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ON A FAMILY OF QUOTIENTS OF FERMAT CURVES

By

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Introduction

Let F_N be the N-th Fermat curve defined by the equation:

$$u^N+v^N=1 .$$

For a pair (r, s) of positive integers such that $r+s\leqq N-1$ and g.c.d. $(r, s, N)=1$, we denote by $F(r, s)$ the quotient of F_N defined by the equation:

$$y^N=x^r(1-x)^s$$

where the projection $F_N\to F(r, s)$ is defined by

$$(x, y)\longmapsto(u^N, u^r v^s).$$

We denote by $\sigma(r, s)$ the automorphism of $F(r, s)$ defined by $\sigma(r, s)^*: (x, y)\mapsto(x, \zeta_N y)$ where ζ_N is a primitive N-th root of unity. The order N of $\sigma(r, s)$ is quite large for the genus $g(r, s)$ of $F(r, s)$. Between them we have a relation:

$$(\sharp)\qquad N\geqq 2g(r, s)+1 .$$

Conversely the inequality ($\sharp$) characterize the quotients $F(r, s)$. In fact we have the following (cf. Theorem 2.2):

THEOREM. *Let X be a complete non-singular curve of genus g over an algebraically closed field k of characteristic* 0, *and let σ be an automorphism of X of order N with $N\geqq 2g+1\geqq 5$. Let H_λ be a hyperelliptic curve of genus g defined by the equation $y^2=(x^{g+1}-1)(x^{g+1}-\lambda)$ with $\lambda\in k\setminus\{0, 1\}$, and let τ_λ be an automorphism of H_λ defined by $\tau_\lambda^*: (x, y)\mapsto(\zeta_{g+1}x, -y)$. Assume that the pair (X, σ) is not isomorphic to $(H_\lambda, \langle\tau_\lambda\rangle)$ for any λ with $N=2g+2$ and g even. Then the pair (X, σ) is isomorphic to $(F(r, s), \sigma(r, s))$, for some (r, s).*

In this paper we are mainly concerned with the curves $F(r, s)$ in which the equality $N=2g(r, s)+1$ holds in ($\sharp$). In a family of these curves there are some interesting curves. For example we have a curve whose group of automor-

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phisms is a cyclic group of maximal order and a Hurwitz curve (for the definition see the section 3.3). The main topics of this paper is to determine isomorphy classes of such curves and their groups of automorphisms completely.

When $N=2g(r, s)+1$ is a prime number, these results are obtained by Seyama [9]. In order to conquer difficulties which arise from the cause that N is not prime, we make use of a technique established by Koblitz-Rohrlich [6].

Let N is very large, then a curve with an automorphism of order N is uniquely determined. In his paper [8], Nakagawa determines curves of genus g with automorphisms of order $N\geqq 3g$.

1. Quotients of Fermat curves

Throughout this paper we fix an algebraically closed field k of characteristic 0. Let $F_N\subset \boldsymbol{P}^2$ denote the Fermat curve of degree N ($N\geqq 3$) defined by the equation

$$U^N+V^N+W^N=0\,.$$

Let u and v be the rational functions on F_N induced by U/W and V/W. For integers r, s such that $1\leqq r$, s we define the differential on F_N by

$$\omega_{r,s}=u^{r-1}v^{s-1}\frac{du}{v^{N-1}}\,.$$

Let

$$A_N=\{(r, s)\in \boldsymbol{Z}^2 \mid 1\leqq r,\ s \text{ and } r+s\leqq N-1\}\,.$$

Then the set $\{\omega_{r,s} \mid (r, s)\in A_N\}$ forms a basis for the space of differentials of the first kind of F_N.

From now on we assume that $(r, s)\in A_N$ satisfies g.c.d. $(r, s, N)=1$. We call such (r, s) a primitive pair. We put

$$x=u^N \quad \text{and} \quad y=u^r v^s\,.$$

Then the equation $u^N+v^N=1$ yields

$$y^N=x^r(1-x)^s\,. \tag{1.1}$$

Let $F(r, s)$ denote the "non-singular model" of the function field $k(x, y)$, so that we have the map $F_N\to F(r, s)$ induced by the inclusion $k(x, y)\subset k(u, v)$.

For $a\in \boldsymbol{Z}/N\boldsymbol{Z}$ or $\boldsymbol{Z}$, we let $\langle a\rangle$ be the integer such that

$$0\leqq \langle a\rangle \leqq N-1 \quad \text{and} \quad \langle a\rangle \equiv a \bmod N\,.$$

Let

$$A(r, s)=\{a\in \boldsymbol{Z}/N\boldsymbol{Z} \mid (\langle ar\rangle, \langle as\rangle)\in A_N\}\,.$$

If $a \in \boldsymbol{Z}/N\boldsymbol{Z}$, then we can regard $\omega_{\langle ar \rangle, \langle as \rangle}$ as a differential on $F(r, s)$ canonically. Then the set $\{\omega_{\langle ar \rangle, \langle as \rangle} \mid a \in A(r, s)\}$ forms a basis for the differentials of the first kind of $F(r, s)$. In particular the genus $g(r, s)$ of $F(r, s)$ is equal to the cardinality of $A(r, s)$. For details, we refer to [7].

Let $\sigma(r, s)$ denote the automorphism of $F(r, s)$ defined by

$$\sigma(r, s)^* x = x \quad \text{and} \quad \sigma(r, s)^* y = \zeta_N y\,. \tag{1.2}$$

We denote by

$$\pi = \pi(r, s) : F(r, s) \longrightarrow \boldsymbol{P}^1 \tag{1.3}$$

the morphism induced by $k(x) \subset k(x, y)$.

THEOREM 1.1. *If $(r, s) \in A_N$ is a primitive pair, then we have*

$$N \geqq 2g(r, s) + 1\,.$$

Equality holds if and only if $(N, r) = (N, s) = (N, r+s) = 1$.

PROOF. We put $e_0 = N/(N, r)$, $e_1 = N/(N, s)$ and $e_\infty = N/(N, r+s)$. Applying the Riemann-Hurwitz relation to the morphism (1.3), we get

$$\frac{2g(r, s) - 2}{N} = 1 - \left(\frac{1}{e_0} + \frac{1}{e_1} + \frac{1}{e_\infty}\right).$$

Hence we have

$$N = 2g(r, s) - 2 + \{(N, r) + (N, s) + (N, r+s)\} \geqq 2g(r, s) + 1\,.$$

Q. E. D.

For later use we shall discuss gap sequences of points where the morphism $\pi : F_{(r,s)} \to \boldsymbol{P}^1$ ramifies. We fix three points P_0, P_1 and P_∞ such that $\pi(P_0) = 0$, $\pi(P_1) = 1$ and $\pi(P_\infty) = \infty$. We denote by $\mathrm{Gap}(P_i)$ the gap sequence of P_i ($i = 0, 1, \infty$), i.e., a positive integer n is contained in $\mathrm{Gap}(P_i)$ means that there exists a differential ω of the first kind with $\mathrm{ord}_{P_i}\omega = n - 1$.

