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ON PERIODIC TAKAHASHI MANIFOLDS*

By

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Abstract. In this paper we show that periodic Takahashi 3-manifolds are cyclic coverings of the connected sum of two lens spaces (possibly cyclic coverings of S^3), branched over knots. When the base space is a 3-sphere, we prove that the associated branching set is a two-bridge knot of genus one, and we determine its type. Moreover, a geometric cyclic presentation for the fundamental groups of these manifolds is obtained in several interesting cases, including the ones corresponding to the branched cyclic coverings of S^3.

1. Introduction

Takahashi manifolds are closed orientable 3-manifolds introduced in [21] by Dehn surgery on S^3, with rational coefficients, along the $2n$-component link $\mathscr{L}_{2n}$ depicted in Figure 1. These manifolds have been intensively studied in [11], [19], and [22]. In the latter two papers, a nice topological characterization of all Takahashi manifolds as two-fold coverings of S^3, branched over the closure of certain rational 3-string braids, is given.

A Takahashi manifold is called *periodic* when the surgery coefficients have the same cyclic symmetry of order n of the link $\mathscr{L}_{2n}$, i.e. the coefficients are p/q and r/s alternately. Several important classes of 3-manifolds, such as (fractional) Fibonacci manifolds [7, 22] and Sieradsky manifolds [2, 20], represent notable examples of periodic Takahashi manifolds.

In this paper we show that each periodic Takahashi manifold is an n-fold cyclic covering of the connected sum of two lens spaces, branched over a knot.

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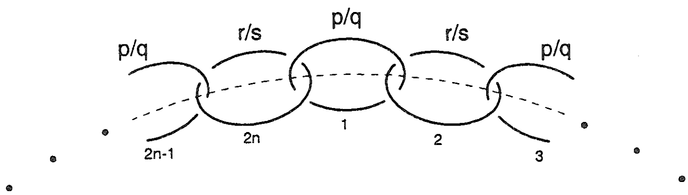

Figure 1: Surgery along $\mathscr{L}_{2n}$ yielding $M_n(p/q, r/s)$.

This knot arises from a component of the Borromean rings, by performing a surgery with coefficients p/q and r/s along the other two components.

For particular values of the surgery coefficients (including the classes of manifolds cited above), the periodic Takahashi manifolds turn out to be n-fold cyclic coverings of S^3, branched over two-bridge knots of genus one[1], whose parameters are obtained using Kirby-Rolfsen calculus [18] (compare the analogous result of [11], obtained by a different approach). Observe that in [19] a characterization of all periodic Takahashi manifolds as n-fold cyclic coverings of S^3, branched over the closure of certain rational 3-string braids, is presented, but the result is incorrect, as we show in Remark 1.

For many interesting periodic Takahashi manifolds—including the ones corresponding to branched cyclic coverings of S^3—a cyclic presentation for the fundamental group is provided and proved to be geometric, i.e. arising from a Heegaard diagram, or, equivalently, from a canonical spine[2] [16].

2. Main results

We denote by $M(p_1/q_1, \dots, p_n/q_n; r_1/s_1, \dots, r_n/s_n)$ the Takahashi manifold obtained by Dehn surgery on S^3 along the $2n$-component link $\mathscr{L}_{2n}$ of Figure 1, with surgery coefficients $p_1/q_1, r_1/s_1, \dots, p_n/q_n, r_n/s_n \in \tilde{Q} = Q \cup \{\infty\}$ respectively, cyclically associated to the components of $\mathscr{L}_{2n}$.

A Takahashi manifold is periodic when $p_i/q_i = p/q$ and $r_i/s_i = r/s$, for every $i = 1, \dots, n$. Denote by $M_n(p/q, r/s)$ the periodic Takahashi manifold $M(p/q, \dots, p/q; r/s, \dots, r/s)$. From now on, without loss of generality, we can always suppose that: $\gcd(p,q) = 1$, $\gcd(r,s) = 1$ and $p, r \geq 0$. Moreover, if $\alpha, \beta \in Z$ with $\alpha \geq 0$ and $\gcd(\alpha, \beta) = 1$, we shall denote by $L(\alpha, \beta)$ the lens space of type (α, β). As usual, $L(0,1)$ is homeomorphic to $S^1 \times S^2$ and $L(1, \beta)$ is homeomorphic to S^3, for all β (including $\beta = 0$).

[1] For notation and properties about two-bridge knots and links we refer to [1]. For the characterization of two-bridge knots of genus one, see [5].
[2] A canonical spine is a 2-dimensional cell complex with a single vertex.

