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Rida T. Farouki
Department of Mechanical and Aerospace Engineering,
University of California, Davis, CA 95616, USA.

Francesca Pelosi
Dipartimento di Matematica, Università di Roma "Tor Vergata,"
Via della Ricerca Scientifica, 00133 Roma, Italy.

Maria Lucia Sampoli
Dipartimento di Ingegneria dell'Informazione e
Scienze Matematiche, Università di Siena, San Niccolò,
Via Roma 56, 53100 Siena, Italy.

Abstract

In the construction and analysis of a planar Pythagorean–hodograph (PH) quintic curve $\mathbf{r}(t)$, $t \in [\,0,1\,]$ using the complex representation, it is convenient to invoke a translation/rotation/scaling transformation so $\mathbf{r}(t)$ is in *canonical form* with $\mathbf{r}(0) = 0$, $\mathbf{r}(1) = 1$ and possesses just two complex degrees of freedom. By choosing two of the five control–polygon legs of a quintic PH curve as these free complex parameters, the remaining three control–polygon legs can be expressed in terms of them and the roots of a quadratic or quartic equation. Consequently, depending on the chosen two control–polygon legs, there exist either two or four distinct quintic PH curves that are consistent with them. A comprehensive analysis of all possible pairs of chosen control polygon legs is developed, and examples are provided to illustrate this control–polygon paradigm for the construction of planar quintic PH curves.

Keywords: Pythagorean–hodograph curves; complex polynomials; control–polygon constraints; quadratic and quartic equations.

e–mail: farouki@ucdavis.edu, pelosi@mat.uniroma2.it, marialucia.sampoli@unisi.it

1 Introduction

The origins of the ubiquitous control–polygon paradigm for constructing and manipulating free–form curves can be traced to the pioneering ideas of Paul de Casteljau and Pierre Bézier, two French engineers who were employed by the automotive companies Citroën and Renault, respectively. Although their original approaches differed from modern formulations based on the Bernstein polynomial basis (and its B–spline extension to piecewise–polynomial forms), they exhibit clear links with them. De Casteljau's *courbes et surfaces à pôles*, based on using "pilot points" to design curves and surfaces [1, 2], is closer in spirit to the modern formulation than Bézier's original approach [3, 4]. The focus of the present study is to elucidate use of the control–polygon paradigm in the context of the planar Pythagorean–hodograph (PH) curves.

The algebraic structure of the PH curves facilitates an exact computation of various properties (arc lengths, offset curves, rotation–minimizing frames, etc.) that otherwise necessitate numerical approximations [11]. However, the non–linear nature of the PH curves makes their construction, consistent with prescribed geometrical constraints, more challenging — as evident in diverse algorithms addressing this task [7, 12, 13, 17, 18, 19, 22, 25, 26, 27, 28].

Since they can match first–order Hermite data and exhibit inflections [6], planar quintic PH curves have sufficient shape flexibility to satisfy free–form design requirements. In order to simplify the construction and shape analysis of a planar quintic PH curve $\mathbf{r}(t)$, $t \in [\,0,1\,]$ it is advantageous to first invoke a translation/rotation/scaling transformation to eliminate all non–essential degrees of freedom, so that $\mathbf{r}(0) = (0,0)$ and $\mathbf{r}(1) = (1,0)$ — the planar PH quintic $\mathbf{r}(t)$ is then said to be in *canonical form*.

The canonical form has been employed to facilitate construction of planar [7] and spatial [8] quintic PH curve interpolants to G^1 data (end points and unit tangents) with prescribed arc lengths, and also in identifying the planar quintic PH curve that is closest to any given "ordinary" Bézier curve [9]. The canonical form also provides a convenient point of departure in characterizing the intrinsic shape features of planar PH quintic Hermite interpolants.

Alternative PH curve control–polygon formulations have been proposed that, unlike the Bézier control polygon, represent the actual number of shape freedoms. The Gauss–Legendre (or rectifying) control polygon matches the end points (but not tangents) of a PH curve, and the polygon length coincides with the curve arc length [20, 24]. The Gauss–Lobatto control polygon [21] matches both the end points and end tangents, but relinquishes the arc length property. However, these formulations forfeit some desirable features of the Bézier form (e.g., the convell hull and variation–diminishing properties, and the association of a unique curve with any given control polygon).

Employing the complex representation [5], wherein a planar PH quintic is generated from a quadratic pre–image polynomial $\mathbf{w}(t)$ by integrating the expression $\mathbf{r}'(t) = \mathbf{w}^2(t)$, a canonical–form quintic PH curve embodies just two complex degrees of freedom. Identifying two of the five control–polygon legs of a PH quintic as free parameters, it is shown herein that the other three control–polygon legs can be determined by simple algorithms involving only the solution of a complex quadratic or quartic equation, whose coefficients depend on the two prescribed control–polygon legs.

The remainder of this paper is organized as follows. After a brief review of the complex representation and basic properties of planar quintic PH curves in Section 2, the system of control–polygon constraints that identify the PH quintics among all planar quintic Bézier curves is summarized in Section 3. Based on these constraints, algorithms are developed in Section 4 that allow for any two of the five control–polygon legs of a canonical–form PH quintic to be prescribed, with an automatic "filling in" of the remaining three legs. Some computed examples are presented in Section 5 to illustrate results from these algorithms. Finally, Section 6 summarizes the key results of the present study, and identifies some aspects that deserve further investigation.

2 Planar quintic PH curves

In terms of the Bernstein basis on $t \in [\,0,1\,]$ defined by

$$b_i^n(t) \,=\, \binom{n}{i}(1-t)^{n-i}t^i\,, \quad i = 0,\ldots,n\,,$$

a planar PH quintic $\mathbf{r}(t)$ may be constructed [5] from a quadratic polynomial

$$\mathbf{w}(t) \,=\, \mathbf{w}_0\,b_0^2(t) + \mathbf{w}_1\,b_1^2(t) + \mathbf{w}_2\,b_2^2(t) \tag{1}$$

with complex coefficients $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ by integrating $\mathbf{r}'(t) = \mathbf{w}^2(t)$. This yields the complex control points $\mathbf{p}_0, \ldots, \mathbf{p}_5$ of the Bézier form

$$\mathbf{r}(t) \,=\, \sum_{k=0}^{5} \mathbf{p}_k\,b_k^5(t)$$

as

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{p}_0 + \frac{1}{5}\,\mathbf{w}_0^2\,, \\
\mathbf{p}_2 &= \mathbf{p}_1 + \frac{1}{5}\,\mathbf{w}_0\mathbf{w}_1\,, \\
\mathbf{p}_3 &= \mathbf{p}_2 + \frac{1}{5}\,\frac{2\,\mathbf{w}_1^2 + \mathbf{w}_0\mathbf{w}_2}{3}\,, \\
\mathbf{p}_4 &= \mathbf{p}_3 + \frac{1}{5}\,\mathbf{w}_1\mathbf{w}_2\,, \\
\mathbf{p}_5 &= \mathbf{p}_4 + \frac{1}{5}\,\mathbf{w}_2^2\,,
\end{aligned} \tag{2}$$

where $\mathbf{p}_0$ is a freely–chosen integration constant.

The parametric speed $\sigma(t) = |\mathbf{r}'(t)| = |\mathbf{w}(t)|^2$, i.e., the derivative $\mathrm{d}s/\mathrm{d}t$ of arc length s with respect to the curve parameter t, is a *polynomial* in the parameter. We focus on curves $\mathbf{r}(t) = (x(t), y(t))$ with *primitive* hodographs satisfying $\gcd(x'(t), y'(t)) = 1$, that generate *regular* curves with $\sigma(t) \neq 0$ for $t \in [\,0, 1\,]$. With $\mathbf{w}(t) = u(t) + \mathrm{i}\,v(t)$, regular curves are characterized by the condition $\gcd(u(t), v(t)) = 1$. Note also that, since $\mathbf{r}(t)$ has the end–point derivatives $\mathbf{r}'(0) = \mathbf{w}_0^2$ and $\mathbf{r}'(1) = \mathbf{w}_2^2$, we must have $\mathbf{w}_0 \neq 0$ and $\mathbf{w}_2 \neq 0$.

Remark 1. One may also consider primitive Pythagorean hodographs of the form $\mathbf{r}'(t) = f(t)\,\mathbf{w}^2(t)$ where $\gcd(u(t), v(t)) = 1$ and $f(t)$ is a real square–free polynomial, that generate regular PH curves when $f(t)$ has no roots on $[\,0, 1\,]$. Suck curves possess less shape freedom than PH curves with $f(t) = 1$ of the same degree, since $f(t)$ influences the *magnitude* of $\mathbf{r}'(t)$ but not its *direction.* For example, PH quintics with $\deg(\mathbf{w}(t)) = 1$ and $\deg(f(t)) = 2$ cannot have inflections, unlike the PH quintics with $\deg(\mathbf{w}(t)) = 2$ and $f(t) = 1$.

The curvature of a PH curve $\mathbf{r}(t)$ generated by the pre–image polynomial $\mathbf{w}(t)$ may be expressed [6] as

$$\kappa(t) \,=\, 2\,\frac{\mathrm{Im}(\overline{\mathbf{w}}(t)\mathbf{w}'(t))}{|\mathbf{w}(t)|^4}\,. \tag{3}$$

In the case of a PH quintic, the numerator in this expression is the quadratic polynomial

$$2\,\mathrm{Im}(\overline{\mathbf{w}}_0\mathbf{w}_1)\,b_0^2(t) - \mathrm{Im}(\overline{\mathbf{w}}_2\mathbf{w}_0)\,b_1^2(t) + 2\,\mathrm{Im}(\overline{\mathbf{w}}_1\mathbf{w}_2)\,b_2^2(t)\,, \tag{4}$$

whose (odd–multiplicity) real roots, if any, identify *inflections* of $\mathbf{r}(t)$. There are two inflections — not necessarily within the $[\,0,1\,]$ parameter domain — if the discriminant

$$\Delta \,=\, \mathrm{Im}^2(\overline{\mathbf{w}}_2\mathbf{w}_0) - 4\,\mathrm{Im}(\overline{\mathbf{w}}_0\mathbf{w}_1)\,\mathrm{Im}(\overline{\mathbf{w}}_1\mathbf{w}_2) \tag{5}$$

of (4) is positive, and none if it is negative. If $\Delta = 0$, the expression (4) has a double root, which identifies a point where $\kappa(t) = \kappa'(t) = 0$. The end–point curvatures are $\kappa(0) = 4\,\mathrm{Im}(\overline{\mathbf{w}}_0\mathbf{w}_1)/|\mathbf{w}_0|^4$ and $\kappa(1) = 4\,\mathrm{Im}(\overline{\mathbf{w}}_1\mathbf{w}_2)/|\mathbf{w}_2|^4$.

Choosing the integration constant $\mathbf{r}(0) = 0$ on integrating $\mathbf{r}'(t) = \mathbf{w}^2(t)$, the canonical–form end point $\mathbf{r}(1) = 1$ is achieved by satisfying the condition

$$\int_0^1 \mathbf{r}'(t)\,\mathrm{d}t \,=\, \mathbf{r}(1) - \mathbf{r}(0) \,=\, 1\,, \tag{6}$$

which reduces [7] to the quadratic equation

$$2\,\mathbf{w}_1^2 + 3\,(\mathbf{w}_0 + \mathbf{w}_2)\mathbf{w}_1 + 3\,(\mathbf{w}_0^2 + \mathbf{w}_2^2) + \mathbf{w}_0\mathbf{w}_2 - 15 \,=\, 0\,. \tag{7}$$

3 Control-polygon constraints

Given a complex pre–image polynomial $\mathbf{w}(t)$ of degree m, a planar PH curve $\mathbf{r}(t)$ of degree $n = 2m+1$ is generated by integrating $\mathbf{r}'(t) = \mathbf{w}^2(t)$. Choosing the integration constant $\mathbf{r}(0) = 0$ and imposing the condition (6) yields a curve $\mathbf{r}(t)$ in canonical form with $\mathbf{r}(1) = 1$, which may be expressed in terms of control points $\mathbf{p}_0, \ldots, \mathbf{p}_n$ as

$$\mathbf{r}(t) \,=\, \sum_{i=0}^{n} \mathbf{p}_i\, b_i^n(t)\,,$$

where $\mathbf{p}_0 = 0$ and $\mathbf{p}_n = 1$. The complex values

$$\mathbf{L}_i \,=\, \mathbf{p}_i - \mathbf{p}_{i-1}\,, \quad i = 1, \ldots, n$$

identify the control–polygon "legs" of $\mathbf{r}(t)$, which satisfy $\mathbf{L}_1 + \cdots + \mathbf{L}_n = 1$. Whereas a control polygon uniquely specifies a PH curve constructed in this manner, not all choices for $\mathbf{L}_1, \ldots, \mathbf{L}_n$ will define a PH curve.

