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RUSCHEWEYH DERIVATIVE AND STRONGLY STARLIKE FUNCTIONS

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Abstract. Let A denote the class of analytic functions $f(z)$ defined in the unit disc satisfying the condition $f(0)=f'(0)-1=0$. Let $\bar{S}^*(\beta,\gamma)$ be the class of strongly starlike functions of order β and type γ, and let $\bar{C}(\beta,\gamma)$ denote the class of strongly convex functions of order β and type γ. Certain new classes $\bar{S}_\alpha^*(\beta,\gamma)$ and $\bar{C}_\alpha(\beta,\gamma)$ are introduced by virtue of Ruscheweyh derivative and some properties of $\bar{S}_\alpha^*(\beta,\gamma)$ and $\bar{C}_\alpha(\beta,\gamma)$ are discussed.

1. Introduction

Let A be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. A function $f(z)$ belonging to A is said to be starlike of order γ if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \quad (z \in E) \tag{1.2}$$

for some γ $(0 \le \gamma < 1)$. We denote by $S^*(\gamma)$ the subclass of A consisting of functions which are starlike of order γ in E. Also, a function $f(z)$ in A is said to be convex of order γ if it satisfies $zf'(z) \in S^*(\gamma)$, or

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma \quad (z \in E) \tag{1.3}$$

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for some γ $(0 \leq \gamma < 1)$. We denote by $C(\gamma)$ the subclass of A consisting of all functions which are convex of order γ in E.

If $f(z) \in A$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta \quad (z \in E) \tag{1.4}$$

for some γ $(0 \leq \gamma < 1)$ and β $(0 < \beta \leq 1)$, then $f(z)$ is said to be strongly starlike of order β and type γ in E, and denoted by $f(z) \in \bar{S}^*(\beta, \gamma)$. If $f(z) \in A$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta \quad (z \in E) \tag{1.5}$$

for some γ $(0 \leq \gamma < 1)$ and β $(0 < \beta \leq 1)$, then we say that $f(z)$ is strongly convex of order β and type γ in E, and we denote by $\bar{C}(\beta, \gamma)$ the class of all such functions. It is obvious that $f(z) \in A$ belongs to $\bar{C}(\beta, \gamma)$ if and only if $zf'(z) \in \bar{S}^*(\beta, \gamma)$. Also, we note that $\bar{S}^*(1, \gamma) = S^*(\gamma)$ and $\bar{C}(1, \gamma) = C(\gamma)$.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A$, then the Hadamard product (or convolution product) $(f * g)(z)$ of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.6}$$

By the Hadamard product, we define

$$D^\alpha f(z) = \frac{z}{(1 - z)^{1+\alpha}} * f(z) \quad (\alpha \geq -1) \tag{1.7}$$

for $f(z) \in A$. $D^\alpha f(z)$ is called the Ruscheweyh derivative and was introduced by Ruscheweyh in [1].

We now introduce the following classes:

$$\bar{S}_\alpha^*(\beta, \gamma) = \left\{f(z) \in A : D^\alpha f(z) \in \bar{S}^*(\beta, \gamma), \alpha \geq -1 \text{ and } \frac{z(D^\alpha f(z))'}{D^\alpha f(z)} \neq \gamma \quad \text{for } z \in E\right\}$$

and

$$\bar{C}_\alpha(\beta, \gamma) = \left\{f(z) \in A : D^\alpha f(z) \in \bar{C}(\beta, \gamma), \alpha \geq -1 \text{ and } 1 + \frac{z(D^\alpha f(z))''}{(D^\alpha f(z))'} \neq \gamma \quad \text{for } z \in E\right\}$$

In this note, we shall investigate some properties of $\bar{S}_\alpha^*(\beta, \gamma)$ and $\bar{C}_\alpha(\beta, \gamma)$.

2. Main Results

We shall need the following lemma.