If $a \in \boldsymbol{Z}/N\boldsymbol{Z}$, then we have

$$\mathrm{ord}_{P_0}\omega_{\langle ar \rangle, \langle as \rangle} = \langle ar \rangle - (N, r)\,,$$

$$\mathrm{ord}_{P_1}\omega_{\langle ar \rangle, \langle as \rangle} = \langle as \rangle - (N, s)$$

and

$$\mathrm{ord}_{P_\infty}\omega_{\langle ar \rangle, \langle as \rangle} = \langle -a(r+s) \rangle - (N, r+s)\,.$$

PROPOSITION 1.2. *Let (r, s) be a pair in A_N with $(N, r) = 1$ (resp. $(N, s) = 1$). Then the map*

$$A(r, s) \longrightarrow \mathrm{Gap}(P_0) \quad (\textit{resp.}\ \mathrm{Gap}(P_1))$$

$$a \longmapsto \langle ar\rangle \quad (resp.\ \langle as\rangle)$$

is bijective.

PROOF. Since both of $A(r, s)$ and $\mathrm{Gap}(P_i)$ have the same cardinality, it suffices to show the injectiviュy. It is easy to show it. Q.E.D.

2. A characterization of quotients of Fermat curves

Let X be a complete non-singular algebraic curve of genus $g \geqq 2$ defined over k. Such a curve is simply called a curve of genus g. Let σ be an automorphism of X of order N. We denote by $X/\langle\sigma\rangle$ the quotient of X by the cyclic group $\langle\sigma\rangle$ generated by σ and $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ the set of points in $X/\langle\sigma\rangle$ over which the projection $\pi: X \to X/\langle\sigma\rangle$ ramifies. The automorphism said to be of type $(g_0; e_1, e_2, \cdots, e_n)$ if the genus of $X/\langle\sigma\rangle$ is g_0 and the ramification index at P_i is e_i, where P_i is any point in X such that $\pi(P_i)=\lambda_i$. Then we have the following fact which is proved by Harvey [3] using a topological method.

LEMMA 2.1. *Let M be the l.c.m. of $\{e_1, e_2, \cdots, e_n\}$. Then the following are satisfied:*

(1) *l.c.m. $\{e_1, \cdots, \hat{e}_i, \cdots, e_n\}=M$ for all i, where $\hat{e}_i$ denotes the omission of e_i;*

(2) *M divides N, and if $g_0=0$, $M=N$;*

(3) *$n \neq 1$, and if $g_0=0$, $n \geqq 3$;*

(4) *If $2^r \| M$, i.e., 2^r divides M and 2^{r+1} does not divide M, then the number of e_i's with $2^r \| e_i$ is even.*

PROOF. Suppose $n=1$. If p is a prime divisor of N, then the covering $X/\langle\sigma^p\rangle \to X/\langle\sigma\rangle$ has the only one ramification point. This contradicts a theorem of Lewittes (cf. [2]) which says that the number of the fixed points $\geqq 2$ for an automorphism of prime order. If $g_0=0$, then we have $n \geqq 3$ by the Riemann-Hurwitz formula. Thus we have (3). (2) follows immediately since all the e_i divide N. If $g_0=0$, we have an unramified covering $X/\langle\sigma^{N/M}\rangle \to X/\langle\sigma\rangle$, hence $N=M$.

We put l.c.m. $\{e_1, \cdots, \hat{e}_i, \cdots, e_n\}=M_i$. Consider the covering $\pi: X/\langle\tau\rangle \to X/\langle\sigma\rangle$ where $\tau=\sigma^{N/M_i}$. If $e_i \nmid M_i$, π_i ramifies only over λ_i. This contradicts (3). Thus we have $e_i | M_i$ and $M=M_i$.

For (4) we consider the covering $X/\langle\sigma^{N/2}\rangle \to X/\langle\sigma\rangle$ of degree 2. It ramifies only over λ_i's such that $2^r | e_i$. The number of ramification points of a covering of degree 2 is even. Q.E.D.

Let H_λ be a hyperelliptic curve of genus g defined by the equation

$$y^2=(x^{g+1}-1)(x^{g+1}-\lambda), \qquad \lambda\in k\setminus\{0, 1\}$$

and let τ_λ be an automorphism of H_λ defined by

$$\tau_\lambda^* : (x, y)\longmapsto(\zeta_{g+1}x, -y)$$

where ζ_{g+1} is primitive $(g+1)$-th root of unıty.

Two pairs $(X, \langle\sigma\rangle)$ and $(Y, \langle\tau\rangle)$ of algebraic curves and cyclic groups generated by σ, τ are said to be isomorphic, if there exists an isomorphism $f: X\to Y$ such that $f^{-1}\cdot\langle\tau\rangle\cdot f=\langle\sigma\rangle$.

THEOREM 2.2. *Let $(X, \langle\sigma\rangle)$ be a pair of an algebraic curve X of genus $g\geqq 2$ and a cyclic group generated by an automorphism σ of X of order N. Assume $N\geqq 2g+1$. Then $(X, \langle\sigma\rangle)$ is isomorphic to either $(F(r, s), \langle\sigma(r, s)\rangle)$ for some primitive pair $(r, s)\in A_N$, or $(H_\lambda, \langle\tau_\lambda\rangle)$ for some $\lambda\in k\setminus\{0, 1\}$ with $N=2g+2$ and g even.*

PROOF. Let $(g_0; e_1, e_2, \cdots, e_n)$ denote the type of the automorphism σ, i.e., g_0 is the genus of $X/\langle\sigma\rangle$ and $\{e_1, e_2, \cdots, e_n\}$ is the set of ramification indices for the projection $X\to X/\langle\sigma\rangle$.

We may assume $e_1\leqq e_2\leqq\cdots\leqq e_n$. In this case the Riemann-Hurwitz formula asserts

$$\frac{2g-2}{N}=2g_0-2+\sum_{i=1}^{n}\left(1-\frac{1}{e_i}\right). \tag{2.1}$$

Then we have the following:

(i) $g_0=0$;

(ii) If N is odd, then $n=3$;

(iii) If N is even, then either $n=3$, or the type of σ is $(0; 2, 2, g+1, g+1)$ and g is even.

