cover has degree 4, the bundle $\mathscr{E}^\vee$ has rank 3 and $\mathscr{L}$ is a polarization of type $(1, 3)$.

The results of the paper can be summarized as follows; see Theorem 2.1.

Main Theorem. *There exists a four-dimensional family $\mathcal{M}_\Phi$ of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 6$ such that, for the general element $\hat{S} \in \mathcal{M}_\Phi$, the canonical class $K_{\hat{S}}$ is ample and the Albanese map $\hat{\alpha}\colon \hat{S} \to \hat{A}$ is a finite cover of degree* 4.

*The Tschirnhausen bundle $\mathscr{E}^\vee$ associated with $\hat{\alpha}$ is isomorphic to $\Phi^{\mathscr{P}}(\mathscr{L}^{-1})$, where $\mathscr{L}$ is a polarization of type $(1, 3)$ on $A$.*

*The family $\mathcal{M}_\Phi$ provides an irreducible component of the moduli space $\mathcal{M}^{\mathrm{can}}_{2,2,6}$ of canonical models of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 6$. Such a component*

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*is generically smooth and contains the two-dimensional family of product–quotient surfaces constructed by the first author in* [**21**].

The Main Theorem is obtained by extending the construction given in [**11**] to the much more complicated case of quadruple covers. More precisely, in order to build a quadruple cover $\hat{\alpha}\colon \hat{S} \to \hat{A}$ with Tschirnhausen bundle $\mathscr{E}^\vee$, we first build a quadruple cover $\alpha\colon S \to A$ with Tschirnhausen bundle $\phi^*_{\mathscr{L}^{-1}}\mathscr{E}^\vee$ (here $\phi_{\mathscr{L}}\colon A \to \hat{A}$ denotes the group homomorphism sending $x \in A$ to $t_x^*\mathscr{L}^{-1} \otimes \mathscr{L} \in \hat{A}$) and then, by using the Schrödinger representation of the finite Heisenberg group $\mathscr{H}_3$ on $H^0(A, \mathscr{L})$, we identify those covers of this type that descend to a quadruple cover $\hat{\alpha}\colon \hat{S} \to \hat{A}$. For the general surface $\hat{S}$, the branch divisor $\hat{B} \subset \hat{A}$ of $\alpha\colon \hat{S} \to \hat{A}$ is a curve in the linear system $|\hat{\mathscr{L}}^{\otimes 2}|$, where $\hat{\mathscr{L}}$ is a polarization of type $(1, 3)$ on $\hat{A}$, with six ordinary cusps and no other singularities; such a curve is $\mathscr{H}_3$-equivariant and can be associated with the dual of a member of the Hesse pencil of plane cubics in $\mathbb{P}^2$.

Let us explain now the way in which this paper is organized.

In Section 1, we set up notation and terminology and we present some preliminaries. In particular, we recall the theory of quadruple covers developed by Casnati–Ekedahl [**6**] and Hahn–Miranda [**15**] and we briefly describe the geometry of the Hesse pencil.

Sections 2–4 are devoted to the proof of the Main Theorem.

In Section 2, we present our construction, we prove that a general surface $\hat{S}$ in our family is smooth and we compute its invariants (Propositions 2.6 and 2.7).

In Section 3, we examine the subset of the moduli space corresponding to our surfaces, showing that it is generically smooth, of dimension 4 and that its closure $\mathcal{M}_\Phi$ provides an irreducible component of $\mathcal{M}^{\mathrm{can}}_{2,2,6}$ (Proposition 3.4). We also observe that the general surface in $\mathcal{M}_\Phi$ admits no pencil over a curve of strictly positive genus (Proposition 3.5).

In Section 4, we prove that the moduli space $\mathcal{M}^{\mathrm{can}}_{2,2,6}$ contains a three-dimensional singular locus (Corollary 5.3). Moreover, we show that the irreducible component $\mathcal{M}_\Phi$ contains the two-dimensional family of product–quotient examples constructed by the first author in [**21**] (Proposition 5.4).

Finally, in Section 6 we present some open problems.

The paper also contains two appendices. In Appendix A, we show the following technical result needed in the proof of the Main Theorem: for a general choice of the pair $(A, \mathscr{L})$, the three distinguished divisors coming from the Schrödinger representation of the Heisenberg group $\mathscr{H}_3$ on $H^0(A, \mathscr{L})$ are smooth and intersect transversally. Appendix B contains the computer algebra script we used to compute the equation of the branch curve $B$ of $\alpha\colon S \to A$; it is written in the computer algebra system `MAGMA`; see http://magma.maths.usyd.edu.au/magma/.

*Notation and conventions.* We work over the field $\mathbb{C}$ of complex numbers.

If $A$ is an abelian variety, then we call $\hat{A} := \mathrm{Pic}^0(A)$ its dual abelian variety.

If $\mathscr{L}$ is a line bundle on $A$, then we denote by $\phi_{\mathscr{L}}$ the morphism $\phi_{\mathscr{L}}\colon A \to \hat{A}$ given by $x \mapsto t_x^*\mathscr{L} \otimes \mathscr{L}^{-1}$. If $\mathscr{L}$ is non-degenerate, then $\phi_{\mathscr{L}}$ is an isogeny, whose kernel is denoted by $K(\mathscr{L})$. In this case, the *index* of $\mathscr{L}$ is the unique integer $i(\mathscr{L})$ such that $H^j(A, \mathscr{L}) = 0$ unless $j = i(\mathscr{L})$.

Throughout the paper, we use italic letters for line bundles and capital letters for the corresponding Cartier divisors, so we write for instance $\mathscr{L} = \mathscr{O}_S(L)$. The corresponding complete linear system is denoted either by $|\mathscr{L}|$ or by $|L|$.

If $L$ is an ample divisor on $A$, then it defines a positive line bundle $\mathscr{L} = \mathscr{O}_A(L)$, whose first Chern class is a polarization on $A$. By abuse of notation, we consider both the line bundle $\mathscr{L}$ and the divisor $L$ as polarizations.

By *surface*, we mean a projective, non-singular surface $S$, and for such a surface $\omega_S = \mathscr{O}_S(K_S)$ denotes the canonical class, $p_g(S) = h^0(S, \omega_S)$ is the *geometric genus*, $q(S) = h^1(S, \omega_S)$ is the *irregularity* and $\chi(\mathscr{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler–Poincaré characteristic*.

## 1. *Preliminaries*

### 1.1. *Quadruple cover of algebraic varieties*

The two papers [6, 15] deal with the quadruple covers of algebraic varieties; the former only considers the Gorenstein case, whereas the latter develops the theory in full generality. The main results are the following.

THEOREM 1.1 [6, Theorems 1.6 and 4.4]. *Let $Y$ be a smooth algebraic variety. Any quadruple Gorenstein cover $f\colon X \to Y$ is determined by a locally free $\mathscr{O}_Y$-module $\mathscr{E}$ of rank 3, a locally free $\mathscr{O}_Y$-module $\mathscr{F}$ of rank 2 with $\bigwedge^2 \mathscr{F} \cong \bigwedge^3 \mathscr{E}^\vee$ and a general section $\eta \in H^0(Y, S^2\mathscr{E}^\vee \otimes \mathscr{F}^\vee)$.*

THEOREM 1.2 [15, Theorem 1.2]. *Let $Y$ be a smooth algebraic variety. Any quadruple cover $f\colon X \to Y$ is determined by a locally free $\mathscr{O}_Y$-module $\mathscr{E}$ of rank 3 and a totally decomposable section $\eta \in H^0(Y, \bigwedge^2 S^2\mathscr{E}^\vee \otimes \bigwedge^3 \mathscr{E})$.*

In order to make the notation consistent, in Theorem 1.1 we called $\mathscr{E}^\vee$ the sheaf which is called $\mathscr{E}$ in [6]. In Theorem 1.2, *totally decomposable* means that, for all $y \in Y$, the image of

$$\eta|_y\colon \left(\bigwedge^3 \mathscr{E}^\vee\right)_y \longrightarrow \left(\bigwedge^2 S^2\mathscr{E}^\vee\right)_y$$

is totally decomposable in $(\bigwedge^2 S^2\mathscr{E}^\vee)_y$, that is, of the form $\xi_1 \wedge \xi_2$ with $\xi_i \in (S^2\mathscr{E}^\vee)_y$.

The vector bundle $\mathscr{E}^\vee$ is called the *Tschirnhausen bundle* of the cover. We have $f_*\mathscr{O}_X = \mathscr{O}_Y \oplus \mathscr{E}$, so

$$h^i(X, \mathscr{O}_X) = h^i(Y, \mathscr{O}_Y) + h^i(Y, \mathscr{E}). \tag{1}$$

By Casnati–Ekedahl [6, Proposition 5.1], there is an exact sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varphi} S^2\mathscr{E}^\vee \longrightarrow f_*\omega^2_{X|Y} \longrightarrow 0, \tag{2}$$

and the associated Eagon–Northcott complex tensored with $\bigwedge^2 \mathscr{F}^\vee$ yields

$$0 \longrightarrow S^2\mathscr{F} \otimes \bigwedge^2 \mathscr{F}^\vee \longrightarrow S^2\mathscr{E}^\vee \otimes \mathscr{F}^\vee \longrightarrow \bigwedge^2 S^2\mathscr{E}^\vee \otimes \bigwedge^3 \mathscr{E}. \tag{3}$$

The induced map in cohomology

$$H^0(Y, S^2\mathscr{E}^\vee \otimes \mathscr{F}^\vee) \longrightarrow H^0\left(Y, \bigwedge^2 S^2\mathscr{E}^\vee \otimes \bigwedge^3 \mathscr{E}\right)$$

provides the bridge between Theorems 1.1 and 1.2. In fact, a straightforward computation shows that it sends the element $\varphi \in \operatorname{Hom}(\mathscr{F}, S^2\mathscr{E}^\vee) \cong H^0(Y, S^2\mathscr{E}^\vee \otimes \mathscr{F}^\vee)$ to the totally decomposable element $2\varphi \wedge \varphi \in \operatorname{Hom}(\bigwedge^2 \mathscr{F}, \bigwedge^2 S^2\mathscr{E}^\vee) \cong H^0(Y, \bigwedge^2 S^2\mathscr{E}^\vee \otimes \bigwedge^3 \mathscr{E})$.

PROPOSITION 1.3 [6, Proposition 5.3]. *Let $X$ and $Y$ be smooth, connected, projective surfaces, $f\colon X \to Y$ be a cover of degree 4 and $R \subset X$ be the ramification divisor of $f$. Then we have the following formulae:*

(i) $\chi(\mathscr{O}_X) = 4\chi(\mathscr{O}_Y) + \frac{1}{2}c_1(\mathscr{E}^\vee)K_Y + \frac{1}{2}c_1^2(\mathscr{E}^\vee) - c_2(\mathscr{E}^\vee)$;
(ii) $K_X^2 = 4K_Y^2 + 4c_1(\mathscr{E}^\vee)K_Y + 2c_1^2(\mathscr{E}^\vee) - 4c_2(\mathscr{E}^\vee) + c_2(\mathscr{F})$;
(iii) $p_a(R) = 1 + c_1(\mathscr{E}^\vee)K_Y + 2c_1^2(\mathscr{E}^\vee) - 4c_2(\mathscr{E}^\vee) + c_2(\mathscr{F})$.

### 1.2. *Fourier–Mukai transforms of W(IT)-sheaves*

Let $A$ be an abelian variety of dimension $g$ and $\hat{A} := \mathrm{Pic}^0(A)$ be its dual abelian variety. We have the following definitions.