LEMMA. (see [2] [3]). *Let a function* $p(z) = 1 + b_1 z + \cdots$ *be analytic in* E *and* $p(z) \neq 0$ $(z \in E)$. *If there exists a point* $z_0 \in E$ *such that*

$$|\arg(p(z))| < \frac{\pi}{2}\beta \ (|z| < |z_0|) \quad \text{and} \quad |\arg(p(z_0))| = \frac{\pi}{2}\beta \quad (0 < \beta \leq 1),$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \left(\text{when } \arg(p(z_0)) = \frac{\pi}{2}\beta\right),$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \left(\text{when } \arg(p(z_0)) = -\frac{\pi}{2}\beta\right),$$

and $(p(z_0))^{1/\beta} = \pm ia$ $(a > 0)$.

THEOREM 1. $\bar{S}^*_{\alpha+1}(\beta, \gamma) \subset \bar{S}^*_\alpha(\beta, \gamma)$ *for* $\alpha \geq -\gamma$ *and* $0 \leq \gamma < 1$.

PROOF. Let $f(z) \in \bar{S}^*_{\alpha+1}(\beta, \gamma)$. Then we set

$$\frac{z(D^\alpha f(z))'}{D^\alpha f(z)} = \gamma + (1 - \gamma)p(z), \tag{2.1}$$

where $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in E and $p(z) \neq 0$ for all $z \in E$. According to the well known identity (see [1] [4])

$$z(D^\alpha f(z))' = (\alpha + 1)D^{\alpha+1} f(z) - \alpha D^\alpha f(z), \tag{2.2}$$

we have

$$\frac{D^{\alpha+1} f(z)}{D^\alpha f(z)} = \frac{1}{\alpha + 1}\left[\frac{z(D^\alpha f(z))'}{D^\alpha f(z)} + \alpha\right] = \frac{1}{\alpha + 1}[(1 - \gamma)p(z) + \gamma + \alpha]. \tag{2.3}$$

Differentiating both sides of (2.3) logarithmically, it follows that

$$\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)} = \frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)} + \frac{(1-\gamma)zp'(z)}{(1-\gamma)p(z)+\gamma+\alpha}$$
$$= (1-\gamma)p(z)+\gamma+\frac{(1-\gamma)zp'(z)}{(1-\gamma)p(z)+\gamma+\alpha},$$

or

$$\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)} - \gamma = (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{(1-\gamma)p(z)+\gamma+\alpha}. \tag{2.4}$$

Suppose that there exists a point $z_0 \in E$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\beta \quad (|z| < |z_0|) \quad \text{and} \quad |\arg(p(z_0))| = \frac{\pi}{2}\beta.$$

Then, applying the Lemma, we can write that $z_0 p'(z_0)/p(z_0) = ik\beta$ and $(p(z_0))^{1/\beta} = \pm ia$ $(a > 0)$.

Therefore, if $\arg(p(z_0)) = \frac{\pi}{2}\beta$, then

$$\frac{z_0(D^{\alpha+1}f(z_0))'}{D^{\alpha+1}f(z_0)} - \gamma = (1-\gamma)p(z_0)\left[1 + \frac{z_0p'(z_0)/p(z_0)}{(1-\gamma)p(z_0)+\gamma+\alpha}\right]$$
$$= (1-\gamma)a^{\beta}e^{i\pi\beta/2}\left[1 + \frac{ik\beta}{(1-\gamma)a^{\beta}e^{i\pi\beta/2}+\gamma+\alpha}\right].$$

This implies that

$$\arg\left\{\frac{z_0(D^{\alpha+1}f(z_0))'}{D^{\alpha+1}f(z_0)} - \gamma\right\}$$
$$= \frac{\pi}{2}\beta + \arg\left\{1 + \frac{ik\beta}{(1-\gamma)a^{\beta}e^{i\pi\beta/2}+\gamma+\alpha}\right\}$$
$$= \frac{\pi}{2}\beta + Tan^{-1}$$
$$\times \left\{\frac{k\beta\left(\gamma+\alpha+(1-\gamma)a^{\beta}\cos\left(\frac{\pi}{2}\beta\right)\right)}{(\gamma+\alpha)^2+2(\gamma+\alpha)(1-\gamma)a^{\beta}\cos((\pi/2)\beta)+(1-\gamma)^2a^{2\beta}+k\beta(1-\gamma)a^{\beta}\sin((\pi/2)\beta)}\right\}$$
$$\geq \frac{\pi}{2}\beta. \quad \left(\text{where } k \geq \frac{1}{2}\left(a+\frac{1}{a}\right) > 1\right),$$

which contradicts the hypothesis that $f(z) \in \bar{S}^*_{\alpha+1}(\beta, \gamma)$.

