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AFFINE INNER AUTOMORPHISMS OF $SU(2)$

By

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Abstract. We show which inner automorphisms of $(SU(2), g)$ with an arbitrary left invariant metric g into itself are affine transformations, and obtain affine transformations of $(SU(2), g)$ which are not harmonic, and study geodesics of $(SU(2), g)$ with some conditions.

0. Introduction

It is interesting to show which diffeomorphisms between two Riemannian manifolds are affine transformations. In this paper, we treat the case $(SU(2), g)$ with a left invariant Riemannian metric g. It is well known that every inner automorphism of G a compact connected semisimple Lie group into itself is both affine and harmonic with respect to a bi-invariant Riemannian metric g_0 on G. However, we here deal with an arbitrary left invariant metric g on $SU(2)$, and show which inner automorphisms of $SU(2)$ are affine transformations of $(SU(2), g)$ into itself.

On the other hand we study geodesics in $(SU(2), g)$. In case of naturally reductive homogeneous space, it is well known that geodesics are orbits of 1-parameter subgroups. R. Dohira (cf. [1]) studied geodesics in reductive homogeneous spaces satisfying certain conditions. Using Dohira's Theorem, we give a complete description of geodesics in $(SU(2), g)$ satisfying some conditions.

In § 1, we obtain necessary and sufficient conditions for inner automorphisms A_x, $(x \in SU(2))$, of $(SU(2), g)$, to be affine transformations (cf. Proposition 1.3–1.5). Moreover, in Theorem 1.7 and 1.8, we show that for any left invariant but not bi-invariant Riemannian metric g on $SU(2)$, there always exist on $(SU(2), g)$

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both a non-affine inner automorphism, and a non-harmonic but affine inner automorphism.

In §2, using R. Dohira's Theorem we give a complete description of geodesics in $(SU(2), g)$ satisfying certain conditions (cf. Theorem 2.1). Finally we get necessary and sufficient conditions for arbitrary given geodesics in $(SU(2), g)$ with certain left invariant metric g to be closed (cf. Theorem 2.3).

§1. Affine Inner Automorphisms of $(SU(2), g)$

Let B be the Killing form of the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$. Then the Killing form satisfies

$$B(X, Y) = 4\,\mathrm{Trace}(XY), \quad (X, Y \in \mathfrak{su}(2)). \tag{1.1}$$

We define an inner product $\langle\,,\rangle_0$ on $\mathfrak{su}(2)$ by

$$\langle\,,\rangle_0 := -B(X, Y), \quad (X, Y \in \mathfrak{su}(2)). \tag{1.2}$$

The following lemma is known (cf. [5, Lemma 1.1, p. 154]):

LEMMA 1.1. *Let g be a left invariant Riemannian metric. Let $\langle\,,\rangle$ be an inner product on $\mathfrak{su}(2)$ defined by $\langle X, Y\rangle := g_e(X_e, Y_e)$, where $X, Y \in \mathfrak{su}(2)$ and e is the identity matrix of $SU(2)$. Then there exist an orthonormal basis (X_1, X_2, X_3) of $\mathfrak{su}(2)$ with respect to $\langle\,,\rangle_0$ such that*

$$\begin{cases} [X_1, X_2] = (1/\sqrt{2})X_3, & [X_2, X_3] = (1/\sqrt{2})X_1, \\ [X_3, X_1] = (1/\sqrt{2})X_2, & \langle X_i, X_j\rangle = \delta_{ij}a_i^2, \end{cases} \tag{1.3}$$

where a_i, $(1 \leqq i \leqq 3)$, are positive real numbers determined by the given left invariant Riemmannian metric g of $SU(2)$.

Let ∇ be the Riemannian connection on $(SU(2), g)$. Here g is an arbitrary given left invariant Riemannian metric in $SU(2)$. Let (X_1, X_2, X_3) be left invariant vector fields related to B and g which appear in Lemma 1.1.

