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# ON TRIVIAL EXTENSIONS WHICH ARE QUASI-FROBENIUS ONES

By

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Recently Y. Kitamura has characterized a trivial extension which is a Frobenius extension in [2]. In this paper we characterize a trivial extension which is a quasi-Frobenius extension.

Let $R$ be a ring with an identity and $M$ an $(R, R)$-bimodule. The trivial extension $S=(R, M)$ of $R$ by $M$ is the direct sum of additive groups $R$ and $M$ with the multiplication $(r_1, m_1)(r_2, m_2)=(r_1r_2, r_1m_2+m_1r_2)$ for $(r_i, m_i)\in S$. $S$ is a ring containing $R$ with the identification $r\rightarrow(r, 0)$ for $r\in R$. Let $*S$ be the dual space of $S$ as a left $R$-module. Then $*S$ is isomorphic to the direct sum of $R$ and $*M=\mathrm{Hom}\,({}_RM, {}_RR)$ : $*S=[R, *M]$. The action of an element $[a, h]\in *S$ on $S$ is given by $[a, h]((r, m))=ra+h(m)$ for $(r, m)\in S$. $*S$ has the structure of an $(S, R)$-bimodule. This is given by $(r, m)[a, h]=[ra+h(m), rh]$ and $[a, h]r=[ar, hr]$ for $(r, m)\in S$, $[a, h]\in *S$ and $r\in R$.

Following to [3] a ring extension $S$ over $R$ is called a left quasi-Frobenius extension when $S$ is left $R$-finitely generated projective and a direct summand of a finite direct sum of $*S$ as an $(S, R)$-bimodule.

Let $S$ be the trivial extension of $R$ by $M$, and assume that $S$ is a left quasi-Frobenius extension of $R$. Then there exist $(S, R)$-homomorphisms $\Phi: S\rightarrow *S\oplus\cdots\oplus *S$ and $\Psi: *S\oplus\cdots\oplus *S\rightarrow S$ such that $\Psi\circ\Phi=1_S$. Let $\Phi((1, 0))=([a_1, h_1], \cdots, [a_n, h_n])$. Then it is easily seen that $h_i$ is contained in $\mathrm{Hom}\,({}_RM_R, {}_RR_R)$ for all $i$. Next, we consider homomorphisms from $*S$ to $S$. Since $S$ is left $R$-finitely generated projective, we have following isomorphisms

$$\begin{aligned}\mathrm{Hom}\,({}_S*S_R, {}_SS_R)&=\mathrm{Hom}\,({}_S\mathrm{Hom}\,({}_RS, {}_RR)_R, {}_SS_R)\\&\cong\{\mathrm{Hom}\,(R_R, S_R)\otimes_RS\}^S\cong\{S\otimes_RS\}^S\end{aligned}$$

where $\{S\otimes_RS\}^S$ means the set of elements in $S\otimes_RS$ commuting to the elements of $S$. Explicitely, the correspondence is given by $\sum(s_1\otimes s_2)(f)=\sum s_1f(s_2)$ for $\sum s_1\otimes s_2\in\{S\otimes_RS\}^S$ and $f\in *S$. Let $\Psi_i$ be the restriction of $\Psi$ to $i$-th component of $*S\oplus\cdots\oplus *S$ and $\Sigma_j(b_{ij}, m_{ij})\otimes(c_{ij}, n_{ij})$ the corresponding element in $\{S\otimes_RS\}^S$. Then, for $[a, h]\in *S$, we have

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$$\Psi_i([a, h])=(b_i a+h(n_i), m_i a+\Sigma_j m_{ij}h(n_{ij}))$$

where $b_i=\Sigma_j b_{ij}c_{ij}$, $n_i=\Sigma_j b_{ij}n_{ij}$ and $m_i=\Sigma_j m_{ij}c_{ij}$. Using the fact that $\Psi_i$ is a left $S$-homomorphism, we see easily that $m_i\in M^R=\{m\in M \mid rm=mr,$ for any $r\in R\}$, $h(rn_i-n_ir)=0$ for any $h\in{}^*M$ and $r\in R$, and $mh(n_i)=m_ih(m)$ for any $h\in{}^*M$ and $m\in M$. Further, from $(0, m)=\Psi\circ\Phi((0, m))$ we have $m=\Sigma_i m_ih_i(m)$ for all $m\in M$. This means that $M$ is a direct summand of a finite direct sum of $R$ as an $(R, R)-$bimodule: ${}_RM_R<\oplus_R(R\oplus\cdots\oplus R)_R$. From this and $h(rn_i-n_ir)=0$ above, $n_i$ is in $M^R$ for all $i$. We have proved the half direction of the next proposition.

PROPOSITION 1. *Let $S$ be the trivial extension of $R$ by $M$. Then $S$ is a left quasi-Frobenius extension of $R$ if and only if $M$ is an $(R, R)$-direct summand of a finite direct sum of $R$, and for a system of projective bases $\{m_i, h_i\}$ of $M$ there exist $n_i$s in $M^R$ such that, for all $i$, $mh(n_i)=m_ih(m)$ hold for any $m\in M^*$ and $h\in{}^*M$.*

PROOF. We prove the converse. Assume that there are given $\{m_i, h_i\}$ and $n_i$ described in the proposition. Set $e=\Sigma_ih_i(n_i)$. Then $e$ is in the centre $C$ of $R$. Further we have $me=\Sigma_imh_i(n_i)=\Sigma_im_ih_i(m)=m$ for any $m\in M$. In particular, since $h_i(n_i)=h_i(n_ie)=h_i(n_i)e$, $e$ is a central idempotent.