Similarly, if $\arg(p(z_0)) = -(\pi/2)\beta$, then we obtain that

$$\arg\left\{\frac{z_0(D^{\alpha+1}f(z_0))'}{D^{\alpha+1}f(z_0)} - \gamma\right\} \leq -\frac{\pi}{2}\beta,$$

which also contradicts the hypothesis that $f(z) \in S^*_{\alpha+1}(\beta, \gamma)$.

Thus the function $p(z)$ has to satisfy $|\arg(p(z))| < \frac{\pi}{2}\beta$ $(z \in E)$. This shows that

$$\left|\arg\left\{\frac{z(D^\alpha f(z))'}{D^\alpha f(z)} - \gamma\right\}\right| < \frac{\pi}{2}\beta \quad (z \in E),$$

or $f(z) \in \bar{S}^*_\alpha(\beta, \gamma)$.

THEOREM 2. *Let* $\alpha \geq -\gamma$ *and* $0 \leq \gamma < 1$, *then* $\bar{C}_{\alpha+1}(\beta, \gamma) \subset \bar{C}_\alpha(\beta, \gamma)$.

PROOF. $f(z) \in \bar{C}_{\alpha+1}(\beta, \gamma) \Leftrightarrow D^{\alpha+1}f(z) \in \bar{C}(\beta, \gamma) \Leftrightarrow z(D^{\alpha+1}f(z))' \in \bar{S}^*(\beta, \gamma)$

$$\Leftrightarrow D^{\alpha+1}(zf'(z)) \in \bar{S}^*(\beta, \gamma) \Leftrightarrow zf'(z) \in \bar{S}^*_{\alpha+1}(\beta, \gamma)$$

$$\Rightarrow zf'(z) \in \bar{S}^*_\alpha(\beta, \gamma) \Leftrightarrow D^\alpha(zf'(z)) \in \bar{S}^*(\beta, \gamma)$$

$$\Leftrightarrow z(D^\alpha f(z))' \in \bar{S}^*(\beta, \gamma) \Leftrightarrow D^\alpha f(z) \in \bar{C}(\beta, \gamma)$$

$$\Leftrightarrow f(z) \in \bar{C}_\alpha(\beta, \gamma).$$

For $c > -1$, and $f(z) \in A$, we define the integral operator $L_c(f)$ as

$$L_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt. \tag{2.5}$$

The operator $L_c(f)$ when $c \in N = \{1, 2, 3, \ldots\}$ was studied by Bernardi [6]. For $c = 1, L_1(f)$ was investigated by Libera [5].

THEOREM 3. *Let* $c > -\gamma$ *and* $0 \leq \gamma < 1$. *If* $f(z) \in \bar{S}^*_\alpha(\beta, \gamma)$ *with* $z(D^\alpha(L_c(f)))'/(D^\alpha(L_c(f))) \neq \gamma$ *for all* $z \in E$, *then we have* $L_c(f) \in \bar{S}^*_\alpha(\beta, \gamma)$.