(i) A coherent sheaf $\mathscr{F}$ on $A$ is an *IT-sheaf of index $i$* (or, equivalently, *$\mathscr{F}$ satisfies IT of index $i$*) if

$$H^j(A, \mathscr{F} \otimes \mathscr{Q}) = 0 \quad \text{for all } \mathscr{Q} \in \mathrm{Pic}^0(A) \text{ and } j \neq i.$$

(ii) A coherent sheaf $\mathscr{F}$ on $A$ is a *WIT-sheaf of index $i$* (or, equivalently, *$\mathscr{F}$ satisfies WIT of index $i$* ) if

$$R^j\pi_{\hat{A}*}(\mathscr{P} \otimes \pi_A^*\mathscr{F}) = 0 \quad \text{for all } j \neq i,$$

where $\mathscr{P}$ is the normalized Poincaré bundle on $A \times \hat{A}$.

If $\mathscr{F}$ is a WIT-sheaf of index $i$, then the coherent sheaf

$$\Phi^{\mathscr{P}}(\mathscr{F}) := R^i\pi_{\hat{A}*}(\mathscr{P} \otimes \pi_A^*\mathscr{F})$$

is called the *Fourier–Mukai transform* of $\mathscr{F}$.

For simplicity of notation, given any WIT-sheaf $\mathscr{G}$ of index $i$ on $\hat{A}$ we use the same symbol $\Phi^{\mathscr{P}}$ for its Fourier–Mukai transform

$$\Phi^{\mathscr{P}}(\mathscr{G}) := R^i\pi_{A*}(\mathscr{P} \otimes \pi_{\hat{A}}^*\mathscr{G}).$$

By the Base Change Theorem (see [**18**, Chapter II]), it follows that $\mathscr{F}$ satisfies IT of index $i$ if and only if it satisfies WIT of index $i$ and $\Phi^{\mathscr{P}}(\mathscr{F})$ is locally free. In particular, any non-degenerate line bundle $\mathscr{L}$ of index $i$ on $A$ is an IT-sheaf of index $i$ and its Fourier–Mukai transform $\Phi^{\mathscr{P}}(\mathscr{L})$ is a vector bundle of rank $h^i(A, \mathscr{L})$ on $\hat{A}$.

PROPOSITION 1.4 [**4**, Theorem 14.2.2; **17**]. *Let $\mathscr{F}$ be a WIT-sheaf of index $i$ on $A$. Then $\Phi^{\mathscr{P}}(\mathscr{F})$ is a WIT-sheaf of index $g-i$ on $\hat{A}$ and*

$$\Phi^{\mathscr{P}} \circ \Phi^{\mathscr{P}}(\mathscr{F}) = (-1)_A^*\mathscr{F}.$$

REMARK 1.5. In general, the Fourier–Mukai transform induces an equivalence of categories between the two derived categories $D(A)$ and $D(\hat{A})$, such that for all $\mathscr{F} \in D(A)$ and $\mathscr{G} \in D(\hat{A})$ one has

$$\Phi^{\mathscr{P}} \circ \Phi^{\mathscr{P}}(\mathscr{F}) = (-1)_A^*(\mathscr{F})[-g] \quad \text{and} \quad \Phi^{\mathscr{P}} \circ \Phi^{\mathscr{P}}(\mathscr{G}) = (-1)_{\hat{A}}^*(\mathscr{G})[-g],$$

where $[-g]$ means ‘shift the complex $g$ places to the right’. When $\mathscr{F}$ is a WIT-sheaf, then the complex $\Phi^{\mathscr{P}}(\mathscr{F})$ can be identified with a vector bundle, since it is different from zero at most at one place.

COROLLARY 1.6. *Let $\mathscr{F}$ be a WIT-sheaf on $A$. Then*

$$\Phi^{\mathscr{P}}((-1)_A^*\mathscr{F}) = (-1)_{\hat{A}}^*\Phi^{\mathscr{P}}(\mathscr{F}).$$

*Proof.* Set $\mathscr{F}' := \Phi^{\mathscr{P}}((-1)_A^*\mathscr{F})$; then by Proposition 1.4,

$$\mathscr{F} = \Phi^{\mathscr{P}} \circ \Phi^{\mathscr{P}}((-1)_A^*\mathscr{F}) = \Phi^{\mathscr{P}}(\mathscr{F}'),$$

hence

$$\Phi^{\mathscr{P}}(\mathscr{F}) = \Phi^{\mathscr{P}} \circ \Phi^{\mathscr{P}}(\mathscr{F}') = (-1)_{\hat{A}}^*\mathscr{F}' = (-1)_{\hat{A}}^*\Phi^{\mathscr{P}}((-1)_A^*\mathscr{F}). \qquad \square$$

### 1.3. *The Hesse pencil and the family of its dual curves*

In the sequel, we will use some classical facts about the Hesse pencil of cubic curves, that are summarized here for the reader's convenience. We follow the treatment given in [1].

The *Hesse pencil* is the one-dimensional linear system of plane cubic curves defined by

$$E_{t_0,t_1}\colon t_0(x^3+y^3+z^3)+t_1xyz=0, \quad [t_0:t_1]\in\mathbb{P}^1.$$

Its nine base points are the inflection points of any smooth curve in the pencil. There are four singular members in the Hesse pencil and each one is the union of three lines:

$$\begin{aligned}
&E_{0,1}\colon xyz=0,\\
&E_{1,-3}\colon (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)=0,\\
&E_{1,-3\omega}\colon (x+\omega y+z)(x+\omega^2 y+\omega^2 z)(x+y+\omega z)=0,\\
&E_{1,-3\omega^2}\colon (x+\omega^2 y+z)(x+\omega y+\omega z)(x+y+\omega^2 z)=0.
\end{aligned}$$

We call the singular members the *triangles*.

The dual curve $\mathbf{B}_{m_0,\,3m_1}$ of a smooth member $E_{m_0,3m_1}$ of the Hesse pencil is a plane curve of degree 6 with nine cusps, whose equation in the dual plane $(\mathbb{P}^2)^\vee$ is

$$\begin{aligned}
&m_0^4(X^6+Y^6+Z^6)-m_0(2m_0^3+32m_1^3)(X^3Y^3+X^3Z^3+Y^3Z^3)\\
&\quad -24m_0^2m_1^2XYZ(X^3+Y^3+Z^3)-(24m_0^3m_1+48m_1^4)X^2Y^2Z^2=0.
\end{aligned} \tag{4}$$

Note that the dual of a triangle becomes a triangle taken with multiplicity 2. For any pair $(m_0,m_1)$, there is a unique cubic $\mathbf{C}_{m_0,3m_1}$ passing through the nine cusps of $\mathbf{B}_{m_0,3m_1}$. The general element of the pencil generated by $\mathbf{B}_{m_0,3m_1}$ and $2\mathbf{C}_{m_0,3m_1}$ is an irreducible curve of degree 6 with nine nodes at the nine cusps of $\mathbf{B}_{m_0,3m_1}$. Such a pencil is called the *Halphen pencil* associated with $\mathbf{B}_{m_0,3m_1}$; see [16].

## 2. *The construction*

The aim of this section and the next one is to prove the main result of the paper, namely the following theorem.

THEOREM 2.1. *There exists a four-dimensional family $\mathcal{M}_\Phi$ of minimal surfaces of general type with $p_g=q=2$ and $K^2=6$ such that, for the general element $\hat{S}\in\mathcal{M}_\Phi$, the canonical class $K_{\hat{S}}$ is ample and the Albanese map $\hat{\alpha}\colon\hat{S}\to\hat{A}$ is a finite cover of degree 4.*

*The Tschirnhausen bundle $\mathscr{E}^\vee$ associated with $\hat{\alpha}$ is isomorphic to $\Phi^{\mathscr{P}}(\mathscr{L}^{-1})$, where $\mathscr{L}$ is a polarization of type $(1,3)$ on $A$.*

*The family $\mathcal{M}_\Phi$ provides an irreducible component of the moduli space $\mathcal{M}^{\mathrm{can}}_{2,2,6}$ of canonical models of minimal surfaces of general type with $p_g=q=2$ and $K^2=6$. Such a component is generically smooth.*

We first outline the main idea of our construction, which is inspired by the one used in [11] in the simpler case where the Albanese map has degree 3. In order to build a quadruple cover $\hat{\alpha}\colon\hat{S}\to\hat{A}$ with Tschirnhausen bundle $\mathscr{E}^\vee$, we first build a quadruple cover $\alpha\colon S\to A$ with Tschirnhausen bundle $\phi^*_{\mathscr{L}^{-1}}\mathscr{E}^\vee$ and then we identify those covers of this type that descend to a quadruple cover $\hat{\alpha}\colon\hat{S}\to\hat{A}$ (Propositions 2.3 and 2.4). Furthermore, we prove that for a general such a cover the surface $\hat{S}$ is smooth and we compute its invariants (Propositions 2.6 and 2.7). Finally, in Section 3 we examine the subset $\mathcal{M}_\Phi$ of the moduli space corresponding to our surfaces.

We start by considering a $(1,3)$-polarized abelian surface $(A,\mathscr{L})$. For all $\mathscr{Q}\in \mathrm{Pic}^0(A)$, we have

$$h^0(A,\mathscr{L}\otimes\mathscr{Q})=3, \quad h^1(A,\mathscr{L}\otimes\mathscr{Q})=0, \quad h^2(A,\mathscr{L}\otimes\mathscr{Q})=0,$$

so by Serre duality the line bundle $\mathscr{L}^{-1}$ satisfies IT of index 2. Its Fourier–Mukai transform $\mathscr{E}^\vee := \Phi^{\mathscr{P}}(\mathscr{L}^{-1})$ is a rank-3 vector bundle on $\hat{A}$ which satisfies WIT of index 0; see Proposition 1.4. The isogeny

$$\phi := \phi_{\mathscr{L}^{-1}}\colon A \longrightarrow \hat{A}, \quad \phi(x)=t_x^*\mathscr{L}^{-1}\otimes\mathscr{L}$$

has kernel $K(\mathscr{L}^{-1})=K(\mathscr{L})\cong(\mathbb{Z}/3\mathbb{Z})^2$ and by Mukai [**17**, Proposition 3.11] we have

$$\phi^*\mathscr{E}^\vee=\mathscr{L}\oplus\mathscr{L}\oplus\mathscr{L}. \tag{5}$$

Since $\Phi^{\mathscr{P}}(\mathscr{E}^\vee)=\Phi^{\mathscr{P}}\circ\Phi^{\mathscr{P}}(\mathscr{L}^{-1})=(-1)^*_A\mathscr{L}^{-1}$ is locally free, it follows that $\mathscr{E}^\vee$ is actually an IT-sheaf of index 0. By (5), we have

$$c_1(\phi^*\mathscr{E}^\vee)=3L, \quad c_2(\phi^*\mathscr{E}^\vee)=18, \tag{6}$$

hence

$$c_1(\mathscr{E}^\vee)=\hat{L}, \quad c_2(\mathscr{E}^\vee)=2, \tag{7}$$

where $\hat{L}$ is a polarization of type $(1,3)$ on $\hat{A}$. Therefore, the Hirzebruch–Riemann–Roch theorem yields $\chi(\hat{A},\mathscr{E}^\vee)=1$, which in turn implies

$$h^0(\hat{A},\mathscr{E}^\vee)=1, \quad h^1(\hat{A},\mathscr{E}^\vee)=0, \quad h^2(\hat{A},\mathscr{E}^\vee)=0. \tag{8}$$

Now we want to construct a quadruple cover $\hat{\alpha}\colon\hat{S}\to\hat{A}$ with Tschirnhausen bundle $\mathscr{E}^\vee$. By Hahn–Miranda [**15**], it suffices to find those totally decomposable elements in

$$H^0\left(A, \bigwedge^2 S^2(\phi^*\mathscr{E}^\vee)\otimes\bigwedge^3(\phi^*\mathscr{E})\right)\cong H^0(A,\mathscr{L})^{\oplus 15}$$

that are $K(\mathscr{L}^{-1})$-invariant and therefore descend to elements in $H^0(\hat{A},\bigwedge^2 S^2\mathscr{E}^\vee\otimes\bigwedge^3\mathscr{E})$. This will be done in Propositions 2.3 and 2.4.

