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WHEN IS AN ORDERED FIELD A METRIC SPACE?

Dedicated to the memory of my colleague, W. Harold "Harry" Row, Jr.

By

David E. DOBBS

Abstract. Let $(F, \leq)$ be an ordered field. With respect to the order topology, F is a Tychonoff uniform space. F is metrizable if and only if there is a countable set $\{b_1, \ldots, b_n, \ldots\}$ of positive elements of F such that if b is any positive element of F, there exists $n \geq 1$ such that $0 < b_n < b$. If F is denumerable or Archimedean, then this metrizability condition is satisfied. For each uncountable cardinal number $\aleph$, there exist ordered fields, F_1 and F_2, each of cardinality $\aleph$, such that the order topology on F_1 (resp, F_2) is (resp., is not) metrizable.

1. Introduction

Our starting point is the observation that the set $\boldsymbol{R}$ of real numbers has many compatible structures. For instance, $\boldsymbol{R}$ is both an ordered field and a metric space. Some other familiar ordered fields, such as the field $\boldsymbol{Q}$ of rational numbers, also have structures as metric spaces, and so it seems natural to ask if every ordered field is a metric space. More precisely, one may ask, given an ordered field $(F, \leq)$, if the order topology on F (in the sense of [7]) is metrizable, in which case one would say that *F is metrizable*. Despite the evidence afforded by $\boldsymbol{R}$ and $\boldsymbol{Q}$, the answer is in the negative, for Theorem 2.6 establishes that for each uncountable cardinal number $\aleph$, there exist ordered fields, F_1 and F_2, each of cardinality $\aleph$, such that F_1 is metrizable and F_2 is not metrizable. This result is best-possible, since Corollary 2.4 establishes that each countable ordered field is metrizable. Moreover, as suggested by the examples of $\boldsymbol{R}$ and $\boldsymbol{Q}$, Corollary 2.5 establishes that each Archimedean field is metrizable. (Background on the

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"Archimedean" concept will be recalled in Section 2. For the moment, we recall that each ordered field F has characteristic zero and, hence, contains an isomorphic copy of $\boldsymbol{Q}$. It is well known that F is Archimedean if and only if $\boldsymbol{Q}$ is order-theoretically dense in F.) Despite the examples of $\boldsymbol{R}$ and $\boldsymbol{Q}$, Remark 2.7 (a) shows that the order topology of a metrizable ordered field need not be separable. One may conclude, by comparing Corollary 2.5 and Remark 2.7 (a), that the metrizability of the usual topology on $\boldsymbol{R}$ is essentially due to the *order-theoretic* density of $\boldsymbol{Q}$ in $\boldsymbol{R}$, rather than the topological density of $\boldsymbol{Q}$ in $\boldsymbol{R}$. In fact, Theorem 2.2 gives the following order-theoretic characterization of metrizable ordered fields $(F, \leq)$: there is a countable set $\{b_1, \dots, b_n, \dots\}$ of positive elements of F such that if b is any positive element of F, there exists $n \geq 1$ such that $0 < b_n < b$.

Theorem 2.6 and Remark 2.7 are essentially algebraic. In these results, one constructs linear orders on monomials involving possibly infinitely many variables, and the ordered fields that result from these constructions are certain function fields or fields of formal Laurent series. On the other hand, Theorem 2.2 results from an analysis of the order topology on any *arbitrary* ordered field. Although this topology need not be metrizable, it does produce a Tychonoff uniform space: see Lemma 2.1 (b) for a proof of uniformity that uses results on topological groups and Remark 2.3 for the sketch of a proof that uses only general topology.

For background on uniform spaces and other aspects of general topology, we refer the reader to [7]. For the rudiments on ordered fields, see [1] and [3]. As for notation, it will be convenient to let $|S|$ denote the cardinality of a set S; and if $(F, \leq)$ is an ordered field, to let F^+ denote $\{a \in F \mid 0 < a\}$, the set of positive elements in F.

2. Results

Let $(F, \leq)$ be an ordered field. In particular, $(F, \leq)$ is a partially ordered (in fact, linearly ordered) set, and so $\leq$ induces an order topology $\mathscr{T} = \mathscr{T}(F)$ on (the set underlying) F. According to [7, Exercise I, page 57], a subbase for $\mathscr{T}$ consists of the sets $(-\infty, b) := \{c \in F \mid c < b\}$ and $(a, \infty) := \{c \in F \mid a < c\}$, as a and b vary over the elements of F. It is then easy to see that a basis for $\mathscr{T}$ consists of all the sets having one of the forms $(a, b) := \{c \in F \mid a < c < b\}$, $(-\infty, b)$, and (a, ∞), as a and b vary over the elements of F. It will be convenient to define the (*canonical*) *uniformity for* F to be $\mathscr{U} := \mathscr{U}(F) := \{U \subseteq F \times F \mid \text{There exists } b \in F^+ \text{ such that } \{(x, y) \in F \times F : |x - y| < b\} \subseteq U\}$. In Lemma 2.1 (b), we show that, in the terminology of [7], $\mathscr{U}$ is a uniformity and the topology of this uniformity coincides with the order topology $\mathscr{T}$. We assume familiarity with this terminology, as well as the notion of a "base" of a uniformity [7, page 177].