By the assumption the left hand side of the equation (2.1) is small than 1. Suppose $g_0\geqq 1$. Since $n\geqq 2$ by Lemma 2.1(3), it follows that the right hand side of (2.1)>1. This is a contradiction. Thus we have (i). Now we prove (ii). Obviously we have $n\geqq 3$ and that e_i is odd for any i. We consider the following four cases: (a) $n\geqq 5$, (b) $n=4$, $e_1\geqq 5$, (c) $n=4$, $e_1=3$, $e_2\geqq 5$, (d) $n=4$, $e_1=e_2=3$, $e_3\geqq 7$. Then the right hand side of (2.1)>1 for any case. If $n=4$, $e_1=e_2=e_3=3$, then $e_4=3$ and $N=3$ by Lemma 2.1(1, 2). If $n=4$, $e_1=e_2=3$, $e_3=5$, then $e_4=5$ or 15 and $N=15$ by Lemma 2.1 (1, 2). By (2.1), we have $g=8$ or 9; hence we have $N<2g+1$. Thus we have (ii). By arguments similar to these, we have (iii). It is easy and tiresome to pursue it, so we shall omit it.

If $n=3$, then $X\to X/\langle\sigma\rangle$ is a cyclic covering of degree N having three branch points. Therefore $(X, \langle\sigma\rangle)$ is isomorphic to $(F(r, s), \langle\sigma(r, s)\rangle)$ for some primitive $(r, s)\in A_N$.

Assume that $N=2g+2$ with g even and the type of σ is $(0; 2, 2, g+1, g+1)$. Then we may assume that the set of the branch points for $\pi: X\to X/\langle\sigma\rangle$ is α, 0, 1, ∞ with $\alpha\in k\backslash\{0, 1\}$ and that

$$\pi^{-1}(\alpha)=\{P, \sigma(P), \cdots, \sigma^g(P)\}, \qquad \pi^{-1}(1)=\{Q, \sigma(Q), \cdots, \sigma^g(Q)\},$$
$$\pi^{-1}(0)=\{P_0, \sigma(P_0)\}, \qquad \pi^{-1}(\infty)=\{P_\infty, \sigma(P_\infty)\}.$$

We put $\sigma^{g+1}=\tau$. Then the set of points invariant under τ is $\{P, \sigma(P), \cdots, \sigma^g(P), Q, \sigma(Q), \cdots, \sigma^g(Q)\}$. Applying the Riemann-Hurwitz formula for $X\to X/\langle\tau\rangle$, we have the genus of $X/\langle\tau\rangle=0$; hence X is a hyperelliptic curve. We denote by $\mathcal{L}=\mathcal{L}(P_\infty+\sigma(P_\infty))$ the vector space of rational functions f such that $\mathrm{div}(f)+P_\infty+\sigma(P_\infty)$ is a positive divisor. Then there is a function $x\in\mathcal{L}$ such that $\mathrm{div}(x)=P_0+\sigma(P_0)-P_\infty-\sigma(P_\infty)$. Moreover we have a function y such that

$$\mathrm{div}(y)=P+\cdots+\sigma^g(P)+Q+\cdots+\sigma^g(Q)-(g+1)(P_\infty+\sigma(P_\infty)).$$

Therefore we have $\mathrm{div}(y^2)=\mathrm{div}(\prod_{i=0}^{g}(x-a_i)(x-b_i))$ where $x(\sigma^{i-1}(P))=a_i$ and $x(\sigma^{i-1}(Q))=b_i$. Since $\sigma^*x\in\mathcal{L}$ and $(\sigma^{g+1})^*x=x$, it follows that $\sigma^*x=\zeta_{g+1}x$ for some primitive $(g+1)$-th root ζ_{g+1} of unity. Moreover we have $\mathrm{div}(\sigma^*(x-a_i))=\sigma(\mathrm{div}(x-a_i))=\mathrm{div}(x-a_{i+1})$. Arranging the constants we have

$$y^2=(x^{g+1}-1)(x^{g+1}-\lambda), \qquad \lambda\in k\backslash\{0, 1\}$$

and σ is induced by $\sigma^*: (x, y)\mapsto(\zeta_{g+1}x, -y)$. This completes the proof. Q.E.D.

REMARK 2.1. The exceptional curve H_λ has the following interesting proparty: Let σ_i $(i=1, 2)$ be the automorphism of H_λ defined by

$$\sigma_i^*(x, y)=(\mu^2x^{-1}, (-1)^i\mu^{g+1}x^{-(g+1)}y),$$

where μ satisfies $\mu^{2(g+1)}=\lambda$. Then we have

$$\mathrm{Jac}(H_\lambda)\cong\mathrm{Jac}(H_\lambda/\langle\sigma_1\rangle)\times\mathrm{Jac}(H_\lambda/\langle\sigma_2\rangle)$$

as abelian varieties (cf. [1]).

3. Algebraic curves of genus g with automorphisms of order $2g+1$

In this section we shall be concerned with a pair $(X, \langle\sigma\rangle)$ of an algebraic curve X of genus $g\geqq 2$ and a cyclic group generated by an automorphism σ of order $N=2g+1$. By Theorem 2.2 and Theorem 1.1, we know that it is isomorphic to a pair $(F(r, s), \langle\sigma(r, s)\rangle)$:

$$F(r, s): y^{2g+1}=x^r(1-x)^s,$$

$$\sigma(r, s)^*: (x, y)\longmapsto(x, \zeta_N y),$$

where $(r, s)\in A_N$ is primitive pair and $(N, r)=(N, s)=(N, r+s)=1$, and where ζ_N is a primitive N-th root of unity. If $r^{[-1]}$ is an integer such that $r\cdot r^{[-1]}\equiv 1 \bmod N$, then we have $1\leqq\langle s\cdot r^{[-1]}\rangle\leqq N-2$ and $\mathrm{g.c.d.}(N, \langle s\cdot r^{[-1]}\rangle)=1$.

LEMMA 3.1. $(F(r, s), \langle\sigma(r, s)\rangle)\cong(F(1, \langle s\cdot r^{[-1]}\rangle), \langle\sigma(1, \langle s\cdot r^{[-1]}\rangle)\rangle)$.

PROOF. Define a and b by $r\cdot r^{[-1]}=1+Na$ and $s\cdot r^{[-1]}=\langle s\cdot r^{[-1]}\rangle+Nb$. We put

$$Y=\frac{y^{r[-1]}}{x^a(1-x)^b} \quad \text{and} \quad X=x.$$

Then we have $Y^N=X(1-X)^{\langle s\cdot r^{[-1]}\rangle}$. Q.E.D.

Now we shall treat only pairs of the form $(F(1, \langle a\rangle), \langle\sigma(1, \langle a\rangle)\rangle)$ where $a\in(\boldsymbol{Z}/N\boldsymbol{Z})^\times$ (i.e., $\mathrm{g.c.d.}(\langle a\rangle, N)=1$) and $\mathrm{g.c.d.}(\langle a\rangle+1, N)=1$. For simplicity we put $F(1, \langle a\rangle)$, $\sigma(1, \langle a\rangle)$ and $A(1, \langle a\rangle)$ to $F(a)$, $\sigma(a)$ and $A(a)$, respectively. So we shall study the following set:

$$C(N)=\{a\in(\boldsymbol{Z}/N\boldsymbol{Z})^\times \mid \mathrm{g.c.d.}(\langle a\rangle+1, N)=1\}.$$

Then $C(N)$ always contains 1, g and $2g-1=N-2$. In the following for a finite set S we denote by $|S|$ the cardinality of S.