An inner automorphism $A_x : (SU(2), g) \to (SU(2), g)$, $(x \in SU(2))$, is an affine transformation if and only if

$$Ad(x)\nabla_{X_i}X_j = \nabla_{Ad(x)X_i}Ad(x)X_j, \quad (i, j = 1, 2, 3). \tag{1.4}$$

With respect to the Riemannian connection, we have

$$\begin{aligned} 2 \cdot g(\nabla_X Y, Z) &= X \cdot g(Y, Z) + Y \cdot g(X, Z) - Z \cdot g(X, Y) \\ &\quad + g([X, Y], Z) + g([Z, X], Y) + g(X, [Z, Y]) \end{aligned} \tag{1.5}$$

for all vector fields X, Y, Z. In this section, for simplicity we put

$$(1.6)\qquad \begin{cases} F_1 := (2\sqrt{2})^{-1}(a_1^2 - a_2^2)a_3^{-2}, & F_2 := (2\sqrt{2})^{-1}(a_2^2 - a_3^2)a_1^{-2}, \\ F_3 := (2\sqrt{2})^{-1}(a_3^2 - a_1^2)a_2^{-2}, \end{cases}$$

From (1.3) and (1.5), we get

$$(1.7)\qquad \begin{cases} \nabla_{X_i} X_i = 0 \quad (i = 1, 2, 3), & \nabla_{X_1} X_2 = \{(2\sqrt{2})^{-1} - F_1\}X_3, \\ \nabla_{X_1} X_3 = -\{(2\sqrt{2})^{-1} + F_3\}X_2, & \nabla_{X_2} X_3 = \{(2\sqrt{2})^{-1} - F_2\}X_1. \end{cases}$$

We put

$$(1.8)\qquad Y_i := 2\sqrt{2}X_i, \quad (i = 1, 2, 3).$$

Then, from (1.3) and (1.8) we have

$$(1.9)\qquad [Y_1, Y_2] = 2Y_3, \quad [Y_2, Y_3] = 2Y_1, \quad [Y_3, Y_1] = 2Y_1.$$

In order to prove the following Propositions, we get:

LEMMA 1.2. *For an inner automorphism* A_x, $(x \in SU(2))$,

$$\nabla_{Ad(x)X_i} Ad(x)X_j = Ad(x)(\nabla_{X_i} X_j)$$

if and only if

$$\nabla_{Ad(x)X_j} Ad(x)X_i = Ad(x)(\nabla_{X_j} X_i), \quad (i, j = 1, 2, 3).$$

PROPOSITION 1.3. *An inner automorphism* A_x, $(x = \exp(rY_1),\ r \in \boldsymbol{R})$, *of* $(SU(2), g)$ *is an affine transformation if and only if* $a_2 = a_3$ *or* $\sin(2r) = 0$, *that is,*

$$(1.10)\qquad a_2 = a_3 \quad \text{or} \quad r \in \{(n\pi)/2 \mid n \text{ is an integer}\}.$$

PROOF. Using (1.3), (1.8) and (1.9), we have

$$(1.11)\qquad \begin{cases} Ad(x)X_1 = X_1, \\ Ad(x)X_2 = \cos(2r)X_2 + \sin(2r)X_3, \\ Ad(x)X_3 = \cos(2r)X_3 - \sin(2r)X_2. \end{cases}$$

Putting $\phi := Ad(x)$, from (1.6), (1.7) and (1.11) we get

$$(1.12)\quad \begin{cases} \phi^{-1}(\nabla_{\phi X_1}\phi X_1) = 0, \ \ \phi^{-1}(\nabla_{\phi X_2}\phi X_2) = -\phi^{-1}(\nabla_{\phi X_3}\phi X_3) = -F_2\sin(4r)X_1, \\ \phi^{-1}(\nabla_{\phi X_1}\phi X_2) = -2^{-1}(F_1+F_2)\sin(4r)X_2 \\ \qquad\qquad + \{(2\sqrt{2})^{-1} + F_3\sin^2(2r) - F_1\cos^2(2r)\}X_3, \\ \phi^{-1}(\nabla_{\phi X_1}\phi X_3) = \{F_1\sin^2(2r) - F_3\cos^2(2r) - (2\sqrt{2})^{-1}\}X_2 \\ \qquad\qquad + 2^{-1}(F_1+F_3)\sin(4r)X_3, \\ \phi^{-1}(\nabla_{\phi X_2}\phi X_3) = \{(2\sqrt{2})^{-1} - F_2\cos(4r)\}X_1. \end{cases}$$

