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NON-COMPACT SIMPLE LIE GROUP $E_{8(8)}$

By

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It is known that there exist three simple Lie groups of type E_8 up to local isomorphism, one of them is compact and the others are non-compact. We have shown in [8] that the group

$$E_8=\{\alpha\in \mathrm{Iso}_C(\mathfrak{e}_8^C, \mathfrak{e}_8^C)\,|\,\alpha[R_1, R_2]=[\alpha R_1, \alpha R_2],\ \langle\alpha R_1, \alpha R_2\rangle=\langle R_1, R_2\rangle\}$$

is a simply connected compact simple Lie group of type E_8, in [9] that the group

$$E_{8,\iota_1}=\{\alpha\in \mathrm{Iso}_C(\mathfrak{e}_8^C, \mathfrak{e}_8^C)\,|\,\alpha[R_1, R_2]=[\alpha R_1, \alpha R_2],\ \langle\alpha R_1, \alpha R_2\rangle_{\iota_1}=\langle R_1, R_2\rangle_{\iota_1}\}$$

is a connected non-compact simple Lie group of type $E_{8(-24)}$ and its polar decomposition is given by

$$E_{8,\iota_1}\simeq(SU(2)\times E_7)/\boldsymbol{Z}_2\times \boldsymbol{R}^{112}.$$

In the present paper, we show that the group

$$E_8'=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{e}_8', \mathfrak{e}_8')\,|\,\alpha[R_1, R_2]=[\alpha R_1, \alpha R_2]\}$$

(where $\mathfrak{e}_8'$ is a simple Lie algebra of type $E_{8(8)}$) is a connected non-compact simple Lie group of type $E_{8(8)}$ and its polar decomposition is given by

$$E_8'\simeq Ss(16)\times \boldsymbol{R}^{128}.$$

1. Preliminaries.

1.1. Notations.

Throughout this paper, we use the following notations. $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$: the fields of real, complex and quaternionic numbers, respectively. $M(n, K)$, $K=\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$: all of $n\times n$ matrices with entries in K. E: the $n\times n$ unit matrix (n is arbitrary).

$$J=\begin{pmatrix} J' & & \\ & \ddots & \\ & & J'\end{pmatrix}\in M(8, \boldsymbol{C}) \text{ or } \in M(16, \boldsymbol{R}) \text{ where } J'=\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$$

$$L=\begin{pmatrix} L' & & \\ & \ddots & \\ & & L'\end{pmatrix}\in M(16, R) \text{ where } L'=\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}.$$

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$\mathfrak{J}(4, \boldsymbol{H})=\{H \in M(4, \boldsymbol{H}) \mid H^*=H\}$. $\mathfrak{J}(4, \boldsymbol{H})_0=\{H \in \mathfrak{J}(4, \boldsymbol{H}) \mid \operatorname{tr}(H)=0\}$.

$\mathfrak{S}(8, \boldsymbol{C})=\{S \in M(8, \boldsymbol{C}) \mid {}^tS=-S\}$.

$SO(16)=\{B \in M(16, \boldsymbol{R}) \mid {}^tBB=E, \det B=1\}$: a special orthogonal group.

$Ss(16)=Spin(16)/\boldsymbol{Z}_2$(not $SO(16)$): a semispnor group.

$SU(8)=\{D \in M(8, \boldsymbol{C}) \mid D^*D=E, \det D=1\}$: a special unitary group.

$Sp(4)=\{C \in M(4, \boldsymbol{H}) \mid C^*C=E\}$: a symplectic group.

$\mathfrak{so}(16)=\{B \in M(16, \boldsymbol{R}) \mid {}^tB=-B\}$.

$\mathfrak{u}(8)=\{D \in M(8, \boldsymbol{C}) \mid D^*=-D\}$, $\mathfrak{su}(8)=\{D \in \mathfrak{u}(8) \mid \operatorname{tr}(D)=0\}$.

$\mathfrak{sp}(4)=\{C \in M(8, \boldsymbol{H}) \mid C^*=-C\}$.

The identity mapping of a set is always denoted by 1.

1.2. The split Cayley algebra $\mathfrak{C}'$, the split exceptional Jordan algebra $\mathfrak{J}'$ and the Freudenthal vector space $\mathfrak{P}'$.

