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SPACELIKE SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE IN PSEUDO-RIEMANNIAN SPACE FORMS

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1. Introduction

In the last years, several authors have studied spacelike hypersurfaces with constant mean curvature in Lorentzian spaces of constant curvature, see for instance [2], [10], [11]. When the codimension of the spacelike submanifold is greater that one, the natural generalization, that is, the case of parallel mean curvature vector in the normal bundle, has been dealt in [1], [3], [5]. From a technical point of view, the closest case to that of spacelike hypersurfaces is when the codimension is equal to the index of the ambient space, so that, when the normal bundle is negative definite [1], [5], [9]. Under this assumption, it has been mainly used as a tool classical Simons' formula for the Laplacian of the length of the second fundamental form. However, this technique does not seen to be useful when the normal bundle is not definite. In [3], a different method has been introduced to study compact spacelike submanifolds with parallel mean curvature vector in de Sitter spaces, with non-definite normal bundle. As any index for the normal bundle was allowed, an assumption on the Ricci curvature (automatically satisfied in the definite case) was shown to be necessary. In this paper we will study compact spacelike submanifolds with (non-zero) parallel mean curvature vector in a pseudo-Riemannian space form with Lorentzian normal bundle (of signature $(1,p)$). We use the same approach as in [3]; however no assumption on the curvature of the submaifold is now made, and the family of ambient spaces is extended in order to consider flat and negatively curved pseudo-Riemannian space forms. Our study was first motivated by the following easy fact: consider a totally umbilical and non-totally geodesic hypersurface M^n of a Riemannian space form of sectional curvature $c\in \boldsymbol{R}$, $N^{n+1}(c)$. Embedding $N^{n+1}(c)$ as a totally geodesic submanifold in an

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$(n+p+1)$-dimensional pseudo-Riemannian space form $N_p^{n+p+1}(c)$, of index $p \geq 1$, we can see M^n as a pseudo-umbilical spacelike submanifold with non-zero parallel mean curvature in $N_p^{n+p+1}(c)$. So, it is natural to ask the converse:

When can the codimension of a spacelike submanifold with non-zero parallel mean curvature be reduced to conclude that it must be lying as a totally umbilical hypersurface in the corresponding Riemannian space form $N^{n+1}(c) \subset N_p^{n+p+1}(c)$?

Note that necessary conditions are that the submanifold be assumed to be pseudo-umbilical and that the mean curvature vector be spacelike. In fact, we will obtain, Theorem 3.1, that the codimension can be reduced in the quoted sense, assuming compactness for M^n, if and only if the submanifold is pseudo-umbilical and the (parallel) mean curvature vector is spacelike. The assumption on the signature of the normal bundle is essential in Theorem 3.1, not only by technical reasons but also by the examples shown in Remark 3.2, which prevent that an analogous to this result can be stated under more general assumption that Lorentzian normal bundle. In the case of 2-dimensional submanifolds, as in [8], a topological assumption permits us to give as a consequence of Theorem 3.1 the following result (Corollary 3.4)

The only topological 2-spheres which are spacelike surfaces in $N_p^{3+p}(c)$ *with non-zero spacelike parallel mean curvature vector are the totally umbilical ones in* $N^3(c) \subset N_p^{3+p}(c)$.

The remainder of this paper is mainly devoted to analyze, in the same previous line, the case in which the mean curvature vector has another causal character, and to study the particular case in which the ambient space is flat. First we give a non-existence result, Proposition 4.1, which asserts that a compact pseudo-umbilical spacelike submanifold in $\mathbf{R}_q^{n+p+1}$ with non-zero mean curvature vector has necessarily spacelike mean curvature. Next, Theorem 3.1 is sharped when the ambient space is flat obtaining, Theorem 4.3.

The only complete pseudo-umbilical spacelike submanifolds in $\mathbf{R}_p^{n+p+1}$ *with non-zero spacelike parallel mean curvature vector are the round n-spheres in* $\mathbf{R}^{n+1} \subset \mathbf{R}_p^{n+p+1}$.