LEMMA 2.1. *Let $(F, \mathscr{T})$ be the topological space arising from an ordered field F and its order topology $\mathscr{T}$. Then*:

(a) *$(F, \mathscr{T})$ is a Hausdorff topological group (with respect to addition).*

(b) *$(F, \mathscr{T})$ is a uniform space with uniformity $\mathscr{U}$.*

(c) *$(F, \mathscr{T})$ is a completely regular space.*

PROOF. (a) If $a < b \in F$, put $\delta := (b-a)/2$, and observe that $a \in (a-\delta, a+\delta)$, $b \in (b-\delta, b+\delta)$, and $(a-\delta, a+\delta) \cap (b-\delta, b+\delta) = \varnothing$. It follows that $\mathscr{T}$ is a Hausdorff topology on F. We show next that, with this topology, F (under addition) is a topological group.

Consider the additive inverse map $- : F \to F$, given by $a \mapsto -a$ for all $a \in F$. Under this map, the inverse image of (a,b) is $(-b,-a)$, the inverse image of $(-\infty, b)$ is $(-b, \infty)$, and the inverse image of (a, ∞) is $(-\infty, -a)$. Thus, the inverse image of each basic open set is open, and so $-$ is a continuous function. It remains to prove that the addition map $+ : F \times F \to F$, $(a,b) \mapsto a+b$, is also continuous.

We shall show that $+$ is continuous at each $(\alpha, \beta) \in F \times F$. Suppose that $+(\alpha, \beta) \in (a,b)$ for some $a < b$ in F; that is, $a < \alpha + \beta < b$. Put $\varepsilon := \min(b - (\alpha+\beta), (\alpha+\beta) - a)$. Now, if $\delta > 0$ with $\alpha_1, \beta_1 \in F$ satisfying $|\alpha_1 - \alpha| < \delta$ and $|\beta_1 - \beta| < \delta$, then we see, by the triangle inequality [3, (iv), page 8], that

$$|+(\alpha_1, \beta_1) - +(\alpha, \beta)| = |(\alpha_1 + \beta_1) - (\alpha + \beta)| \leq |\alpha_1 - \alpha| + |\beta_1 - \beta| < \delta + \delta = 2\delta.$$

Thus, if we take $\delta := \varepsilon/2$, we find that $|+(\alpha_1, \beta_1) - +(\alpha, \beta)| < \varepsilon$, so that $+(\alpha_1, \beta_1) \in (a,b)$. The above argument applies formally as well if one supposes that either $+(\alpha, \beta) \in (-\infty, b)$ or $+(\alpha, \beta) \in (a, \infty)$. In view of the above description of a basis for $\mathscr{T}$, this establishes that $+$ is continuous and completes the proof that F is a topological group.

(b) We claim that a fundamental basis of the $\mathscr{T}$-neighborhoods of 0 in F is given by all the sets of the form $(-\delta, \delta)$ as δ varies over the elements of F^+. Indeed, if $0 \in (a,b)$, then $\delta := \min(-a, b)$ satisfies $0 \in (-\delta, \delta) \subseteq (a,b)$. Similarly, given $0 \in (a, \infty)$, use $\delta := -a$; and given $0 \in (-\infty, b)$, use $\delta := b$. This proves the claim.

Since (a) ensures that $(F, \mathscr{T})$ is a topological group, we see, by combining the above claim with [5, Proposition 5, page 53], that $(F, \mathscr{T})$ is a uniform space and that a base for its uniformity $\mathscr{W}$ is given by all the sets of the form $L(\delta) := \{(x, y) \in F \times F \mid x - y \in (-\delta, \delta)\}$ as δ varies over the elements of F^+. According to [7, page 177, lines 21–23], this base determines $\mathscr{W}$ entirely, namely, $\mathscr{W} = \{U \subseteq F \times F \mid \text{There exists } \delta \in F^+ \text{ such that } L(\delta) \subseteq U\}$. In view of the definitions of $L(\delta)$ and $\mathscr{U}(F)$, it follows that $\mathscr{W} = \mathscr{U}(F)$. In particular, $\mathscr{U} = \mathscr{U}(F)$ is a uniformity. Moreover, [5, Proposition 5, page 53] also establishes that the topology of the uniformity $\mathscr{W}$ (that is, of $\mathscr{U}$), is the topology of the topological group $(F, \mathscr{T})$, namely, $\mathscr{T}$. This completes the proof of (b).