A planar PH curve of degree n is generated from a pre–image polynomial $\mathbf{w}(t)$ of degree $m = \frac{1}{2}(n-1)$ that has $m+1 = \frac{1}{2}(n+1)$ complex coefficients.

Hence, the control–polygon legs $\mathbf{L}_1, \ldots, \mathbf{L}_n$ must be subject to $n - \frac{1}{2}(n+1) = \frac{1}{2}(n-1)$ complex constraints. However, for a canonical–form planar PH curve, also imposing the end–point condition (6) increases the number of constraints to $\frac{1}{2}(n+1)$, and reduces the number of complex degrees of freedom to $\frac{1}{2}(n-1)$ — namely, one for a PH cubic, two for a PH quintic, etc.

The simplest non–trivial PH curves are the cubics, which are identified [5] by the constraint

$$\mathbf{L}_2^2 \,=\, \mathbf{L}_1\mathbf{L}_3 \tag{8}$$

on the control–polygon legs. The simplicity of this relation reflects the fact that all PH cubics are merely translated/scaled/rotated segments of a unique non–inflectional curve — the *Tschirnhaus cubic* [15].

The quintics are the lowest–order PH curves that are generally considered to be suitable for free–form design applications. For a planar PH quintic $\mathbf{r}(t)$ in canonical form, the coefficients of the pre–image polynomial (1) satisfy the constraint (7), and from (2) the control–polygon legs are related to the coefficients $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ of (1) by the equations

$$\left(\mathbf{w}_0^2, \mathbf{w}_0\mathbf{w}_1, \frac{2\,\mathbf{w}_1^2 + \mathbf{w}_0\mathbf{w}_2}{3}, \mathbf{w}_1\mathbf{w}_2, \mathbf{w}_2^2\right) \,=\, 5\,(\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_4, \mathbf{L}_5)\,. \tag{9}$$

It was shown in Proposition 5.1 of [5] that — for the generic case with $\mathbf{L}_i \neq 0$, $i = 1, \ldots, 5$ — the satisfaction of

$$\mathbf{L}_1\mathbf{L}_4^2 \,=\, \mathbf{L}_5\mathbf{L}_2^2 \tag{10}$$

and any one of the four equations

$$3\,\mathbf{L}_1\,\mathbf{L}_2\,\mathbf{L}_3 - \mathbf{L}_1^2\,\mathbf{L}_4 - 2\,\mathbf{L}_2^3 \,=\, 0\,, \tag{11}$$

$$3\,\mathbf{L}_5\,\mathbf{L}_4\,\mathbf{L}_3 - \mathbf{L}_5^2\,\mathbf{L}_2 - 2\,\mathbf{L}_4^3 \,=\, 0\,, \tag{12}$$

$$3\,\mathbf{L}_1\,\mathbf{L}_4\,\mathbf{L}_3 - \mathbf{L}_5\,\mathbf{L}_1\,\mathbf{L}_2 - 2\,\mathbf{L}_2^2\,\mathbf{L}_4 \,=\, 0\,, \tag{13}$$

$$3\,\mathbf{L}_5\,\mathbf{L}_2\,\mathbf{L}_3 - \mathbf{L}_1\,\mathbf{L}_5\,\mathbf{L}_4 - 2\,\mathbf{L}_4^2\,\mathbf{L}_2 \,=\, 0\,, \tag{14}$$

is sufficient and necessary for a quintic Bézier curve to be a PH curve. For a canonical–form curve, the control–polygon legs must also satisfy

$$\mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 + \mathbf{L}_4 + \mathbf{L}_5 \,=\, 1\,. \tag{15}$$

Remark 2. The canonical–form constraint (15) is not appropriate to curves with $\mathbf{r}(1) = \mathbf{r}(0)$ that are intended to form closed loops, and must be replaced by $\mathbf{L}_1+\mathbf{L}_2+\mathbf{L}_3+\mathbf{L}_4+\mathbf{L}_5=0$ in such cases. An arc length constraint may be imposed [7] to ensure that such curves do not degenerate to a single point. Their construction through control–polygon constraints is closely analogous to that developed below using (15) in the case $\mathbf{r}(1)\neq\mathbf{r}(0)$.

Note that equations (11) and (12) are equivalent under a reverse–ordering of the control–polygon leg indices (which amounts to the reparameterization $t\to 1-t$), and this also holds for equations (13) and (14). Moreover, each of the equations (11)–(14) furnishes a different expression for $\mathbf{L}_3$ in terms of $\mathbf{L}_1,\mathbf{L}_2,\mathbf{L}_4,\mathbf{L}_5$ — the satisfaction of equation (10) guarantees consistency of these different expressions. It is possible to identify further constraints that are invariant with respect reverse–ordering of the indices, for example

$$9\,\mathbf{L}_1\mathbf{L}_5\mathbf{L}_3^2 \;=\; (2\,\mathbf{L}_2\mathbf{L}_4+\mathbf{L}_1\mathbf{L}_5)^2\,, \tag{16}$$

but this is quartic, rather than cubic, in $\mathbf{L}_1,\mathbf{L}_2,\mathbf{L}_3,\mathbf{L}_4,\mathbf{L}_5$.

Remark 3. In the degenerate case $\mathbf{L}_2=\mathbf{L}_4=0$, there are only four distinct control points $\mathbf{p}_0,\mathbf{p}_1=\mathbf{p}_2,\mathbf{p}_3=\mathbf{p}_4,\mathbf{p}_5$ and all of the equations (10)–(14) are trivially satisfied. This case occurs [5] when the condition

$$9\,\mathbf{L}_3^2 \;=\; \mathbf{L}_1\mathbf{L}_5 \tag{17}$$

is satisfied — any two of $\mathbf{L}_1,\mathbf{L}_3,\mathbf{L}_5$ may be chosen, the remaining one being determined from (17). Equation (17) is analogous to the PH cubic condition (8), so the control polygons of these PH quintics have identical interior angles and sides satisfying $3\,L_3=\sqrt{L_1L_5}$, by analogy with the PH cubic case [5].

Since canonical–form quintic PH curves satisfy three complex constraints — namely, equations (10) and (15) and any one of equations (11)–(14) — on the control–polygon legs $\mathbf{L}_1,\ldots,\mathbf{L}_5$, they incorporate only two free complex parameters. Thus, if we choose any two of the control–polygon legs $\mathbf{L}_1,\ldots,\mathbf{L}_5$ the remaining three should be expressible in terms of the chosen two.

The existence of control–polygon constraints for PH curves complicates their design through manipulation of control points, as in the standard Bézier curve paradigm. However, we show here that any two of the control polygon legs $\mathbf{L}_1,\ldots,\mathbf{L}_5$ may be freely assigned, with the remaining three being then automatically "filled in" through a simple algorithm. This may be considered a more "geometrical" approach to constructing PH quintics than first–order Hermite interpolation [13], which is largely algebraic in nature.

4 Construction of quintic PH curves

We now present a comprehensive analysis of how the quintic PH curves can be constructed by choosing any two of the five control–polygon legs $\mathbf{L}_1, \ldots, \mathbf{L}_5$ with the remaining three being determined by solving a quadratic or quartic equation whose coefficients depend on the chosen two. As is typical with PH curve constructions [13], this yields a multiplicity of formal solutions, among which the "good" solution must be identified by a suitable shape measure.

There are ten possible ways to choose two among the five control–polygon legs, of which only two — $\mathbf{L}_1, \mathbf{L}_5$ and $\mathbf{L}_2, \mathbf{L}_4$ — have symmetric indices. The remaining eight cases can be assembled into four pairs, whose members are equivalent under reverse–ordering of the control–polygon leg indices.

4.1 Symmetric choices

We consider first the two symmetric choices ($\mathbf{L}_1, \mathbf{L}_5$ and $\mathbf{L}_2, \mathbf{L}_4$) of the control–polygon legs, and treat the asymmetric cases in Section 4.2.

4.1.1 $\mathbf{L}_1, \mathbf{L}_5$ as free parameters

Pre–assigning $\mathbf{L}_1$ and $\mathbf{L}_5$ is equivalent to the first–order Hermite interpolation problem with canonical–form PH quintics, since $\mathbf{r}'(0) = 5\,\mathbf{L}_1$ and $\mathbf{r}'(1) = 5\,\mathbf{L}_5$. Although established algorithms for this problem are already available [13], the present approach offers a rather different perspective.

Remark 4. For any complex number $\mathbf{x} \neq 0$, the expression $\sqrt{\mathbf{x}}$ henceforth denotes the *principal value* of its square root, satisfying $\mathrm{Re}(\sqrt{\mathbf{x}}\,) \geq 0$ if $\mathbf{x}$ is not a negative real number, and $\mathrm{Im}(\sqrt{\mathbf{x}}\,) \geq 0$ if it is. The two square roots of $\mathbf{x}$ can then be specified by $s\sqrt{\mathbf{x}}$ with $s = \pm 1$.

Proposition 1. *The control polygon of a canonical–form PH quintic can be expressed in terms of* $\mathbf{L}_1$ *and* $\mathbf{L}_5$ *as free parameters through the relations*

$$\mathbf{L}_2 \,=\, \mathbf{z}\, s_1 \sqrt{\mathbf{L}_1}\,, \quad \mathbf{L}_4 \,=\, \mathbf{z}\, s_2 \sqrt{\mathbf{L}_5}\,, \tag{18}$$

$$\mathbf{L}_3 \,=\, 1 - \mathbf{L}_1 - \mathbf{L}_5 - \mathbf{z}\,(s_1\sqrt{\mathbf{L}_1} + s_2\sqrt{\mathbf{L}_5}\,)\,, \tag{19}$$

where $s_1, s_2 = \pm 1$ *are independent sign choices, and* $\mathbf{z}$ *is a root of the complex quadratic equation*

$$2\,\mathbf{z}^2 + 3\,(s_1\sqrt{\mathbf{L}_1} + s_2\sqrt{\mathbf{L}_5}\,)\,\mathbf{z} + 3\,(\mathbf{L}_1 + \mathbf{L}_5 - 1) + s_1 s_2 \sqrt{\mathbf{L}_1}\sqrt{\mathbf{L}_5} \,=\, 0\,. \tag{20}$$

Proof : For non–zero $\mathbf{L}_1, \mathbf{L}_5$ equation (10) implies that $\mathbf{L}_2, \mathbf{L}_4$ must be of the form (18) for some complex number $\mathbf{z}$. Substituting $\mathbf{L}_3 = 1 - \mathbf{L}_1 - \mathbf{L}_2 - \mathbf{L}_4 - \mathbf{L}_5$ from (15) and the expressions (18) for $\mathbf{L}_2, \mathbf{L}_4$ into (11) yields the quadratic equation (20) for $\mathbf{z}$, and $\mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_4$ are then determined in terms of $\mathbf{L}_1, \mathbf{L}_5$ and a root $\mathbf{z}$ of (20) by the expressions (18)–(19). ∎

By Proposition 1, the control points of a canonical–form PH quintic with prescribed control–polygon legs $\mathbf{L}_1, \mathbf{L}_5$ can be expressed in terms of them and a root $\mathbf{z}$ of equation (20) as

$$\mathbf{p}_0 \,=\, 0\,, \quad \mathbf{p}_1 \,=\, \mathbf{L}_1\,, \quad \mathbf{p}_2 \,=\, \mathbf{L}_1 + \mathbf{z}\, s_1 \sqrt{\mathbf{L}_1}\,,$$

$$\mathbf{p}_3 \,=\, 1 - \mathbf{L}_5 - \mathbf{z}\, s_2 \sqrt{\mathbf{L}_5}\,, \quad \mathbf{p}_4 \,=\, 1 - \mathbf{L}_5\,, \quad \mathbf{p}_5 \,=\, 1\,.$$