Let us consider the Heisenberg group

$$\mathscr{H}_3 := \{(k,t,l)\mid k\in\mathbb{C}^*,\ t\in\mathbb{Z}/3\mathbb{Z},\ l\in\widehat{\mathbb{Z}/3\mathbb{Z}}\},$$

whose group law is

$$(k,t,l)\cdot(k',t',l')=(kk'l'(t),t+t',l+l').$$

By Birkenhake–Lange [**4**, Chapter 6], there exists a canonical representation, known as the Schrödinger representation, of $\mathscr{H}_3$ on $H^0(A,\mathscr{L})$, where the latter space is identified with the vector space $V:=\mathbb{C}(\mathbb{Z}/3\mathbb{Z})$ of all complex-valued functions on the finite group $\mathbb{Z}/3\mathbb{Z}$. Such an action is given by

$$(k,t,l)f(x)=kl(x)f(t+x).$$

Let $\{X,Y,Z\}$ be the basis of $H^0(A,\mathscr{L})$ corresponding to the characteristic functions of $0,1,2$ in $V$.

PROPOSITION 2.2. *For a general choice of the pair $(A,\mathscr{L})$, the three effective divisors defined by $X$, $Y$, $Z\in H^0(A,\mathscr{L})$ are smooth and intersect transversally.*

*Proof.* See Appendix A, in particular Proposition A.3. □

PROPOSITION 2.3. *The $K(\mathscr{L}^{-1})$-invariant subspace of $H^0(A, \bigwedge^2 S^2(\phi^*\mathscr{E}^\vee) \otimes \bigwedge^3(\phi^*\mathscr{E}))$ can be identified with*

$$\begin{aligned} \phi^* H^0\left(\hat{A}, \bigwedge^2 S^2\mathscr{E}^\vee \otimes \bigwedge^3 \mathscr{E}\right) = \{&(aZ, bY, cY, dX, -cZ, eX, -bX, \\ & - eZ, -dY, -dZ, eY, -aX, aY, cX, -bZ) \mid \\ & a, b, c, d, e \in \mathbb{C}\} \subset H^0(A, \mathscr{L})^{\oplus 15}. \end{aligned} \tag{9}$$

*Proof.* With respect to the basis $\{X, Y, Z\}$, the Schrödinger representation can be written as

$$(1,0,0) \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (1,0,1) \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (1,0,2) \longmapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$(1,1,0) \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad (1,1,1) \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix}, \quad (1,1,2) \longmapsto \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix},$$

$$(1,2,0) \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad (1,2,1) \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega^2 \\ \omega & 0 & 0 \end{pmatrix}, \quad (1,2,2) \longmapsto \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix},$$

where $\omega := e^{2\pi i/3}$. There is an induced representation of $\mathscr{H}_3$ on the 45-dimensional vector space $\bigwedge^2 S^2 V^\vee \otimes \bigwedge^3 V \otimes V$, and a long but straightforward computation shows that this gives in turn a representation of $K(\mathscr{L}^{-1}) \cong (\mathbb{Z}/3\mathbb{Z})^2$. Denoting by $\{\hat{X}, \hat{Y}, \hat{Z}\}$ the dual basis of $\{X, Y, Z\}$, one checks that the space of $K(\mathscr{L}^{-1})$-invariant vectors in $\bigwedge^2 S^2 V^\vee \otimes \bigwedge^3 V \otimes V$ has dimension 5 and is generated by

$$\begin{aligned} v_1 &= -(\hat{X}\hat{Z} \wedge \hat{Z}^2) \otimes (X \wedge Y \wedge Z) \otimes X + (\hat{Y}^2 \wedge \hat{Y}\hat{Z}) \otimes (X \wedge Y \wedge Z) \otimes Y \\ &\quad + (\hat{X}^2 \wedge \hat{X}\hat{Y}) \otimes (X \wedge Y \wedge Z) \otimes Z, \\ v_2 &= -(\hat{X}\hat{Y} \wedge \hat{Y}^2) \otimes (X \wedge Y \wedge Z) \otimes X + (\hat{X}^2 \wedge \hat{X}\hat{Z}) \otimes (X \wedge Y \wedge Z) \otimes Y \\ &\quad + (\hat{Z}^2 \wedge \hat{Y}\hat{Z}) \otimes (X \wedge Y \wedge Z) \otimes Z, \\ v_3 &= (\hat{Y}^2 \wedge \hat{Z}^2) \otimes (X \wedge Y \wedge Z) \otimes X + (\hat{X}^2 \wedge \hat{Y}^2) \otimes (X \wedge Y \wedge Z) \otimes Y \\ &\quad - (\hat{X}^2 \wedge \hat{Z}^2) \otimes (X \wedge Y \wedge Z) \otimes Z, \\ v_4 &= (\hat{X}^2 \wedge \hat{Y}\hat{Z}) \otimes (X \wedge Y \wedge Z) \otimes X - (\hat{X}\hat{Y} \wedge \hat{Z}^2) \otimes (X \wedge Y \wedge Z) \otimes Y \\ &\quad - (\hat{X}\hat{Z} \wedge \hat{Y}^2) \otimes (X \wedge Y \wedge Z) \otimes Z, \\ v_5 &= (\hat{X}\hat{Y} \wedge \hat{X}\hat{Z}) \otimes (X \wedge Y \wedge Z) \otimes X + (\hat{X}\hat{Z} \wedge \hat{Y}\hat{Z}) \otimes (X \wedge Y \wedge Z) \otimes Y \\ &\quad - (\hat{X}\hat{Y} \wedge \hat{Y}\hat{Z}) \otimes (X \wedge Y \wedge Z) \otimes Z. \end{aligned}$$

Therefore, any $K(\mathscr{L}^{-1})$-invariant vector can be written as $av_1 + bv_2 + cv_3 + dv_4 + ev_5$ for some $a$, $b$, $c$, $d$, $e \in \mathbb{C}$. Now the claim follows by choosing for $\bigwedge^2 S^2 V^\vee$ the basis

$$\begin{aligned} \{&\hat{X}^2 \wedge \hat{X}\hat{Y},\ \hat{X}^2 \wedge \hat{X}\hat{Z},\ \hat{X}^2 \wedge \hat{Y}^2,\ \hat{X}^2 \wedge \hat{Y}\hat{Z},\ \hat{X}^2 \wedge \hat{Z}^2, \\ &\hat{X}\hat{Y} \wedge \hat{X}\hat{Z},\ \hat{X}\hat{Y} \wedge \hat{Y}^2,\ \hat{X}\hat{Y} \wedge \hat{Y}\hat{Z},\ \hat{X}\hat{Y} \wedge \hat{Z}^2,\ \hat{X}\hat{Z} \wedge \hat{Y}^2, \\ &\hat{X}\hat{Z} \wedge \hat{Y}\hat{Z},\ \hat{X}\hat{Z} \wedge \hat{Z}^2,\ \hat{Y}^2 \wedge \hat{Y}\hat{Z},\ \hat{Y}^2 \wedge \hat{Z}^2,\ \hat{Y}\hat{Z} \wedge \hat{Z}^2\}. \end{aligned}$$ □

Proposition 2.4. *A section $\eta \in \phi^* H^0(\hat{A}, \bigwedge^2 S^2\mathscr{E}^\vee \otimes \bigwedge^3 \mathscr{E})$ corresponding to a vector as in (9) is totally decomposable if and only if one of the following conditions applies.*

(i) $b = -a$, $d = 0$ and $e = -a^2/c$.
(ii) $a = b = d = e = 0$.
(iii) $a = b = c = d = 0$.

*In other words, the totally decomposable sections (up to a multiplicative constant) are in one-to-one correspondence with the points of the smooth conic of equations*

$$a^2 + ce = 0, \quad b + a = 0, \quad d = 0$$

*in the projective space $\mathbb{P}^4$ with homogeneous coordinates $[a : b : c : d : e]$.*

*Proof.* Proposition 2.3 allows one to identify the building data $c_{ij}$ in [**15**, p. 7] as follows:

$$\begin{aligned} &c_{12} = aZ, \quad c_{13} = bY, \quad c_{14} = cY, \quad c_{15} = dX, \quad c_{16} = -cZ, \\ &c_{23} = eX, \quad c_{24} = -bX, \quad c_{25} = -eZ, \quad c_{26} = -dY, \quad c_{34} = -dZ, \\ &c_{35} = eY, \quad c_{36} = -aX, \quad c_{45} = aY, \quad c_{46} = cX, \quad c_{56} = -bZ. \end{aligned} \tag{10}$$

By Hahn–Miranda [**15**, Theorem 3.1], the corresponding section $\eta \in \phi^* H^0(\hat{A}, \bigwedge^2 S^2\mathscr{E}^\vee \otimes \bigwedge^3 \mathscr{E})$ is totally decomposable if and only if the $c_{ij}$ satisfy the Plücker relations:

$$\begin{aligned} &c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23} = 0, \quad c_{12}c_{35} - c_{13}c_{25} + c_{15}c_{23} = 0, \\ &c_{12}c_{36} - c_{13}c_{26} + c_{16}c_{23} = 0, \quad c_{12}c_{45} - c_{14}c_{25} + c_{15}c_{24} = 0, \\ &c_{12}c_{46} - c_{14}c_{26} + c_{16}c_{24} = 0, \quad c_{12}c_{56} - c_{15}c_{26} + c_{16}c_{25} = 0, \\ &c_{13}c_{45} - c_{14}c_{35} + c_{15}c_{34} = 0, \quad c_{13}c_{46} - c_{14}c_{36} + c_{16}c_{34} = 0, \\ &c_{13}c_{56} - c_{15}c_{36} + c_{16}c_{35} = 0, \quad c_{14}c_{56} - c_{15}c_{46} + c_{16}c_{45} = 0, \\ &c_{23}c_{45} - c_{24}c_{35} + c_{25}c_{34} = 0, \quad c_{23}c_{46} - c_{24}c_{36} + c_{26}c_{34} = 0, \\ &c_{23}c_{56} - c_{25}c_{36} + c_{26}c_{35} = 0, \quad c_{24}c_{56} - c_{25}c_{46} + c_{26}c_{45} = 0, \\ &c_{34}c_{56} - c_{35}c_{46} + c_{36}c_{45} = 0. \end{aligned} \tag{11}$$

Substituting in (11) the values given in (10), we get the result. □

Let us take now a point $[a : c] \in \mathbb{P}^1$. By Theorem 1.2 and Proposition 2.4, there exists a quadruple cover $\alpha\colon S \to A$ induced by the totally decomposable section $\eta \in \phi^* H^0(\hat{A}, \bigwedge^2 S^2\mathscr{E}^\vee \otimes \bigwedge^3 \mathscr{E})$ represented by the point $[a : -a : c : 0 : -a^2/c] \in \mathbb{P}^4$; note that cases (ii) and (iii) in Proposition 2.4 correspond to $[a : c] = [0 : 1]$ and $[a : c] = [1 : 0]$, respectively. By construction, the cover $\alpha$ is $K(\mathscr{L}^{-1})$-equivariant, so it induces a quadruple cover $\hat{\alpha}\colon \hat{S} \to \hat{A}$ that fits into a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\psi"] \arrow[d, "\alpha"'] & \hat{S} \arrow[d, "\hat{\alpha}"] \\ A \arrow[r, "\phi_{\mathscr{L}^{-1}}"] & \hat{A} \end{tikzcd} \tag{12}$$

where $\psi$ is an étale $(\mathbb{Z}/3\mathbb{Z})^2$-cover.

We denote by $B$ and $R$ the branch divisor and the ramification divisor of $\alpha\colon S \to A$ and by $\hat{B}$ and $\hat{R}$ those of $\hat{\alpha}\colon \hat{S} \to \hat{A}$.