Now, it is assumed that the mean curvature to be lightlike. In this case, we only have to consider as ambient space a pseudo-Euclidean sphere $\mathbf{S}_p^{n+p+1}$, Proposition 4.1 and 4.7. Several examples of totally umbilical isometric (not totally geodesic) embeddings from $\mathbf{S}^n(1)$ in $\mathbf{S}_1^{n+2}(1)$, Example 4.8, are shown to be unique in the lightlike case of $\mathbf{H}$, Proposition 4.9. Finally, it is noted that in the timelike case, no result in this direction can be given, as it is shown in Example 4.10.

2. Preliminaries

Let $\boldsymbol{R}_t^m$ be the m-dimensional pseudo-Euclidean space with metric tensor $\langle,\rangle$ of index t given by

$$\langle v, w\rangle = \sum_{i=1}^{m-t} v_i w_i - \sum_{j=m-t+1}^{m} v_j w_j,$$

where $v = (v_1, \dots v_m)$, $w = (w_1, \dots w_m)$, and let us denote by $N_q^{n+p+1}(c)$ the standard model of $(n+p+1)$-dimensional space of index $q \geq 1$ and constant sectional curvature c, which can be assumed, without loss of generality, to be $c = 0, 1, -1$. That is, $N_q^{n+p+1}(c)$ is the pseudo-Euclidean space $\boldsymbol{R}_q^{n+p+1}$ when $c = 0$, the pseudo-Euclidean sphere $\boldsymbol{S}_q^{n+p+1} \subset \boldsymbol{R}_q^{n+p+2}$ when $c = 1$ and the pseudo-Euclidean hyperbolic space $\boldsymbol{H}_q^{n+p+1} \subset \boldsymbol{R}_{q+1}^{n+p+2}$ when $c = -1$. Generically, let us represent by $\boldsymbol{R}_{q+s}^{n+p+k}$ the corresponding pseudo-Euclidean space where $N_q^{n+p+1}(c)$ is lying, and by $N^{n+1}(c)$ a complete totally geodesic spacelike submanifold contained in $N_q^{n+p+1}(c)$, so that $N^{n+1}(c)$ equals to $\boldsymbol{R}^{n+1}$, $\boldsymbol{S}^{n+1}$, or $\boldsymbol{H}^{n+1}$ when $c = 0, 1$, or -1, respectively.

Let us consider $x : M^n \to N_q^{n+p+1}(c) \subset \boldsymbol{R}_{q+s}^{n+p+k}$ a spacelike submanifold in $N_q^{n+p+1}(c)$. Throughout this paper we will denote by ∇^0, $\bar{\nabla}$ and ∇ the Levi-Civita connections of $\boldsymbol{R}_{q+s}^{n+p+k}$, $N_q^{n+p+1}(c)$ and M^n, respectively, and $\nabla^{\perp}$ will be the normal connection of M^n in $N_q^{n+p+1}(c)$. Then, from the Gauss and Weingarten formulas of M^n in $N_q^{n+p+1}(c)$ we have

$$\nabla_X^0 Y = \bar{\nabla}_X Y - c\langle X, Y\rangle x = \nabla_X Y + \sigma(X, Y) - c\langle X, Y\rangle x \tag{2.1}$$

and

$$\nabla_X^0 \xi = \bar{\nabla}_X \xi = -A_\xi X + \nabla_X^{\perp} \xi, \tag{2.2}$$

for all tangent vector fields $X, Y \in \chi(M)$ and normal vector field $\xi \in \chi^{\perp}(M)$, where σ stands for the second fundamental form of M^n in $N_q^{n+p+1}(c)$ and A_ξ is the Weingarten endomorphism associated to ξ.

Let $a \in \boldsymbol{R}_{q+s}^{n+p+k}$ be a fixed arbitrary vector and put

$$a = a^T + a^N + c\langle a, x\rangle x, \tag{2.3}$$

where $a^T \in \chi(M)$ is tangent to M^n (a^T is the gradient of $\langle a, x\rangle$ on M^n) and $a^N \in \chi^{\perp}(M)$ is normal to M^n in $N_q^{n+p+1}(c)$. By taking covariant derivative in (2.3) and using (2.1) and (2.2), it is not difficult to get from $\nabla^0 a = 0$ that

$$\nabla_X a^T = A_{a^N} X - c\langle a, x\rangle X \tag{2.4}$$

and

$$\nabla^{\perp}_X a^N = -\sigma(a^T, X), \tag{2.5}$$

for all $X \in \chi(M)$. Now, directly from (2.4) we get

$$\operatorname{div}(a^T) = \operatorname{tr}(A_{a^N}) - nc\langle a, x\rangle = n\langle a, \mathbf{H}\rangle - nc\langle a, x\rangle, \tag{2.6}$$