LEMMA 3.2. *Let $N=p_1^{e_1}\cdots p_n^{e_n}$ be the decomposition into prime factors. Then we have*

$$|C(N)|=\prod_{i=1}^n p_i^{e_i-1}(p_i-2).$$

PROOF. If $N=N_1N_2$ and $\mathrm{g.c.d.}(N_1, N_2)=1$, then the map $(r \bmod N)\mapsto(r \bmod N_1, r \bmod N_2)$ gives a bijection $C(N)\cong C(N_1)\times C(N_2)$. Since $|C(p^e)|=p^{e-1}(p-2)$, we get the result. Q.E.D.

As in (1.3), let $\pi=\pi(a): F(a)\to F(a)/\langle\sigma(a)\rangle\cong\boldsymbol{P}^1$ denote the projection induced by the inclusion $k(x)\subset k(x, y)$. We denote by $\mathrm{Fix}(\sigma(a))$ the set of points fixed under $\sigma(a)$, which consists of three points:

$$\pi^{-1}(0)=P_0^{(a)}, \qquad \pi^{-1}(1)=P_1^{(a)}, \qquad \pi^{-1}(\infty)=P_\infty^{(a)}.$$

Sometimes we omit the superscript (a) from the notation.

3.1. Automorphisms φ and ψ of $C(N)$.

We define φ and ψ by

$$\varphi(a)=-a(1+a)^{-1} \quad \text{and} \quad \psi(a)=a^{-1}, \qquad a\in C(N).$$

We denote by G the group of automorphisms of $C(N)$ generated by φ and ψ. Then we have

$$G=\{1, \varphi, \psi, \psi\varphi, \psi\varphi\psi, (\psi\varphi)^2\}$$

and an isomorphism ρ of G to the symmetric group of three letters $\{0, 1, \infty\}$ such that

$$\rho(\varphi)=\begin{pmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho(\psi)=\begin{pmatrix} 0 & 1 & \infty \\ 1 & 0 & \infty \end{pmatrix}.$$

Let G_a denote the stabilizer subgroup of G at $a\in C(N)$. Then we have the following:

(1) $|G_a|=1$, 2 or 3;

(2) $|G_a|=2$ if and only if $a\in\{1, g, 2g-1\}$;

(3) $|G_a|=3$ if and only it $a^2+a+1=0$.

LEMMA 3.3. *For any $\theta\in G$ and $a\in C(N)$, there is an isomorphism*:

$$\theta_a : (F(a), \langle\sigma(a)\rangle) \longrightarrow (F(\theta(a)), \langle\sigma(\theta(a))\rangle)$$

such that

$$\theta_a(P_i^{(a)})=P_{\rho(\theta)(i)}{}^{(\theta(a))}, \qquad i=0, 1, \infty.$$

PROOF. It suffices to prove the lemma for $\theta=\varphi$ and ψ. We denote by $k(x, y)$ (resp. $k(u, v)$) the rational function field of $F(a)$ (resp. $F(1, \theta(a))$) such that

$$y^N=x(1-x)^{\langle a\rangle} \quad (\text{resp. } v^N=u(1-u)^{\langle\theta(a)\rangle}).$$

For $\theta=\varphi$, let

$$(\varphi_a)^*(u)=x^{-1} \quad \text{and} \quad (\varphi_a)^*(v)=\frac{\zeta y^\alpha}{x(1-x)^{\alpha-\beta-1}}$$

where α, β and ζ are defined by the equations $\alpha=N-\langle\varphi(a)\rangle-1$, $\{N-(\langle a\rangle+1)\}\alpha=1+\beta N$ and $\zeta^N=(-1)^{\langle\varphi(a)\rangle}$. Then φ_a is a required one. On the other hand, for $\theta=\psi$, let

$$(\psi_a)^*(u)=1-x \quad \text{and} \quad (\psi_a)^*(v)=\frac{y^{\langle\psi(a)\rangle}}{(1-x)^\alpha}$$

where a is defined by $\langle a\rangle\cdot\langle\psi(a)\rangle=1+N\alpha$. Then ψ_a is a required one. Q.E.D.

3.2. Hyperelliptic curves.

The following gives a characterization of hyperelliptic curves of genus $g\geqq 2$ with an automorphism of order $N=2g+1$.

THEOREM 3.4. *$F(1)$, $F(g)$ and $F(2g-1)$ are hyperelliptic curves isomorphic to each ether and if $F(a)$, $a\in C(N)$, is a hyperelliptic curve then $a\in\{1, g, 2g-1\}$.*

PROOF. Obviously $F(1)$ is hyperelliptic. Since $\varphi(1)=g$ and $\psi(2g-1)=g$, it follows that the orbit of $1\in C(N)$ under the action of G is the set $\{1, g, 2g-1\}$. By Lemma 3.3, we have $F(1)\cong F(g)\cong F(2g-1)$.

Assume $F(a)$ is hyperelliptic. Since $(\psi\varphi\psi)(a)=-a-1$ and $\langle -a-1\rangle=N-\langle a\rangle-1$, we may assume $a\leqq g$, i.e., $a\leqq g-1$. The defining equation of $F(a)$ is $y^N=x(1-x)^a$. We put $\mathrm{Fix}(\sigma(a))=\{P_0, P_1, P_\infty\}$. Since the rational function y is contained in $\mathcal{L}((a+1)P_\infty)$, the gap sequence of P_∞ is not equal to $\{1, 2, \cdots, g\}$, that is, P_∞ is a Weierstrass point (cf. section 1). Since $F(a)$ is hyperelliptic, we have

$$\mathrm{Gap}(P_\infty)=\{1, 3, 5, \cdots, 2g-1\}.$$

Let $z\in\mathcal{L}(2P_\infty)$ be a rational function such that

$$\mathrm{div}(z)=P_0+P_0'-2P_\infty,$$

where "$'$" means the hyperelliptic involution. Then the set $\{1, z, \cdots, z^{(a+1)/2}\}$ forms a linear basis for $\mathcal{L}((a+1)P_\infty)$. Since $y(P_0)=z(P_0)=0$, we can put

$$y=zF(z), \tag{3.2}$$

where

$$F(z)=\alpha_1+\alpha_2 z+\cdots+\alpha_{(a+1)/2}z^{(a-1)/2}.$$

Comparing the divisors of both sides of (3.2), we have

$$P_0+aP_1-(a+1)P_\infty=P_0+P_0'-2P_\infty+\mathrm{div}(F(z)).$$

It follows that we have $P_0'=P_1$ and $\mathrm{div}(F(z))=(a-1)(P_1-P_\infty)$. If $a>1$, then $F(z)(P_1)=\alpha_1=0$. Hence we have $y=z^2(\alpha_2+\cdots)$. Then we have $P_0=P_1$. This is a contradiction. Q.E.D.