Since equation (20) has two solutions for each of the four possible s_1, s_2 combinations, it might seem that there are in total eight solutions. However, these solutions occur in pairs that identify identical curves, so there are only four *distinct* PH quintics consistent with any prescribed $\mathbf{L}_1, \mathbf{L}_5$. This may be verified as follows. The complete set of roots of equation (20) is specified by

$$\mathbf{z} \,=\, \frac{-\,3\,(s_1\sqrt{\mathbf{L}_1} + s_2\sqrt{\mathbf{L}_5}\,) + s_3\sqrt{\boldsymbol{\Delta}}}{4}\,, \tag{21}$$

where $s_3 = \pm 1$, and the discriminant $\boldsymbol{\Delta}$ of (20) may be written as

$$\boldsymbol{\Delta} \,=\, 24 - 15\,(\mathbf{L}_1 + \mathbf{L}_5) + 10\, s_1 s_2 \sqrt{\mathbf{L}_1}\sqrt{\mathbf{L}_5}\,.$$

Now the control–polygon legs (18)–(19) depend only on the quantities

$$\mathbf{z}\, s_1 \,=\, \frac{-\,3\,(\sqrt{\mathbf{L}_1} + s_1 s_2\sqrt{\mathbf{L}_5}\,) + s_1 s_3\sqrt{\boldsymbol{\Delta}}}{4}\,,$$

$$\mathbf{z}\, s_2 \,=\, \frac{-\,3\,(s_1 s_2\sqrt{\mathbf{L}_1} + \sqrt{\mathbf{L}_5}\,) + s_2 s_3\sqrt{\boldsymbol{\Delta}}}{4}\,.$$

Hence $\boldsymbol{\Delta}$ and $\mathbf{z}\, s_1, \mathbf{z}\, s_2$ — and consequently also the control polygon — remain unchanged on replacing any choice of signs (s_0, s_1, s_2) by $(-s_0, -s_1, -s_2)$.

To generate the four distinct solutions without replication, it suffices to fix $s_3 = 1$ in (21) and exercise only the two sign choices s_1, s_2. An analogous situation holds in the first–order PH quintic Hermite interpolation problem [13], although the specific details in that context are rather different.

Corollary 1. *For a canonical–form PH quintic with specified control–polygon legs* $\mathbf{L}_1, \mathbf{L}_5$ *and a root* $\mathbf{z}$ *of equation* (20)*, the coefficients of the pre–image polynomial* (1) *may be identified as*

$$\mathbf{w}_0 \,=\, s_1\sqrt{5\,\mathbf{L}_1}\,, \quad \mathbf{w}_1 \,=\, \sqrt{5}\,\mathbf{z}\,, \quad \mathbf{w}_2 \,=\, s_2\sqrt{5\,\mathbf{L}_5}\,. \tag{22}$$

Proof : From the first and last of equations (9), we have $\mathbf{w}_0 = s_1\sqrt{5\,\mathbf{L}_1}$ and $\mathbf{w}_2 = s_2\sqrt{5\,\mathbf{L}_5}$, where $s_1, s_2 = \pm 1$ are independent sign choices. Substituting these expressions into the second and fourth of equations (9), and invoking equations (18), then yields $\mathbf{w}_1 = \sqrt{5}\,\mathbf{z}$. It remains to check that these values are consistent with the third of equations (9). Substituting from (18)–(19) and (22) into that equation and simplifying yields equation (20), so the third of equations (9) is also satisfied when $\mathbf{z}$ is a root of (20). ∎

One of the four solutions is a curve of fair shape, and the remaining three typically exhibit tight loops at one or both ends of the curve — the "good" solution is usually identified as that with the smallest *absolute rotation index* [13], defined by

$$R_{\mathrm{abs}} \,=\, \frac{1}{2\pi}\int_0^1 |\kappa(t)|\,\sigma(t)\,\mathrm{d}t\,, \tag{23}$$

where $\kappa(t)$ is the curvature (3) and $\sigma(t) = |\mathbf{w}(t)|^2$ is the derivative $\mathrm{d}s/\mathrm{d}t$ of arc length s with respect to the parameter t. R_{abs} indicates the total turning of the curve tangent (without cancelling clockwise and anti–clockwise rotation), and admits exact evaluation [13] for quintic PH curves. An alternative to (23) for identifying the good interpolant is based on comparing the four solutions with the "ordinary" cubic interpolant to the given Hermite data. It has been shown [23] that a specific choice of signs identifies the good interpolant when $\mathbf{L}_1, \mathbf{L}_5$ have positive real parts and magnitudes satisfying $|\mathbf{L}_1|, |\mathbf{L}_5| < 3/5$.

4.1.2 $\mathbf{L}_2, \mathbf{L}_4$ as free parameters

As a novel approach to constructing planar PH quintics, we now consider pre–assigning the control–polygon legs $\mathbf{L}_2, \mathbf{L}_4$ — this case admits a fairly simple analysis, but (unlike the $\mathbf{L}_1, \mathbf{L}_5$ case) there are only *two* distinct solutions.

Proposition 2. *The control polygon of a canonical–form PH quintic can be expressed in terms of* $\mathbf{L}_2$ *and* $\mathbf{L}_4$ *as free parameters through the relations*

$$\mathbf{L}_1 \,=\, \mathbf{z}\,\mathbf{L}_2^2\,, \quad \mathbf{L}_3 \,=\, 1 - (\mathbf{L}_2 + \mathbf{L}_4) - \mathbf{z}\,(\mathbf{L}_2^2 + \mathbf{L}_4^2)\,, \quad \mathbf{L}_5 \,=\, \mathbf{z}\,\mathbf{L}_4^2\,, \tag{24}$$

where $\mathbf{z}$ *is either root of the complex quadratic equation*

$$(3\,\mathbf{L}_2^2 + 3\,\mathbf{L}_4^2 + \mathbf{L}_2\mathbf{L}_4)\,\mathbf{z}^2 + 3\,(\mathbf{L}_2 + \mathbf{L}_4 - 1)\,\mathbf{z} + 2 \,=\, 0\,. \tag{25}$$

Proof : For non–zero $\mathbf{L}_2, \mathbf{L}_4$ equation (10) implies that $\mathbf{L}_1, \mathbf{L}_5$ must satisfy, for some complex number $\mathbf{z}$, the expressions specified in (24). Substituting $\mathbf{L}_3 = 1 - \mathbf{L}_1 - \mathbf{L}_2 - \mathbf{L}_4 - \mathbf{L}_5$ from (15) and the expressions for $\mathbf{L}_1, \mathbf{L}_5$ into (11) and simplifying then yields the quadratic equation (25) for $\mathbf{z}$, with $\mathbf{L}_1, \mathbf{L}_3, \mathbf{L}_5$ defined in terms of $\mathbf{L}_2, \mathbf{L}_4$ by (24). ∎

Thus, by Proposition 2, the control points of a canonical–form PH quintic with prescribed control–polygon legs $\mathbf{L}_2, \mathbf{L}_4$ can be written as

$$\mathbf{p}_0 \,=\, 0\,, \quad \mathbf{p}_1 \,=\, \mathbf{z}\,\mathbf{L}_2^2\,, \quad \mathbf{p}_2 \,=\, \mathbf{z}\,\mathbf{L}_2^2 + \mathbf{L}_2\,,$$
$$\mathbf{p}_3 \,=\, 1 - \mathbf{z}\,\mathbf{L}_4^2 - \mathbf{L}_4\,, \quad \mathbf{p}_4 \,=\, 1 - \mathbf{z}\,\mathbf{L}_4^2\,, \quad \mathbf{p}_5 \,=\, 1\,,$$

where $\mathbf{z}$ is a root of equation (25). Since equation (25) has (in general) two distinct complex roots, there are usually two distinct canonical–form quintic PH curves consistent with the specified control polygon legs $\mathbf{L}_2, \mathbf{L}_4$.

Corollary 2. *For a canonical–form PH quintic with specified control–polygon legs* $\mathbf{L}_2, \mathbf{L}_4$ *the coefficients of the pre–image polynomial* (1) *are determined by*

$$\mathbf{w}_0 \,=\, \sqrt{5\,\mathbf{z}}\,\mathbf{L}_2\,, \quad \mathbf{w}_1 \,=\, \sqrt{5/\mathbf{z}}\,, \quad \mathbf{w}_2 \,=\, \sqrt{5\,\mathbf{z}}\,\mathbf{L}_4\,. \tag{26}$$

Proof : From the first and fifth of equations (9), together with the relations (24), we obtain $\mathbf{w}_0 = s_1\sqrt{5\,\mathbf{z}}\,\mathbf{L}_2$ and $\mathbf{w}_2 = s_2\sqrt{5\,\mathbf{z}}\,\mathbf{L}_4$, where $s_1, s_2 = \pm 1$ are independent sign choices. Substituting these expressions into the second and fourth of equations (9), we obtain $\mathbf{w}_1 = s_1\sqrt{5/\mathbf{z}}$ and $\mathbf{w}_1 = s_2\sqrt{5/\mathbf{z}}$, and for consistency we must have $s_1 = s_2$. Since $(\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2)$ and $(-\mathbf{w}_0, -\mathbf{w}_1, -\mathbf{w}_2)$ generate the same hodograph $\mathbf{r}'(t) = \mathbf{w}^2(t)$, we set $s_1 = s_2 = 1$ without loss of generality. This gives the expressions (26), and substituting from (24) and (26) into the third of equations (9) yields equation (25) for $\mathbf{z}$. ∎

Example 1. Consider the choices $\mathbf{L}_2 = 0.20 + 0.12\,\mathrm{i}$ and $\mathbf{L}_4 = 0.20 - 0.12\,\mathrm{i}$. In this case, the coefficients of the quadratic equation (25) are all real, and it has a positive discriminant, so the two roots $\mathbf{z}$ are both real. Together with the corresponding $\mathbf{L}_1, \mathbf{L}_3, \mathbf{L}_5$ values, we obtain

$$\begin{aligned}
\mathbf{z} \,&=\, 1.30916253\,, \\
\mathbf{L}_1 \,&=\, 0.03351456 + 0.06283980\,\mathrm{i}\,, \\
\mathbf{L}_3 \,&=\, 0.53297088 + 0.00000000\,\mathrm{i}\,, \\
\mathbf{L}_5 \,&=\, 0.03351456 - 0.06283980\,\mathrm{i}\,,
\end{aligned}$$

$$
\begin{aligned}
\mathbf{z} &= 7.34468362\,, \\
\mathbf{L}_1 &= 0.18802390 + 0.35254481\,\mathrm{i}\,, \\
\mathbf{L}_3 &= 0.22395220 + 0.00000000\,\mathrm{i}\,, \\
\mathbf{L}_5 &= 0.18802390 - 0.35254481\,\mathrm{i}\,.
\end{aligned}
$$

The two PH quintics incorporating the specified $\mathbf{L}_2, \mathbf{L}_4$ control–polygon legs are illustrated in Figure 1.

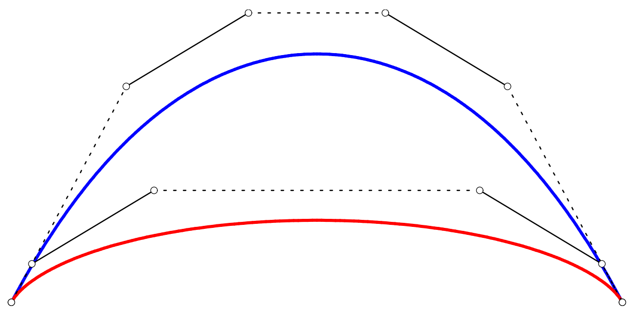

Figure 1: The quintic PH curves that incorporate the two prescribed control–polygon legs $\mathbf{L}_2 = 0.20 + 0.12\,\mathrm{i}$ and $\mathbf{L}_4 = 0.20 - 0.12\,\mathrm{i}$ in Example 1. $\mathbf{L}_2, \mathbf{L}_4$ are indicated as solid lines, and the "filled in" legs $\mathbf{L}_1, \mathbf{L}_3, \mathbf{L}_5$ as dotted lines.