(c) According to [5, Theorem 5, page 49], any Hausdorff topological group is completely regular. Apply (a). □

Let F be an ordered field. Then Lemma 2.1 (a), (c) ensure that F, in its order topology $\mathscr{T}$, is a completely regular T_1-space; that is, a Tychonoff space, to use terminology as introduced in [7, page 117]. In this regard, it is natural to consider metrizability, for the celebrated metrization theorem of Urysohn [7, Theorem 16, page 125] implies that any second-countable Tychonoff space is metrizable. We shall say that F is *(pseudo-)metrizable* in case $(F, \mathscr{T})$ is (pseudo-)metrizable in the sense of [7, page 124]; that is, in case $\mathscr{T}$ is induced by some (pseudo-)metric on F. Metrizability results for uniform spaces are classical (cf. the Alexandroff-Urysohn metrization theorem [7, page 186]), as are metrizability results for topological groups (cf. [6]). While much of Lemma 2.1 works for any Abelian topological group, we next use the field structure of the ordered field F (specifically, that $0 < 1/2 < 1$) to characterize when F is (pseudo-)metrizable.

Theorem 2.2. *Let F be an ordered field, with canonical uniformity $\mathscr{U}$. Then the following conditions are equivalent*:

(1) *F is metrizable*;

(2) *$\mathscr{U}$ has a countable base*;

(3) *There exists a countable set $\{b_1, b_2, \ldots\} \subseteq F^+$ such that for each $b \in F^+$, there exists $n \geq 1$ so that $0 < b_n < b$.*

Proof. By Lemma 2.1 (b), F (with its order topology) is a uniform space. Hence, by [7, Metrization Theorem 13, page 186], (2) holds if and only if F is pseudo-metrizable. Since Lemma 2.1 (a) ensures that F is Hausdorff, it then follows that (2) $\Leftrightarrow$ (1). (See the comments in [7, page 186] concerning the metrization theorem of Alexandroff-Urysohn.) It remains to show that (2) $\Leftrightarrow$ (3).

For each $b \in F^+$, let $U_b := \{(x, y) \in F \times F : |x - y| < b\}$. Thus, the above definition of the uniformity $\mathscr{U}$ may be rewritten as $\mathscr{U} = \{U \subseteq F \times F \mid \text{There exists}$

$b \in F^+$ such that $U_b \subseteq U\}$. In particular, $U_b \in \mathscr{U}$ for each $b \in F^+$. Moreover, if $0 < b_n < b$ in F, then $U_{b_n} \subseteq U_b$. Thus, if $\{b_1, b_2, \ldots\}$ is as in (3), then $\{U_{b_1}, U_{b_2}, \ldots\}$ forms a (countable) base for $\mathscr{U}$. Therefore, (3) $\Rightarrow$ (2).

Finally, we show that (2) $\Rightarrow$ (3). Suppose that $\mathscr{B} = \{B_1, B_2, \ldots\}$ is a countable base for the uniformity $\mathscr{U}$. Let $b \in F^+$. Since $U_b \in \mathscr{U}$, it follows from the definition of "base" that $B_n \subseteq U_b$ for some n. However, by the definition of $\mathscr{U}$, there exists $c_n \in F^+$ such that $U_{c_n} \subseteq B_n$. (Note that c_n depends on n but not on b.) Hence, $|x - y| < c_n \Rightarrow (x, y) \in U_{c_n} \Rightarrow (x, y) \in B_n \Rightarrow (x, y) \in U_b \Rightarrow |x - y| < b$ in F. Taking $y := 0$, we conclude that $|x| < c_n \Rightarrow |x| < b$ in F. If $b < c_n$, then $x := (b + c_n)/2$ satisfies $|x| = x < c_n$ and $|x| > b$, a contradiction. (We have just used that $1/2 > 0$, which is, of course, valid in any ordered field. Two sentences hence, we shall use the fact that $1/2 < 1$.) Therefore, $c_n \leq b$. Hence, $b_n := c_n/2$ satisfies $0 < b_n < c_n \leq b$. It follows that the set $\{b_1, b_2, \ldots\}$ is as in (3). □

Although it was convenient to use material concerning topological groups from [5] in proving Lemma 2.1, we pause next to sketch how a direct proof of Lemma 2.1 (b) may be accomplished by using only background on uniform spaces from [7]. Of course, a purely topological approach (as in the proof of Theorem 2.2) has natural overlaps with an approach that invokes results on topological groups. For instance, the reader will have noticed that for $b \in F^+$, the set U_b in the proof of Theorem 2.2 is the same as the set $L(b)$ in the proof of Lemma 2.1. We used the notation $L(b)$ because this corresponds to the notation of Husain [5, page 52] that supports the results invoked from [5]: his $L(U)$ is just our $L(b)$ in case $U = (-b, b)$.