Let $\mathfrak{C}'=\boldsymbol{H} \oplus \boldsymbol{H}e'$ denote the split Cayley algebra with the multiplication

$$(a+be')(c+de')=(ac+\bar{d}b)+(b\bar{c}+da)e'.$$

and the conjugation $\overline{a+be'}=\bar{a}-be'$. Let γ: $\mathfrak{C}' \to \mathfrak{C}'$ be the involutive automorphism defined by

$$\gamma(a+be')=a-be'.$$

Let $\mathfrak{J}'=\{X \in M(3, \mathfrak{C}') \mid X^*=X\}$ denote the split exceptional Jordan algebra with the multiplication $X \circ Y=\frac{1}{2}(XY+YX)$. The above involution γ: $\mathfrak{C}' \to \mathfrak{C}'$ is naturally extended to the involutive automorphism γ: $\mathfrak{J}' \to \mathfrak{J}'$. In $\mathfrak{P}'$, the inner product $(X, Y)'$, the positive definite inner product (X, Y), the Freudenthal multiplication $X \times Y$, the trilinear form $(X, Y, Z)'$ and the determinant $\det X$ are defined respectively by

$$(X, Y)'=\operatorname{tr}(X \circ Y), \quad (X, Y)=(\gamma X, Y)',$$

$$X \times Y=\frac{1}{2}(2X \circ Y-\operatorname{tr}(X)Y-\operatorname{tr}(Y)X+(\operatorname{tr}(X)\operatorname{tr}(Y)-(X, Y)')E),$$

$$(X, Y, Z)'=(X, Y \times Z)', \quad \det X=\frac{1}{3}(X, X, X)'.$$

Finally consider the vector space $\mathfrak{P}'=\mathfrak{J}' \oplus \mathfrak{J}' \oplus \boldsymbol{R} \oplus \boldsymbol{R}$ called the Freudenthal vector space. We define linear transformations γ, ι and v of $\mathfrak{P}'$ respectively (γ is used the same notation as above) by

$$\gamma(X, Y, \xi, \eta)=(\gamma X, \gamma Y, \xi, \eta), \quad \iota(X, Y, \xi, \eta)=(Y, -X, \eta, -\xi), \quad v=\gamma\iota=\iota\gamma.$$

In $\mathfrak{P}'$, the inner product $(P, Q)'$, the positive definite inner product (P, Q) and the skew-symmetric inner product $\{P, Q\}'$ are defined respectively by

$$(P, Q)'=(X, Z)'+(Y, W)'+\xi\zeta+\eta\omega, \quad (P, Q)=(\gamma P, Q)',$$
$$\{P, Q\}'=(X, W)'-(Z, Y)'+\xi\omega-\zeta\eta=(P, \iota Q)'=-(\iota P, Q)'$$

where $P=(X, Y, \xi, \eta)$, $Q=(Z, W, \zeta, \omega)\in\mathfrak{P}'$.

1.3. The Lie group $E_{6(6)}$ and the subgroup $Sp(4)/Z_2$

We have shown in [11] that the group

$$E_{6(6)}=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}')\mid \det \alpha X=\det X\}$$

is a connected non-compact simple Lie group of type $E_{6(6)}$, its Lie algebra is

$$\mathfrak{e}_6'=\mathfrak{e}_{6(6)}=\{\phi\in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}')\mid (\phi X, X, X)'=0\}$$

and found a subgroup of type C_4 in $E_{6(6)}$. To find this subgroup, we use a linear isomorphism $f\colon \mathfrak{J}'\to\mathfrak{J}(4, H)_0$,

$$f\begin{pmatrix} \xi_1 & a_3+b_3e' & \bar{a}_2-b_2e' \\ \bar{a}_3-b_3e' & \xi_2 & a_1+b_1e' \\ a_2+b_2e' & \bar{a}_1-b_1e' & \xi_3 \end{pmatrix}=\begin{pmatrix} \lambda_0 & b_1 & b_2 & b_3 \\ \bar{b}_1 & \lambda_1 & a_3 & \bar{a}_2 \\ \bar{b}_2 & \bar{a}_3 & \lambda_2 & a_1 \\ \bar{b}_3 & a_2 & \bar{a}_1 & \lambda_3 \end{pmatrix}$$

where $\xi_i\in\boldsymbol{R}$, a_i, $b_i\in\boldsymbol{H}$, $i=1, 2, 3$ and $\lambda_0=\frac{1}{2}(\xi_1+\xi_2+\xi_3)$, $\lambda_1=\frac{1}{2}(\xi_1-\xi_2-\xi_3)$, $\lambda_2=\frac{1}{2}(\xi_2-\xi_1-\xi_3)$, $\lambda_3=\frac{1}{2}(\xi_3-\xi_1-\xi_2)$. Then f satisfies

$$fX\circ fY=f\,(\gamma(X\times Y))+\frac{1}{4}(X, Y)E, \quad X, Y\in\mathfrak{J}'$$

where the multiplication $H_1\circ H_2$ is $\mathfrak{J}(4, \boldsymbol{H})$ is defined by $H_1\circ H_2=\frac{1}{2}(H_1H_2+H_2H_1)$. Now, a subgroup $(E_{6(6)})_K$ of the group $E_{6(6)}$,