where div denotes the divergence on M^n and $\mathbf{H}$ stands for the mean curvature vector of M^n in $N_q^{n+p+1}(c)$, that is, $\mathbf{H} = \frac{1}{n}\operatorname{tr}(\sigma)$. Using now the Codazzi equation and the Gauss equation, as well as (2.4) and (2.5), a straightforward computation leads to

$$\begin{aligned}\operatorname{div}(A_{a^N} a^T) = {} & n\langle \nabla^{\perp}_{a^T}\mathbf{H}, a^N\rangle + \operatorname{tr}(A^2_{a^N}) - c\langle a, x\rangle \operatorname{tr}(A_{a^N}) \\ & - \sum_{i=1}^{n} \langle \sigma(a^T, e_i), \sigma(a^T, e_i)\rangle,\end{aligned} \tag{2.7}$$

where $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame on M^n.

From (2.6) we obtain

$$\operatorname{div}\left[\left(\frac{1}{n}\operatorname{tr} A_{a^N}\right) a^T\right] = \frac{1}{n} a^T(\operatorname{tr} A_{a^N}) + \frac{1}{n}\operatorname{tr}^2(A_{a^N}) - c\operatorname{tr}(A_{a^N})\langle a, x\rangle. \tag{2.8}$$

On the other hand, the Ricci tensor of M^n, Ricc, satisfies

$$\begin{aligned}\operatorname{Ricc}(X, Y) = {} & c(n-1)\langle X, Y\rangle + n\langle \sigma(X, Y), \mathbf{H}\rangle \\ & - \sum_{i=1}^{n} \langle \sigma(X, e_i), \sigma(Y, e_i)\rangle,\end{aligned} \tag{2.9}$$

for all $X, Y \in \chi(M)$.

Next, from equations (2.7), (2.8) and (2.9) and integrating on M^n, which is now assumed to be compact, we obtain the following integral formula,

$$\int_M \left\{(n-1)\langle \nabla^{\perp}_{a^T}\mathbf{H}, a\rangle + T(a^T, a^T) + \operatorname{tr}(A^2_{a^N}) - \frac{1}{n}\operatorname{tr}^2(A_{a^N})\right\} dV = 0, \tag{2.10}$$

where we are putting

$$T(X, X) = \operatorname{Ricc}(X, X) - c(n-1)|X|^2 - (n-1)\langle \sigma(X, X), \mathbf{H}\rangle, \tag{2.11}$$

for all $X \in \chi(M)$.

3. Main Results

In this section we will use formula (2.10) to study compact spacelike submanifolds with non-zero spacelike parallel mean curvature vector $\mathbf{H}$. Recall that a submanifold is said to be *pseudo-umbilical* if $\mathbf{H}$ is umbilical, and in this case we have $A_{\mathbf{H}} = \langle \mathbf{H}, \mathbf{H} \rangle I$, where I is the identity transformation.

For the case of normal bundle of signature $(1,p)$ we can state the following result.

THEOREM 3.1. *Let M^n be a compact spacelike submanifold in $N_p^{n+p+1}(c)$. Then, M^n is pseudo-umbilical with non-zero spacelike parallel mean curvature vector if and only if M^n is a totally umbilical and non totally geodesic hypersurface in $N^{n+1}(c) \subset N_p^{n+p+1}(c)$.*

PROOF. Under our assumptions, the integral formula (2.10) can be written as

$$\int_M \{T(a^T, a^T) + u_a\}\, dV = 0, \tag{3.1}$$

where $u_a = \operatorname{tr}(A_{a^N}^2) - \frac{1}{n}\operatorname{tr}^2(A_{a^N})$. Note that from the Schwartz inequality, the function u_a is non-negative everywhere and $u_a \equiv 0$ if and only if a^N is an umbilical direction.