Remark 2.3. A proof of Lemma 2.1 (b) that avoids citing results on topological groups can proceed as follows. First, one shows directly that $\mathscr{U} = \mathscr{U}(F)$ is a uniformity. To do so, one must verify conditions (a)–(e) in the definition of "uniformity" in [7, page 176]. Condition (a) follows because $0 < b$ for all $b \in F^+$; (b) follows because $|x - y| = |y - x|$ for all $x, y \in F$; (c) follows essentially by the triangle inequality, because $U_b \subseteq U \subseteq F \times F$ and $c := b/2$ imply that $U_c \circ U_c \subseteq U$; (d) follows because $U_b \subseteq U \subseteq F \times F$ and $U_c \subseteq V \subseteq F \times F$ imply that $U_{\min(b,c)} \subseteq U \cap V$; and (e) follows immediately from the definition of $\mathscr{U}$.

Since $\mathscr{U}$ is a uniformity, it induces a topology $\mathfrak{J}$ on F. According to [7, page 178], this topology is given by $\mathfrak{J} = \{T \subseteq F \mid \text{for all } z \in T, \text{ there exists } U \in \mathscr{U} \text{ such that } \{y \in F \mid (z, y) \in U\} \subseteq T\}$. We proceed to show that $\mathfrak{J} = \mathscr{T}$.

First, we show that each $\mathscr{T}$-subbasic open set is open in $\mathfrak{J}$. We give the proof for sets of the form $(-\infty, b)$, leaving the similar proof for the sets (a, ∞) to the reader. Fix $b \in F$. It is enough to show that if $z < b$ in F, then there exists $d \in F^+$ such that if $y \in F$ and $|z - y| < d$, then $y < b$. Observe that $d := b - z$ works, for $-d < z - y < d$ and $y < z + d = b$.

Next, we prove that each $\mathfrak{I}$-open set T is open in $\mathscr{T}$. Fix $z \in T$. It suffices to find a $\mathscr{T}$-basic open set V such that $z \in V \subseteq T$. Now, because $T \in \mathfrak{I}$, we can choose $U \in \mathscr{U}$ such that $\{y \in F \mid (z, y) \in U\} \subseteq T$. Then, by the definition of $\mathscr{U}$, there exists $b \in F^+$ such that $\{(u, v) \in F \times F : |u - v| < b\} \subseteq U$. Evidently, $V := (z - b, z + b)$ is a $\mathscr{T}$-basic open set such that $z \in V$. Moreover, $V \subseteq T$. In fact, $V \subseteq \{y \in F \mid (z, y) \in U\}$. Indeed, if $y \in V$, then $(z, y) \in U$ because $|z - y| < b$. This completes the proof that $\mathfrak{I} = \mathscr{T}$.

Since $\mathscr{T} = \mathfrak{I}$ is the topology of the uniformity $\mathscr{U}$, we can conclude that $(F, \mathscr{T})$ is a uniform space with uniformity $\mathscr{U}$. In other words, we have completed the alternate proof of Lemma 2.1 (b).

The next two corollaries show that many familiar ordered fields are metrizable.

COROLLARY 2.4. *Each countable ordered field is metrizable.*

PROOF. Let F be a countable ordered field. Then F is metrizable by Theorem 2.3, since $\{b_1, b_2, \ldots\} := F^+$ satisfies condition (3) in Theorem 2.3. □

Recall that an ordered field F is said to be *Archimedean* if for each $a \in F^+$, there exists a positive integer n such that $na > 1$. It is known that if F is an ordered field, then: F is Archimedean $\Leftrightarrow$ F is order-isomorphic to a subfield of the field $\boldsymbol{R}$ of all real numbers $\Leftrightarrow$ $\boldsymbol{Q}$ is dense in F (in the sense that, whenever $a < b$ in F, there exists $c \in \boldsymbol{Q} \subseteq F$ such that $a < c < b$). Accessible references are available for what we need of the forgoing, as follows. See [1, Theorem 2, page 92] for a proof that $\boldsymbol{R}$ is Archimedean; and adapt the proof of [1, Theorem 3, page 93] to see that $\boldsymbol{Q}$ is dense in any Archimedean field.

COROLLARY 2.5. *Each Archimedean field is metrizable.*

PROOF. Let F be an Archimedean field. As noted in the Introduction, we may view $\boldsymbol{Q}$ as a subfield of F and, as such, $\boldsymbol{Q}$ is dense in F, by the above remarks. It follows that F is metrizable by Theorem 2.3, since $\{b_1, b_2, \ldots\} := \boldsymbol{Q}^+$ satisfies condition (3) in Theorem 2.3. □

It should be noted that neither Corollary 2.4 nor Corollary 2.5 includes the other. For instance, $\boldsymbol{R}$ is an Archimedean field which is not countable. On the other hand, by changing the field of coefficients from $\boldsymbol{R}$ to $\boldsymbol{Q}$ in the argument supporting [3, (4), pages 15–16], we see that the (countable) field $\boldsymbol{Q}(X)$ of rational functions in one variable over $\boldsymbol{Q}$ can be given the structure of an ordered field which is not Archimedean. Some of the orders introduced in Theorem 2.6 will extend this construction to arbitrarily many variables.