Remark 5. Note that in this case the control–polygon legs $\mathbf{L}_2, \mathbf{L}_4$ determine the *relative* orientation of the curve end tangents, since $\mathbf{r}'(0) = 5\,\mathbf{z}\,\mathbf{L}_2^2$ and $\mathbf{r}'(1) = 5\,\mathbf{z}\,\mathbf{L}_4^2$, and consequently

$$
\arg(\mathbf{r}'(1)) - \arg(\mathbf{r}'(0)) \;=\; \arg\left(\frac{\mathbf{r}'(1)}{\mathbf{r}'(0}\right) \;=\; \arg\left(\frac{\mathbf{L}_4^2}{\mathbf{L}_2^2}\right) \;=\; 2\,[\,\arg(\mathbf{L}_4) - \arg(\mathbf{L}_2)\,]\,.
$$

The absolute orientations of the end tangents, relative to the real line segment $[\,0,1\,]$ is determined also by the factor $\mathbf{z}$ in (24).

4.2 Asymmetric choices

Consider now asymmetric choices of two of the control–polygon legs $\mathbf{L}_1, \ldots, \mathbf{L}_5$ as the free parameters. The principal cases, together with their counterparts under reverse–ordering of the control–polygon legs, are:

- $\mathbf{L}_1, \mathbf{L}_2$ or $\mathbf{L}_4, \mathbf{L}_5$ as free parameters;

- $\mathbf{L}_1, \mathbf{L}_3$ or $\mathbf{L}_3, \mathbf{L}_5$ as free parameters;
- $\mathbf{L}_2, \mathbf{L}_3$ or $\mathbf{L}_3, \mathbf{L}_4$ as free parameters.
- $\mathbf{L}_1, \mathbf{L}_4$ or $\mathbf{L}_2, \mathbf{L}_5$ as free parameters.

The first case admits a fairly straightforward characterization, incurring only a quadratic equation. The remaining three cases are more involved, resulting in a quartic equation, and the `Maple` computer algebra system was employed to develop their characterizations. It should be noted that, in general, there is no unique solution to the problem of expressing three of the control–polygon legs in terms of the other two, under the constraints specified by the equations (10)–(15). The formulations presented below are the simplest found through testing several different approaches to this problem.

4.2.1 $\mathbf{L}_1, \mathbf{L}_2$ or $\mathbf{L}_4, \mathbf{L}_5$ as free parameters

Consider the asymmetric cases in which the first or last two control–polygon legs are free parameters. The case $\mathbf{L}_1, \mathbf{L}_2$ may be characterized as follows.

Proposition 3. *The control polygon of a canonical–form PH quintic can be expressed in terms of* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *as free parameters through the relations*

$$\mathbf{L}_3 \,=\, 1-(1+\mathbf{z}^2)\,\mathbf{L}_1-(1+\mathbf{z})\,\mathbf{L}_2\,, \quad \mathbf{L}_4 \,=\, \mathbf{z}\,\mathbf{L}_2\,, \quad \mathbf{L}_5 \,=\, \mathbf{z}^2\,\mathbf{L}_1\,, \tag{27}$$

where $\mathbf{z}$ *is any root of the complex quadratic equation*

$$3\,\mathbf{L}_1^2\,\mathbf{z}^2+(\mathbf{L}_1+3\,\mathbf{L}_2)\,\mathbf{L}_1\,\mathbf{z}+3\,\mathbf{L}_1(\mathbf{L}_1+\mathbf{L}_2-1)+2\,\mathbf{L}_2^2 \,=\, 0\,. \tag{28}$$

Proof : For any non–zero $\mathbf{L}_2$, we may set $\mathbf{L}_4=\mathbf{z}\,\mathbf{L}_2$ for some complex value $\mathbf{z}$. Substituing this in equation (10) then yields $\mathbf{L}_5=\mathbf{z}^2\,\mathbf{L}_1$, so we have the relations $\mathbf{L}_4=\mathbf{z}\,\mathbf{L}_2$ and $\mathbf{L}_5=\mathbf{z}^2\,\mathbf{L}_1$ between the first and last two control–polygon legs. Substituting $\mathbf{L}_3=1-\mathbf{L}_1-\mathbf{L}_2-\mathbf{L}_4-\mathbf{L}_5$ from (15) and these expressions for $\mathbf{L}_4, \mathbf{L}_5$ into (11) then yields the quadratic equation (28) for $\mathbf{z}$, with $\mathbf{L}_3, \mathbf{L}_4, \mathbf{L}_5$ defined in terms of $\mathbf{L}_1, \mathbf{L}_2$ by the expressions (27). ∎

In this case, the control points are defined in terms of $\mathbf{L}_1, \mathbf{L}_2$ and $\mathbf{z}$ as

$$\mathbf{p}_0 \,=\, 0\,, \quad \mathbf{p}_1 \,=\, \mathbf{L}_1\,, \quad \mathbf{p}_2 \,=\, \mathbf{L}_1+\mathbf{L}_2\,,$$

$$\mathbf{p}_3 \,=\, 1-\mathbf{z}^2\,\mathbf{L}_1-\mathbf{z}\,\mathbf{L}_2\,, \quad \mathbf{p}_4 \,=\, 1-\mathbf{z}^2\,\mathbf{L}_1\,, \quad \mathbf{p}_5 \,=\, 1\,.$$

Corollary 3. *For a canonical–form PH quintic with specified control–polygon legs* $\mathbf{L}_1, \mathbf{L}_2$ *the coefficients of the pre–image polynomial* (1) *are*

$$\mathbf{w}_0 \,=\, \sqrt{5\,\mathbf{L}_1}\,, \quad \mathbf{w}_1 \,=\, \sqrt{5/\mathbf{L}_1}\,\mathbf{L}_2\,, \quad \mathbf{w}_2 \,=\, \mathbf{z}\,\sqrt{5\,\mathbf{L}_1}\,. \tag{29}$$

Proof : From the first and fifth of equations (9), together with the relations (27), we obtain $\mathbf{w}_0 = s_1\sqrt{5\,\mathbf{L}_1}$ and $\mathbf{w}_2 = s_2\,\mathbf{z}\,\sqrt{5\,\mathbf{L}_1}$, where $s_1, s_2 = \pm 1$ are independent sign choices. Substituting these expressions into the second and fourth of equations (9), we obtain $\mathbf{w}_1 = s_1\sqrt{5/\mathbf{L}_1}\,\mathbf{L}_2$ and $\mathbf{w}_1 = s_2\sqrt{5/\mathbf{L}_1}\,\mathbf{L}_2$, and consequently we must have $s_1 = s_2$. As in Corollary 2, we set $s_1 = s_2 = 1$ without loss of generality. This yields the expressions (29) for $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ and by substituting from (27) and (29) into the third of equations (9) we obtain the quadratic equation (28) for $\mathbf{z}$. ∎

When $\mathbf{L}_4, \mathbf{L}_5$ are chosen as the free parameters, we obtain a counterpart to the above results through the substitutions $\mathbf{L}_1 \leftrightarrow \mathbf{L}_5$ and $\mathbf{L}_2 \leftrightarrow \mathbf{L}_4$.

Remark 6. Choosing $\mathbf{L}_1$ and $\mathbf{L}_2$ allows the initial curvature $\kappa(0)$ of $\mathbf{r}(t)$ to be assigned, since

$$\kappa(0) \,=\, \frac{(\mathbf{r}'(0) \times \mathbf{r}''(0)) \cdot \mathbf{k}}{|\,\mathbf{r}'(0)\,|^3} \,=\, \frac{4}{5}\,\frac{(\mathbf{L}_1 \times \mathbf{L}_2) \cdot \mathbf{k}}{|\,\mathbf{L}_1\,|^3}\,, \tag{30}$$

where complex values are interpreted as vectors in $\mathbb{R}^2$ and $\mathbf{k}$ is a unit vector orthogonal to the plane.

Example 2. For the choices $\mathbf{L}_2 = 0.15 + 0.12\,\mathrm{i}$ and $\mathbf{L}_4 = 0.30 + 0.18\,\mathrm{i}$, the roots of equation (28) and corresponding $\mathbf{L}_3, \mathbf{L}_3, \mathbf{L}_5$ values are

$$\begin{aligned}
\mathbf{z} &= 0.14546403 - 0.98211688\,\mathrm{i}\,, \\
\mathbf{L}_3 &= 0.43680178 + 0.12451760\,\mathrm{i}\,, \\
\mathbf{L}_4 &= 0.22042025 - 0.26845154\,\mathrm{i}\,, \\
\mathbf{L}_5 &= -\,0.10722202 - 0.15606606\,\mathrm{i}\,,
\end{aligned}$$

$$\begin{aligned}
\mathbf{z} &= -2.28367541 + 1.22601932\,\mathrm{i}\,, \\
\mathbf{L}_3 &= 0.22701936 + 0.13775883\,\mathrm{i}\,, \\
\mathbf{L}_4 &= -\,0.90578610 - 0.04325578\,\mathrm{i}\,, \\
\mathbf{L}_5 &= 1.22876674 - 0.39450305\,\mathrm{i}\,.
\end{aligned}$$

The two quintic PH curves that incorporate the given $\mathbf{L}_1, \mathbf{L}_2$ control–polygon legs are illustrated in Figure 2 — there is one "good" solution, and the other solution with a tight loop should be discarded.

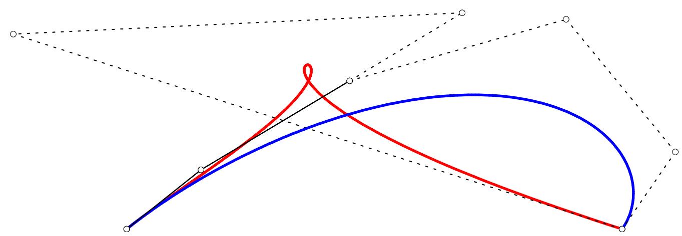

Figure 2: The quintic PH curves that incorporate the two prescribed control–polygon legs $\mathbf{L}_1 = 0.15 + 0.12\,\mathrm{i}$ and $\mathbf{L}_2 = 0.30 + 0.18\,\mathrm{i}$ in Example 2. $\mathbf{L}_1, \mathbf{L}_2$ are indicated as solid lines, and the "filled in" legs $\mathbf{L}_3, \mathbf{L}_4, \mathbf{L}_5$ as dotted lines.

We omit proofs for the control–polygon characterizations in the remaining three cases with asymmetric choices of the prescribed legs, as they are closely analogous to the proofs in the preceding cases.

4.2.2 $\mathbf{L}_1, \mathbf{L}_3$ or $\mathbf{L}_3, \mathbf{L}_5$ as free parameters

The case with $\mathbf{L}_1$ and $\mathbf{L}_3$ as free parameters may be characterized as follows.