Since $\mathbf{H} \neq 0$ is a spacelike normal vector field to M^n, we can choose a local orthonormal frame $\{\xi_1, \ldots, \xi_{p+1}\}$ in $\chi^\perp(M)$ such that ξ_1 is collinear to $\mathbf{H}$. Taking into account that M^n is pseudo-umbilical we obtain

$$\langle \sigma(a^T, a^T), \mathbf{H} \rangle = |A_{\xi_1} a^T|^2. \tag{3.2}$$

Then, from (2.11) and (3.2) we deduce

$$\begin{aligned} T(a^T, a^T) &= \langle \sigma(a^T, a^T), \mathbf{H} \rangle - |A_{\xi_1} a^T|^2 + \sum_{j=2}^{p+1} |A_{\xi_j} a^T|^2 \\ &= \sum_{j=2}^{p+1} |A_{\xi_j} a^T|^2 \geq 0. \end{aligned} \tag{3.3}$$

Therefore, from the integral formula (3.1) we have that $u_a \equiv 0$ and $T(a^T, a^T) \equiv 0$, for all vector $a \in \mathbf{R}_{p+s}^{n+p+k}$. This means that every normal direction is umbilical and thus M^n is totally umbilical in $N_p^{n+p+1}(c)$. Moreover, from $T(a^T, a^T) \equiv 0$ and (3.3), we obtain $A_{\xi_j} \equiv 0$ for every $j = 2, \ldots, p+1$. This implies that the first normal space $N_1 = \{\xi \in \chi^\perp(M) : A_\xi = 0\}^\perp$ is parallel and (non-degenerate) onedimensional. Reasoning now as in [6], Theorem 1.1. or [7], Proposition 4.1, the codimension can be reduced, obtaining that M^n is a totally umbilical (and non-totally geodesic) hypersurface in $N^{n+1}(c)$.

Conversely, let $\Psi : M^n \to N^{n+1}(c)$ be a totally umbilical hypersurface in $N^{n+1}(c)$ which is not totally geodesic, and let $j : N^{n+1}(c) \to N_p^{n+p+1}(c)$ be the natural inclusion. Taking into account that $N^{n+1}(c)$ is totally geodesic in $N_p^{n+p+1}(c)$, it is easy to see that $x = j \circ \Psi : M^n \to N_p^{n+p+1}(c)$ is a pseudo-umbilical submanifold with parallel mean curvature vector $\mathbf{H}$ such that $\langle \mathbf{H}, \mathbf{H} \rangle > 0$. ■

REMARK 3.2. It should be noticed that when the signature of the normal bundle is not $(1,p)$, an analogous to Theorem 3.1 cannot be stated. In fact, let $\psi_1 : M^2 \to S_1^4(k)$ be a non-totally geodesic compact maximal surface in a 4-dimensional pseudo-Euclidean sphere of curvature k, $0 < k < 1$, and let ψ_2 be the standard embedding of $S_1^4(k)$ in $S_1^5(1)$ as a totally umbilical and non-totally geodesic hypersurface. It follows that the isometric immersion $\psi = \psi_2 \circ \psi_1 : M^2 \to S_1^5(1)$ is pseudo-umbilical with non-zero spacelike parallel mean curvature vector, but it is not totally umbilical.

We will see now that Theorem 3.1 can be improved when $n = 2$, using the following fact.

LEMMA 3.3. *Let M^2 be a spacelike surface with non-zero parallel mean curvature vector in $N_p^{3+p}(c)$, such that M^2 is a topological sphere. Then, it is pseudo-umbilical.*

PROOF. Let ω be the quadratic differential on M^2 locally given by

$$\omega = \langle \sigma(\partial_z, \partial_z), \mathbf{H} \rangle \, dz^2,$$

where $z = x + iy$ and (x, y) are local isothermal parameters on M^2. Then, ω is well defined and $\omega \equiv 0$ if and only if M^2 is pseudo-umbilical (see, for example, Section 2 in [8]). Now, from the Codazzi equation it follows that

$$\partial_{\bar{z}} \langle \sigma(\partial_z, \partial_z), \mathbf{H} \rangle = \frac{\lambda}{4} \, \partial_z \langle \mathbf{H}, \mathbf{H} \rangle + \langle \sigma(\partial_z, \partial_z), \nabla_{\partial_{\bar{z}}}^{\perp} \mathbf{H} \rangle,$$

where $\lambda = \langle \partial_x, \partial_x \rangle = \langle \partial_y, \partial_y \rangle$, and then, if $\nabla^{\perp} \mathbf{H} \equiv 0$, we deduce that ω is holomorphic, but M^2 being a topological sphere it implies $\omega \equiv 0$ and M^2 is pseudo-umbilical. ■