Following [**15**], over an affine open subset $U \subset A$ we can describe the quadruple cover $\alpha\colon S \to A$ as

$$\operatorname{Spec} \frac{\mathscr{O}_U[u, v, w]}{(F_1, \ldots, F_6)},$$

with

$$
\begin{aligned}
F_1 &= u^2 - (a_1u + a_2v + a_3w + b_1),\\
F_2 &= uv - (a_4u + a_5v + a_6w + b_2),\\
F_3 &= uw - (a_7u + a_8v + a_9w + b_3),\\
F_4 &= v^2 - (a_{10}u + a_{11}v + a_{12}w + b_4),\\
F_5 &= vw - (a_{13}u + a_{14}v + a_{15}w + b_5),\\
F_6 &= w^2 - (a_{16}u + a_{17}v + a_{18}w + b_6).
\end{aligned} \tag{13}
$$

Here

$$
\begin{aligned}
a_1 &= \tfrac{1}{2}c_{23}, \quad a_2 = -c_{13}, \quad a_3 = c_{12},\\
a_4 &= \tfrac{1}{4}c_{25} - \tfrac{1}{2}c_{34}, \quad a_5 = -\tfrac{1}{2}c_{15} - \tfrac{1}{4}c_{23}, \quad a_6 = c_{14},\\
a_7 &= \tfrac{1}{2}c_{26} - \tfrac{1}{4}c_{35}, \quad a_8 = -c_{16}, \quad a_9 = \tfrac{1}{2}c_{15} - \tfrac{1}{4}c_{23},\\
a_{10} &= c_{45}, \quad a_{11} = -\tfrac{1}{2}c_{25}, \quad a_{12} = c_{24},\\
a_{13} &= c_{46}, \quad a_{14} = -\tfrac{1}{2}c_{26} - \tfrac{1}{4}c_{35}, \quad a_{15} = \tfrac{1}{4}c_{25} + \tfrac{1}{2}c_{34},\\
a_{16} &= c_{56}, \quad a_{17} = -c_{36}, \quad a_{18} = \tfrac{1}{2}c_{35}
\end{aligned}
$$

and

$$
\begin{aligned}
b_1 &= -a_1a_5 + a_2a_4 - a_2a_{11} - a_3a_{14} + a_5^2 + a_6a_8,\\
b_2 &= a_2a_{10} + a_3a_{13} - a_4a_5 - a_6a_7,\\
b_3 &= a_2a_{13} + a_3a_{16} - a_4a_8 - a_7a_9,\\
b_4 &= -a_1a_{10} + a_4^2 - a_4a_{11} + a_5a_{10} + a_6a_{13} - a_7a_{12},\\
b_5 &= -a_5a_{13} + a_8a_{10} + a_{12}a_{17} - a_{14}a_{15},\\
b_6 &= -a_1a_{16} - a_4a_{17} + a_7^2 - a_7a_{18} + a_8a_{13} + a_9a_{16},
\end{aligned}
$$

where the $c_{ij}$ are given in (10).

Using the `MAGMA` script in Appendix B, we can check that the branch locus $B$ of $\alpha\colon S \to A$ is the element in $|6L|$ of equation

$$
\begin{aligned}
&a^8c^4(X^6+Y^6+Z^6) + a^2c\left(-\frac{4}{27}a^9 - \frac{2}{9}a^6c^3 - \frac{64}{9}a^3c^6 + \frac{256}{27}c^9\right)(X^3Y^3 + X^3Z^3 + Y^3Z^3)\\
&\quad + a^4c^2\left(-\frac{2}{3}a^6 + \frac{16}{3}a^3c^3 - \frac{32}{3}c^6\right)XYZ(X^3+Y^3+Z^3)\\
&\quad + \left(-\frac{1}{27}a^{12} - \frac{92}{27}a^9c^3 + \frac{112}{9}a^6c^6 + \frac{256}{27}a^3c^9 - \frac{256}{27}c^{12}\right)X^2Y^2Z^2 = 0.
\end{aligned} \tag{14}
$$

We can also see (14) as the equation of a sextic curve in the dual projective plane $(\mathbb{P}^2)^\vee$ with homogeneous coordinates $[X : Y : Z]$; we shall denote such a curve by $\mathbf{B}$. Varying the point $[a : c] \in \mathbb{P}^1$, the curves $\mathbf{B}$ form a (non-linear) pencil in $\mathbb{P}^{2\vee}$, which turns out to be the pencil of the dual curves of members of the Hesse pencil. In fact, $\mathbf{B}$ is precisely the dual of the curve $E_{m_0,\,3m_1}$ with $m_0 = a^2c$ and $m_1 = \frac{1}{6}a^3 - \frac{2}{3}c^3$; see Subsection 1.3.

Remark 2.5. The $\mathscr{H}_3$-equivariant sextic $\mathbf{B}$ also appears, for some special choices of the pair $(A, \mathscr{L})$, as a component of the branch locus of the 6-fold cover $\varphi_{|\mathscr{L}|}\colon A \to \mathbb{P}H^0(A, \mathscr{L})^\vee$. See, for instance, [**3**, **5**].

The group $K(\mathscr{L}^{-1})$ acts on $\mathbf{B}$ and this induces an action on $B$. The quotient of $B$ by this action is a curve $\hat{B} \in |2\hat{L}|$, where $\hat{L}$ is the polarization of type $(1,3)$ on $\hat{A}$ appearing in (7); the curve $\hat{B}$ is precisely the branch locus of the quadruple cover $\hat{\alpha}\colon \hat{S} \to \hat{A}$.

Now set

$$T_1 := \{[1:1], [\omega:1], [\omega^2:1], [1:0]\},$$
$$T_2 := \{[-2:1], [-2\omega:1], [-2\omega^2:1], [0:1]\}.$$

If $[a:c] \notin T_1 \cup T_2$, then $\mathbf{B}$ has nine ordinary cusps as the only singularities; since $X, Y, Z \in H^0(A, \mathscr{L})$ and $L^2 = 6$, by Proposition 2.2 it follows that, for a general choice of the pair $(A, \mathscr{L})$, the curve $B$ has 54 ordinary cusps as the only singularities. In this case, a `MAGMA` calculation (see again the script in Appendix B) shows that $S$ is smooth; since $\psi\colon S \to \hat{S}$ is étale, the surface $\hat{S}$ is also smooth. Moreover, the nine cusps of $\mathbf{B}$ belong to a single $K(\mathscr{L}^{-1})$-orbit and the stabilizer of each cusp is the identity. Therefore, the 54 cusps of $B$ fall in precisely six orbits and consequently $\hat{B}$ is an irreducible curve with six cusps as the only singularities.

If $[a:c] \in T_1 \cup T_2$, then $\mathbf{B} = 2\mathbf{B}'$, where $\mathbf{B}'$ is a triangle. Consequently, $B = 2B'$, where $B'$ has 18 ordinary double points as the only singularities. In this case, the `MAGMA` script shows that $S$ is smooth if and only if $[a:c] \notin T_1$; if instead $[a:c] \in T_1$, then the singular locus of $S$ coincides with the preimage of $B$. Moreover, the group $K(\mathscr{L}^{-1})$ acts on the three sides of the triangle $\mathbf{B}'$; each side has stabilizer isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and there is only one orbit; consequently, $\hat{B} = 2\hat{B}'$, where $\hat{B}'$ is an irreducible curve with two ordinary double points as the only singularities.

In any case, the preimage of a general point in the branch divisor $\hat{B}$ consists of *three* distinct points. This shows that the quadruple cover $\alpha\colon \hat{S} \to \hat{A}$ cannot factor through a double cover, hence it must be the Albanese map of $\hat{S}$.

Summing up, we have proved the following proposition.

Proposition 2.6. *Let $(A, \mathscr{L})$ be a general $(1,3)$-polarized abelian surface. Then the following holds.*

(i) *The surface $\hat{S}$ is smooth precisely when $[a:c] \notin T_1$, whereas if $[a:c] \in T_1$, then it has a one-dimensional singular locus.*

(ii) *If $\hat{S}$ is smooth and $[a:c] \notin T_2$, then the branch locus $\hat{B}$ of $\hat{\alpha}\colon \hat{S} \to \hat{A}$ is an irreducible curve in $|2\hat{L}|$, where $\hat{L}$ is a polarization of type $(1,3)$ on $\hat{A}$, with six ordinary cusps as the only singularities.*

(iii) *If $\hat{S}$ is smooth and $[a:c] \in T_2$, then $\hat{B} = 2\hat{B}'$, where $\hat{B}'$ is an irreducible curve in $|\hat{L}|$ with two ordinary double points as the only singularities.*

(iv) *The quadruple cover $\hat{\alpha}\colon \hat{S} \to \hat{A}$ is the Albanese map of $\hat{A}$.*

We can now compute the invariants of $\hat{S}$.

Proposition 2.7. *If $[a:c] \notin T_1$ and $\hat{A}$ is general, then $\hat{S}$ is a minimal surface of general type with $p_g = q = 2$ and $K^2 = 6$. The canonical class $K_{\hat{S}}$ is ample and the general element of $|K_{\hat{S}}|$ is smooth and irreducible.*

*Proof.* Using (1) and (8), we obtain $p_g(\hat{S}) = q(\hat{S}) = 2$. By the Hurwitz formula, we get

$$K_{\hat{S}} = \hat{\alpha}^* K_{\hat{A}} + \hat{R} = \hat{R},$$

so $\hat{R} \in |K_{\hat{S}}|$. If $\hat{S}$ is general, then we can check by our `MAGMA` script that the restriction $\hat{\alpha}\colon \hat{R} \to \hat{B}$ is the normalization map, that is, $\hat{R}$ is a smooth curve of genus 7 (see Proposition 2.6). Thus the genus formula gives

$$2K_{\hat{S}}^2 = \hat{R}(K_{\hat{S}} + \hat{R}) = 2g(\hat{R}) - 2 = 12,$$

hence $K_{\hat{S}}^2 = 6$. Moreover, by Bertini's theorem the general element of $|K_{\hat{S}}|$ is smooth and irreducible, because $\hat{R}$ is so. Since $\hat{\alpha}\colon \hat{S} \to \hat{A}$ is a finite map onto an abelian variety, the surface $\hat{S}$ contains no rational curves. In particular, $\hat{S}$ is a minimal model and $K_{\hat{S}}$ is ample. □

REMARK 2.8. The six cusps of the branch curve $\hat{B}$ are not in general position: in fact, by the results of Subsection 1.3 it follows that there exists a unique element $\hat{C} \in |\hat{L}|$ containing them. The general element of the pencil generated by $\hat{B}$ and $2\hat{C}$ is an irreducible curve in $|2\hat{L}|$ with six nodes at the six cusps of $\hat{B}$ and no further singularities. Hence the cuspidal curve $\hat{B} \in |2\hat{L}|$ can be obtained as the limit of nodal curves belonging to the same linear system.

In the sequel, we denote by $\mathscr{A}_{1,3}$ the moduli space of $(1, 3)$-polarized abelian surfaces. It is a quasi-projective variety of dimension 3; see [**4**, Chapter 8]. The following result will be used in the next section.

PROPOSITION 2.9. *For a general choice of $\hat{A} \in \mathscr{A}_{1,3}$ and $[a : c] \in \mathbb{P}^1$, the curve $\hat{R}$ is not hyperelliptic. In particular, for any line bundle $\mathscr{N}$ on $\hat{R}$ with $\deg(\mathscr{N}) = 6$ we have $h^0(\hat{R}, \mathscr{N}) \leqslant 3$.*

*Proof.* This follows from a rigidity result of hyperelliptic curves on general abelian varieties; see [**24**]. More precisely, one can show that the only hyperelliptic deformations of a (possibly singular) hyperelliptic curve on a fixed simple abelian variety are the translations. This implies that a linear system on a general abelian surface contains at most a finite number of hyperelliptic curves. In our situation, we have an equisingular, one-dimensional family $\{\hat{B}\}$ of cuspidal curves in the linear system $|2\hat{L}|$, parametrized by the points $[a : c] \in \mathbb{P}^1$. Then, if $\hat{A}$ is simple, the general curve $\hat{B}$ is non-hyperelliptic, that is, its normalization $\hat{R}$ is non-hyperelliptic.