In general we have the following:

THEOREM 3.5. *Let (r, s) be a primitive pair in A_N for $N\geqq 5$. If $F(r, s)$ is a hyperelliptic curve, then the pair $(F(r, s), \langle\sigma(r, s)\rangle)$ is isomorphic to one of the following:*

(1) $N=2g+1$ *and* $(F(1, 1), \langle\sigma(1, 1)\rangle)$;

(2) $N=2g+2$ *with* g *even and* $(H_\lambda, \langle\tau_\lambda\rangle)$, $\lambda\in k\setminus\{0, 1\}$ (*cf. section* 2);

(3) $N=4g$ *and* $(H(4g), \langle\sigma)4g)\rangle)$ *which are defined by*

$$y^2=x(x^{2g}-1) \quad \text{and} \quad \sigma(4g)^*(x, y)=(\zeta_{4g}{}^2x, \zeta_{4g}y).$$

(4) $N=4g+2$ *and* $(H(4g+2), \langle\sigma(4g+2)\rangle)$ *which are defined by*

$$y^2=x^{2g+1}-1 \quad \text{and} \quad \sigma(4g+2)^*(x, y)=(\zeta_{2g+1}x, -y).$$

PROOF. We denote by "$'$" the hyperelliptic involution, which is contained in the center of the group of all automorphisms. For simplicity's sake we put $F(r, s)=F$ and $\sigma(r, s)=\sigma$. If P is a Weierstrass point of F, i.e., $P=P'$, then so is $\sigma(P)$. If there is a Weierstrass point which is not a ramification point for $\pi: F\to F/\langle\sigma\rangle\cong \boldsymbol{P}^1$, it follows that

$$\{P, \sigma(P), \cdots, \sigma^{N-1}(P)\}\subset \text{the set of Weierstrass points};$$

hence we have $N\leqq 2g+2$. Assume that any Weierstrass point is a ramification point. Then we have

$$\frac{N}{e_0}+\frac{N}{e_1}+\frac{N}{e_\infty}\geqq 2g+2,$$

where $e_0=N/(N, r)$, $e_1=N/(N, s)$ and $e_\infty=N/(N, r+s)$. By the Riemann-Hurwitz formula:

$$\frac{2g-2}{N}=1-\left(\frac{1}{e_0}+\frac{1}{e_1}+\frac{1}{e_\infty}\right), \tag{3.3}$$

we have $N\geqq 4g$.

The case $N\leqq 2g+2$ comes from Theorem 2.2 and Theorem 3.4. Now we assume $N\geqq 4g$. Then by (3.3) we have

$$\frac{1}{e_0}+\frac{1}{e_1}+\frac{1}{e_\infty}\geqq 1-\frac{2g-2}{4g}=\frac{2g+2}{4g}.$$

By Lemma 2.1, we have

$$(e_0, e_1, e_\infty)=\begin{cases} (2, 4g, 4g), & N=4g, \\ (2, 2g+1, 4g+2), & N=4g+2. \end{cases}$$

If $N=4g$, then we may assume that $F(r, s)$ is defined by

$$y^N=x^r(1-x)^{2g},$$

where $1\leqq r<2g$ and $(2g, r)=1$. We put $\pi^{-1}(0)=P_0$, $\pi^{-1}(\infty)=P_\infty$. Take a point P_1 such that $\pi(P_1)=1$. Then we have

$$\operatorname{div}(x)=N\cdot P_0-N\cdot P_\infty$$

and

$$\operatorname{div}(y)=P_1+\sigma(P_1)+\cdots+\sigma^{2g-1}(P_1)+rP_0-(2g+r)P_\infty .$$

Since the projection $F(r, 2g)\to F(r, 2g)/\langle\sigma^{2g}\rangle$ ramifies at P_0, P_∞ and $\sigma^i(P_1)$, $i=0, 1, \cdots, 2g-1$, it follows that the genus of $F(r, 2g)/\langle\sigma^{2g}\rangle$ is 0 (hence $F(r, 2g)$ is necessarily hyperelliptic). Take a function u on $F(r, 2g)$ such that

$$\operatorname{div}(u)=2P_0-2P_\infty, \qquad \operatorname{div}(u-1)=2P_1-2P_\infty .$$

Then we have

$$v^2=(u^{2g}-1)u$$

where $v=y\cdot u^{-(r-1)/2}$. By the same way as above we can prove the case $N=4g+2$, so we shall omit its proof. Q.E.D.

REMARK 3.1. In this proof, we have proved that if $N\geqq 4g$, then $(F(r, s), \sigma(r, s))$ is isomorphic to $(H(4g), \sigma(4g))$ or $(H(4g+2), \sigma(4g+2))$. This fact is, already, proved by Nakagawa ([8] Theorem 1, Theorem 2).

REMARK 3.2. We have $(F(1, 1), \langle\sigma(1, 1)\rangle)\cong(H(4g+2), \langle\sigma(4g+2)^2\rangle)$.

3.3. Hurwitz curves.

Let (a, b) be a pair of relatively prime positive integers. The Hurwitz curve, which we denote by $H(a, b)$, of index (a, b) is a non-singular model of the plane curve defined by the equation:

$$x^b y^{a+b}+y^b z^{a+b}+z^b x^{a+b}=0 .$$

In particular $H(2, 1)$ is the Klein curve, i.e., the algebraic curve of genus $g=3$ whose group of automorphisms has the order $168=84(g-1)$. Let

$$N=a^2+ab+b^2 .$$

Then we have $(N, a)=(N, b)=1$. If we regard a and b as elements of $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$, then we have $ab^{-1}\in C(N)$, i.e., g.c.d.$(N, 1+\langle ab^{-1}\rangle)=1$ and $(ab^{-1})^2+(ab^{-1})+1\equiv 0 \bmod N$.