The announced improvement of Theorem 3.1 is the following

COROLLARY 3.4. *The only topological 2-spheres which are spacelike surfaces in $N_p^{3+p}(c)$ with non-zero spacelike parallel mean curvature vector are the totally umbilical ones in $N^3(c) \subset N_p^{3+p}(c)$.*

4. Non-spacelike mean curvature and several remarks

Now we will consider that the mean curvature vector has another causal character. First we give the following non-existence result.

PROPOSITION 4.1. *Let M^n be a compact pseudo-umbilical spacelike submanifold in $\boldsymbol{R}_q^{n+p+1}$ with non-zero parallel mean curvature vector* $\mathbf{H}$. *Then,* $\mathbf{H}$ *is spacelike.*

PROOF. If $\langle \mathbf{H}, \mathbf{H} \rangle = 0$, $\mathbf{H} \neq 0$, then $A_{\mathbf{H}} = 0$ and (2.2) allows us to say that $\nabla^0 \mathbf{H} = 0$, i.e. $\mathbf{H}$ is a fixed vector in $\boldsymbol{R}_q^{n+p+1}$. On the other hand, compactness and (2.6) give $\langle a, \mathbf{H} \rangle = 0$ for all $a \in \boldsymbol{R}_q^{n+p+1}$, which contradict $\mathbf{H} \neq 0$.

Assume next $\langle \mathbf{H}, \mathbf{H} \rangle < 0$ and put $b = x + (1/\langle \mathbf{H}, \mathbf{H} \rangle)\mathbf{H}$. From (2.2) we easily get $\nabla^0 b = 0$, that is, b is a fixed vector in $\boldsymbol{R}_q^{n+p+1}$. If we set $y = x - b$, then, from (2.6), it follows that $\Delta y = -\langle \mathbf{H}, \mathbf{H} \rangle n y$. But last equality is incompatible with the compactness of the submanifold. ∎

REMARK 4.2. It is worth pointing out that no compact (connected) spacelike submanifold M^n with mean curvature vector $\mathbf{H}$, non-zero everywhere, satisfies $\langle \mathbf{H}, \mathbf{H} \rangle \leq 0$, in Lorentz-Minkowski space $\boldsymbol{L}^m$. This easily follows from (2.6) which gives

$$\int_M \langle a, \mathbf{H} \rangle \, dV = 0,$$

for all vector $a \in \boldsymbol{L}^m$. Choose now a timelike vector $a \in \boldsymbol{L}^m$. At any point $p \in M^n$, we have that H_p is either lightlike or timelike, and in both cases $\langle a, \mathbf{H}_p \rangle \neq 0$. Therefore either $\langle a, \mathbf{H} \rangle < 0$ everywhere or $\langle a, \mathbf{H} \rangle > 0$ everywhere too, which is not possible from this integral formula. In particular, if a compact spacelike submanifold of $\boldsymbol{L}^m$ has non-zero parallel mean curvature vector, then it always has spacelike mean curvature vector.

Although we have assumed compacteness on the submanifold in the previous section to prove Theorem 3.1, we are able to change this assumption by completeness whenever the ambient space be flat as shows the following result.

THEOREM 4.3. *The only complete pseudo-umbilical spacelike submanifolds in* R_p^{n+p+1} *with non-zero spacelike parallel mean curvature vector are the round n-spheres in* $R^{n+1} \subset R_p^{n+p+1}$.

PROOF. We put $d = x + (1/\langle \mathrm{H}, \mathrm{H} \rangle)\mathrm{H}$, where x is the immersion of the corresponding submanifold M^n in R_p^{n+p+1}. From (2.2) easily follows $\nabla^0 d = 0$, i.e. d is a fixed vector in R_p^{n+p+1}. Thus we have $\langle x - d, x - d \rangle = 1/\langle \mathrm{H}, \mathrm{H} \rangle > 0$, which means that M^n lies as a maximal submanifold in an $(n+p)$-dimensional pseudo-Euclidean sphere with index p and curvature $\langle \mathrm{H}, \mathrm{H} \rangle$. Using now Theorem 1.1 in [9] we obtain that $x(M^n)$ is an n-dimensional sphere of radius $1/(\langle \mathrm{H}, \mathrm{H} \rangle)^{1/2}$ in R^{n+1}. ■