If $\mathscr{N}$ is any line bundle on $\hat{R}$ with $\deg(\mathscr{N}) = 6$, then the Riemann–Roch theorem yields $h^0(\hat{R}, \mathscr{N}) - h^1(\hat{R}, \mathscr{N}) = 0$; in particular, if $\mathscr{N}$ is non-special, then we obtain $h^0(\hat{R}, \mathscr{N}) = 0$. If instead $\mathscr{N}$ is special, since $\hat{R}$ is non-hyperelliptic Clifford's theorem implies

$$h^0(\hat{R}, \mathscr{N}) - 1 < \tfrac{1}{2}\deg(\mathscr{N}),$$

that is, $h^0(\hat{R}, \mathscr{N}) \leqslant 3$. □

## 3. *The moduli space*

DEFINITION 3.1. We denote by $\mathcal{M}_{\Phi}^0$ the family of canonical models $\hat{X}$ of minimal surfaces of general type with $p_g = q = 2$, $K^2 = 6$ such that the following conditions apply.

(i) The Albanese map $\hat{\alpha}\colon \hat{X} \to \hat{A}$ is a finite, quadruple cover.

(ii) The Tschirnhausen bundle $\mathscr{E}^\vee$ associated with $\hat{\alpha}\colon \hat{X} \to \hat{A}$ is of the form $\mathscr{E}^\vee = \Phi^{\mathscr{P}}(\mathscr{L}^{-1})$, where $\mathscr{L}$ is a polarization of type $(1, 3)$ on $A$.

Note that $\mathcal{M}_{\Phi}^0$ coincides with the family of canonical models of surfaces $\hat{S}$ constructed in Section 2, and that such a family depends on four parameters (three parameters from $\hat{A} \in \mathscr{A}_{1,3}$ and one parameter from $[a : c] \in \mathbb{P}^1$). More precisely, there is a generically finite, dominant map

$$\mathbf{P}^0 \longrightarrow \mathcal{M}_{\Phi}^0,$$

where $\mathbf{P}^0$ is a Zariski-dense subset of a $\mathbb{P}^1$-bundle over $\mathscr{A}_{1,3}$. Then $\mathcal{M}_\Phi^0$ is irreducible and $\dim \mathcal{M}_\Phi^0 = 4$.

Proposition 3.2. *If $\hat{S} \in \mathcal{M}_\Phi^0$, then the Tschirnhausen bundle $\mathscr{E}^\vee$ is stable with respect to the polarization $\hat{L}$. In particular, it is simple, that is, $H^0(\hat{A}, \mathscr{E} \otimes \mathscr{E}^\vee) = \mathbb{C}$.*

*Proof.* By definition of $\mathscr{E}^\vee$ and since the Fourier–Mukai transform gives an equivalence of derived categories, we have

$$H^0(\hat{A}, \mathscr{E} \otimes \mathscr{E}^\vee) \cong \operatorname{Hom}_{\hat{A}}(\mathscr{E}^\vee, \mathscr{E}^\vee) \cong \operatorname{Hom}_A(\mathscr{L}^{-1}, \mathscr{L}^{-1}) = \mathbb{C},$$

that is, $\mathscr{E}^\vee$ is simple. Then, by (5) and [**4**, Exercise 2, p. 476], it follows that $\mathscr{E}^\vee$ is semi-homogeneous. Finally, any simple, semi-homogeneous vector bundle on an abelian variety is stable with respect to any polarization; see [**4**, Exercise 1, p. 476]. $\square$

Definition 3.3. We denote by $\mathcal{M}_\Phi$ the closure of $\mathcal{M}_\Phi^0$ in the moduli space $\mathcal{M}_{2,6,6}^{\mathrm{can}}$ of canonical models of minimal surfaces of general type with $p_g = q = 2$, $K^2 = 6$.

By the previous considerations, $\mathcal{M}_\Phi$ is irreducible and $\dim \mathcal{M}_\Phi = 4$. Now we want to prove that $\mathcal{M}_\Phi$ provides an irreducible component of the moduli space $\mathcal{M}_{2,2,6}^{\mathrm{can}}$ and that such a component is generically smooth. In order to do this, we must prove that for the general surface $\hat{S} \in \mathcal{M}_\Phi$ one has $h^1(\hat{S}, T_{\hat{S}}) = \dim \mathcal{M}_\Phi = 4$.

By Sernesi [**26**, p. 262], we have an exact sequence

$$0 \longrightarrow T_{\hat{S}} \xrightarrow{d\hat{\alpha}} \hat{\alpha}^* T_{\hat{A}} \longrightarrow \mathscr{N}_{\hat{\alpha}} \longrightarrow 0, \tag{15}$$

where $\mathscr{N}_{\hat{\alpha}}$ is a coherent sheaf supported on $\hat{R}$ and called the normal sheaf of $\hat{\alpha}$; we denote by $\varphi = \hat{\alpha}|_{\hat{R}} \colon \hat{R} \to \hat{B} \subset \hat{A}$ the normalization map of $\hat{B}$.

Let $\Delta \subset \hat{R}$ be the divisor formed by all points $p$ such that $\varphi(p)$ is a cusp of $\hat{B}$, each counted with multiplicity 1, and let $i \colon \hat{R} \to \hat{S}$ be the inclusion of $\hat{R}$ in $\hat{S}$; then $\hat{\alpha} \circ i = \varphi$. By Ciliberto–Flamini [**12**, Section 7], there are two commutative diagrams

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathscr{O}_\Delta \arrow[d] & \\
0 \arrow[r] & T_{\hat{R}} \arrow[r, "d\varphi"] \arrow[d] & \varphi^* T_{\hat{A}} \arrow[r] \arrow[d, equal] & \mathscr{N}_\varphi \arrow[r] \arrow[d] & 0 \\
 & i^* T_{\hat{S}} \arrow[r, "i^*(d\hat{\alpha})"] & \varphi^* T_{\hat{A}} \arrow[r] & i^* \mathscr{N}_{\hat{\alpha}} \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd} \tag{16}$$

and

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ & 0 \arrow[d] & & \mathscr{O}_{\Delta} \arrow[d] & \\ 0 \arrow[r] & T_{\hat{R}} \arrow[r, "d\varphi"] \arrow[d] & \varphi^* T_{\hat{A}} \arrow[r] \arrow[d, equal] & \mathscr{N}_{\varphi} \arrow[r] \arrow[d] & 0 \\ & T_{\hat{R}} \otimes \mathscr{O}_{\hat{R}}(\Delta) \arrow[r, "d\varphi"] & \varphi^* T_{\hat{A}} \arrow[r] & \mathscr{N}'_{\varphi} \arrow[r] \arrow[d] & 0 \\ & & & 0 & \end{tikzcd} \tag{17}$$

where $\mathscr{N}'_{\varphi}$ is a line bundle on $\hat{R}$ satisfying

$$\begin{aligned} \deg(\mathscr{N}'_{\varphi}) &= \deg(\mathscr{N}_{\varphi}) - \deg(\Delta) = \deg(\varphi^* T_{\hat{A}}) - \deg(T_{\hat{R}}) - \deg(\Delta) \\ &= \deg(\mathscr{O}_{\hat{R}}^{\oplus 2}) + \deg(K_{\hat{R}}) - \deg(\Delta) = 6. \end{aligned}$$

The fact that $\mathscr{N}_{\hat{\alpha}}$ is supported on $\hat{R}$, together with (16) and (17), implies

$$h^i(\hat{S}, \mathscr{N}_{\hat{\alpha}}) = h^i(\hat{R}, i^* \mathscr{N}_{\hat{\alpha}}) = h^i(\hat{R}, \mathscr{N}'_{\varphi}), \quad i = 0, 1, 2,$$

so for a general choice of the abelian variety $\hat{A} \in \mathscr{A}_{1,3}$ and of the point $[a : c] \in \mathbb{P}^1$ one has

$$h^0(\hat{S}, \mathscr{N}_{\hat{\alpha}}) = h^0(\hat{R}, \mathscr{N}'_{\varphi}) \leqslant 3; \tag{18}$$

see Proposition 2.9. Now we can prove the desired result.

Proposition 3.4. *If $\hat{S}$ is a general element of $\mathcal{M}_{\Phi}$, then $h^1(\hat{S}, T_{\hat{S}}) = 4$. Hence $\mathcal{M}_{\Phi}$ provides an irreducible component of the moduli space $\mathcal{M}^{\mathrm{can}}_{2,2,6}$ of canonical models of minimal surfaces with $p_g = q = 2$ and $K^2 = 6$. Such a component is generically smooth, of dimension 4.*

*Proof.* Since $\dim \mathcal{M}_{\Phi} = 4$, it is sufficient to show that for a general choice of $\hat{S}$ one has

$$h^1(\hat{S}, T_{\hat{S}}) \leqslant 4. \tag{19}$$

The surface $\hat{S}$ is of general type, so we have $H^0(\hat{S}, T_{\hat{S}}) = 0$ and (15) yields the following exact sequence in cohomology:

$$0 \longrightarrow H^0(\hat{S}, \hat{\alpha}^* T_{\hat{A}}) \longrightarrow H^0(\hat{S}, \mathscr{N}_{\hat{\alpha}}) \longrightarrow H^1(\hat{S}, T_{\hat{S}}) \xrightarrow{\varepsilon} H^1(\hat{S}, \hat{\alpha}^* T_{\hat{A}}). \tag{20}$$

The same argument used in [**23**, Section 3] shows that the image of $\varepsilon\colon H^0(\hat{S}, T_{\hat{S}}) \to H^1(\hat{S}, \hat{\alpha}^* T_{\hat{A}})$ has dimension 3 (this is essentially a consequence of the fact that when one deforms $\hat{S}$ the Albanese torus $\hat{A}$ remains algebraic). Then (19) follows from (18) and (20). $\square$

Proposition 3.5. *The general element $\hat{S}$ of $\mathcal{M}_{\Phi}$ admits no pencil $p\colon \hat{S} \to T$ over a curve $T$ with $g(T) \geqslant 1$.*

*Proof.* Assume that $\hat{A}$ is a simple abelian surface. In this case, the set

$$V^1(\hat{S}) := \{\mathscr{Q} \in \mathrm{Pic}^0(\hat{S}) \mid h^1(\hat{S}, \mathscr{Q}^{\vee}) > 0\}$$

cannot contain any component of positive dimension, so $\hat{S}$ does not admit any pencil over a curve $T$ with $g(T) \geqslant 2$; see [**14**, Theorem 2.6]. If instead $g(T) = 1$, then the universal property of the Albanese map yields a surjective morphism $\hat{A} \to T$, contradicting the fact that $\hat{A}$ is simple. □