Proposition 4. *The control polygon of a canonical–form PH quintic can be expressed in terms of* $\mathbf{L}_1$ *and* $\mathbf{L}_3$ *as free parameters through the relations*

$$\mathbf{L}_2 \,=\, \mathbf{z}\,\mathbf{L}_1\,, \quad \mathbf{L}_4 \,=\, -\,2\,\mathbf{z}^3\,\mathbf{L}_1 + 3\,\mathbf{z}\,\mathbf{L}_3$$

$$\mathbf{L}_5 \,=\, 2\,\mathbf{z}^3\,\mathbf{L}_1 - \mathbf{z}\,(\mathbf{L}_1 + 3\,\mathbf{L}_3) - \mathbf{L}_1 - \mathbf{L}_3 + 1\,,$$

where $\mathbf{z}$ *is any root of the complex quartic equation*

$$4\,\mathbf{L}_1^2\,\mathbf{z}^4 - 2\,\mathbf{L}_1^2\,\mathbf{z}^3 - 12\,\mathbf{L}_1\mathbf{L}_3\,\mathbf{z}^2 + (\mathbf{L}_1 + 3\,\mathbf{L}_3)\,\mathbf{L}_1\,\mathbf{z} + 9\,\mathbf{L}_3^2 + (\mathbf{L}_1 + \mathbf{L}_3 - 1)\,\mathbf{L}_1 \,=\, 0\,.$$

The control points are determined in terms of $\mathbf{L}_1, \mathbf{L}_3$ and $\mathbf{z}$ as

$$\mathbf{p}_0 \,=\, 0\,, \quad \mathbf{p}_1 \,=\, \mathbf{L}_1\,, \quad \mathbf{p}_2 \,=\, (1 + \mathbf{z})\,\mathbf{L}_1\,,$$

$$\mathbf{p}_3 \,=\, (1 + \mathbf{z})\,\mathbf{L}_1 + \mathbf{L}_3\,, \quad \mathbf{p}_4 \,=\, (1 + \mathbf{z} - 2\,\mathbf{z}^3)\,\mathbf{L}_1 + (1 + 3\,\mathbf{z})\,\mathbf{L}_3\,, \quad \mathbf{p}_5 \,=\, 1\,.$$

In this case, $\mathbf{z}$ is a root of a quartic equation — which admits a closed–form solution through Ferrari's method [29]. Consequently, we may in general expect four distinct PH quintics consistent with prescribed control–polygon legs $\mathbf{L}_1, \mathbf{L}_3$. A result analogous to Proposition 4 holds if $\mathbf{L}_3, \mathbf{L}_5$ are the free parameters, through the substitutions $\mathbf{L}_1 \leftrightarrow \mathbf{L}_5$ and $\mathbf{L}_2 \leftrightarrow \mathbf{L}_4$.

Figure 3 illustrates the four distinct PH quintics with prescribed control–polygon legs $\mathbf{L}_1 = 0.2 + 0.2\,\mathrm{i}$ and $\mathbf{L}_3 = 0.20$. The two the solutions one the left may be considered fair curves, with reasonable values of the absolute rotation index (23), while the other two solutions has an undesirable tight loop.

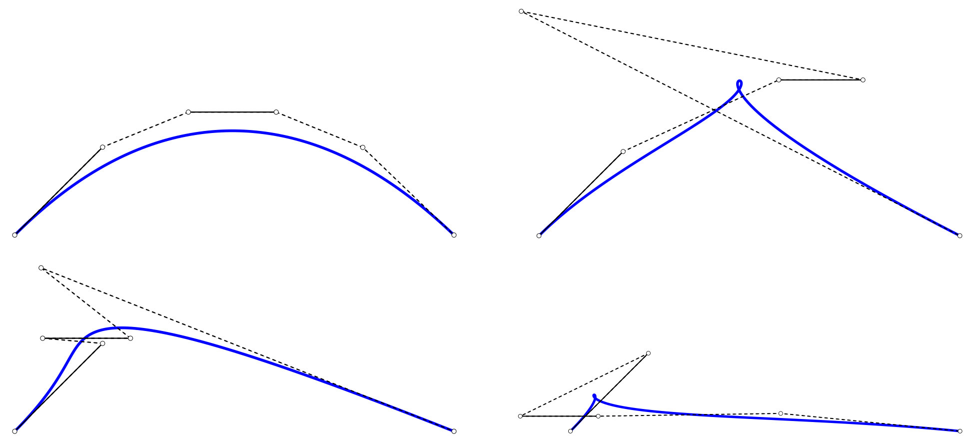

Figure 3: The quintic PH curves that incorporate the two prescribed control–polygon legs $\mathbf{L}_1 = 0.2 + 0.2\,\mathrm{i}$ and $\mathbf{L}_3 = 0.20$. $\mathbf{L}_1, \mathbf{L}_3$ are indicated as solid lines, and the "filled in" legs $\mathbf{L}_2, \mathbf{L}_4, \mathbf{L}_5$ as dotted lines.

4.2.3 $\mathbf{L}_2, \mathbf{L}_3$ or $\mathbf{L}_3, \mathbf{L}_4$ as free parameters

Consider now the case with $\mathbf{L}_2$ and $\mathbf{L}_3$ chosen as the free parameters.

Proposition 5. *The control polygon of a canonical–form PH quintic can be expressed in terms of* $\mathbf{L}_2$ *and* $\mathbf{L}_3$ *as free parameters through the relations*

$$\mathbf{L}_1 \,=\, \mathbf{z}\,, \quad \mathbf{L}_4 \,=\, (3\,\mathbf{z}\,\mathbf{L}_3 - 2\,\mathbf{L}_2^2)\,\mathbf{L}_2/\mathbf{z}^2\,, \quad \mathbf{L}_5 \,=\, (3\,\mathbf{z}\,\mathbf{L}_3 - 2\,\mathbf{L}_2^2)^2/\mathbf{z}^3\,,$$

where $\mathbf{z}$ *is any root of the complex quartic equation*

$$\mathbf{z}^4 + (\mathbf{L}_2 + \mathbf{L}_3 - 1)\,\mathbf{z}^3 + 3\,(\mathbf{L}_2 + 3\,\mathbf{L}_3)\,\mathbf{L}_3\,\mathbf{z}^2 - 2\,(\mathbf{L}_2 + 6\,\mathbf{L}_3)\,\mathbf{L}_2^2\,\mathbf{z} + 4\,\mathbf{L}_2^4 \,=\, 0\,.$$

The control points can then be expressed in terms of $\mathbf{L}_2, \mathbf{L}_3$ and $\mathbf{z}$ as

$$\mathbf{p}_0 \,=\, 0\,, \quad \mathbf{p}_1 \,=\, \mathbf{z}\,, \quad \mathbf{p}_2 \,=\, \mathbf{z} + \mathbf{L}_2\,,$$

$$\mathbf{p}_3 \,=\, \mathbf{z} + \mathbf{L}_2 + \mathbf{L}_3 \,, \quad \mathbf{p}_4 \,=\, 1 - (3\,\mathbf{z}\,\mathbf{L}_3 - 2\,\mathbf{L}_2^2)^2/\mathbf{z}^3 \,, \quad \mathbf{p}_5 \,=\, 1 \,.$$

Figure 4 illustrates the four distinct PH quintics with prescribed control–polygon legs $\mathbf{L}_2 = 0.25 + 0.25\,\mathrm{i}$ and $\mathbf{L}_3 = 0.20 + 0.15\,\mathrm{i}$. Three of the solutions may be considered fair curves, with reasonable values of the absolute rotation index (23), while the fourth solution has an undesirable tight loop.

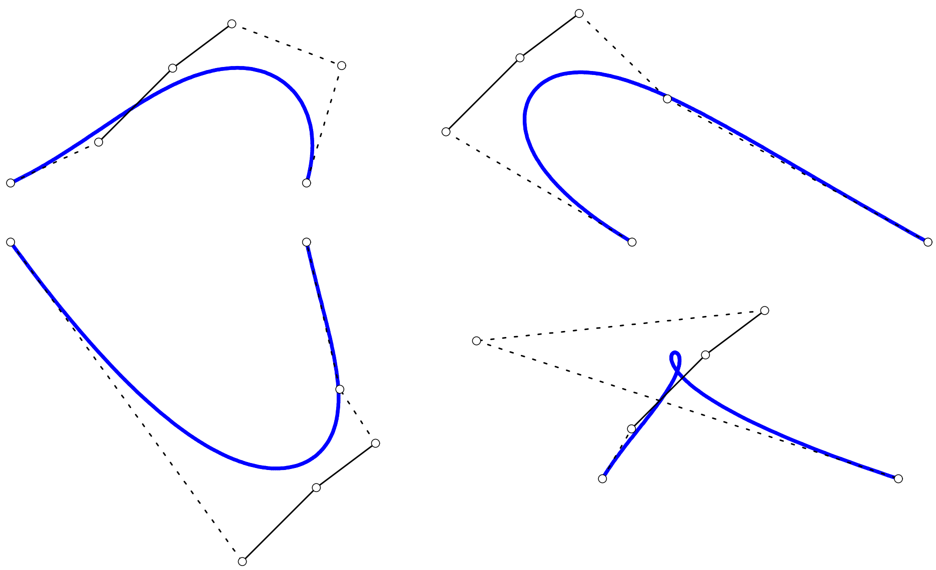

Figure 4: The four quintic PH curves that incorporate $\mathbf{L}_2 = 0.25 + 0.25\,\mathrm{i}$ and $\mathbf{L}_3 = 0.20 + 0.15\,\mathrm{i}$ as prescribed control–polygon legs — $\mathbf{L}_2, \mathbf{L}_3$ are indicated as solid lines, and the remaining "filled in" legs $\mathbf{L}_1, \mathbf{L}_4, \mathbf{L}_5$ as dotted lines.

When $\mathbf{L}_3, \mathbf{L}_4$ are chosen as the free parameters, we obtain a counterpart to the above result through the substitutions $\mathbf{L}_1 \leftrightarrow \mathbf{L}_5$ and $\mathbf{L}_2 \leftrightarrow \mathbf{L}_4$.

4.2.4 $\mathbf{L}_1, \mathbf{L}_4$ or $\mathbf{L}_2, \mathbf{L}_5$ as free parameters

The final case concerns the choice of $\mathbf{L}_1, \mathbf{L}_4$ as free parameters.

Proposition 6. *The control polygon of a canonical–form PH quintic can be expressed in terms of* $\mathbf{L}_1$ *and* $\mathbf{L}_4$ *as free parameters through the relations*

$$\mathbf{L}_2 \,=\, \mathbf{z}\,\mathbf{L}_4 \,, \quad \mathbf{L}_3 \,=\, (2\,\mathbf{z}^3\,\mathbf{L}_4^2 + \mathbf{L}_1^2)/(3\,\mathbf{z}\,\mathbf{L}_1) \,, \quad \mathbf{L}_5 \,=\, \mathbf{L}_1/\mathbf{z}^2 \,, \tag{31}$$

where $\mathbf{z}$ *is any root of the complex quartic equation*

$$2\,\mathbf{L}_4^2\,\mathbf{z}^4 + 3\,\mathbf{L}_1\mathbf{L}_4\,\mathbf{z}^3 + 3\,(\mathbf{L}_1 + \mathbf{L}_4 - 1)\,\mathbf{L}_1\,\mathbf{z}^2 + \mathbf{L}_1^2\,\mathbf{z} + 3\,\mathbf{L}_1^2 \,=\, 0 \,. \tag{32}$$

The control points can be expressed in terms of $\mathbf{L}_1, \mathbf{L}_4$ and $\mathbf{z}$ as

$$\mathbf{p}_0 \,=\, 0\,, \quad \mathbf{p}_1 \,=\, \mathbf{L}_1\,, \quad \mathbf{p}_2 \,=\, \mathbf{L}_1 + \mathbf{z}\,\mathbf{L}_4\,,$$

$$\mathbf{p}_3 \,=\, 1 - \mathbf{L}_1/\mathbf{z}^2 - \mathbf{L}_4\,, \quad \mathbf{p}_4 \,=\, 1 - \mathbf{L}_1/\mathbf{z}^2\,, \quad \mathbf{p}_5 \,=\, 1\,.$$

Figure 5 illustrates the four distinct PH quintics with prescribed control–polygon legs $\mathbf{L}_1 = 0.25 - 0.5\,\mathrm{i}$ and $\mathbf{L}_4 = 0.20 + 0.15\,\mathrm{i}$.