Notice that $M_n(p/q, -p/q)$ is the Fractional Fibonacci manifold $M_n^{p/q}$ defined in [22] and, in particular, $M_n(1, -1)$ is the Fibonacci manifold M_n studied in [7]. Moreover, $M_n(1, 1)$ is the Sieradsky manifold M_n introduced in [20] and studied in [2]. Because of the symmetries of $\mathscr{L}_{2n}$, the homeomorphisms

$$M_n(p/q, r/s) \cong M_n(-p/q, -r/s) \cong M_n(r/s, p/q) \cong M_n(-r/s, -p/q)$$

can easily be obtained for all $n \geq 1$ and $p/q, r/s \in \tilde{\boldsymbol{Q}}$.

A balanced presentation of the fundamental group of every Takahashi manifold is given in [21], and in [19] it is shown that this presentation is geometric, i.e. it arises from a Heegaard diagram (or, equivalently, from a canonical spine). As a consequence, $\pi_1(M_n(p/q, r/s))$ admits the following geometric presentation with $2n$ generators and $2n$ relators:

$$\langle x_1, \ldots, x_{2n} \mid x_{2i-1}^q x_{2i}^{-r} x_{2i+1}^{-q}, x_{2i}^s x_{2i+1}^p x_{2i+2}^{-s};\ i = 1, \ldots, n \rangle,$$

where the subscripts are mod $2n$.

When $r = 1$, we can easily get a cyclic presentation [9] with n generators:[3]

$$\pi_1(M_n(p/q, 1/s)) = \langle z_1, \ldots, z_n \mid z_i^p (z_i^{-q} z_{i+1}^q)^s (z_i^{-q} z_{i-1}^q)^s;\ i = 1, \ldots, n \rangle, \qquad (1)$$

where the subscripts are mod n.

Proposition 1. *For all* $p/q \in \tilde{\boldsymbol{Q}}$ *and* $s \in \boldsymbol{Z}$, *the cyclic presentation* (1) *of* $\pi_1(M_n(p/q, 1/s))$ *is geometric.*

Proof. If $s = 0$ then $M_n(p/q, 1/s)$ is homeomorphic to the connected sum of n copies of $L(p/q)$, and therefore the statement is straightforward. If $s > 0$, the presentation becomes

$$\langle z_1, \ldots, z_n \mid z_i^{p-q} (z_{i+1}^q z_i^{-q})^s (z_{i-1}^q z_i^{-q})^{s-1} z_{i-1}^q;\ i = 1, \ldots, n \rangle. \qquad (1')$$

Figure 2 shows an RR-system which induces (1′), and so, by [17], this presentation is geometric. If $s < 0$, the presentation becomes

$$\langle z_1, \ldots, z_n \mid z_i^{p+q} (z_{i+1}^{-q} z_i^q)^{-s} (z_{i-1}^{-q} z_i^q)^{-s-1} z_{i-1}^{-q};\ i = 1, \ldots, n \rangle. \qquad (1'')$$

Therefore, if we replace q with $-q$, Figure 2 also gives an RR-system inducing (1″). ∎

Since the link $\mathscr{L}_2$ is a two-component trivial link, we immediately get the following results:

[3] Alternatively, a similar cyclic presentation can be obtained when $p = 1$.