By Corollary 2.4, *all* countable ordered fields are metrizable; and by Corollary 2.5, at least *some* uncountable ordered fields are metrizable. (Recall that the Introduction raised the question of generalizing the fact that the most familiar uncountable field, $\boldsymbol{R}$, is metrizable. Corollary 2.5 recovers this fact.) On the other hand, the next result shows, in particular, that some uncountable ordered fields are *not* metrizable. We explicitly assume the Axiom of Choice, in order to be able to use the standard facts about the arithmetic of infinite cardinal numbers (cf. [4, pages 96–98]).

THEOREM 2.6. *For each uncountable cardinal number* $\aleph$, *there exist ordered fields,* F_1 *and* F_2, *each of cardinality* $\aleph$, *such that* F_1 *is metrizable and* F_2 *is not metrizable.*

PROOF. We begin with some general observations that will be used repeatedly. Let $(K, \leq)$ be an ordered field and choose a set I of cardinality $\aleph$. Well-order I; by *abus de langage*, let $\leq$ denote a well-ordering of I. Let $\{Y_i | i \in I\}$ denote a set of algebraically independent indeterminates indexed by I. Let $R := K[\{Y_i | i \in I\}]$, the ring of polynomials in the variables Y_i with coefficients in K; and let $L := K(\{Y_i | i \in I\})$, the field of rational functions over K in the variables Y_i. Since L is the quotient field of R, every structure of R as an ordered (integral) domain can be uniquely extended to give L the structure of an ordered field [1, Theorem 12, page 49]. Before creating some ordered structures for R and L, we discuss (lexicographic and reverse lexicographic) orders on monomials. As terminology in this area varies in the literature and our set I is typically infinite, we shall do this in some detail.

We shall say that a monomial $Y_{i_1}^{m_1} \cdots Y_{i_n}^{m_n}$ in R is in *canonical form* if $i_1 < \cdots < i_n$ in the well-ordering $\leq$ on I and $m_1, \ldots, m_n$ are positive integers. Suppose that $u = Y_{i_1}^{m_1} \cdots Y_{i_n}^{m_n}$ and $v = Y_{j_1}^{p_1} \cdots Y_{j_k}^{p_k}$ are distinct monomials in canonical form. We say that $u \prec v$ if one of the following sets of conditions holds: $i_1 < j_1$ (in I); $i_1 = j_1$, $m_1 < p_1$; $i_1 = j_1$, $m_1 = p_1$, $i_2 < j_2$; $i_1 = j_1$, $m_1 = p_1$, $i_2 = j_2$, $m_2 < p_2$; etc. Notice that each nonzero polynomial $w \in R$ can be written uniquely as $w = \alpha_1 w_1 + \cdots + \alpha_d w_d$, where $\alpha_1, \ldots, \alpha_d \in K \backslash \{0\}$ and the w_i are monomials (in canonical form) satisfying $w_1 \prec \cdots \prec w_d$. (By convention, we take $1 \in K$ to be the "empty monomial" and $1 \prec w$ for every other monomial w.) In stating the "key fact" below, it will be convenient to call $\alpha_1 w_1$ the *miminal term of* w and $\alpha_d w_d$ the *maximal term of* w.

The key fact is that if w_1 and w_2 are nonzero polynomials in R, then the minimal (resp., maximal) term of $w_1 w_2$ is the product of the minimal (resp., maximal) term of w_1 and the minimal (resp., maximal) term of w_2. This is easily seen by applying the following observation to the descriptions of w_1 and w_2 as sums of scalar multiples of monomials in ascending $\prec$-order. If u, v, and w are monomials such that $u \prec v$, then $uw \prec vw$. We next proceed to define two useful ordered domain structures on R.