PROOF. Set

$$\frac{z(D^\alpha(L_c(f)))'}{D^\alpha(L_c(f))} = \gamma + (1-\gamma)p(z), \tag{2.6}$$

where $p(z)$ is analytic in E, $p(0) = 1$ and $p(z) \neq 0$ $(z \in E)$. From (2.5), we have

(2.7) $$z(D^{\alpha}(L_c(f)))' = (c+1)D^{\alpha}f - cD^{\alpha}(L_c(f)).$$

Using (2.6) and (2.7), we get

(2.8) $$(c+1)\frac{D^{\alpha}f}{D^{\alpha}(L_c(f))} = c + \gamma + (1-\gamma)p(z).$$

Differentiating (2.8) logarithmically, we obtain

$$\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)} - \gamma = (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{c+\gamma+(1-\gamma)p(z)}.$$

Suppose that there exists a point $z_0 \in E$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\beta \quad (|z| < |z_0|) \quad \text{and} \quad |\arg(p(z_0))| = \frac{\pi}{2}\beta.$$

Then, applying the Lemma, we can write that $z_0p'(z_0)/p(z_0) = ik\beta$ and

$$(p(z_0))^{1/\beta} = \pm ia \quad (a > 0).$$

If $\arg(p(z_0)) = -(\pi/2)\beta$, then

$$\begin{aligned}\frac{z_0(D^{\alpha}f(z_0))'}{D^{\alpha}f(z_0)} - \gamma &= (1-\gamma)p(z_0)\left[1 + \frac{z_0p'(z_0)/p(z_0)}{c+\gamma+(1-\gamma)p(z_0)}\right] \\ &= (1-\gamma)a^{\beta}e^{-i\pi\beta/2}\left[1 + \frac{ik\beta}{c+\gamma+(1-\gamma)a^{\beta}e^{-i\pi\beta/2}}\right].\end{aligned}$$

This shows that

$$\begin{aligned}&\arg\left\{\frac{z_0(D^{\alpha}f(z_0))'}{D^{\alpha}f(z_0)} - \gamma\right\} \\ &= -\frac{\pi}{2}\beta + \arg\left\{1 + \frac{ik\beta}{c+\gamma+(1-\gamma)a^{\beta}e^{-i\pi\beta/2}}\right\} \\ &= -\frac{\pi}{2}\beta + Tan^{-1} \\ &\quad \times \left\{\frac{k\beta\left(c+\gamma+(1-\gamma)a^{\beta}\cos\left(\frac{\pi}{2}\beta\right)\right)}{(c+\gamma)^2+2(c+\gamma)(1-\gamma)a^{\beta}\cos((\pi/2)\beta)+(1-\gamma)^2a^{2\beta}-k\beta(1-\gamma)a^{\beta}\sin((\pi/2)\beta)}\right\} \\ &\le -\frac{\pi}{2}\beta \quad \left(\text{where } k \le -\frac{1}{2}\left(a+\frac{1}{a}\right) < -1\right),\end{aligned}$$

which contradicts the condition $f(z) \in \bar{S}_{\alpha}^{*}(\beta, \gamma)$.

Similarly, we can prove the case $\arg(p(z_0)) = (\pi/2)\beta$. Thus we conclude that the function $p(z)$ has to satisfy $|\arg(p(z))| < (\pi/2)\beta$ for all $z \in E$. This gives that

$$\left|\arg\left\{\frac{z(D^\alpha(L_c(f)))'}{D^\alpha(L_c(f))} - \gamma\right\}\right| < \frac{\pi}{2}\beta \quad (z \in E),$$

or $L_c(f) \in \bar{S}_\alpha^*(\beta, \gamma)$.

THEOREM 4. *Let* $c > -\gamma$ *and* $0 \le \gamma < 1$. *If* $f(z) \in \bar{C}_\alpha(\beta, \gamma)$ *and* $1 + z(D^\alpha(L_c(f)))''/(D^\alpha(L_c(f)))' \neq \gamma$ *for all* $z \in E$, *then we have* $L_c(f) \in \bar{C}_\alpha(\beta, \gamma)$.

PROOF. $f \in \bar{C}_\alpha(\beta, \gamma) \Leftrightarrow zf' \in \bar{S}_\alpha^*(\beta, \gamma) \Rightarrow L_c(zf') \in \bar{S}_\alpha^*(\beta, \gamma)$

$$\Leftrightarrow z(L_c(f))' \in \bar{S}_\alpha^*(\beta, \gamma) \Leftrightarrow L_c(f) \in \bar{C}_\alpha(\beta, \gamma).$$

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