Hence, we find from (1.6), (1.7), (1.12) and Lemma 1.2 that A_x is an affine transformation if and only if

$$(1.13)\qquad a_2 = a_3 \quad \text{or} \quad \sin(2r) = 0. \qquad \text{q.e.d.}$$

Proposition 1.4. *An inner automorphism* A_x, $(x = \exp(rY_2), r \in \boldsymbol{R})$, *of* $(SU(2), g)$ *is an affine transformation if and only if*

$$(1.14)\qquad a_3 = a_1 \quad \textit{or} \quad r \in \{(n\pi)/2 \mid n \textit{ is an integer}\}.$$

Proof. Using (1.3), (1.8) and (1.9), we have

$$(1.15)\quad \begin{cases} Ad(x)X_1 = \cos(2r)X_1 - \sin(2r)X_3, \\ Ad(x)X_2 = X_2, \quad Ad(x)X_3 = \sin(2r)X_1 + \cos(2r)X_3. \end{cases}$$

From (1.6), (1.7) and (1.15), we have

$$(1.16)\quad \begin{cases} \phi^{-1}(\nabla_{\phi X_1}\phi X_1) = -\phi^{-1}(\nabla_{\phi X_3}\phi X_3) = F_3\sin(4r)X_2, \phi^{-1}(\nabla_{\phi X_2}\phi X_2) = 0, \\ \phi^{-1}(\nabla_{\phi X_1}\phi X_2) = 2^{-1}(F_1+F_2)\sin(4r)X_1 \\ \qquad\qquad + \{(2\sqrt{2})^{-1} + F_2\sin^2(2r) - F_1\cos^2(2r)\}X_3, \\ \phi^{-1}(\nabla_{\phi X_1}\phi X_3) = -\{(2\sqrt{2})^{-1} + F_3\cos(4r)\}X_2, \\ \phi^{-1}(\nabla_{\phi X_2}\phi X_3) = \{(2\sqrt{2})^{-1} + F_1\sin^2(2r) - F_2\cos^2(2r)\}X_1 \\ \qquad\qquad - 2^{-1}(F_1+F_2)\sin(4r)X_3, \end{cases}$$

where $\phi := Ad(x)$. We know from (1.6), (1.7), (1.16) and Lemma 1.2 that A_x is an affine transformation if and only if

$$(1.17)\qquad a_1 = a_3 \quad \text{or} \quad \sin(2r) = 0. \qquad \text{q.e.d.}$$

PROPOSITION 1.5. *An inner automorphism* A_x, $(x = \exp(rY_3),\ r \in \boldsymbol{R})$, *is an affine transformation if and only if*

$$a_1 = a_2 \quad or \quad r \in \{(n\pi)/2 \mid n \text{ is an integer}\}. \tag{1.18}$$

PROOF. We get from (1.3), (1.8) and (1.9)

$$\begin{cases} Ad(x)(X_1) = \cos(2r)X_1 + \sin(2r)X_2, \\ Ad(x)(X_2) = \cos(2r)X_2 - \sin(2r)X_1, \quad \mathrm{Ad}(x)(X_3) = X_3. \end{cases} \tag{1.19}$$

From (1.6), (1.7) and (1.19), we obtain

$$\begin{cases} \phi^{-1}(\nabla_{\phi X_1}\phi X_1) = -\phi^{-1}(\nabla_{\phi X_2}\phi X_2) = -F_1 \sin(4r)X_3, \\ \phi^{-1}(\nabla_{\phi X_3}\phi X_3) = 0, \ \phi^{-1}(\nabla_{\phi X_1}\phi X_2) = \{(2\sqrt{2})^{-1} - F_1\cos(4r)\}X_3, \\ \phi^{-1}(\nabla_{\phi X_1}\phi X_3) = -2^{-1}(F_2 + F_3)\sin(4r)X_1 \\ \qquad\qquad + \{F_2\sin^2(2r) - F_3\cos^2(2r) - (2\sqrt{2})^{-1}\}X_2, \\ \phi^{-1}(\nabla_{\phi X_2}\phi X_3) = \{(2\sqrt{2})^{-1} + F_3\sin^2(2r) - F_2\cos^2(2r)\}X_1 \\ \qquad\qquad + 2^{-1}(F_2 + F_3)\sin(4r)X_2, \end{cases} \tag{1.20}$$

where $\phi := Ad(x)$. Using (1.6), (1.7) and Lemma 1.2, we obtain this proposition. q.e.d.