## 4. *Quadruple covers with simple Tschirnhausen bundle*

PROPOSITION 4.1. *Let $(\hat{A}, \hat{\mathscr{L}})$ be a $(1,3)$-polarized abelian surface and $\mathscr{E}^\vee$ be a simple rank-3 vector bundle on $\hat{A}$ with*

$$h^0(\hat{A}, \mathscr{E}^\vee) = 1, \quad h^1(\hat{A}, \mathscr{E}^\vee) = 0, \quad h^2(\hat{A}, \mathscr{E}^\vee) = 0,$$

$$c_1(\mathscr{E}^\vee) = \hat{\mathscr{L}}, \quad c_2(\mathscr{E}^\vee) = 2.$$

*Then there exists a polarization $\mathscr{L}$ of type $(1,3)$ on $A$ such that*

$$\mathscr{E}^\vee = \Phi^{\mathscr{P}}(\mathscr{L}^{-1}).$$

*Proof.* Since $\mathscr{E}^\vee$ is simple and $2c_1^2(\mathscr{E}^\vee) - 6c_2(\mathscr{E}^\vee) = 0$, by [Birkenhake–Lange–Oka, **4**, Exercise 2, p. 476; **19**, Corollary, p. 249] there exist an abelian surface $Z$, a line bundle $\mathscr{G}$ on $Z$ and an isogeny $g\colon Z \to \hat{A}$ such that $g^*\mathscr{E}^\vee = \mathscr{G}^{\oplus 3}$. Hence we obtain

$$3c_1(\mathscr{G}) = c_1(g^*\mathscr{E}^\vee) = g^*c_1(\mathscr{E}^\vee) = g^*\hat{\mathscr{L}},$$

that is, $\mathscr{G}$ is ample. For any $\mathscr{Q} \in \mathrm{Pic}^0(\hat{A})$, we have $g^*\mathscr{Q} \in \mathrm{Pic}^0(Z)$ and $g^*(\mathscr{E}^\vee \otimes \mathscr{Q}) = (\mathscr{G} \otimes g^*\mathscr{Q})^{\oplus 3}$, so the ampleness of $\mathscr{G}$ implies

$$H^i(Z, g^*(\mathscr{E}^\vee \otimes \mathscr{Q})) = H^i(Z, \mathscr{G} \otimes g^*\mathscr{Q})^{\oplus 3} = 0, \quad i = 1, 2,$$

for all $\mathscr{Q} \in \mathrm{Pic}^0(\hat{A})$. Since $g$ is a finite map, we get

$$H^i(\hat{A}, \mathscr{E}^\vee \otimes \mathscr{Q}) \cong g^*H^i(\hat{A}, \mathscr{E}^\vee \otimes \mathscr{Q}) \subseteq H^i(Z, g^*(\mathscr{E}^\vee \otimes \mathscr{Q})) = 0, \quad i = 1, 2,$$

that is, $\mathscr{E}^\vee$ satisfies IT of index 0. Since $\mathrm{rank}(\mathscr{E}^\vee) = 3$ and $h^0(\hat{A}, \mathscr{E}^\vee) = 1$, the Fourier–Mukai transform $\Phi^{\mathscr{P}}(\mathscr{E}^\vee)$ is a line bundle of type $(1,3)$ on $A$ that we denote by $\mathscr{M}^{-1}$. Therefore, we have

$$(-1)^*_{\hat{A}}\mathscr{E}^\vee = \Phi^{\mathscr{P}} \circ \Phi^{\mathscr{P}}(\mathscr{E}^\vee) = \Phi^{\mathscr{P}}(\mathscr{M}^{-1}),$$

so $\mathscr{E}^\vee = \Phi^{\mathscr{P}}((-1)^*_A\mathscr{M}^{-1})$ by Corollary 1.6. Setting $\mathscr{L}^{-1} := (-1)^*_A\mathscr{M}^{-1}$, we are done. □

The following corollary can be seen as a converse of Proposition 3.2.

COROLLARY 4.2. *Let $\hat{S}$ be a surface of general type with $p_g = q = 2$, $K^2_{\hat{S}} = 6$ such that the Albanese map of its canonical model $\hat{X}$ is a finite, quadruple cover $\hat{\alpha}\colon \hat{X} \to \hat{A}$, where $\hat{A}$ is a $(1,3)$-polarized abelian surface. Assume in addition that the Tschirnhausen bundle $\mathscr{E}^\vee$ associated with $\hat{\alpha}$ is simple. Then $\hat{X}$ belongs to $\mathcal{M}_\Phi$.*

## 5. *Two remarkable subfamilies of $\mathcal{M}_\Phi$*

### 5.1. *A three-dimensional family of surfaces contained in the singular locus of $\mathcal{M}^{\mathrm{can}}_{2,2,6}$*

If we choose $[a : c] = [0 : 1]$ in the construction of Section 2, then by (10) we obtain

$$c_{12} = c_{13} = c_{15} = c_{23} = c_{24} = c_{25} = c_{26} = c_{34} = c_{35} = c_{36} = c_{45} = c_{56} = 0. \tag{21}$$

Then (13) implies that the local equations of the surface $S$ are

$$\begin{aligned} u^2 &= YZ, \\ uv &= Yw, \\ uw &= Zv, \\ v^2 &= XY, \\ vw &= Xu, \\ w^2 &= XZ, \end{aligned} \tag{22}$$

and this shows that the Albanese map $\alpha\colon S \to A$ is a *bidouble cover*, that is, a Galois cover with Galois group $(\mathbb{Z}/2\mathbb{Z})^2$. Conversely, relations (21) are precisely the conditions ensuring that $\alpha\colon S \to A$ is a bidouble cover; see [**15**, pp. 25–27].

We use the theory of bidouble covers developed in [**7**]. The cover $\alpha\colon S \to A$ is branched over the three divisors $D_X$, $D_Y$, $D_Z \in |L|$ corresponding to the distinguished sections $X$, $Y$, $Z \in H^0(A, \mathscr{L})$; for simplicity of notation, we write $D_1$, $D_2$, $D_3$ instead of $D_X$, $D_Y$, $D_Z$, respectively. Then the building data of $\alpha\colon S \to A$ consist of three line bundles $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ on $A$, with $\mathscr{L}_i = \mathscr{O}_A(L_i)$, such that

$$\begin{aligned} 2L_1 &\cong D_2 + D_3, \\ 2L_2 &\cong D_1 + D_3, \\ 2L_3 &\cong D_1 + D_2, \end{aligned} \tag{23}$$

and $\alpha_*\mathscr{O}_S = \mathscr{O}_A \oplus \mathscr{L}_1^{-1} \oplus \mathscr{L}_2^{-1} \oplus \mathscr{L}_3^{-1}$. On the other hand, by (5) we obtain $\alpha_*\mathscr{O}_S = \mathscr{O}_A \oplus \phi^*\,\mathscr{E} = \mathscr{O}_A \oplus \mathscr{L}^{-1} \oplus \mathscr{L}^{-1} \oplus \mathscr{L}^{-1}$; therefore, [**2**] implies $\mathscr{L}_1 = \mathscr{L}_2 = \mathscr{L}_3 = \mathscr{L}$.

By Catanese [**7**, p. 497], there is an exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(S, \alpha^*T_A) \longrightarrow \bigoplus_{i=1}^{3} H^0(\mathscr{O}_{D_i}(D_i) \oplus \mathscr{O}_{D_i}) \xrightarrow{\ \partial\ } H^1(S, T_S) \\ \xrightarrow{\ \varepsilon\ } H^1(S, \alpha^*T_A) \longrightarrow \cdots, \end{aligned} \tag{24}$$

whose meaning is the following: the image of $\partial$ consists of the first-order deformations given by the so-called natural deformations (see [**7**, Definition 2.8, p. 494]) and such first-order deformations are trivial if they are induced by the automorphisms of $A$. Moreover, if one considers the map

$$\partial'\colon \bigoplus_{i=1}^{3} H^0(\mathscr{O}_A(D_i) \oplus \mathscr{O}_A) \to H^1(S, T_S),$$

obtained as the composition of the direct sum of the restriction maps with $\partial$, then for any $\bigoplus_i (a_i \oplus \delta_i) \in \bigoplus_i H^0(\mathscr{O}_A(D_i) \oplus \mathscr{O}_A)$ the element $\partial'(\bigoplus_i (a_i \oplus \delta_i)) \in H^1(S,\, T_S)$ is the Kodaira–Spencer class of the corresponding natural deformation of $S$. More explicitly, taking

$$a_i = \alpha_i X + \beta_i Y + \gamma_i Z, \quad \alpha_i, \beta_i, \gamma_i \in \mathbb{C},$$

the local equations of the associated natural deformation of the bidouble cover (22) are

$$\begin{aligned} u^2 &= (Y + \alpha_3 X + \beta_3 Y + \gamma_3 Z + \delta_3 w)(Z + \alpha_2 X + \beta_2 Y + \gamma_2 Z + \delta_2 v), \\ uv &= (Y + \alpha_3 X + \beta_3 Y + \gamma_3 Z)w + \delta_3 w^2, \\ uw &= (Z + \alpha_2 X + \beta_2 Y + \gamma_2 Z)v + \delta_2 v^2, \\ v^2 &= (X + \alpha_1 X + \beta_1 Y + \gamma_1 Z + \delta_1 u)(Y + \alpha_3 X + \beta_3 Y + \gamma_3 Z + \delta_3 w), \\ vw &= (X + \alpha_1 X + \beta_1 Y + \gamma_1 Z)u + \delta_1 u^2, \\ w^2 &= (X + \alpha_1 X + \beta_1 Y + \gamma_1 Z + \delta_1 u)(Z + \alpha_2 X + \beta_2 Y + \gamma_2 Z + \delta_2 v). \end{aligned} \tag{25}$$

By using the restriction exact sequence

$$0 \longrightarrow H^0(A, \mathscr{O}_A) \longrightarrow H^0(A, \mathscr{O}_A(D_i)) \xrightarrow{\rho_i} H^0(D_i, \mathscr{O}_{D_i}(D_i)) \longrightarrow H^1(A, \mathscr{O}_A) \longrightarrow 0,$$

we obtain

$$H^0(D_i, \mathscr{O}_{D_i}(D_i)) = \operatorname{im} \rho_i \oplus H^1(A, \mathscr{O}_A), \tag{26}$$

where

$$\begin{aligned} \operatorname{im} \rho_1 &= \operatorname{span}(Y/X, Z/X), \\ \operatorname{im} \rho_2 &= \operatorname{span}(X/Y, Z/Y), \\ \operatorname{im} \rho_3 &= \operatorname{span}(X/Z, Y/Z). \end{aligned} \tag{27}$$

There is an action of $(\mathbb{Z}/3\mathbb{Z})^2 = \langle r, s \,|\, r^3 = s^3 = [r, s] = 1\rangle$ on $S$ given by

$$\begin{aligned} r(u, v, w, X, Y, Z) &:= (u, \omega v, \omega^2 w, X, \omega^2 Y, \omega Z), \\ s(u, v, w, X, Y, Z) &:= (v, w, u, Z, X, Y), \end{aligned} \tag{28}$$

and the corresponding quotient map is precisely $\psi \colon S \to \hat{S}$; note that the Albanese map $\hat{\alpha} \colon \hat{S} \to \hat{A}$ is *not* a Galois cover. We shall denote by $\chi_0$ the trivial character of $(\mathbb{Z}/3\mathbb{Z})^2$ and by $\chi$ any non-trivial character; moreover, if $V$ is a vector space with a $(\mathbb{Z}/3\mathbb{Z})^2$-representation, then we indicate by $V^{\chi_0}$ the invariant eigenspace and by $V^{\chi}$ the eigenspace relative to $\chi$. For instance, since $q(A) = q(\hat{A})$ and $q(S) = q(\hat{S})$, it follows

$$\begin{aligned} H^i(A, \mathscr{O}_A)^{\chi_0} &= H^i(A, \mathscr{O}_A), \quad i = 0, 1, \\ H^i(A, \mathscr{O}_A)^{\chi} &= 0, \quad \chi \neq \chi_0, \quad i = 0, 1, \end{aligned} \tag{29}$$

$$\begin{aligned} H^i(S, \mathscr{O}_S)^{\chi_0} &= H^i(S, \mathscr{O}_S), \quad i = 0, 1, \\ H^i(S, \mathscr{O}_S)^{\chi} &= 0, \quad \chi \neq \chi_0, \quad i = 0, 1. \end{aligned} \tag{30}$$

Since $\alpha^* T_A = \mathscr{O}_S^{\oplus 2}$, we also have

$$\begin{aligned} H^i(S, \alpha^* T_A)^{\chi_0} &= H^i(S, \alpha^* T_A), \quad i = 0, 1, \\ H^i(S, \alpha^* T_A)^{\chi} &= 0, \quad \chi \neq \chi_0, \quad i = 0, 1. \end{aligned} \tag{31}$$

LEMMA 5.1. *The action* (28) *induces a natural action of* $(\mathbb{Z}/3\mathbb{Z})^2$ *on* $\operatorname{im} \partial$ *such that*

$$\dim(\operatorname{im} \partial)^{\chi_0} = 5.$$

*Proof.* Using (26), (27) and (29), by a tedious but straightforward computation one proves that there is an induced action of $(\mathbb{Z}/3\mathbb{Z})^2$ on the vector space $\bigoplus_i H^0(\mathscr{O}_{D_i}(D_i) \oplus \mathscr{O}_{D_i}) \cong \mathbb{C}^{15}$, such that

$$\dim \left( \bigoplus_i H^0(\mathscr{O}_{D_i}(D_i) \oplus \mathscr{O}_{D_i}) \right)^{\chi_0} = 7,$$

whereas the eight remaining eigensheaves are all one-dimensional. By (31), the two-dimensional subspace $\ker \partial = H^0(S, \alpha^* T_A)$ is contained in the invariant eigenspace, so the claim follows. □

PROPOSITION 5.2. *Assume that* $\alpha \colon S \to A$ *is a bidouble cover, that is,* $[a : c] = [0 : 1]$. *Then* $h^1(\hat{S}, T_{\hat{S}}) = 8$.