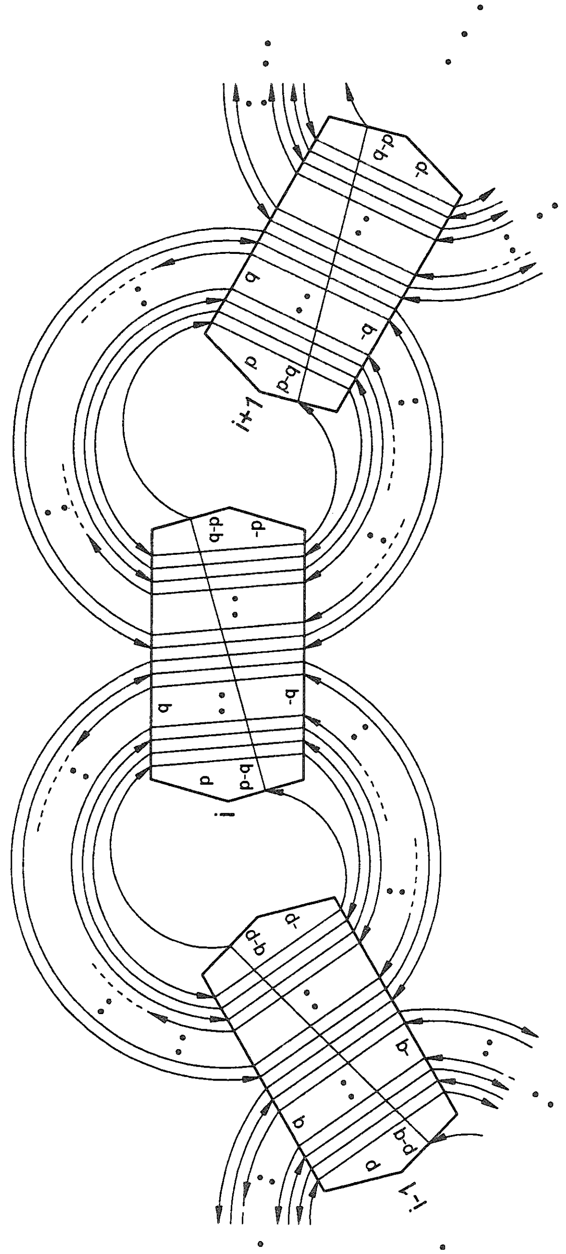

Figure 2: An RR-system for the cyclic presentation (1′).

Lemma 2. *For all* $p/q, r/s \in \tilde{Q}$, *the manifold* $M_1(p/q, r/s)$ *is homeomorphic to the connected sum of lens spaces* $L(p,q) \sharp L(r,s)$. *In particular,* $M_1(p/q, 1/s)$ *is homeomorphic to the lens space* $L(p,q)$ *and* $M_1(1/q, 1/s)$ *is homeomorphic to* S^3.

Proof. $M_1(p/q, r/s)$ is obtained by Dehn surgery on S^3, with coefficients p/q and r/s, along the trivial link with two components $\mathscr{L}_2$. ■

Now we prove the main result of the paper:

Theorem 3. *For all* $p/q, r/s \in \tilde{Q}$ *and* $n > 1$, *the periodic Takahashi manifold* $M_n(p/q, r/s)$ *is the n-fold cyclic covering of the connected sum of lens spaces* $L(p,q) \sharp L(r,s)$, *branched over a knot K which does not depend on n. Moreover, K arises from a component of the Borromean rings, by performing a surgery with coefficients* p/q *and* r/s *along the other two components.*

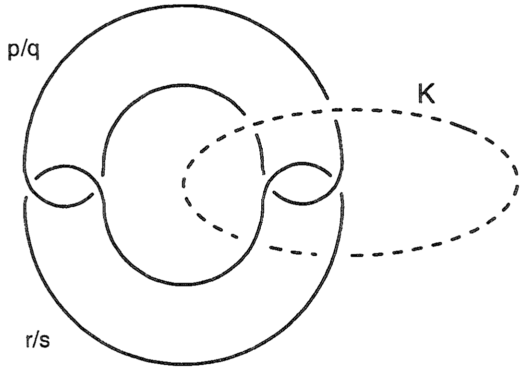

Figure 3: The branching set K (dashed line).

Proof. Both the link $\mathscr{L}_{2n}$ and the surgery coefficients defining $M_n(p/q, r/s)$ are invariant with respect to the rotation ρ_n of S^3, which sends the i-th component of $\mathscr{L}_{2n}$ onto the $(i+2)$-th component (mod $2n$). Let $\mathscr{G}_n$ be the cyclic group of order n generated by ρ_n. Observe that the fixed-point set of the action of $\mathscr{G}_n$ on S^3 is a trivial knot disjoint from $\mathscr{L}_{2n}$. Therefore, we have an action of $\mathscr{G}_n$ on $M_n(p/q, r/s)$, with a knot K_n as fixed-point set. The quotient $M_n(p/q, r/s)/\mathscr{G}_n$ is precisely the manifold $M_1(p/q, r/s)$, which is homeomorphic to $L(p,q) \sharp L(r,s)$ by Lemma 2, and $K_n/\mathscr{G}_n$ is obviously a knot $K \subset M_1(p/q, r/s)$, which only depends on p/q and r/s. Moreover, $K \cup \mathscr{L}_2$ is the Borromean rings, as showed in Figure 3. This proves the statement. ■

We can give another description of the branching set K, as the inverse image of a trivial knot in a certain two-fold branched covering.