Given distinct polynomials u and v in R, write $w := v - u$ as above; namely, $w = \alpha_1 w_1 + \cdots + \alpha_d w_d$, where $\alpha_1, \ldots, \alpha_d \in K \backslash \{0\}$ and the w_i are monomials satisfying $w_1 \prec \cdots \prec w_d$. We say that $u <_1 v \Leftrightarrow \alpha_d > 0$ in the given order on K; and that $u <_2 v \Leftrightarrow \alpha_1 > 0$ in the given order on K. Of course, one then obtains binary relations $\leq_1$ and $\leq_2$ on R by interpreting $\leq_i$ as "$<_i$ or $=$". We claim that $\leq_1$ and $\leq_2$ each give R the structure of an ordered domain. Indeed, the above "key fact" shows that the product of $\leq_i$-positive elements is $\leq_i$-positive; and by an easy case analysis, we can check directly that the sum of $\leq_i$-positive elements is also $\leq_i$-positive. Passing to the quotient field by [1, Theorem 12, page 49], we then extend these orders to obtain the ordered field structures $(L, \leq_1)$ and $(L, \leq_2)$. It will be convenient to refer to $\leq_1$ as the *maximal order on* L and to $\leq_2$ as the *minimal order on* L. With these preliminaries in hand, we can now proceed to construct the required ordered fields F_1 and F_2.

The field F_1 will take the form $\boldsymbol{Q}(\{Y_i | i \in I\})(X)$ and it will be constructed by using both types of orders, $\leq_1$ and $\leq_2$, that were introduced above. To begin the construction, take I to be a set of cardinality $\aleph$. With $K := \boldsymbol{Q}$ in the above, we obtain the $\leq_2$-ordered field structure on the field $K_1 = \boldsymbol{Q}(\{Y_i | i \in I\})$. (New notation is needed for the following reason. Although K_1 has, to this point, played the role of L, it is about to play the role of K as we continue to apply the above preliminaries.) Observe that $|K_1| = \max(|\boldsymbol{Q}|, |I|) = \max(\aleph_0, \aleph) = \aleph$. Next, choose X to be an indeterminate over K_1. Then, with K_1 playing the role of K and $\{X\}$ playing the role of $\{Y_i | i \in I\}$, we obtain the $\leq_1$-ordered field structure on the field $F_1 := K_1(X) = \boldsymbol{Q}(\{Y_i | i \in I\})(X)$. (The previous considerations apply, as the construction of the field does not require the assumption that the set of variables has a specific cardinality.) Observe that $|F_1| = \max(|K_1|, \aleph_0) = \aleph$. It remains to show that F_1 is metrizable. We shall do so by verifying condition (3) in Theorem 2.2.

Consider a nonzero element $f \in F_1$. Write $f = g/h$, where g and h are nonzero elements in $K_1[X]$, the ring of polynomials in the variable X with coefficients in K_1. By the definition of $\leq_1$, $f \in F_1^+$ if and only if the terms of maximal degree in g and h have coefficients (in K_1) with the same sign (without loss of generality, both positive). Now, suppose that $f \in F_1^+$ (with g and h each having a positive leading coefficient). Put $n := \max(\deg(g), \deg(h))$. By the above comments, $1/X^{n+1} \in F_1^+$. Moreover, by the definition of $\leq_1$, we have that $h <_1 gX^{n+1}$, since the leading coefficient of $gX^{n+1} - h$ is the leading coefficient of g, which is positive in K_1. It follows that $0 < 1/X^{n+1} <_1 gh^{-1} = f$. Therefore, $\{b_1, b_2, \ldots\} := \{1/X, 1/X^2, 1/X^3, \ldots\}$ satisfies condition (3) in Theorem 2.2, and so F_1 is metrizable, as asserted.

We pause to give a different construction of a satisfactory F_1. Unlike the above two-step construction, we now simply consider the field $F^* = K((X))$ of formal Laurent series in an analytic indeterminate X with coefficients in an ordered field K, where K is assumed to have cardinality $\aleph$. (For example, K could be the above field K_1.) Observe that the ring $S := K[[X]]$ of formal power series over K has cardinality equal to $|K|^{\aleph_0} = \aleph^{\aleph_0}$. Since $\aleph > \aleph_0$, it follows from the *GCH* (Generalized Continuum Hypothesis) [4, page 102] that $|S| = \aleph$. (We assume the *GCH* here in order to proceed with the alternate construction of F_1.) It follows that F^*, which is the quotient field of S, also has cardinality $\aleph$.

Next, we define a binary relation on S. If u and v are distinct elements of S, there exists a nonnegative integer n and elements $a_0, a_1, a_2, \ldots \in K$ such that $a_0 \neq 0$ and $v - u = X^n(a_0 + a_1X + a_2X^2 + \cdots)$. We say that $u < v$ in S if and only if $a_0 > 0$ in K. It is easy to verify that S acquires the structure of an ordered domain in this way. We extend this in the only possible way to an ordered field structure on F^*. It follows that when a nonzero element $w \in F^*$ is expressed (uniquely) as $w = X^n(a_0 + a_1X + a_2X^2 + \cdots)$, for some integer n and elements $a_i \in K$ with $a_0 \neq 0$, then: $w > 0$ in $F^* \Leftrightarrow a_0 > 0$ in K. Therefore, if $w = X^n(a_0 + a_1X + a_2X^2 + \cdots) > 0$, we have that $0 < X^{n+1} < w$ in F^*, since $w - X^{n+1} = X^n(a_0 + (a_1 - 1)X + a_2X^2 + \cdots) > 0$ in F^*. Hence, $\{b_1, b_2, \ldots\} := \{X^n \in F^* \mid n \in \mathbf{Z}\}$ satisfies condition (3) in Theorem 2.2. Therefore, F^* is metrizable, thus completing the proof that F^* has all the asserted properties of F_1.