*Proof.* By (24), we have

$$H^1(S, T_S) = \operatorname{im} \partial \oplus \operatorname{im} \varepsilon,$$

and, as remarked in the proof of Proposition 3.4, we can prove that $\operatorname{im}\varepsilon \subset H^1(S, \alpha^* T_A)$ has dimension 3. Using Lemma 5.1 and the fact that $H^1(S, \alpha^* T_A)$ is contained in the invariant eigenspace (see (31)), it follows that the dimension of $H^1(S, T_S)^{\chi_0}$ equals $5 + 3 = 8$. Finally, the $(\mathbb{Z}/3\mathbb{Z})^2$-cover $\psi\colon S \to \hat{S}$ is unramified, so by Pardini [20, Proposition 4.1] we infer $H^1(S, T_S)^{\chi_0} = H^1(\hat{S}, T_{\hat{S}})$ and we are done. □

Since the irreducible component $\mathcal{M}_\Phi$ of $\mathcal{M}^{\mathrm{can}}_{2,2,6}$ has dimension 4, Proposition 5.2 implies that if $\alpha\colon S \to A$ is a bidouble cover, then the canonical model of $\hat{S}$ yields a singular point of the moduli space. So we obtain the following corollary.

COROLLARY 5.3. *The moduli space $\mathcal{M}^{\mathrm{can}}_{2,2,6}$ contains a three-dimensional singular locus.*

## 5.2. *A two-dimensional family of product–quotient surfaces*

In [21, Theorem 4.15], the first author constructed a two-dimensional family of product–quotient surfaces (having precisely two ordinary double points as singularities) with $p_g = q = 2$, $K^2 = 6$ and whose Albanese map is a generically finite quadruple cover. We will now recall the construction and show that this family is actually contained in $\mathcal{M}_\Phi$.

Let us denote by $\mathfrak{A}_4$ the alternating group on four symbols and by $V_4$ its Klein subgroup, namely

$$V_4 = \langle \mathrm{id},\ (12)(34),\ (13)(24),\ (14)(23)\rangle \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

The subgroup $V_4$ is normal in $\mathfrak{A}_4$ and the quotient $H := \mathfrak{A}_4/V_4$ is a cyclic group of order 3. By using Riemann's existence theorem, it is possible to construct two smooth curves $C_1$, $C_2$ of genus 4 endowed with an action of $\mathfrak{A}_4$ such that the only non-trivial stabilizers are the elements of $V_4$. Then we can check the following facts.

(i) The curve $E_i' := C_i/\mathfrak{A}_4$ is an elliptic curve.
(ii) The $\mathfrak{A}_4$-cover $f_i\colon C_i \to E_i'$ is branched at exactly one point of $E_i'$, with branching order 2.

It follows that the product–quotient surface

$$\hat{X} := (C_1 \times C_2)/\mathfrak{A}_4,$$

where $\mathfrak{A}_4$ acts diagonally, has two rational double points of type $\frac{1}{2}(1,1)$ and no other singularities. It is straightforward to check that the desingularization $\hat{S}$ of $\hat{X}$ is a minimal surface of general type with $p_g = q = 2$, $K^2_{\hat{S}} = 6$ and that $\hat{X}$ is the canonical model of $\hat{S}$.

The $\mathfrak{A}_4$-cover $f_i\colon C_i \to E_i'$ factors through the bidouble cover $g_i\colon C_i \to E_i$, where $E_i := C_i/V_4$. Note that $E_i$ is again an elliptic curve, so there is an isogeny $E_i \to E_i'$, which is a triple Galois cover with Galois group $H$. Consequently, we have an isogeny

$$p\colon E_1 \times E_2 \longrightarrow \hat{A} := (E_1 \times E_2)/H,$$

where the group $H$ acts diagonally, and a commutative diagram

$$\begin{tikzcd} C_1 \times C_2 \arrow[r, "\tilde{p}"] \arrow[d, "g_1 \times g_2"'] & (C_1 \times C_2)/\mathfrak{A}_4 = \hat{X} \arrow[d, "\hat{\alpha}"] \\ E_1 \times E_2 \arrow[r, "p"] & (E_1 \times E_2)/H = \hat{A} \end{tikzcd} \tag{32}$$

In this way, one constructs a two-dimensional family $\mathcal{M}_{PQ}$ of product–quotient surfaces $\hat{X}$ which are canonical models of surfaces of general type with $p_g = q = 2$ and $K^2 = 6$. The morphism $\hat{\alpha}\colon \hat{X} \to \hat{A}$ is the Albanese map of $\hat{X}$; it is a finite, non-Galois quadruple cover.

Proposition 5.4. *The family $\mathcal{M}_{PQ}$ is contained in $\mathcal{M}_{\Phi}$.*

*Proof.* We must show that $\hat{X}$ belongs to $\mathcal{M}_{\Phi}$. From the construction, it follows that the Tschirnhausen bundle of $\hat{\alpha}\colon \hat{X} \to \hat{A}$ is of the form

$$\mathscr{E}^{\vee} = p_* \mathscr{D},$$

where $\mathscr{D}$ is a principal polarization of product type on $E_1 \times E_2$. Since $K(\mathscr{D}) = 0$, one clearly has $t_x^* \mathscr{D} \neq \mathscr{D}$ for all $x \in \mathrm{Ker}(p) \setminus \{0\}$, so by Oda [**19**, Theorem 2, p. 248] the canonical injection $\mathscr{O}_{\hat{A}} \to \mathscr{E} \otimes \mathscr{E}^{\vee}$ induces isomorphisms

$$h^i(\hat{A}, \mathscr{O}_{\hat{A}}) \longrightarrow h^i(\hat{A}, \mathscr{E} \otimes \mathscr{E}^{\vee}), \quad i = 0, 1.$$

In particular, $H^0(\hat{A}, \mathscr{E} \otimes \mathscr{E}^{\vee}) \cong \mathbb{C}$, that is, $\mathscr{E}^{\vee}$ is simple. Then the claim follows from Corollary 4.2. □

Remark 5.5. The branch locus of the Albanese map $\hat{\alpha}\colon \hat{X} \to \hat{A}$ is a curve $\hat{B} = 2(\hat{B}_1 + \hat{B}_2)$, where $\hat{B}_1$ and $\hat{B}_2$ are the images via $p$ of two elliptic curves belonging to the two natural fibrations of $E_1 \times E_2$; then $\hat{B}_1 \hat{B}_2 = 3$. Note that $\hat{B}$ is not of the form described in Proposition 2.6; the reason is that the (1, 3)-polarized abelian surface $\hat{A} = (E_1 \times E_2)/H$ is not general, since it is an étale $\mathbb{Z}/3\mathbb{Z}$-quotient of a product of elliptic curves.

## 6. *Open problems*

(i) Is $\mathcal{M}_{\Phi}$ a *connected* component of $\mathcal{M}^{\mathrm{can}}_{2,2,6}$? This is equivalent to asking whether it is *open* therein; in other words, given a smooth family $\mathscr{X} \to \Delta$ over a small disk such that $X_0 \in \mathcal{M}_{\Phi}$, is $X_t \in \mathcal{M}_{\Phi}$ for $t$ small enough?
Note that any surface which is deformation equivalent to a surface in $\mathcal{M}_{\Phi}$ must have Albanese map of degree 4, since the Albanese degree of a surface with $q = 2$ and maximal Albanese dimension is a topological invariant; see [**8**, **9**, Section 5]. This leads naturally to the next question.
(ii) What are the possible degrees for the Albanese map of a minimal surface with $p_g = q = 2$ and $K^2 = 6$?
And, more generally we have the following question.
(iii) What are the irreducible/connected components of $\mathcal{M}^{\mathrm{can}}_{2,2,6}$?

## Appendix A. *The divisors corresponding to $X$, $Y$, $Z \in H^0(A, \mathscr{L})$*

Let $D_X$, $D_Y$, $D_Z$ be the three divisors on $A$ which correspond to the distinguished sections $X$, $Y$, $Z \in H^0(A, \mathscr{L})$ defined by the Schrödinger representation of the Heisenberg group $\mathscr{H}_3$; see Section 2. In this appendix, we show that, for a general choice of the pair $(A, \mathscr{L})$, the curves $D_X$, $D_Y$, $D_Z$ are smooth and intersect transversally. We believe that this fact is well known to the experts, however, we give a proof for lack of a suitable reference.

Let us start with a couple of auxiliary results.

Lemma A.1. *Let $(B, \mathscr{M})$ be a principally polarized abelian surface and $K_1 \cong \mathbb{Z}/3\mathbb{Z}$ be a subgroup of the group $B[3]$ of points of order 3 on $B$. Then there exist a $(1, 3)$-polarized abelian surface $(A, \mathscr{L})$ and an isogeny $f\colon A \to B$ of degree 3 such that the following conditions apply:*

(i) $\mathscr{L} = f^* \mathscr{M}$;
(ii) $K(\mathscr{L}) = \ker f \oplus f^{-1}(K_1)$.

*Proof.* In a suitable basis, the period matrix for $B$ is

$$\begin{pmatrix} z_{11} & z_{12} & 1 & 0 \\ z_{21} & z_{22} & 0 & 1 \end{pmatrix},$$

where $Z := \left(\begin{smallmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{smallmatrix}\right)$ satisfies ${}^tZ = Z$ and $\operatorname{Im} Z > 0$. Then $B = \mathbb{C}^2/\Lambda'$, with

$$\Lambda' = \lambda_1\mathbb{Z} \oplus \lambda_2\mathbb{Z} \oplus \mu_1\mathbb{Z} \oplus \mu_2\mathbb{Z}$$

and

$$\lambda_1 := \begin{pmatrix} z_{11} \\ z_{21} \end{pmatrix}, \quad \lambda_2 := \begin{pmatrix} z_{12} \\ z_{22} \end{pmatrix}, \quad \mu_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mu_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Up to a translation, we may assume $K_1 = \langle\overline{\lambda_2/3}\rangle$, where the symbol '$\bar{\ }$' denotes the image in the complex torus. The lattice

$$\Lambda := \lambda_1\mathbb{Z} \oplus \lambda_2\mathbb{Z} \oplus \mu_1\mathbb{Z} \oplus 3\mu_2\mathbb{Z}$$

verifies $[\Lambda' : \Lambda] = 3$, hence setting $A := \mathbb{C}^2/\Lambda$ the identity map $\mathbb{C}^2 \to \mathbb{C}^2$ induces a degree-3 isogeny $f\colon A \to B$ such that $\ker f = \langle\bar{\mu}_2\rangle$.