Since the metric g of $(SU(2), g)$ is bi-invariant iff $a_1 = a_2 = a_3$, we obtain from Proposition 1.3, 1.4 and 1.5:

THEOREM 1.6. *An inner automorphism* A_x *of* $(SU(2), g)$ *for any* $x \in SU(2)$ *is an affine transformation if and only if the metric* g *of* $(SU(2), g)$ *is bi-invariant.*

Harmonic maps of a compact Riemannian manifold into another Riemannian manifold are the extrema (cf. [2, 6]). In the case of $(SU(2), g)$, the following Lemma (cf. [4]) is known:

LEMMA 1.7. *A necessary and sufficient condition for an inner automorphism* A_x *of* $(SU(2), g)$ *to be harmonic is*

$$\begin{cases} a_2 = a_3 \ \textit{or} \ \sin(4r) = 0, & \textit{in case of } x = \exp(rY_1) \textit{ and } r \in \boldsymbol{R}, \\ a_1 = a_3 \ \textit{or} \ \sin(4r) = 0, & \textit{in case of } x = \exp(rY_2) \text{ and } r \in \boldsymbol{R}, \\ a_1 = a_2 \ \textit{or} \ \sin(4r) = 0, & \textit{in case of } x = \exp(rY_3) \textit{ and } r \in \boldsymbol{R}. \end{cases}$$

From Propositions 1.3–1.5 and Lemma 1.7, we get:

THEOREM 1.8. *Assume that a left invariant metric g of $(SU(2), g)$ is not bi-invariant. Then, there always exists harmonic inner automorphisms A_x of $(SU(2), g)$ which are not affine transformations.*

REMARK. An affine transformation between two Riemannian manifolds is harmonic.

Moreover, from (1.11), (1.15), (1.19) and Propositions 1.3–1.5, we have

COROLLARY 1.9. *If an inner automorphism A_x for $x \in SU(2)$ such that*

$$\begin{cases} \exp(rY_1) & \textit{if } a_2 \neq a_3, \\ \exp(rY_2) & \textit{if } a_3 \neq a_1, \\ \exp(rY_3) & \textit{if } a_1 \neq a_2, \end{cases}$$

is an affine transformation, then A_x is an isometry.

§ 2. Geodesics in $(SU(2), g)$

We retain the notations as in § 1. R. Dohira's Theorem and Corollary which appear in [1] can be stated in our case $(SU(2), g)$ as follows:

THEOREM 2.1. *Assume $a_2 = a_3$. Let $\sigma(t)$ be a geodesic in $(SU(2), g)$ such that*

$$\sigma(0) = e, \quad \dot{\sigma}(0) = \sum_{i=1}^{3} k_i Y_i \quad (\textit{each } k_i \in \boldsymbol{R}).$$

Then

$$\sigma(t) = \exp(t(k_2 Y_2 + k_3 Y_3 + a_1^2 a_2^{-2} k_1 Y_1)) \exp(t(1 - a_1^2 a_2^{-2}) k_1 Y_1). \tag{2.1}$$

PROOF. We put $\{Y_2, Y_3\}_R =: \mathfrak{m}_1$ and $\{Y_1\}_R =: \mathfrak{m}_2$. Then

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_2, \qquad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1, \tag{2.2}$$

$$g([X, Y], Z) + a_1^2 a_2^{-1} g(X, [Z, Y]) = 0 \tag{2.3}$$

for each $X, Y \in \mathfrak{m}_1$, $Z \in \mathfrak{m}_2$. In view of Dohira's Theorem (cf. [1]), we can get this Theorem.