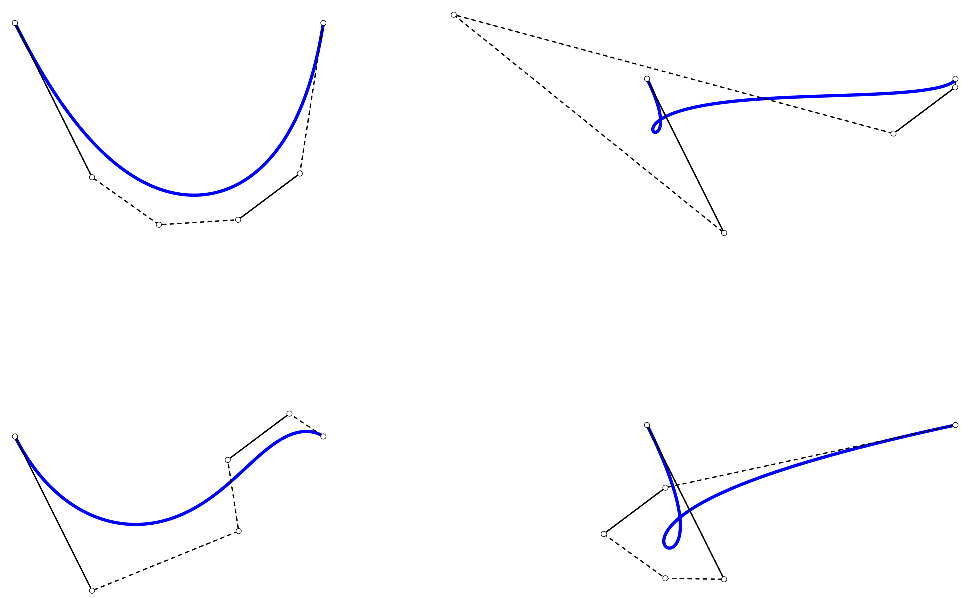

Figure 5: The quintic PH curves that incorporate the two prescribed control–polygon legs $\mathbf{L}_1 = 0.25 - 0.5\,\mathrm{i}$ and $\mathbf{L}_4 = 0.20 + 0.15\,\mathrm{i}$. $\mathbf{L}_1, \mathbf{L}_4$ are indicated as solid lines, and the "filled in" legs $\mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_5$ as dotted lines.

When $\mathbf{L}_2, \mathbf{L}_5$ are chosen as the free parameters, we obtain a counterpart to the above result through the substitutions $\mathbf{L}_1 \leftrightarrow \mathbf{L}_5$ and $\mathbf{L}_2 \leftrightarrow \mathbf{L}_4$.

5 Control–polygon design of PH quintics

We now present some preliminary examples showing how the control–polygon constraint paradigm can be exploited in designing quintic PH curves. These examples focus primarily on the simpler cases with $\mathbf{L}_1$ and $\mathbf{L}_5$, $\mathbf{L}_2$ and $\mathbf{L}_4$, or $\mathbf{L}_1$ and $\mathbf{L}_2$ as the free parameters (which incur only quadratic equations), but an example with $\mathbf{L}_1$ and $\mathbf{L}_4$ as free parameters also included. The examples are only illustrative in nature — a detailed analysis of the possibilities offered by this approach deserves a separate comprehensive treatment.

5.1 Symmetric convex curves defined by $\mathbf{L}_2, \mathbf{L}_4$

Suppose $\mathbf{L}_2, \mathbf{L}_4$ are specified as complex conjugates: $\mathbf{L}_2 = L\,\mathrm{e}^{\mathrm{i}\,\theta}$, $\mathbf{L}_4 = L\,\mathrm{e}^{-\mathrm{i}\,\theta}$. Then equation (25) has real coefficients, namely

$$(6\cos 2\,\theta + 1)L^2\,\mathbf{z}^2 + 3\,(2L\cos\theta - 1)\,\mathbf{z} + 2 \,=\, 0\,, \tag{33}$$

and if it has real roots $\mathbf{z}$, the expressions (24) indicate that $\mathbf{L}_1$ and $\mathbf{L}_5$ are also conjugates, and $\mathbf{L}_3$ is real. In this case, the control polygon is symmetric and it defines a convex PH quintic. For real roots, the discriminant Δ of equation (33) must be non–negative. By setting $\cos 2\,\theta = 2\cos^2\theta - 1$ and simplifying, we obtain

$$\Delta(L,\theta) \,=\, 20\,(2 - 3\cos^2\theta)\,L^2 - 36\,L\,\cos\theta + 9\,,$$

and the condition $\Delta(L,\theta) \geq 0$ defines a domain in the (L,θ) plane identifying symmetric convex planar PH quintics. Figure 6 illustrates this domain over $(L,\theta) \in [\,0,1\,]\times[\,0,\frac{1}{2}\pi\,]$, with four sub–domains $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ identified that correspond to distinctive properties of the resulting PH curves.

Figure 7 illustrates some examples of the quintic PH curves corresponding to each of the four sub–domains $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ of the domain $\mathcal{D}$ over which $\Delta(L,\theta) \geq 0$. By choosing $(L,\theta) \in \mathcal{D}_1 \cup \mathcal{D}_2$, both the PH quintic solutions are observed to be well–behaved curves.

5.2 Quintic PH curves with $\mathbf{L}_2 = \mathbf{L}_4 = 0$

As noted in Remark 3, equations (10)–(14) are trivially satisfied if $\mathbf{L}_2 = \mathbf{L}_4 = 0$, and the corresponding PH quintics are characterized by the condition (17) instead. This condition implies that $\mathbf{L}_1, \mathbf{L}_5$ must be of the form

$$\mathbf{L}_1 \,=\, 3\,\mathbf{z}\,\mathbf{L}_3 \quad \text{and} \quad \mathbf{L}_5 \,=\, 3\,\mathbf{L}_3/\mathbf{z}$$

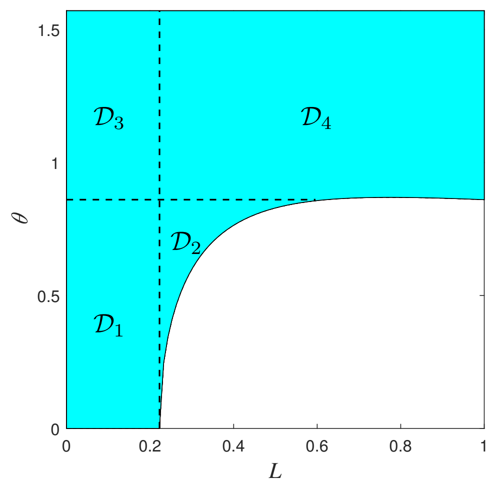

Figure 6: The domain $\mathcal{D}$ defined by $\Delta(L, \theta) > 0$ (shaded area) where equation (33) admits real roots. Some representative quintic PH curves corresponding to of each the four sub–domains $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ are illustrated in Figure 7.

for some complex number $\mathbf{z}$. Substituting these expressions and $\mathbf{L}_2 = \mathbf{L}_4 = 0$ into (15) then results in the quadratic equation

$$3\,\mathbf{L}_3\,\mathbf{z}^2 + (\mathbf{L}_3 - 1)\,\mathbf{z} + 3\,\mathbf{L}_3 \;=\; 0$$

for $\mathbf{z}$, dependent on the single complex parameter $\mathbf{L}_3$. The control points

$$\mathbf{p}_0 = 0\,, \quad \mathbf{p}_1 = \mathbf{p}_2 \;=\; 3\,\mathbf{z}\,\mathbf{L}_3\,, \quad \mathbf{p}_3 = \mathbf{p}_4 \;=\; (1 - 3/\mathbf{z})\,\mathbf{L}_3\,, \quad \mathbf{p}_5 = 1$$

define a family of "cubic–like" quintic PH curves, and by the analogy between conditions (8) and (17) they are all convex curves.

Example 3. It is interesting to compare the shapes of true PH cubics and "cubic–like" PH quintics. To accomplish this, we first note that a canonical–form PH cubic has control–polygon legs of the form

$$\mathbf{L}_1 \;=\; \mathbf{z}\,\mathbf{L}_2\,, \quad \mathbf{L}_2\,, \quad \mathbf{L}_3 \;=\; \mathbf{L}_2/\mathbf{z}\,,$$

where $\mathbf{z}$ is a root of the quadratic equation

$$\mathbf{L}_2\,\mathbf{z}^2 + (\mathbf{L}_2 - 1)\,\mathbf{z} + \mathbf{L}_2 \;=\; 0\,.$$

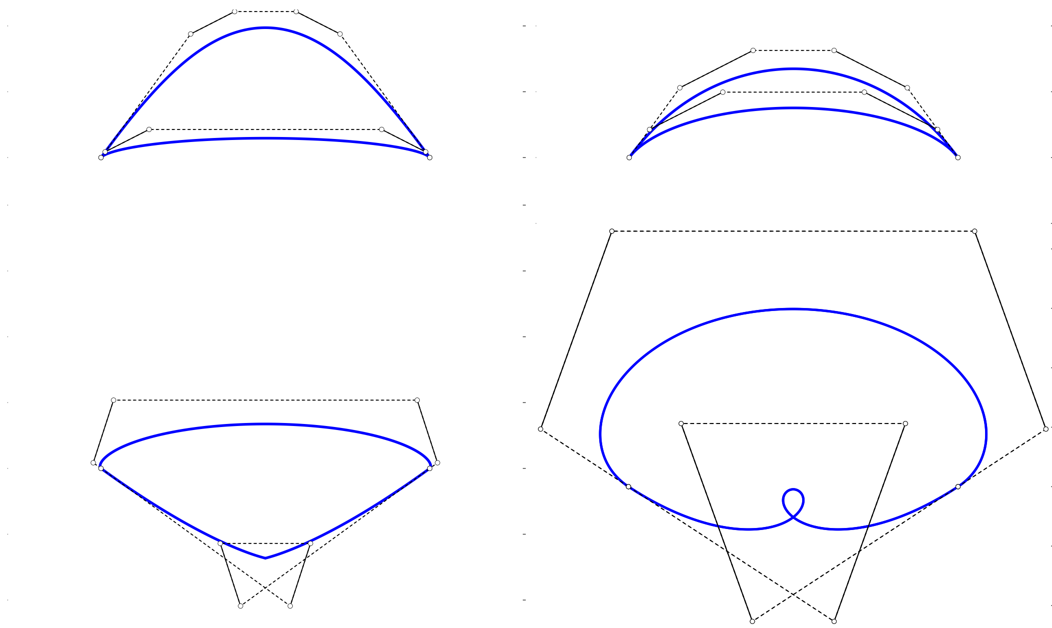

Figure 7: Quintic PH curves with prescribed control–polygon legs $\mathbf{L}_2, \mathbf{L}_4$ as in Section 5.1, with (L, θ) values $(0.15, 0.15\,\pi) \in \mathcal{D}_1$ upper left; $(0.25, 0.15\,\pi) \in \mathcal{D}_2$ upper right; $(0.2, 0.4\,\pi) \in \mathcal{D}_3$ lower left; $(0.7, 0.375\,\pi) \in \mathcal{D}_4$ lower right.

Comparing equation (8) with equation (17) written in the form

$$\left(\frac{\mathbf{L}_3}{3}\right)^2 = \mathbf{L}_1\mathbf{L}_5\,,$$

we see that, upon choosing $\mathbf{L}_2 = \mathbf{L}$ for the PH cubic, we should take $\mathbf{L}_3 = \mathbf{L}/3$ for the "cubic–like" PH quintics to achieve comparable control polygons,[1] as shown in Figure 8 for $\mathbf{L} = 0.5, 0.6, \ldots, 1$.

Figure 9 compares the curvature graphs for the "cubic–like" PH quintics (left) and true PH cubics (right). To facilitate the comparison, the curvature is plotted in term of the fractional arc length $s(t)/s(1)$, $t \in [\,0, 1\,]$ where $s(t)$ is the polynomial arc length function of the PH curves.

Remark 7. Planar PH quintics with $\mathbf{L}_2 = \mathbf{L}_4 = 0$ have previously been used as G^2 blending curves, with associated feedrate functions, to smooth sharp

[1]Real $\mathbf{L}$ values in the intervals $(-1, 1/3)$ and $(-1/5, 1/7)$ should be avoided for the PH cubic and "cubic–like" PH quintic, since the resulting curves degenerate to straight lines.

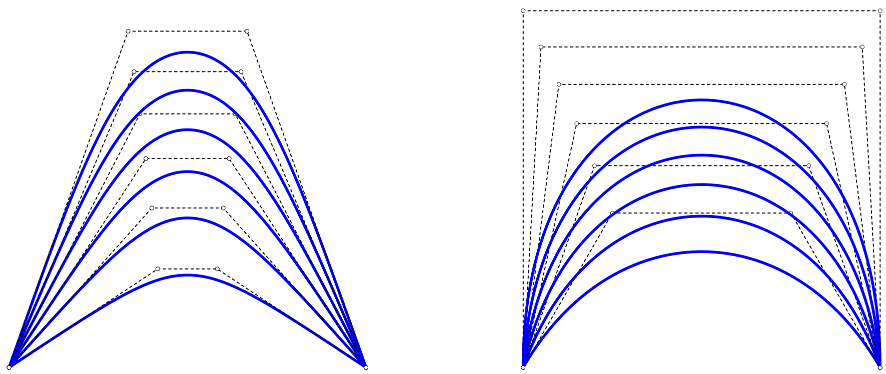

Figure 8: The "cubic–like" PH quintics with $\mathbf{L}_3 = \mathbf{L}/3$ (left) and true PH cubics with $\mathbf{L}_2 = \mathbf{L}$ (right) for $\mathbf{L} = 0.5, 0.6, \ldots, 1$ as described in Example 3.