REMARK 4.4. If we assume that the submanifold is compact in the previous Theorem, then Proposition 4.1 can be claimed, and the assumption on the causal character of H can be omited. On the other hand, in [3], Theorem 3.1, it has been proved that *a complete maximal submanifold in* $S_q^{n+p}(1)$, $1 \leq q \leq p$, *such that its Ricci curvature is greather or equal to* $n-1$, *is totally geodesic.* Thus, using this result, an analogous argument as in Theorem 4.3, permits us to state:

The only complete pseudo-umbilical spacelike submanifolds in R_q^{n+p+1}, $1 \leq q \leq p$, *with non-zero spacelike parallel mean curvature vector and Ricci curvature greather or equal to* $\langle \mathrm{H}, \mathrm{H} \rangle (n-1)$ *are the round n-spheres in* $R^{n+1} \subset R_q^{n+p+1}$.

Recall that there is no compact maximal submanifold in R_s^m. So, from Lemma 3.3, Proposition 4.1 and Theorem 4.3, we get

COROLLARY 4.5. *The only topological* 2*-spheres which are spacelike surfaces in* R_p^{3+p} *with parallel mean curvature vector are the round spheres in* $R^3 \subset R_p^{3+p}$.

REMARK 4.6. As an application of Theorem 4.3 we have that *the only* 1*-type complete spacelike n-submanifolds in* R_p^{n+p+1} *with non-zero spacelike mean curvature vector, are the round n-spheres in* R^{n+1}. This easily follows taking into account that every 1-type submanifold in R_p^{n+p+1} is pseudo-umbilical with parallel mean curvature vector [4]. Taking into account Proposition 4.1 we can also assert: *the round n-spheres in* R^{n+1} *are the only* 1*-type compact spacelike n-submanifolds in* R_p^{n+p+1}.

Next we will examine the lightlike case of H. Proposition 4.1 says, in particular, that, under this assumption, the ambient space must be non-flat. Even more, we have the following result.

PROPOSITION 4.7. *There exists no compact spacelike submanifold which is pseudo-umbilical and with lightlike parallel mean curvature vector in any pseudo-Euclidean hyperbolic space* H_s^m.

PROOF. If $\langle \mathbf{H}, \mathbf{H} \rangle = 0$, $\mathbf{H} \neq 0$, then $A_{\mathbf{H}} = 0$ and therefore, it follows $\nabla^0 \mathbf{H} = 0$, by using (2.2), i.e., $\mathbf{H}$ is a fixed vector in $\mathbf{R}_{s+1}^{m+1}$. If x represents the corresponding immersion, we put $y = x + \mathbf{H}$. From (2.6) we have now $\Delta y = ny$, where n is the dimension of the submanifold, which contradicts the compactness assumption. ■

Taking into account last results, in order to study the lightlike case of $\mathbf{H}$, we only have to consider as ambient space a pseudo-Euclidean sphere $\mathbf{S}_p^{n+p+1}$.

In the following example we construct an isometric immersion of $\mathbf{S}^n$ in $\mathbf{S}_1^{n+2}$, $n \geq 2$, with lightlike parallel mean curvature vector, which is pseudo-umbilical and totally umbilical.

EXAMPLE 4.8. Let us consider the isometric immersion $x_a : \mathbf{S}^n(1) \to \mathbf{S}_1^{n+2}(1)$, $n \geq 2$, defined by $x_a(u) = (a, u, a)$, with $a \neq 0$. Clearly $\xi_1 = (a^2+1)^{-1/2}(1, -au, 0)$ and $\xi_2 = (a^2+1)^{-1/2}(a^2, au, a^2+1)$ are an orthonormal frame of vector fields normal to $\mathbf{S}^n(1)$ in $\mathbf{S}_1^{n+2}(1)$, with $A_{\xi_1} = -A_{\xi_2} = a(a^2+1)^{-1/2}\mathbf{I}_n$, $\mathbf{I}_n$ being the identity transformation. Thus, its mean curvature vector $\mathbf{H} = a(a^2+1)^{-1/2}(\xi_1 + \xi_2)$ is lightlike, parallel and umbilical, and, of course, x_a is totally umbilical.