COROLLARY 2.2. *Assume $a_2 = a_3$. A geodesic in $(SU(2), g)$ which intersects itself is a closed geodesic.*

Using Theorem 2.1 and Corollary 2.2, we obtain

THEOREM 2.3. *Let $r \in \boldsymbol{R}\backslash\{(n\pi)/2 \mid n$ is an integer$\}$ and $x = \exp(rY_1)$. Assume A_x is an affine transformation. Then, a geodesic $\sigma(t)$ in $(SU(2), g)$ with condition $\sigma(0) = e$ and $\dot{\sigma}(0) = \sum_{i=1}^{3} k_i Y_i$ is closed if and only if there exist a real number $L(\in \boldsymbol{R}\backslash\{0\})$ satisfying*

$$(2.4) \qquad \begin{cases} \cos(EL) = \cos((a_1^2 a_2^{-2} - 1)k_1 L), \\ a_1^2 a_2^{-2} k_1 \sin(EL) = E \sin((a_1^2 a_2^{-2} - 1)k_1 L), \\ k_2 \sin(EL) = 0, \quad \text{and} \quad k_3 \sin(EL) = 0, \end{cases}$$

where $E := \sqrt{(k_2^2 + k_3^2 + a_2^{-4} k_1^2)}$.

PROOF. In this proof, we put $c := a_1^2 a_2^{-2}$. Then, from Proposition 1.3 and Theorem 2.1 we have

$$(2.5) \qquad \sigma(t) = \exp(t(k_2 Y_2 + k_3 Y_3 + ck_1 Y_1)) \exp((1-c)k_1 t Y_1).$$

And then, if $\sigma(t)$ is closed, by Corollary 2.2 we know that there exist real numbers $L(\in \boldsymbol{R}\backslash\{0\})$ satisfying

$$(2.6) \qquad \exp(L(k_2 Y_2 + k_3 Y_3 + ck_1 Y_1)) = \exp((c-1)k_1 L Y_1).$$

We may assume (cf. [5, Proof of Lemma 1.1, p. 154]) that (Y_1, Y_2, Y_3), which appears in (1.8), satisfies

$$(2.7) \qquad Y_i^{4n} = e, \quad Y_i^{4n+1} = Y_i, \quad Y_i^{4n+2} = -e, \quad Y_i^{4n+3} = -Y_i, \quad (i = 1, 2, 3),$$

for every non-negative integer n, and

$$(2.8) \qquad Y_1 Y_2 = -Y_2 Y_1 = Y_3, \quad Y_2 Y_3 = -Y_3 Y_2 = Y_1, \quad Y_3 Y_1 = -Y_1 Y_3 = Y_2.$$

Using (2.7) and (2.8), we get

(2.9)

$$\begin{cases} (k_2Y_2+k_3Y_3+ck_1Y_1)^{4n}=(k_2^2+k_3^2+c^2k_1^2)^{2n}e, \\ (k_2Y_2+k_3Y_3+ck_1Y_1)^{4n+1}=(k_2^2+k_3^2+c^2k_1^2)^{2n}(k_2Y_2+k_3Y_3+ck_1Y_1), \\ (k_2Y_2+k_3Y_3+ck_1Y_1)^{4n+2}=-(k_2^2+k_3^2+c^2k_1^2)^{2n+1}e, \\ (k_2Y_2+k_3Y_3+ck_1Y_1)^{4n+3}=-(k_2^2+k_3^2+c^2k_1^2)^{2n+1}(k_2Y_2+k_3Y_3+ck_1Y_1), \end{cases}$$

for every non-zero integer n. From (2.7), we have

(2.10) $$\exp((c-1)k_1LY_1)=\cos((c-1)k_1L)I_2+\sin((c-1)k_1L)Y_1.$$

By the help of (2.9), we obtain

(2.11) $$\begin{aligned}&\exp(L(k_2Y_2+k_3Y_3+ck_1Y_1))\\ &\quad=\cos(EL)I_2+ck_1E^{-1}\sin(EL)Y_1+k_2E^{-1}\sin(EL)Y_2\\ &\qquad+k_3E^{-1}\sin(EL)Y_3.\end{aligned}$$

Comparing (2.10) with (2.11), we can get this Theorem. q.e.d.

From this Theorem, we obtain the following:

Corollary 2.4. *Assume that $k_1k_2k_3\neq 0$ and the metric g in $(SU(2),g)$ with $a_2=a_3$ is not bi-invariant. Then, if $k_1^{-1}(a_1^2a_2^{-2}-1)\sqrt{(k_2^2+k_3^2+a_1^4a_2^{-4}k_1^2)}$ is not a rational number, the geodesic $\sigma(t)$ with $\sigma(0)=e$ and $\dot\sigma(0)=\sum_{i=1}^3 k_iY_i$ is not closed.*

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