Define the map $\Psi_i: {}^*S\to S$ by $\Psi_i([a, h])=((1-e)a+h(n_i), m_ia)$. Then $\Psi_i$ is an $(S, R)$-homomorphism. Set $\Psi=\Sigma_i\Psi_i$, the map from ${}^*S\oplus\cdots\oplus{}^*S$ to $S$. Next, define the map $\Phi: S\to{}^*S\oplus\cdots\oplus{}^*S$ by $\Phi((1, 0))=([1-e, h_1], [0, h_2], \cdots, [0, h_n])$. Then we see that $\Psi\cdot\Phi=1_S$ and this completes the proof.

We continue the consideration. From the equation $mh(n_i)=m_ih(m)$, we have $m_jh_j(n_i)=m_ih_j(m_j)$, and so $n_i=m_it$ with $t=\Sigma_jh_j(m_j)$. Further, as $h_i(n_i)=h_i(m_i)t$ we have $e=t^2$ and $n_it=m_i$.

As $M$ is an $(R, R)$-direct summand of a finite direct sum of $R$, $M$ is isomorphic to $M^R\otimes_CR$ and $M^R$ is $C$-(and also $eC$-) finitely generated projective (faithful) by [1] Theorem 1.2. Further, since there hold following isomorphisms

$$\begin{aligned}{}^*M&=\mathrm{Hom}({}_RM, {}_RR)\cong\mathrm{Hom}({}_R(M^R\otimes_CR), {}_RR)\cong\mathrm{Hom}_C(M^R, \mathrm{Hom}({}_RR, {}_RR))\\&\cong\mathrm{Hom}_C(M^R, R)\cong\mathrm{Hom}_C(M^R, C)\otimes_CR,\end{aligned}$$

we may regard that $\mathrm{Hom}_C(M^R, C)$ is in ${}^*M$. (Note that $\mathrm{Hom}_C(M^R, C)=\mathrm{Hom}_{eC}(M^R, eC)$). Therefore the relation $mh(n_i)=m_ih(m)$ holds for any $m\in M^R$ and $h\in\mathrm{Hom}_C(M^R, C)$. Thus we have $mh(m_i)=m_ih(m)t$. As $M^R=\Sigma_im_iC$, we have $mh(n)=nh(m)t$ for all $m, n\in M^R$ and $h\in\mathrm{Hom}_C(M^R, C)=\mathrm{Hom}_{eC}(M^R, eC)$. On the other hand, since we may consider $mh(n)=(m\otimes h)(n)$ where $m\otimes h\in M^R\otimes_{eC}\mathrm{Hom}_{eC}(M^R, eC)\cong\mathrm{Hom}_{eC}(M^R, M^R)$, we conclude that $\mathrm{Hom}_{eC}(M^R, M^R)\cong eC$. Thus $M^R$ is an $eC$-finitely generated projective module of rank 1.

Conversely, assume that $e$ is a central idempotent of $R$ and $M_0$ is an $eC$-finitely generated projective module of rank 1. Then, as the canonical map $eC \to \mathrm{Hom}_{eC}(M_0, M_0)$ is an isomorphism, for an element $m \otimes h \in M_0 \otimes_{eC} \mathrm{Hom}_{eC}(M_0, eC)$ there exists $a \in eC$ such that $mh(n) = (m \otimes h)(n) = na$ for any $n \in M_0$. Let $\{m_i, h_i\}$ be a system of projective bases for $M_0$. Then, since $mh(m_i) = m_i a$, we obtain $h_i(m_i)a = h_i(m_i a) = h_i(mh(m_i)) = h_i(m)h(m_i) = h(h_i(m)m_i)$. Therefore $ta = h(m)$ where $t = \Sigma_i h_i(m_i)$. Then $nh(m) = nta = mh(n)t$. As this holds for any $n, h$ and $m$, we have $nh(m) = mh(n)t = nh(m)t^2$. Therefore, since $M_0$ is faithful, we have $n = nt^2$ for any $n \in M_0$, and $t^2 = e$. Put $n_i = m_i t$. Then $mh(n_i) = m_i ta = m_i h(m)$ for any $m \in M_0$ and $h \in \mathrm{Hom}_{eC}(M_0, eC)$. Define $M = M_0 \otimes_{eC} R (= M_0 \otimes_C R)$. Then $M$ is an $(R, R)$-direct summand of a finite direct sum of $R$ and it is easily seen that there holds, for each $i$, $mh(n_i) = m_i h(m)$ for any $m \in M$ and $h \in \mathrm{Hom}({}_R M, {}_R R)$. By Proposition 1, we proved the following theorem.

THEOREM 2. *Let $S$ be the trivial extension of $R$ by $M$ and $C$ the centre of $R$. Then $S$ is a left quasi-Frobenius extension of $R$ if and only if $M$ is isomorphic to $M^R \otimes_C R$ and there exists a central idempotent $e$ in $R$ such that $M^R$ is an $eC$-finitely generated projective module of rank* 1.

REMARK. It can be shown that the element $t = \Sigma_i h_i(m_i)$ in the proof of Theorem 2 is equal to $e$.

Since the condition discribed in Theorem 2 is left right symmetric, we have

COROLLARY 3. *On a trivial extension, a left quasi-Frobenius extension is as well as a right quasi-Frobenius extension.*

Y. Kitamura has proved that a trivial extension is a Frobenius extension if and only if $M$ is isomorphic to $eR$ for some central idempotent $e$. As $M$ is isomorphic to $eR$ if and only if $M^R$ is isomorphic to $eC$, we have

COROLLARY 4. *A trivial extension which is a quasi-Frobenius one is a Frobenius extension if and only if $M^R$ in Theorem 2 is $eC$-free of rank 1 for some central idempotent $e$ in $R$.*

## References

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