LEMMA 3.6. *Let N be a positive integer. Then the following are equivalent:*

(1) *There exists $r\in C(N)$ such that $r^2+r+1\equiv 0 \bmod N$;*

(2) *If $N=3^{e_0}p_1^{e_1}p_2^{e_2}\cdots p_n^{e_n}$ is the decomposition into prime factors, then $e_0=0$ or 1 and $p_i\equiv 1 \bmod 3$ for all i.*

PROOF. (1)$\Rightarrow$(2) If the equation

$$X^2+X+1=0 \tag{3.4}$$

has a solution in $(\boldsymbol{Z}/N\boldsymbol{Z})^{\times}$, then it has a solution r in each $(\boldsymbol{Z}/p_i\boldsymbol{Z})^{\times}$ for $i=0, 1, \cdots, n$, where $p_0=3$. Since the subgroup $\langle r\rangle$ generated by r is of order 3 or 1, it follows that $p_i=3$ or 3 divides the order p_i-1 of $(\boldsymbol{Z}/p_i\boldsymbol{Z})^{\times}$. Thus we have $p_i\equiv 1 \bmod 3$. On the other hand the equation (3.3) has no solution in $(\boldsymbol{Z}/9\boldsymbol{Z})^{\times}$. Therefore we have $e_0=0$ or 1.

(2)⇒(1) For each i, we have a solution of (3.4) in $(\boldsymbol{Z}/P_i\boldsymbol{Z})^{\times}$ where $P_i=p_i^{e_i}$. By the isomorphism

$$(\boldsymbol{Z}/N\boldsymbol{Z})^{\times}\cong(\boldsymbol{Z}/P_0\boldsymbol{Z})^{\times}\times\cdots\times(\boldsymbol{Z}/P_n\boldsymbol{Z})^{\times} \tag{3.5}$$

we get a required solution. Q.E.D.

From now on we fix a positive integer

$$N=3^{e_0}p_1^{e_1}\cdots p_n^{e_n}$$

satisfying the condition (2) in Lemma 3.6. Then we have

LEMMA 3.7. *Let*

$$\Omega(N)=\{r\in C(N)\,|\,r^2+r+1=0\}$$

and

$$H(N)=\{(a, b)\in\boldsymbol{N}\times\boldsymbol{N}\,|\,N=a^2+ab+b^2,\ \text{g.c.d.}(N, a)=\text{g.c.d.}(N, b)=1\}.$$

Then the map of $H(N)$ *to* $\Omega(N)$ *defined by* $(a, b)\to ab^{[-1]}$ *is bijective and* $|\Omega(N)| =|H(N)|=2^n$, $b^{[-1]}$ *is an integer such that* $bb^{[-1]}\equiv 1 \bmod N$.

PROOF. We shall show that the injectivity of the map $(a, b)\to ab^{[-1]}$. There are two uniquely determined integers s and r satisfying

$$xs-yr=1$$

and the integer

$$l(x, y)=(2x+y)r+(x+2y)s$$

satisfies

$$l(x, y)^2\equiv -3 \bmod 4N, \qquad 0\leqq l(x, y)<2N. \tag{3.6}$$

(cf. [4] Chapter 11 Theorem 4.1). Then we have

$$\frac{(l(x, y)-1)}{2}=(x+y)r+ys$$

and

$$\frac{x\cdot(l(x, y)-1)}{2}=Nr+y,$$

hence we have

$$\frac{(l(x,\ y)-1)}{2}\equiv x^{[-1]}y \quad \bmod N\,.$$

If $ab^{[-1]}\equiv a'(b')^{[-1]}$, then we have

$$\frac{(l(a,\ b)-1)}{2}\equiv\frac{(l(a',\ b')-1)}{2} \quad \bmod N\,.$$

By (3.6), we have

$$l(a,\ b)=l(a',\ b')\,.$$

It follows that there exists a unit u in the ring of the integers in $Q(\sqrt{-3})$ satisfying

$$a+b\omega=(a'+b'\omega)u$$

where $\omega=(1+\sqrt{-3})/2$ (cf. ibid, Chapter 11 Theorem 4.2). Since a, b, a' and b' are positive, we have $(a,\ b)=(a',\ b')$. This completes the proof. Q.E.D.

LEMMA 3.8. $H(a,\ b)\cong H(b,\ a)\cong F(a,\ b)\cong F(1,\ \langle ab^{[-1]}\rangle)$.

PROOF. The defining equation of the N-th Fermat curve is

$$U^{N}+V^{N}+W^{N}=0\,.$$

We put

$$X=U^{a+b}V^{b},\qquad Y=V^{a+b}W^{b},\qquad Z=W^{a+b}U^{b}\,.$$

Then we have the defining equatiin of the Hurwitz curve of index $(a,\ b)$:

$$X^{b}Y^{a+b}+Y^{b}Z^{a+b}+Z^{b}X^{a+b}=0\,.$$

Moreover we have $k(x,\ y)=k(x,\ u^{N})$ where $x=X/Z$, $y=Y/Z$ and $u=U/W$. In fact we have $x=u^{a}v^{b}$, $y=v^{a+b}u^{-b}$, $u^{N}=x^{a+b}/y^{b}$ and $v^{N}=x^{b}y^{a}$ where $v=V/W$. Therefore y^{a} and $y^{b}\in k(x,\ u^{N})$, because $v^{N}=-(u^{N}+1)\in k(x,\ u^{N})$. Since $(a,\ b)=1$, $y\in k(x,\ u^{N})$.

Now let $r=-u^{N}$ and $s=\xi x$ where $\xi^{N}=(-1)^{a+b}$. Then we have

$$s^{N}=r^{a}(1-r)^{b}\,;$$

hence we have $H(a,\ b)\cong F(a,\ b)$. Q.E.D.

Combining Lemma 3.7 and 3.8, we get

LEMMA 3.9. *Let* $c\in C(N)$. *Then* $F(c)$ *is a Hurwitz curve, i.e., there exists a pair* $(a,\ b)$ *of relatively prime integers such that* $N=a^{2}+ab+b^{2}$ *and* $ab^{[-1]}\equiv c$ mod N *if and only if* $c^{2}+c+1=0$.

Let $a\in\Omega(N)$, i.e., $a^{2}+a+1=0$. Then we have $\phi\varphi(a)=a$, hence we have the automorphism $(\psi\varphi)_a: F(a) \to F(a)$, which we denote $\tau(a)$. By an easy calculatiin (cf. Lemma 3.3), we have

LEMMA 3.10. *$\tau(a)\cdot\sigma(a)=\sigma(a)^{\alpha}\cdot\tau(a)$, where $\alpha=N-\langle a^{-1}\rangle-1\geqq 2$.*

EXAMPLE 3.1. Let $N=39$. Then we have

$$C(N)=\{1, 4, 7, 10, 16, 19, 22, 28, 31, 34, 37\}.$$

We have three orbits of the action of G:

(i) $\{1, 19, 37\}$, $F(1, 1)$ is a hyperelliptic curve;

(ii) $\{4, 7, 10, 28, 31, 34\}$;

(iii) $\{16, 22\}=\Omega(N)$, $F(1, 16)$ is a Hurwitz curve of index $(2, 5)$.

3.4. Isomorphism theorem.

Now we shall prove the main theorem in this paper.