Denote by $\mathscr{L}(p/q, r/s)$ the link depicted in Figure 4. It is composed by the closure of the rational 3-string braid $\sigma_1^{p/q}\sigma_2^{r/s}$, which is the connected sum of the two-bridge knots or links $\boldsymbol{b}(p,q)$ and $\boldsymbol{b}(r,s)$, and by a trivial knot. Moreover, denote: (i) by $\mathcal{O}_n(p/q, r/s) = M_n(p/q, r/s)/\mathscr{G}_n$ the orbifold from the proof of Theorem 3, whose underlying space is $L(p,q)\sharp L(r,s)$ and whose singular set is the knot K, with index n; (ii) by $\boldsymbol{S}^3(\mathscr{K}_n(p/q, r/s))$ the orbifold whose underlying space is $\boldsymbol{S}^3$ and whose singular set is the closure of the rational 3-string braid $(\sigma_1^{p/q}\sigma_2^{r/s})^n$, with index 2; and (iii) by $\boldsymbol{S}^3(\mathscr{L}(p/q, r/s))$ the orbifold whose underlying space is $\boldsymbol{S}^3$ and whose singular set is the link $\mathscr{L}(p/q, r/s)$, with index 2 and n as pointed out in Figure 4.

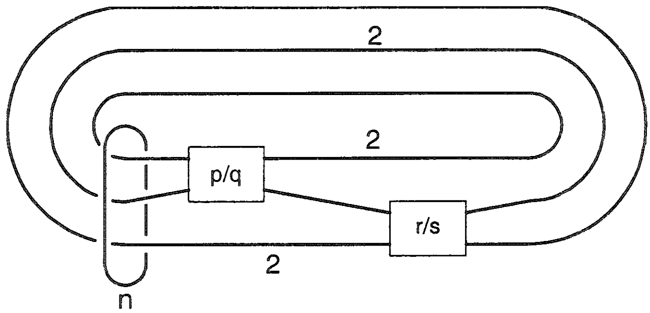

Figure 4: The link $\mathscr{L}(p/q, r/s)$.

PROPOSITION 4. *Assuming the previous notations, the following commutative diagram holds for each periodic Takahashi manifold.*

$$\begin{array}{ccc} & M_n(p/q, r/s) & \\ {\scriptstyle 2}\swarrow & & \searrow{\scriptstyle n} \\ \boldsymbol{S}^3(\mathscr{K}_n(p/q, r/s)) & & \mathcal{O}_n(p/q, r/s) \\ {\scriptstyle n}\searrow & & \swarrow{\scriptstyle 2} \\ & \boldsymbol{S}^3(\mathscr{L}(p/q, r/s)) & \end{array}$$

PROOF. The link $\mathscr{L}_{2n}$ admits an invertible involution τ, whose axis intersects each component in two points (see the dashed line of Figure 1), and the rotation symmetry ρ_n of order n which was discussed in Theorem 3. These symmetries induce symmetries (also denoted by τ and ρ_n) on the periodic Takahashi manifold $M = M_n(p/q, r/s)$, such that $\langle\tau, \rho_n\rangle \cong \langle\tau\rangle \oplus \mathscr{G}_n \cong \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_n$. We have $M/\langle\tau\rangle = \boldsymbol{S}^3(\mathscr{K}_n(p/q, r/s))$ (see [19] and [22]) and $M/\mathscr{G}_n = \mathcal{O}_n(p/q, r/s)$ (see Theorem 3). It is immediate to see that ρ_n induces a symmetry (also denoted by ρ_n) on the orbifold $M/\langle\tau\rangle$, and $(M/\langle\tau\rangle)/\mathscr{G}_n$ is the orbifold $S^3(\mathscr{L}(p/q,r/s))$. As we see from Figure 3, τ induces a strongly invertible involution (also denoted by τ) on the link $\mathscr{L}_2$. Using the Montesinos algorithm we see that $(M/\mathscr{G}_n)/\langle\tau\rangle = S^3(\mathscr{L}(p/q,r/s))$. This concludes the proof. ■

As a consequence, the branching set K of Theorem 3 can be obtained as the inverse image of the trivial component of $\mathscr{L}(p/q,r/s)$ in the two-fold branched covering $\mathcal{O}_n(p/q,r/s) \to S^3(\mathscr{L}(p/q,r/s))$.

From Theorem 3 we can get the following result, which has already been obtained in [11] by a different technique.