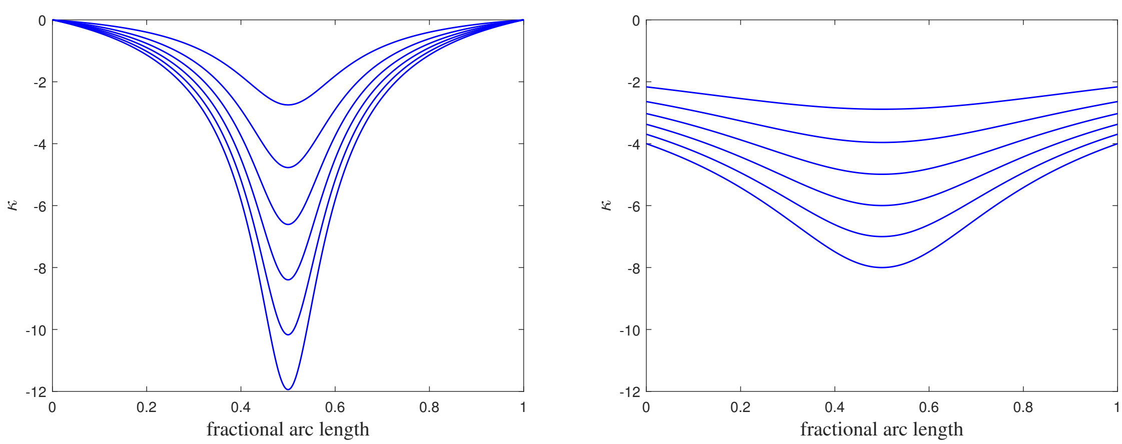

Figure 9: Curvature plots for the "cubic–like" PH quintics with $\mathbf{L}_3 = \mathbf{L}/3$ (left), and the true PH cubics with $\mathbf{L}_2 = \mathbf{L}$ (right), for $\mathbf{L} = 0.5, 0.6, \ldots, 1$.

corners for efficient execution of piecewise–linear tool paths in the context of high–speed machining [14].

5.3 Initial G^2 data determined by $\mathbf{L}_1, \mathbf{L}_2$

By prescribing $\mathbf{L}_1$ and $\mathbf{L}_2$ one can construct families of quintic PH curves that have identical tangents, but varying curvatures at the initial point $\mathbf{r}(0) = 0$. Consider the choices $\mathbf{L}_1 = L\,\mathrm{e}^{\mathrm{i}\,\theta}$, $\mathbf{L}_2 = L\,\mathrm{e}^{\mathrm{i}\,(\theta+\phi)}$ where θ is the initial tangent angle of $\mathbf{r}(t)$, and $L > 0$ and ϕ are free parameters. Then from (30) we have

$$\kappa(0) \;=\; \frac{4}{5}\,\frac{\sin\phi}{L}\,, \tag{34}$$

so $\kappa(0)$ is proportional to $\sin\phi$, and inversely proportional to L. Substituting for $\mathbf{L}_1$ and $\mathbf{L}_2$ under the assumption $L \neq 0$, the quadratic equation (28) can be simplified to obtain

$$3\,L\,\mathbf{z}^2 + (1 + 3\,\mathrm{e}^{\mathrm{i}\,\phi})\,L\,\mathbf{z} + 3\,(L + L\,\mathrm{e}^{\mathrm{i}\,\phi} - \mathrm{e}^{-\,\mathrm{i}\,\theta}) + 2\,L\,\mathrm{e}^{\mathrm{i}\,2\phi} \;=\; 0\,.$$

For any given θ and L, ϕ this equation has two complex roots $\mathbf{z}$ defining the remaining control–polygon legs $\mathbf{L}_3, \mathbf{L}_4, \mathbf{L}_5$ through expressions (27), such that the resulting curves have initial tangent angle θ and curvature $\kappa(0)$ defined by (34). One can vary L and ϕ to achieve any desired $\kappa(0)$ value.[2] Another free parameter can be introduced using distinct magnitudes L_1, L_2 for $\mathbf{L}_1, \mathbf{L}_2$.

The PH quintics in Figure 10 (left) are obtained by varying L from 0.1 to 0.7 for fixed angles $\theta = \pi/4$, $\phi = -\pi/10$, while in Figure 10 (right) the initial curvature is assigned as $\kappa(0) \in [\,-2.50, -0.35\,]$ and L is computed from (34).

5.4 Inflectional curves defined by $\mathbf{L}_1, \mathbf{L}_4$

From (31) we observe that a PH quintic with prescribed control–polygon legs $\mathbf{L}_1, \mathbf{L}_4$ can satisfy $\mathbf{L}_2 = \mathbf{L}_4$ and $\mathbf{L}_5 = \mathbf{L}_1$ if equation (32) has $\mathbf{z} = 1$ as a root. For $\mathbf{z} = 1$ to be a root of equation (32), $\mathbf{L}_1$ and $\mathbf{L}_4$ must satisfy the condition

$$2\,\mathbf{L}_4^2 + 6\,\mathbf{L}_1\mathbf{L}_4 + 7\,\mathbf{L}_1^2 - 3\,\mathbf{L}_1 \;=\; 0\,,$$

which determines $\mathbf{L}_4$ in terms of $\mathbf{L}_1$ as

$$\mathbf{L}_4 \;=\; \frac{-3\,\mathbf{L}_1 \pm \sqrt{(6 - 5\,\mathbf{L}_1)\mathbf{L}_1}}{2}\,. \tag{35}$$

[2] $\kappa(0)$ and $\sin\phi$ must have the same sign if L is derived from them, to ensure that $L > 0$.

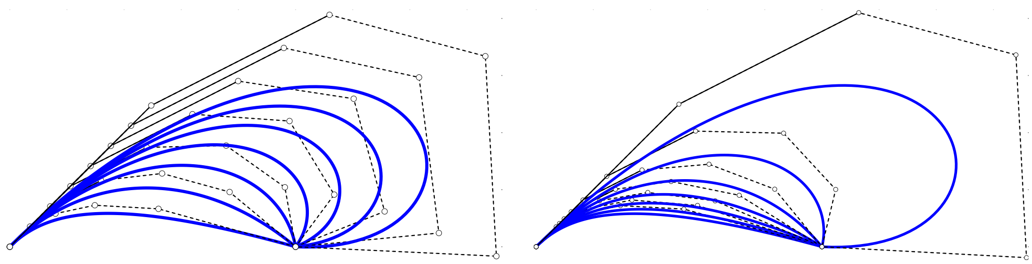

Figure 10: PH quintics with given control–polygon legs $\mathbf{L}_1, \mathbf{L}_2$ as in Section 5.3. Left: $L = 0.1, 0.2, \ldots, 0.7$. Right: $\kappa(0)$ varying uniformly in $[-2.50, -0.35]$ with L computed from (34). Both cases are for $\theta = \pi/4$, $\phi = -\pi/10$. $\mathbf{L}_1, \mathbf{L}_2$ are indicated as solid lines, and the "filled in" legs $\mathbf{L}_3, \mathbf{L}_4, \mathbf{L}_5$ as dotted lines.

For any chosen $\mathbf{L}_1$ and either of the corresponding $\mathbf{L}_4$ values as defined above, we obtain inflectional PH quintics defined by the control–polygon legs

$$\mathbf{L}_1, \mathbf{L}_4, 1 - 2\,\mathbf{L}_1 - 2\,\mathbf{L}_4, \mathbf{L}_4, \mathbf{L}_1\,.$$

Example 4. For the choice $\mathbf{L}_1 = -0.1 + 0.1\,\mathrm{i}$, we determine from (35) the two $\mathbf{L}_4$ values $0.35060983 - 0.28617004\,\mathrm{i}$ and $-0.05060983 - 0.58617004\,\mathrm{i}$. The remaining control leg $\mathbf{L}_3$ assumes the values $0.49878033 - 0.772340084\,\mathrm{i}$ and $1.30121967 + 0.97234008\,\mathrm{i}$. The two quintic PH curves defined by these control polygons are illustrated in Figure 11.

5.5 Construction of Hermite interpolants

The problem of constructing planar PH quintic interpolants to C^1 end–point data admits 4 formal solutions [13], among which the "good" solution (free of extreme curvature variations) can be identified *a posteriori* using the absolute rotation index (23). As noted in Section 4.1, the 4 solutions can be generated without replication by choosing a specific root of the discriminant of equation (20) and utilizing both square roots of $\mathbf{L}_1$ and $\mathbf{L}_5$.

Example 5. Consider the choices $\mathbf{L}_1 = \mathbf{L}_5 = 0.25 + 0.40\,\mathrm{i}$. For the four sign combinations $s_1 = \pm 1$, $s_2 = \pm 1$ the roots of the quadratic equation (20) and

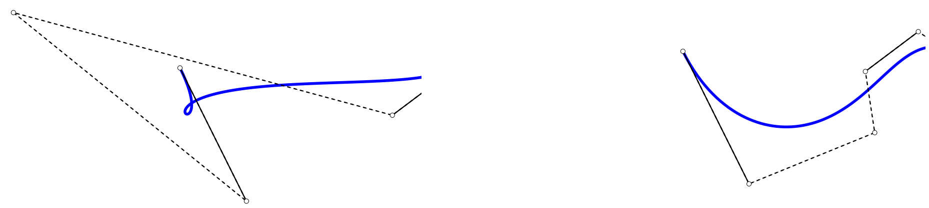

Figure 11: Inflectional quintic PH curves satisfying $\mathbf{L}_1 = \mathbf{L}_5$ and $\mathbf{L}_2 = \mathbf{L}_4$ as in Example 4. The two solutions for $\mathbf{L}_1 = -\,0.1 + 0.1\,\mathrm{i}$ are shown — $\mathbf{L}_1, \mathbf{L}_4$ are indicated as solid lines, and the "filled in" legs $\mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_5$ as dotted lines.

the corresponding control polygon legs $\mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_4$ are

$$
\begin{aligned}
s_1 &= -1\,, \quad s_2 = -1\,, \\
\mathbf{z} &= 2.01370914 + 0.27472185\,\mathrm{i}\,, \\
\mathbf{L}_2 &= -\,1.11818414 - 0.83547327\,\mathrm{i}\,, \\
\mathbf{L}_3 &= 2.73636827 + 0.87094654\,\mathrm{i}\,, \\
\mathbf{L}_4 &= -\,1.11818414 - 0.83547327\,\mathrm{i}\,,
\end{aligned}
$$

$$
\begin{aligned}
s_1 &= -1\,, \quad s_2 = +1\,, \\
\mathbf{z} &= 1.04970668 - 0.47632354\,\mathrm{i}\,, \\
\mathbf{L}_2 &= -\,0.78915421 - 0.06335897\,\mathrm{i}\,, \\
\mathbf{L}_3 &= 0.50000000 - 0.80000000\,\mathrm{i}\,, \\
\mathbf{L}_4 &= 0.78915421 + 0.06335897\,\mathrm{i}\,,
\end{aligned}
$$

$$
\begin{aligned}
s_1 &= +1\,, \quad s_2 = -1\,, \\
\mathbf{z} &= 1.04970668 - 0.47632354\,\mathrm{i}\,, \\
\mathbf{L}_2 &= 0.78915421 + 0.06335897\,\mathrm{i}\,, \\
\mathbf{L}_3 &= 0.50000000 - 0.80000000\,\mathrm{i}\,, \\
\mathbf{L}_4 &= -\,0.78915421 - 0.06335897\,\mathrm{i}\,,
\end{aligned}
$$

$$
\begin{aligned}
s_1 &= +1\,, \quad s_2 = +1\,, \\
\mathbf{z} &= 0.21158657 - 0.72410033\,\mathrm{i}\,, \\
\mathbf{L}_2 &= 0.36818414 - 0.36452673\,\mathrm{i}\,, \\
\mathbf{L}_3 &= -\,0.23636827 - 0.07094654\,\mathrm{i}\,, \\
\mathbf{L}_4 &= 0.36818414 - 0.36452673\,\mathrm{i}\,.
\end{aligned}
$$

The four quintic PH curves incorporating the specified $\mathbf{L}_1, \mathbf{L}_5$ control–polygon legs are illustrated in Figure 12. These curves are found to be identical to those generated [13] by first–order Hermite interpolation with planar PH quintics, for the data $\mathbf{r}(0) = 0$, $\mathbf{r}'(0) = 5\,\mathbf{L}_1$ and $\mathbf{r}(1) = 1$, $\mathbf{r}'(1) = 5\,\mathbf{L}_5$.