THEOREM 3.11. *Let a and b be elements in $C(N)$. Then $F(a)$ and $F(b)$ are isomorphic if and only if there exists an element θ in the group G (cf. the section 3.1) such that $\theta(a)=b$.*

PROOF. "if"-part comes from Lemma 3.3. When $F(a)$ is the Klein curve, then the proof is obvious. So we shall exclude this case. Assume there is an isomorphism

$$f: F(a) \longrightarrow F(b).$$

Then we have $\langle f^{-1}\sigma(b)f\rangle=\langle\sigma(a)\rangle$ and $f(\mathrm{Fix}(\sigma(a)))=\mathrm{Fix}(\sigma(b))$ by Lemma 3.13 in the section 3.5. Now, put $f(P_i^{(a)})=P_{f_i}^{(b)}$ $(i=0, 1, \infty)$, so we can take the element in G corresponding to the permutation $(f_0, f_1, f_\infty)\mapsto(0, 1, \infty)$. It means we may assume

$$f(P_i^{(a)})=P_i^{(b)}, \qquad i=0, 1, \infty.$$

by Lemma 3.3 And we have $\mathrm{Gap}(P_0^{(a)})=\mathrm{Gap}(P_0^{(b)})$; hence we have $A(a)=A(b)$ by Proposition 1.2. We put

$$A(c)^{\times}=A(c)\cap(\boldsymbol{Z}/N\boldsymbol{Z})^{\times} \qquad \text{for } c=a, b.$$

Then the theorem comes from the following:

LEMMA 3.12. *$A(a)^{\times}=A(b)^{\times}$ if and only if $a=b$ or $-b-1$.*

PROOF OF LEMMA. Since we have $A(-b-1)=A(b)$, it follows the proof of "if"-part. We shall now follow a technique of the proof of Theorem 1 in [6] to prove "only if"-part. For any $r\in(\boldsymbol{Z}/N\boldsymbol{Z})^{\times}$, we define an element $G(r)$ in the group algebra $\boldsymbol{Q}[\mathrm{Gal}(\boldsymbol{Q}(\zeta_N)/\boldsymbol{Q})]$, (where $\zeta_N=e^{2\pi i/N}$):

$$G(r)=\sum_{h\in(\boldsymbol{Z}/N\boldsymbol{Z})^{\times}} B_1(hr)\sigma_h$$

where $B_1(s)=\langle s\rangle/N-1/2$ and σ_h is the automorphism of $\boldsymbol{Q}(\zeta_N)$ over $\boldsymbol{Q}$ defined by $\zeta_N\to\zeta_N^h$. If $h\in A(a)^{\times}$ (resp. $h\notin A(a)^{\times}$), then $\langle h\rangle+\langle ha\rangle+\langle h(-a-1)\rangle=N$ (resp. $\langle h\rangle+\langle ha\rangle+\langle h(-a-1)\rangle=2N$). Hence we have

$$G(1)+G(a)+G(-a-1)\sum_{h\notin A(a)^{\times}}\frac{1}{2}\sigma_h-\sum_{h\in A(a)^{\times}}\frac{1}{2}\sigma_h\,.$$

It follows that

$$G(a)+G(-a-1)=G(b)+G(-b-1)\,. \tag{3.7}$$

Applying a character

$$\chi\colon \mathrm{Gal}(\boldsymbol{Q}(\zeta_N)/\boldsymbol{Q})\longrightarrow \boldsymbol{C}^{\times}$$

to both sides of (2.7), we get

$$B_{1,\chi}\bar{\chi}(a)+B_{1,\chi}\bar{\chi}(-a-1)=B_{1,\chi}\bar{\chi}(b)+B_{1,\chi}\bar{\chi}(-b-1)$$

where $B_{1,\chi}$ is the generalized Bernoulli number

$$B_{1,\chi}=\sum_h B_1(h)\chi(h)\,.$$

We fix an odd character χ_0. Then we have

$$\bar{\chi}_0(a)\bar{\psi}(a)+\bar{\chi}_0(-a-1)\bar{\psi}(-a-1)=\bar{\chi}_0(b)\bar{\psi}(b)+\bar{\chi}_0(-b-1)\bar{\psi}(-b-1) \tag{3.8}$$

for all even character ψ with $B_{1,\chi_0\psi}\neq 0$. Now we shall use the following results proved by Koblitz-Rohrlich (cf. ibid. section 2 Proposition, Remark 2 and Lemma):

SUBLEMMA A. *Suppose N is odd. Let $S(N)$ be the set of odd characters of $(\boldsymbol{Z}/N\boldsymbol{Z})^{\times}$, and let*

$$S_0(N)=\{\chi\in S(N)\mid B_{1,\chi}=0\}\,.$$

Then $|S_0(N)|\leqq(1/4)|S(N)|$ and equality holds if and only if $N=39$.

SUBLEMMA B. *Let A be a finite abelian group, S a subset of the group $\hat{A}$ of characters, T a subset of A. If*

$$|S|>\frac{|T|-1}{|T|}|A|$$

then the rows of the matrix

$$(\psi(g))_{(g,\psi)\in T\times S}$$

are linearly independent.

Suppose $N\neq 39$. Let $A=(\boldsymbol{Z}/N\boldsymbol{Z})^{\times}/\{+1, -1\}$. Then $\hat{A}$ can be naturally identified with the set of even characters of $(\boldsymbol{Z}/N\boldsymbol{Z})^{\times}$. We put

$$S=\{\psi\in\hat{A}\mid B_{1,\chi_0\psi}\neq 0\}$$

and

$$T=\{(a), (-a-1), (b), (-b-1)\}$$

where (c) denotes the element of A determined by c. By sublemma A, we have

$$\frac{|S|}{|A|}>\frac{3}{4}.$$

Considering the relations (3.8), we have $a=b$ or $-b-1$ by sublemma B.

When $N=39$, $A(1)$, $A(4)$ and $A(16)$ are distinct from each other (cf. Example 3.1.). This completes the proof of Lemma. Q.E.D.

3.5. The group $\mathrm{Aut}(F(a))$ of automorphisms.

As usual let X be a curve of genus $g\geqq 2$ and let σ be an automorphism of order $N=2g+1$. We denote by $\mathrm{Aut}(X)$ the group of automorphisms of X.

LEMMA 3.13. *Let X be a non-hyperelliptic curve of genus $g\geqq 3$ and let H be a cyclic subgroup of* $\mathrm{Aut}(X)$ *of order* $2g+1$. *Assume X is not isomorphic to the Klein curve*: $H(1, 2)$. *Then H is a normal subgroup of* $\mathrm{Aut}(X)$ *of index* $\leqq 3$.