Proposition 5. *For all $q,s \in \boldsymbol{Z}$ and $n > 1$, the periodic Takahashi manifold $M_n(1/q,1/s)$ is the n-fold cyclic covering of S^3, branched over the two-bridge knot of genus one $\boldsymbol{b}(|4sq-1|,2s) \cong \boldsymbol{b}(|4sq-1|,2q)$.*

Proof. From Theorem 3, $M_n(1/q,1/s)$ is the n-fold cyclic covering of $L(1,q)\sharp L(1,s) \cong S^3$, branched over a knot K which does not depend on n. By isotopy and Kirby-Rolfsen moves it is easy to obtain (see Figure 5) a diagram of K, which is a Conway's normal form of type $[-2q,2s]$. This proves the statement. ■

Proposition 5 covers the results of [2], [7] and [22] concerning n-fold branched cyclic coverings of two-bridge knots. Moreover, for all $p,q \in \boldsymbol{Z}$, the periodic Takahashi manifold $M_n(1/q,1/s)$ is homeomorphic to the Lins-Mandel manifold $S(n,|4sq-1|,2s,1)$ [13, 15], the Minkus manifold $M_n(|4sq-1|,2s)$ [14] and the Dunwoody manifold $M((|4q-1|-1)/2,0,1,s,n,-q_\sigma)$ [3, 6].

Moreover, observe that all cyclic coverings of two-bridge knots of genus one are periodic Takahashi manifolds.

Remark 1. The results of Corollaries 8, 9 and 11 of [19], concerning periodic Takahashi manifolds as n-fold cyclic branched coverings of the closure of certain (rational) 3-string braids, are incorrect. This is evident from the following counterexamples. If $p/q = 3$ and $r/s = -3$ then the first homology group of the 3-fold cyclic branched covering of the closure of the 3-string braid $(\sigma_1^3\sigma_2^{-3})^2$ has order 256, but $|H_1(M_3(3,-3))| = 1296$. If $p/q = 3/2$ and $r/s = 1$ then the first homology group of the 4-fold cyclic branched covering of the closure of the rational 3-string braid $(\sigma_1^{3/2}\sigma_2)^2$ has order 135, but $|H_1(M_4(3/2,1))| = 15$. Note that the corollaries are valid if $p = r = 1$.

The following conjecture is naturally suggested by the previous results.

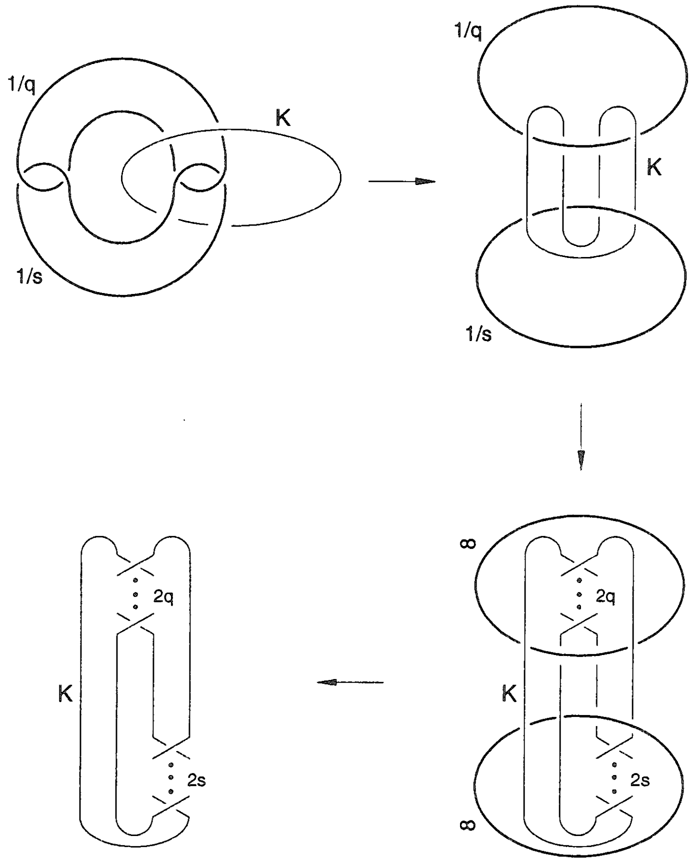

Figure 5:

CONJECTURE. Let $p/q, r/s \in \tilde{Q}$ be fixed. Then, for all $n > 1$, the periodic Takahashi manifolds $T_n = M_n(p/q, r/s)$ are n-fold cyclic coverings of S^3, branched over a knot which does not depend on n, if and only if $p = 1 = r$.

Added in revision—The referee pointed out that it is possible to prove the conjecture for "almost all cases" by using the hyperbolic Dehn surgery theorem and the shortest geodesic arguments by Kojima [12].

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