Now the polarization $\mathscr{L} := f^*\mathscr{M}$ is of type $(1, 3)$ and satisfies

$$K(\mathscr{L}) = \langle\bar{\mu}_2, \overline{\lambda_2/3}\rangle = \ker f \oplus f^{-1}(K_1). \qquad \square$$

LEMMA A.2. *Let $(A, \mathscr{L})$ and $(B, \mathscr{M})$ be as in Lemma A.1. Then, up to a simultaneous translation, the three divisors $D_X$, $D_Y$, $D_Z$ corresponding to $X$, $Y$, $Z \in H^0(A, \mathscr{L})$ are given by*

$$f^*M, \quad f^*t_x^*M, \quad f^*t_{2x}^*M,$$

*where $x \in B$ is a generator of $K_1$ and $M$ is the unique effective divisor in the linear system $|\mathscr{M}|$.*

*Proof.* Up to translation, we may assume that $\mathscr{L}$ is a line bundle of characteristic 0 with respect to the decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$, where $\Lambda_1 = \langle\bar{\lambda}_1, \bar{\lambda}_2\rangle$ and $\Lambda_2 = \langle\bar{\mu}_1, 3\bar{\mu}_2\rangle$. Hence, by the isogeny theorem for finite theta functions (see [**4**, p. 168]), it follows $D_X = f^*M$.

On the other hand, if $y \in A$ is any point such that $f(y) = x$, that is, $y \in f^{-1}(K_1) \subset K(\mathscr{L})$, then one has

$$f^*t_x^*M \in |t_y^*f^*\mathscr{M}| = |t_y^*\mathscr{L}| = |\mathscr{L}|,$$

and the same holds for $f^*t_{2x}^*M$. In other words, $f^*t_x^*M$ and $f^*t_{2x}^*M$ are effective divisors corresponding to sections of $H^0(A, \mathscr{L})$, and straightforward computations as in [**4**, Chapter 6] show that they are $D_Y$ and $D_Z$, respectively. $\square$

Now we can prove the desired result.

PROPOSITION A.3. *For a general choice of the pair $(A, \mathscr{L})$, the three divisors $D_X$, $D_Y$, $D_Z \in |\mathscr{L}|$ are smooth and intersect transversally.*

*Proof.* It is sufficient to exhibit an example in which each claim is satisfied.

First, take $B = J(C)$, the Jacobian of a smooth curve $C$ of genus 2, and let $\mathscr{M}$ be the natural principal polarization. Choose as $K_1$ any cyclic subset of $B[3]$ and construct $(A, \mathscr{L})$ as in Lemma A.1. The unique effective divisor $M \in |\mathscr{M}|$ is smooth, so the same is true for its translates $t_x^*M$ and $t_{2x}^*M$. By Lemma A.2, it follows that $D_X$, $D_Y$, $D_Z$ are smooth, too. Hence they are smooth for a general choice of the pair $(A, \mathscr{L})$.

Next, take $(B, \mathscr{M}) = (E_1 \times E_2, p_1^*\mathscr{M}_1 \otimes p_2^*\mathscr{M}_2)$, where $E_i$ is an elliptic curve, $\mathscr{M}_i$ is a divisor of degree 1 on $E_i$ and $p_i \colon E_1 \times E_2 \to E_i$ is the projection. Now choose a 3-torsion point on $E_1 \times E_2$ of the form $x = (x_1, x_2)$, where $x_i$ is a non-trivial 3-torsion point on $E_i$, take $K_1 = \langle x \rangle$ and construct $(A, \mathscr{L})$ as in Lemma A.1. If $M$ is the unique effective divisor in $|\mathscr{M}|$, then the three divisors $M$, $t_x^*M$, $t_{2x}^*M$ are pairwise without common components and intersect transversally, hence by Lemma A.2 the same is true for $D_X$, $D_Y$, $D_Z$. Therefore, $D_X$, $D_Y$, $D_Z$ intersect transversally for a general choice of the pair $(A, \mathscr{L})$.

This completes the proof. □

Remark A.4. One can show that $(-1)_A \colon A \to A$ acts on $D_X$ as an involution, whereas $D_Y$ and $D_Z$ are exchanged by $(-1)_A$. See [4, Exercise 13, p. 177].

## Appendix B. *The `MAGMA` script used to calculate the equation of the branch divisor of* $\alpha : S \to A$

```
QuadrupleCover:=function(Q,a,g)
//
// Given the field Q and the two parameters a and g of Proposition 2.4, the function QuadrupleCover
// gives:
// (1) the local equations of S "CoverAff5";
// (2) the local equation of the ramification divisor "RamCurve";
// (3) the local equation of the branch locus "AffBranch". Moreover
// (4) it checks whether these objects are singular or not, and it gives, if
// possible, a description of their singularities.
//
R<Y,Z,u,v,w>:=PolynomialRing(Q,5);
// Here Y,Z are sections in H^0(A,L)
A5 <Y,Z,u,v,w>:= AffineSpace(Q,5);
/////////////////////////////////////////////////////////////////////////////
// The local model of the cover S
/////////////////////////////////////////////////////////////////////////////
CoverAff5:=function(A5,R,a,g,T)
//
// Given a 5-dimensional affine space A5, a polynomial ring in the same
// coordinates R and three integers a,g,T (T is a translation), the function "CoverAff5"
// returns an affine scheme in A5 which is the local model of S with local equations in coordinates
// Y,Z,u,v,w.
//
b := -a; d := 0; if g eq 0 then e := 1; else e := -a^2/g; end if;
////////////////////////////////////////////////////////////////////////////
c12:=a*(Z-T); c13:=b*(Y-T); c14:=g*(Y-T); c15:=d; c16:=-g*(Z-T);
c23:=e; c24:=-b; c25:=-e*(Z-T); c26:=-d*(Y-T); c34:=-d*(Z-T);
c35:=e*(Y-T); c36:=-a; c45:=a*(Y-T); c46:=g; c56:=-b*(Z-T);
/////////////////////////////////////////////////////////////////////////////
a1:=1/2*c23; a2:=-c13; a3:=c12; a4:=1/4*c25-1/2*c34;
a5:=-1/2*c15-1/4*c23; a6:=c14; a7:=1/2*c26-1/4*c35; a8:=-c16;
a9:=1/2*c15-1/4*c23; a10:=c45; a11:=-1/2*c25; a12:=c24; a13:=c46;
a14:=-1/2*c26-1/4*c35; a15:=1/4*c25+1/2*c34; a16:=c56; a17:=-c36;
a18:=1/2*c35;
/////////////////////////////////////////////////////////////////////////////
b1:=-a1*a5+a2*a4-a2*a11-a3*a14+(a5)^2+a6*a8;
b2:=a2*a10+a3*a13-a4*a5-a6*a7; b3:=a2*a13+a3*a16-a4*a8-a7*a9;
b4:=-a1*a10+(a4)^2-a4*a11+a5*a10+a6*a13-a7*a12;
b5:=-a5*a13+a8*a10+a12*a17-a14*a15;
b6:=-a1*a16-a4*a17+(a7)^2-a7*a18+a8*a13+a9*a16;
/////////////////////////////////////////////////////////////////////////////
f1:=u^2-(a1*u+a2*v+a3*w+b1); f2:=u*v-(a4*u+a5*v+a6*w+b2);
f3:=u*w-(a7*u+a8*v+a9*w+b3); f4:=v^2-(a10*u+a11*v+a12*w+b4);
f5:=v*w-(a13*u+a14*v+a15*w+b5); f6:=w^2-(a16*u+a17*v+a18*w+b6);
/////////////////////////////////////////////////////////////////////////////
I:=ideal<R|[f1,f2,f3,f4,f5,f6]>;
/////////////////////////////////////////////////////////////////////////////
return Scheme(A5,Generators(I)); end function;
/////////////////////////////////////////////////////////////////////////////
// The ramification curve
/////////////////////////////////////////////////////////////////////////////
RamCurveA5:=function(Cover,A5,R)
```

```
//
// Given a local model of the "Cover", the affine space A5 in which the
// cover sits and a polynomial ring in the same coordinates,
// the function "RamCurveA5" returns an affine scheme in A5 which is
// the ramification curve of \ALPHA.
//
f1:= Generators(Ideal(Cover))[1]; f2:= Generators(Ideal(Cover))[2];
f3:= Generators(Ideal(Cover))[3]; f4:= Generators(Ideal(Cover))[4];
f5:= Generators(Ideal(Cover))[5]; f6:= Generators(Ideal(Cover))[6];
/////////////////////////////////////////////////////////////////////////////
I:=ideal<R|[f1,f2,f3,f4,f5,f6]>;
/////////////////////////////////////////////////////////////////////////////
L:=[Derivative(f1,u),Derivative(f1,v),Derivative(f1,w),
    Derivative(f2,u),Derivative(f2,v),Derivative(f2,w),
    Derivative(f3,u),Derivative(f3,v),Derivative(f3,w),
    Derivative(f4,u),Derivative(f4,v),Derivative(f4,w),
    Derivative(f5,u),Derivative(f5,v),Derivative(f5,w),
    Derivative(f6,u),Derivative(f6,v),Derivative(f6,w)];
/////////////////////////////////////////////////////////////////////////////
J:=Matrix(R,6,3,L); Ms:=Minors(J,3); J3:=ideal<R | Ms>;
RJ3:=Radical(J3); H:=RJ3+I; U:=Generators(H);
/////////////////////////////////////////////////////////////////////////////
return Scheme(A5,U); end function;
/////////////////////////////////////////////////////////////////////////////
// MAIN ROUTINE
/////////////////////////////////////////////////////////////////////////////
a:=a; g:=g; T:=0;
/////////////////////////////////////////////////////////////////////////////
printf "\n --------BEGINNING------- \n ";
printf "\n a=%o,\n g=%o,\n Q=%o \n", a,g,Q;
/////////////////////////////////////////////////////////////////////////////
CoverA5:=CoverAff5(A5,R,a,g,T); printf "\n The cover is a  ";
CoverA5; printf "\n The affine dimension of the cover is ";
Dimension(CoverA5); printf "The cover is singular: ";
IsSingular(CoverA5); if IsSingular(CoverA5) then
SingSchCoverA5:=SingularSubscheme(CoverA5); printf"The dimension of
the singular locus is: "; Dimension(SingSchCoverA5); if
Dimension(SingSchCoverA5) eq 0 then
SingPCoverA5:=SingularPoints(CoverA5); printf"The rational singular
points are: "; SingPCoverA5; else printf"The cover is not normal!";
end if; else RamCurveA5:=RamCurveA5(CoverA5,A5,R); printf "\n The
ramification divisor is a  "; RamCurveA5; printf "The affine
dimension of the ramification divisor is "; Dimension(RamCurveA5);
printf"The ramification divisor is singular: ";
IsSingular(RamCurveA5); printf"\n \n
---------------------------- \n \n";
/////////////////////////////////////////////////////////////////////////////
// The branch curve
/////////////////////////////////////////////////////////////////////////////
A2<q,l>:=AffineSpace(Q,2); f := map< A5 -> A2 | [Y,Z] >;
AffBranch:=f(RamCurveA5); printf"The branch divisor is singular:
"; IsSingular(AffBranch); if IsSingular(AffBranch) then
SingSchBranch:=SingularSubscheme(AffBranch); printf"The dimension
of its singular locus is "; Dimension(SingSchBranch); end if; if
Dimension(SingSchBranch) eq 0 then
SingPBranch:=SingularPoints(AffBranch); printf"The rational
singular points of the branch are "; SingPBranch; printf" The
branch is a "; AffBranch; printf" \n \n
---------------------------- \n \n";
/////////////////////////////////////////////////////////////////////////////
end if; end if; printf" \n ------------THE END---------- \n"; return
0; end function;
/////////////////////////////////////////////////////////////////////////////
```

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