We turn now to the more delicate task of constructing a satisfactory F_2. This field will take the form $F_2 := K(\{Y_i | i \in I\})$ discussed in the preliminaries, and it will be constructed by using the maximal order, $\leq_1$. For the specifics, we take K to be any ordered field of cardinality $\aleph$, and we take I to be the set of all countable ordinal numbers. We pause to explain that I is well-ordered under the usual order relation on ordinal numbers. To see this, it is helpful to view I as $\Omega' \backslash \{\Omega\}$, where Ω denotes the first uncountable ordinal number and Ω' denotes the set of all ordinal numbers that are less than or equal to Ω. Since Ω' is known to be well-ordered [7, Summary 22 (a)] and its maximal element is Ω, one easily concludes that $\Omega' \backslash \{\Omega\}$ is also well-ordered. Next, we show that $(F_2, \leq_1)$ is not metrizable, by proving that if $\{b_1, b_2, \ldots\}$ is any countable subset of F_2^+, then $\{b_1, b_2, \ldots\}$ does not satisfy condition (3) in Theorem 2.2.

For each n, fix a description of b_n as a ratio of polynomials in variables drawn from $\{Y_i | i \in I\}$ and with coefficients in K. Let J be the subset of I consisting of all the indexes i such that Y_i appears (with nonzero coefficient) in either a numerator or a denominator of at least one of the b_n. Since Ω' is well-ordered, it follows from [7, Theorem 9, page 14] that J has a supremum, say j, in Ω'. Since J is countable and $\Omega \notin J$, a fundamental result (and the main reason we termed this construction "delicate" above) [7, Theorem 23, page 30] ensures that $j \neq \Omega$; that is, $j \in I$. Put $k := j + 1 \in I$. It follows from properties of the ordinal numbers that $\lambda < k$ for each $\lambda \in J$.

Consider $g := Y_k \in F_2$, and let $f := g^{-1}$. By the definition of the maximal order, $g > 0$ in F_2; hence, $f > 0$ in F_2. Now, let N and D be the numerator and denominator, respectively, of some b_n. Since $b_n \in F_2^+$, the definition of $\leq_1$ allows us to suppose, without loss of generality, that both N and D have maximal terms with positive coefficients. The next observation fundamentally uses the construction of the maximal order, the above conclusion concerning k, and the "key fact" in the preliminaries. Observe that $D <_1 Y_k = g$ and $1 <_1 Y_k N = gN$, whence $f = g^{-1} <_1 N$. Therefore , $0 <_1 (1/Y_k^2) = (f/g) <_1 (N/D) = b_n$. As k does not depend on n, it follows that $\{b_1, b_2, \ldots\}$ does not satisfy condition (3) in Theorem 2.2. Therefore, by Theorem 2.2, $(F_2, \leq_1)$ is not metrizable.

It remains only to show that $|F_2| = \aleph$. According to the approach in [7, Theorem 119, page 269], the ordinal number Ω *is* the set of all ordinal numbers that are less than Ω; that is, $\Omega = I$. Therefore, $|I| = |\Omega| = \aleph_1$. As $\aleph > \aleph_0$, we have that $\aleph \geq \aleph_1$. Hence, $|F_2| = \max(|K|, |I|) = \max(\aleph, \aleph_1) = \aleph$. □

REMARK 2.7. (a) It is interesting that even if an ordered field is metrizable, the underlying topological space need not be separable. Indeed, a standard metrization theorem [7, Theorem 17, page 125] ensures that any separable metric space has cardinality at most $|[0,1]^{\omega}| = c^{\aleph_0} = 2^{\aleph_0} = c$. Thus, if one takes I in Theorem 2.6 to be such that $|I| \geq 2^c$, then the order $\leq_1$ produces a metrizable ordered field structure on $\boldsymbol{Q}(\{Y_i | i \in I\})(X)$ whose canonical topology is not separable.

(b) If one takes $I = K = \boldsymbol{R}$ in the construction of F_2 in Theorem 2.6, the result is an ordered field structure on $F_2 = \boldsymbol{R}(\{Y_i | i \in \boldsymbol{R}\})$ which is not metrizable.