The following proposition can be viewed as a uniqueness result concerning previous examples.

PROPOSITION 4.9. *Let* $x : M^n \to \mathbf{S}_p^{n+p+1}$ *be a compact spacelike submanifold in a pseudo-Euclidean sphere* $\mathbf{S}_p^{n+p+1}$. *If* M^n *is pseudo-umbilical with lightlike parallel mean curvature vector then it is totally umbilical and* x *coincides, up to a rigid motion, with* $j \circ x_a$, *for some* $a \neq 0$, *where* j *is a totally geodesic embedding of* $\mathbf{S}_1^{n+2}$ *in* $\mathbf{S}_p^{n+p+1}$.

PROOF. Using again (2.10) we deduce that

$$\int_M \{T(a^T, a^T) + u_a\}\, dV = 0.$$

Since $\mathbf{H}$ is lightlike, we can choose a local frame of normal vector fields $\{\eta_1, \eta_2, \xi_1, \ldots, \xi_{p-1}\}$ such that $\langle \eta_1, \eta_1 \rangle = \langle \eta_2, \eta_2 \rangle = 0$, $\langle \eta_1, \eta_2 \rangle = 1$, $\langle \xi_j, \xi_k \rangle = \delta_{jk}$,

$\langle\eta_i,\xi_j\rangle = 0$, $i = 1,2$; $1 \le j$, $k \le p-1$, and $\eta_1 = \mathbf{H}$. Using now that η_1 is an umbilical lightlike direction, we obtain from (2.11) that

$$\begin{aligned} T(a^T,a^T) &= -\sum_{i=1}^{n}\langle\sigma(a^T,e_i),\sigma(a^T,e_i)\rangle \\ &= -\sum_{i=1}^{n} 2\langle A_{\eta_1}a^T,e_i\rangle\langle A_{\eta_2}a^T,e_i\rangle + \sum_{k=1}^{p-1}|A_{\xi_k}a^T|^2 \\ &= \sum_{k=1}^{p-1}|A_{\xi_k}a^T|^2 \ge 0, \end{aligned}$$

and reasoning as in Theorem 3.1 we deduce that M^n is totally umbilical in S_p^{n+p+1}. Moreover, being $\langle\mathbf{H},\mathbf{H}\rangle = 0$, $\mathbf{H} \ne 0$, we have $\nabla^0\mathbf{H} = 0$. Thus, $\mathbf{H}$ is a vector in $\boldsymbol{R}_p^{n+p+2}$ and $\langle x,\mathbf{H}\rangle$ is constant on M^n, which says that $x(M^n)$ is contained in a degenerate hyperplane π of $\boldsymbol{R}_p^{n+p+2}$. Now take a rigid motion Φ of S_p^{n+p+1} such that Φ (contemplated as a rigid motion of $\boldsymbol{R}_p^{n+p+2}$) carries $\mathbf{H}$ to $(1,0,\dots,0,1) \in \boldsymbol{R}_p^{n+p+2}$, and hence π onto the hyperplane $x_1 = x_{n+p+2}$ (in the natural coordinates of $\boldsymbol{R}_p^{n+p+2}$). It follows that $\Phi\circ x$ is also a totally umbilical immersion with mean curvature $\Phi(\mathbf{H})$. Now, if $p:\boldsymbol{R}_p^{n+p+2}\to\boldsymbol{R}_{p-1}^{n+p}$ is the projection onto the coordinates $(x_2,\dots,x_{n+p+1})$ then $y = p\circ\Phi\circ x$ is an isometric immersion of M^n in S_{p-1}^{n+p-1}, which is easily showed to be totally geodesic, and this concludes the proof. ∎

The following example shows us that when the mean curvature vector is timelike, the submanifold is not necessarily totally umbilical and the codimension cannot be reduced.

EXAMPLE 4.10. Let M^2 be any non-totally geodesic compact minimal surface in a sphere $S^3(k)$, of curvature k, $0 < k < 1$, and let us consider $S^3(k)$ as a totally umbilical hypersurface in $S_1^4(1)$. It is easy to see that M^2 is pseudo-umbilical with timelike parallel mean curvature vector, but, clearly, it is not totally umbilical.

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