PROOF. Let $\pi: X\to X/\mathrm{Aut}(X)$ be the projection. The genus of X/H is zero, so is $X/\mathrm{Aut}(X)$. Let $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ be the set of branch points. Take a point P_i such that $\pi(P_i)=\lambda_i$ and put

$$G_i=\{\sigma\in\mathrm{Aut}(X)\mid\sigma(P_i)=P_i\},$$

which is a cyclic subgroup of $\mathrm{Aut}(X)$. We denote by e_i the order of G_i and assume $2\leqq e_1\leqq e_2\leqq\cdots$. H is a subgroup of some G_i. Then $e_i=m(2g+1)$ for some positive integer m. Moreover we have $m=1$ or 2 by Theorem 3.5. If $m=2$, then $X\cong F(1)$ which is a hyperelliptic curve. By the Riemann-Hurwitz formula for π:

$$\frac{2g-2}{|\mathrm{Aut}(X)|}=-2+\sum_{i=1}^{n}\left(1-\frac{1}{e_i}\right), \tag{3.9}$$

we easily have $n=3$. Then we have

$$\frac{2g-2}{|\mathrm{Aut}(X)|}=1-\left(\frac{1}{e_1}+\frac{1}{e_2}+\frac{1}{2g+1}\right). \tag{3.10}$$

By this relation we have $|\mathrm{Aut}(X): H| \leqq 3$ except $(e_1, e_2)=(2, 3)$. In the exceptional case (3.10) becomes

$$\frac{2g-1}{2g-5}=\frac{|\mathrm{Aut}(X): H|}{6}>1;$$

hence we have $|\mathrm{Aut}(X): H| \leqq 12$ and $\equiv 0 \bmod 4$ for $g \geqq 4$. If $|\mathrm{Aut}(X): H|=8$ then $g=7$ and $2g+1=15$. If $|\mathrm{Aut}(X): H|=12$, then $g=4$ and $2g+1=9$. Since $|C(15)|=|C(9)|=3$ by Lemma 3.2, such curves are hyperelliptic. When $g=3$, we have $|\mathrm{Aut}(X): H|=24$. Then X is the Klein curve. Thus we have shown that $|\mathrm{Aut}(X): H| \leqq 3$. Since the order of H is odd, H is a normal subgroup of $\mathrm{Aut}(X)$. Q. E. D.

As we saw in the section 3.2, the hyperelliptic curve $F(1)$ is defined by the equation:

$$x^2=y^{2g+1}-1 .$$

The automorphism $\tilde{\sigma}$ of $F(1)$ defined by $\tilde{\sigma}^*(x, y)=(-x, \zeta_{2g+1}y)$ has the order $4g+2$ and $\tilde{\sigma}^2=\sigma(1)$. Then the following fact is well-known and it is proved by arguments similar to the proof of the preceding lemma, so we shall omit its proof.

LEMMA 3.14. $\mathrm{Aut}(F(1))=\langle\tilde{\sigma}\rangle$.

LEMMA 3.15. *Let a and b be elements in $C(N)$. Assume $F(a)$ is not the Klein curve. If*

$$f: F(a) \longrightarrow F(b)$$

is an isomorphism, then $\langle\sigma(a)\rangle=\langle f^{-1}\sigma(b)f\rangle$. In particular we have

$$f(\mathrm{Fix}(\sigma(a)))=\mathrm{Fix}(\sigma(b)).$$

PROOF. We put $H=\langle\sigma(a)\rangle$ and $H'=\langle f^{-1}\sigma(b)f\rangle$. By Lemma 3.13 and 3.14, we have $|HH': H| \leqq 3$ unless $F(a)$ is the Klein curve. Since the order of H is $N=2g+1 \geqq 5$, we have $|HH': H|=1$ or 3. If $F(a)$ is hyperelliptic then $|HH': H|=1$ and $H=H'$. Otherwise $(f^{-1}\sigma(b)f)^3 \in H$. Therefore we have $\mathrm{Fix}(\sigma(a))=\mathrm{Fix}(f^{-1}\sigma(b)f)$. Since the stabilizer group at $F_0^{(a)}$ is H, we have $H=H'$. Q.E.D.

Let $a \in C(N)$. By the preceding lemma, we see that each automorphism of $F(a)$ induces a permutation of the three points in $\mathrm{Fix}(\sigma(a))=\{P_0, P_1, P_\infty\}$. Therefore we get a homomorphism:

$$p(a): \mathrm{Aut}(F(a)) \longrightarrow \mathrm{Per}(\mathrm{Fix}(\sigma(a))),$$

where $\mathrm{Per}(\mathrm{Fix}(\sigma(a)))$ is the group of permutations.

THEOREM 3.16. *Assume $F(a)$ is not the Klein curve. Then we have an exact sequence:*

$$1 \longrightarrow \langle\sigma(a)\rangle \longrightarrow \mathrm{Aut}(F(a)) \longrightarrow G_a .$$

where G_a is the stabilizer subgroup of G at a.

PROOF. Since the kernel of $p(a)$ is $\langle\sigma(a)\rangle$ (cf. Lemma 3.1 in [9]), it is enough to show $\mathrm{Im}(p(a))\cong G_a$. If $|G_a|=2$, i.e., $F(a)$ is hyperelliptic, then there is only one Weierstrass point in $\mathrm{Fix}(\sigma(a))$. Hence we have $|\mathrm{Im}(p(a))|=2$. If $|G_a|=3$, i.e., $F(a)$ is a Hurwitz curve, then the automorphism $\tau(a)$ induces a permutation of order 3. Assume $|G_a|=1$. Let

$$f: F(a) \longrightarrow F(a)$$

be an automorphism. Then by Lemma 3.3 we have an element $\theta\in G$ such that

$$(f\cdot\theta_a)(P_i^{(a)})=P_i^{(\theta(a))} \qquad \text{for } i=0,\ 1,\ \infty .$$

Then by Lemma 3.12 we have $\theta(a)=a$ or $-a-1$. If $\sigma(a)=a$, we have $\theta=1$ by $G_a=\{1\}$; hence $f\in\langle\sigma(a)\rangle$. Suppose $\theta(a)=-a-1$. Then the composite morphism

$$f'=(\psi\cdot\varphi\cdot\psi)_a{}^{-1}\cdot\theta_a\cdot f: F(a) \longrightarrow F(-a-1) \longrightarrow F(a)$$

satisfies

$$f'(P_0{}^{(a)})=P_0{}^{(a)}, \qquad f'(P_1{}^{(a)})=P_\infty{}^{(a)} .$$

Therefore $(f')^2\in\langle\sigma(a)\rangle$, i.e., the order of f' is $2N=2(2g+1)$. Then $F(a)$ is hyperelliptic by Theorem 3.5; hence $|G_a|=2$. This is a contradiction. Q.E.D.

REMARK 3.3. If $F(a)$ is a Hurwitz curve then the exact sequence in the theorem does not split (cf. Lemma 3.11).

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