In that case, $|F_2| = c = |\boldsymbol{R}|$, and so a bijection $F_2 \to \boldsymbol{R}$ can be used to transfer the structure from F_2 to the set underlying $\boldsymbol{R}$. In this way, the *set* underlying $\boldsymbol{R}$ can be given the structure of an ordered field which is not metrizable. However, the algebraic part of this structure is definitely *not* the usual field structure on $\boldsymbol{R}$. The point is that there is only one way to endow the usual field structure on $\boldsymbol{R}$ with the structure of an ordered field, namely, $a < b$ in $\boldsymbol{R}$ if and only if there exists $d \in \boldsymbol{R}\backslash\{0\}$ such that $b - a = d^2$ (cf. [1, Exercise 9, page 100]); and this structure is the most familiar metrizable structure in mathematics.

The reader will have noticed that both the maximal order $\leq_1$ and the minimal order $\leq_2$ produce ordered field structures that are not Archimedean. Thus, with $\aleph := c$, if one takes F to be either of the fields constructed in Theorem 2.6 to have the properties asserted of F_1, then F is a metrizable field of cardinality c which is not Archimedean. As in the preceding paragraph, we can use a bijection $F \to \boldsymbol{R}$ to transfer the structure from F to the set underlying $\boldsymbol{R}$. In this way, the *set* underlying $\boldsymbol{R}$ can be given the structure of a metrizable ordered field which is not Archimedean. This new structure is not algebraically isomorphic to the usual field structure on $\boldsymbol{R}$, because we have seen that the latter admits only one order, namely, the familiar Archimedean one.

As the minimal order $\leq_2$ saw limited use in Theorem 2.6, we pause to note how it can be used to give an example of the type noted in the preceding paragraph, namely, a metrizable non-Archimedean structure on a field of cardinality c. Let us begin with the base field $\boldsymbol{R}$ and form the function field $L := \boldsymbol{R}(Y)$, equipped with the minimal order, $\leq_2$. By the above comments, we need only verify that L is metrizable. For this, it suffices to show that the set $\{b_1, b_2, \ldots\} := \{f/g \in L \mid f \in \boldsymbol{Q}^+X^n$ for some $n \geq 0, g \in \boldsymbol{Q}^+\}$ satisfies condition (3) in Theorem 2.2. Indeed, by the definition of the minimal order, any positive element in L has a numerator N and a denominator D whose terms of lowest degree have positive coefficients. We seek f, g as above so that $f/g <_2 (N/D)$. It suffices to arrange that $f <_2 N$ and $D <_2 g$. The former is achieved since $\boldsymbol{Q}$ is order-theoretically dense in $\boldsymbol{R}$; the latter is arranged by taking g to be a a rational number which exceeds the coefficient of the term of D of least degree.

(c) Let $(Y, \leq)$ be a partially ordered set. As in [8, page 821], a T_0-topology $\mathscr{T}$ on Y is said to be *order-compatible* (*with* $\leq$) if, for all y_1 and y_2 in $Y : y_1 \leq y_2 \Leftrightarrow y_2 \in \overline{\{y_1\}}$ (where $\bar{S}$ denotes the closure in $\mathscr{T}$ of a set $S \subseteq Y$). Any partially ordered set admits an order-compatible topology. However, if $(F, \leq)$ is an ordered field, then the order topology on F is *not* order-compatible (with $\leq$). Indeed, since Lemma 2.1 (a) ensures that the order topology on F is T_1, we have that $\overline{\{y\}} = \{y\}$ for each $y \in Y$, although $y < y + 1$. Nevertheless, there is more than one T_0-topology on F which is order-compatible with $\leq$. To see this, since $\leq$ is a linear order, we may apply the criterion in [2, Corollary 2.7], noting that no element in F has an "immediate successor" in F. (In detail, if $a \in F$, then $a < 1 + |a|$; and if $a < d$ in F, then $a < (a+d)/2 < d$.)

References

[1] G. Birkhoff and S. Mac Lane, A Survey of Modern Algebra, rev. ed., The Macmillan Co., New York, 1953.
[2] D. E. Dobbs, Posets admitting a unique order-compatible topology, Discrete Math., **41** (1982), 235–240.
[3] B. R. Gelbaum and J. M. H. Olmsted, Counterexamples in Analysis, Holden-Day, Inc., San Francisco, 1964.
[4] P. R. Halmos, Naive Set Theory, D. Van Nostrand Co., Inc., Princeton, 1960.
[5] T. Husain, Introduction to Topological Groups, W. B. Saunders Co., Philadelphia, 1966.
[6] S. Kakutani, Über die Metrisation der topologischen Gruppen, Proc. Imp. Ac. Tokyo, **12** (1936), 82–84.
[7] J. L. Kelley, General Topology, D. Van Nostrand Co., Inc., Princeton, 1955.
[8] W. J. Lewis and J. Ohm, The ordering of Spec R, Canad. J. Math., **28** (1976), 820–835.

Department of Mathematics
University of Tennessee
Knoxville, Tennessee 37996-1300
U.S.A.