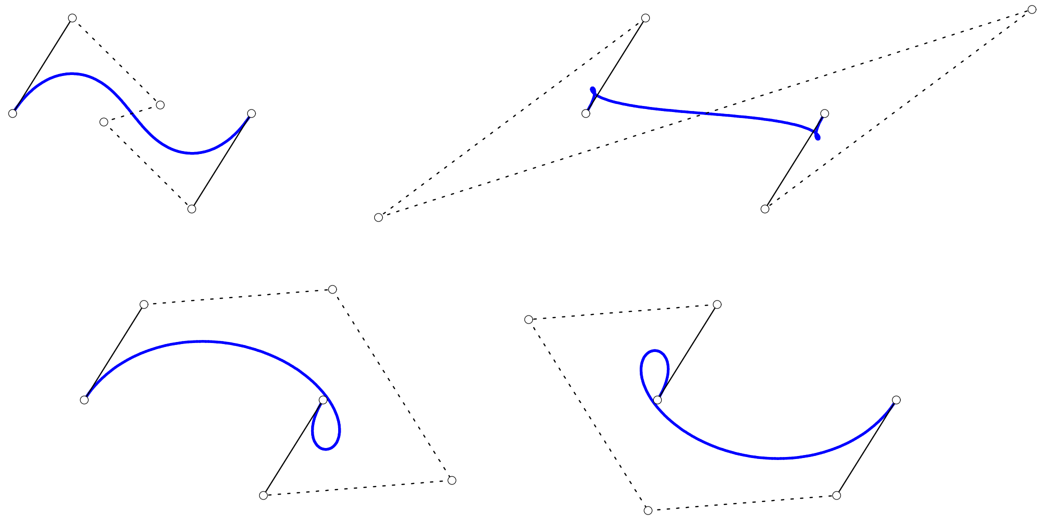

Figure 12: The four distinct quintic PH curves with specified initial and final control–polygon legs $\mathbf{L}_1 = \mathbf{L}_5 = 0.25 + 0.40\,\mathrm{i}$, as in Example 5 — $\mathbf{L}_1, \mathbf{L}_5$ are indicated as solid lines, and the "filled in" legs $\mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_4$ as dotted lines.

Example 6. For the case $\mathbf{L}_5 = \mathbf{L}_1 = (\mathbf{L},$ say), equation (20) using opposite signs s_1, s_2 reduces to

$$2\,\mathbf{z}^2 + 5\,\mathbf{L} - 3 \;=\; 0\,.$$

Setting $\mathbf{c} = \sqrt{(3 - 5\,\mathbf{L})/2}$, we obtain from (18)–(19) the control polygon legs

$$(\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_4, \mathbf{L}_5) \;=\; (\mathbf{L}, \pm\,\mathbf{c}\,\sqrt{\mathbf{L}}, 1 - 2\,\mathbf{L}, \mp\,\mathbf{c}\,\sqrt{\mathbf{L}}, \mathbf{L})\,.$$

Thus, when $\mathbf{L}_5 = \mathbf{L}_1$ and $s_1 s_2 = -1$, we obtain two solutions with $\mathbf{L}_4 = -\,\mathbf{L}_2$, and it is observed that these solutions both have the absolute rotation index

$R_{\text{abs}} = \frac{1}{2}$. In the remaining two solutions (generated when$s_1, s_2 = 1$), it can be seen from (18) that the control–polygon legs satisfy $\mathbf{L}_4 = \mathbf{L}_2$.

Example 7. When $\mathbf{L}_1, \mathbf{L}_5$ are the conjugates $L\mathrm{e}^{\mathrm{i}\,\theta}$ and $L\,\mathrm{e}^{-\mathrm{i}\,\theta}$, equation (20) with identical signs s_1, s_2 has real coefficients, namely

$$2\,\mathbf{z}^2 \pm 6\sqrt{L}\cos\tfrac{1}{2}\theta\,\mathbf{z} + (1 + 6\cos\theta)\,L - 3 \;=\; 0\,,$$

where the $+$ and $-$ signs correspond to $s_1 = s_2 = 1$ and $s_1 = s_2 = -1$, and its roots will be real when the discriminant

$$\Delta(L,\theta) \;=\; 24 + 10\,L(1 - 3\,\cos\theta)$$

is positive. As seen in Figure 13, Δ is positive for $(L,\theta) \in [\,0,1\,] \times [\,-\pi,\pi\,]$.

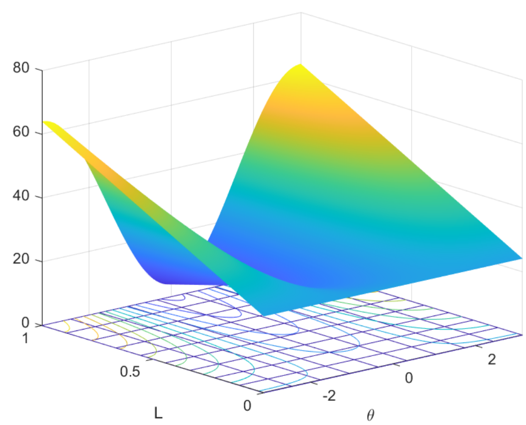

Figure 13: Graph of $\Delta(L,\theta)$ for $(L,\theta) \in [\,0,1\,] \times [\,-\pi,\pi\,]$ in Example 7.

Figure 14 shows the family of PH quintics obtained with $\theta = 2\pi/3$ and $L = 0.1, 0.2, \ldots, 0.7$, and $s_1 = s_2 = 1$. A set of 100,000 randomly–generated examples with $0 \le L \le 1$ and $0 \le \theta < 2\,\pi$ indicate the following properties — (1) the case with $s_1 = s_2 = +1$ always has the smallest R_{abs} value; (2) the two cases with opposite signs s_1, s_2 have equal R_{abs} values; and (3) the two cases with equal signs s_1, s_2 have R_{abs} values that sum to 1.

5.6 Other PH quintic forms

Before concluding, we briefly consider how the methods described above can be adapted to the PH quintics defined (see Remark 1) by a square–free real

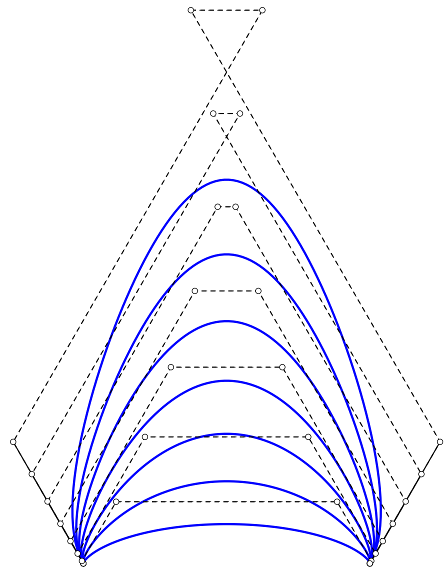

Figure 14: PH quintics with complex conjugate control–polygon legs $\mathbf{L}_1, \mathbf{L}_5$ as in Example 7, for $\theta = 2\pi/3$, $L = 0.1, 0.2, \ldots, 0.7$ and $s_1 = s_2 = 1$ — $\mathbf{L}_1, \mathbf{L}_5$ are indicated as solid lines, and the "filled in" legs $\mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_4$ as dotted lines.

quadratic polynomial $f(t)$ and a complex linear polynomial $\mathbf{w}(t)$ through the expression $\mathbf{r}'(t) = f(t)\,\mathbf{w}^2(t)$. To preclude non–essential free parameters, we set

$$f(t) \;=\; f_0(1-t)^2 + \tfrac{1}{2}(1-f_0-f_2)\,2(1-t)t + f_2t^2\,,$$

which ensures that $f(t)$ is monic, i.e., on expansion the coefficient of t^2 is 1. Together with

$$\mathbf{w}(t) \;=\; \mathbf{w}_0(1-t) + \mathbf{w}_1 t\,,$$

the control–polygon legs are determined in terms of $f_0, f_2, \mathbf{w}_0, \mathbf{w}_1$ through

$$\begin{aligned}
5\,\mathbf{L}_1 &= f_0\mathbf{w}_0^2\,, \\
20\,\mathbf{L}_2 &= (f_0+f_2-1)\mathbf{w}_0^2 + 2f_0\mathbf{w}_0\mathbf{w}_1\,, \\
30\,\mathbf{L}_3 &= 2\,(f_0+f_2-1)\mathbf{w}_0\mathbf{w}_1 + f_0\mathbf{w}_1^2 + f_2\mathbf{w}_0^2\,, \\
20\,\mathbf{L}_4 &= (f_0+f_2-1)\mathbf{w}_1^2 + 2f_2\mathbf{w}_0\mathbf{w}_1\,, \\
5\,\mathbf{L}_5 &= f_2\mathbf{w}_1^2\,.
\end{aligned}$$

To obtain a system of control–polygon constraints analogous to equations (10)–(14), characterizing this type of PH quintic, one can appeal to a Gröbner basis calculation — as in [5] — and identify those basis elements independent of $f_0, f_2, \mathbf{w}_0, \mathbf{w}_1$. However, the Gröbner basis computation is more involved in this case, in part because four variables are to be eliminated, rather than the three variables $\mathbf{w}_1, \mathbf{w}_1, \mathbf{w}_2$ yielding equations (10)–(14), and the result is rather involved. As previously noted, the PH quintics identified in Remark 1 have less shape freedom than those defined by quadratic complex pre–image polynomials, so we shall not further dwell on them.

6 Closure

Since planar PH curves incorporate fewer intrinsic degrees of freedom than are indicated by their Bézier control points, it is customary to construct them through a largely algebraic Hermite interpolation process for prescribed end points and derivatives. Control–polygon constraints that characterize quintic PH curves have long been known [5] but have not been much exploited.

A canonical–form quintic PH curve $\mathbf{r}(t)$ in complex form, with $\mathbf{r}(0) = 0$ and $\mathbf{r}(1) = 1$, embodies two free complex parameters that must be chosen so as to ensure that its five control–polygon legs satisfy the constraints [5] that identify quintic PH curves. In the present study it is shown that, when any two of the control–polygon legs of a PH quintic are specified, the remaining three can be "filled in" by a simple algorithm that requires only the solution of a quadratic or quartic equation with complex coefficients. Several examples are included to illustrate how this approach can be employed in the practical design of planar PH quintics with desired shape features.

Although it seems natural to seek a generalization of the methodology to spatial PH curves, this is not a trivial task. Whereas spatial PH cubics admit a relatively simple characterization in terms of their control polygons [16], no system of control–polygon constraints for the spatial PH quintics (analogous to those [5] for the planar PH quintics) is currently known. This difficulty is further compounded by the fact that, in the quaternion representation, the spatial PH quintic interpolants to given first–order Hermite data comprise a two–parameter family [10] rather than a discrete set as in the planar case.

Acknowledgements

Research on this paper was supported in part by the MIUR Excellence Department Project, awarded to the Department of Mathematics, University of Rome "Tor Vergata" (CUP E83C18000100006F), and by INdAM-GNCS, Gruppo Nazionale per il Calcolo Scientifico which F. Pelosi and M.L. Sampoli are members of.

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