$$(E_{6(6)})_K=\{\alpha\in E_{6(6)}\mid (\alpha X, \alpha Y)=(X, Y)\}$$

is isomorphic to the group $Sp(4)/\boldsymbol{Z}_2$ by the correspondence

$$\phi\colon Sp(4)\longrightarrow (E_{6(6)})_K,\ \phi(C)X=f^{-1}(C(fX)C^*),\ X\in\mathfrak{J}'$$

with $\mathrm{Ker}\,\phi=Z_2=\{E, -E\}$. Therefore the Lie algebra $(\mathfrak{e}_{6(6)})_K$ of the group $(E_{6(6)})_K$,

$$(\mathfrak{e}_6')_K=(\mathfrak{e}_{6(6)})_K=\{\phi\in\mathfrak{e}_6'\mid (\phi X, Y)=-(X, \phi Y)\}$$

is isomorphic to the Lie algebra $\mathfrak{sp}(4)$ by the correspondence

$$\phi_*\colon \mathfrak{sp}(4)\longrightarrow (\mathfrak{e}_6')_K,\ \phi_*(C)X=f^{-1}(C(fX)-(fX)C),\ X\in\mathfrak{J}'.$$

Finally, for $A, B\in\mathfrak{J}'$, $A\vee B\in\mathfrak{e}_6'$ is defined by

$$(A\vee B)X=\frac{1}{2}(B, X)'A+\frac{1}{6}(A, B)'X-2B\times(A\times X),\ X\in\mathfrak{J}'.$$

1.4. The Lie group $E_{7(7)}$ and the subgroup $SU(8)/Z_2$

For $\phi\in\mathfrak{e}_6'$, $A, B\in\mathfrak{J}'$, $\nu\in\boldsymbol{R}$, we define a linear transformation $\Phi(\phi, A, B, \nu)$ of $\mathfrak{P}'$ by

$$\Phi(\phi, A, B, \nu)\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{bmatrix} \phi X - \dfrac{\nu}{3} X + 2B \times Y + \eta A \\ 2A \times X + \phi' Y + \dfrac{\nu}{3} Y + \xi B \\ (A, Y)' + \nu \xi \\ (B, X)' - \nu \eta \end{bmatrix}$$

where $\phi' \in \mathfrak{e}_6'$ denotes the skew-transpose of ϕ with respect to the inner product $(X, Y)'$: $(\phi X, Y)' = -(X, \phi' Y)'$. For $P=(X, Y, \xi, \eta)$, $Q=(Z, W, \zeta, \omega) \in \mathfrak{P}'$, we define a linear transformation $P \times Q$ of $\mathfrak{P}'$ by

$$P \times Q = \Phi(\phi, A, B, \nu), \begin{cases} \phi = -\dfrac{1}{2}(X \vee W + Z \vee Y), \\ A = -\dfrac{1}{4}(2Y \times W - \xi Z - \zeta X), \\ B = \dfrac{1}{4}(2X \times Z - \eta W - \omega Y), \\ \nu = \dfrac{1}{8}((X, W)' + (Z, Y)' - 3(\xi\omega + \zeta\eta)). \end{cases}$$

We have shown in [18] that the group

$$E_{7(7)} = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P}', \mathfrak{P}') \,|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\}$$

is a connected non-compact simple Lie group of type $E_{7(7)}$, its Lie algebra is

$$\mathfrak{e}_7' = \mathfrak{e}_{7(7)} = \{\Phi(\phi, A, B, \nu) \,|\, \phi \in \mathfrak{e}_6', A, B \in \mathfrak{J}', \nu \in \boldsymbol{R}\}$$

and found a subgroup of type A_7 in $E_{7(7)}$. To find this subgroup, we use a linear isomorphism $\chi: \mathfrak{P}' \to \mathfrak{S}(8, \boldsymbol{C})$,

$$\chi(X, Y, \xi, \eta) = \left(k\left(fX - \frac{\xi}{2}E\right) + \mathrm{i}\, k\left(f(\gamma Y) - \frac{\eta}{2}E\right)\right) J$$

where $k: M(4, \boldsymbol{H}) \to M(8, \boldsymbol{C})$ is the algebraic homomorphism defined by $k(a_{ij} + \mathrm{j} b_{ij}) = \begin{pmatrix} a_{ij} & -\bar{b}_{ij} \\ b_{ij} & \bar{a}_{ij} \end{pmatrix}$, $a_{ij}, b_{ij} \in \boldsymbol{C}$ ($\mathrm{i} \in \boldsymbol{C}$, $\mathrm{j} \in \boldsymbol{H}$ are the usual elements: $\mathrm{i}^2 = -1$, $\mathrm{j}^2 = -1$). Now, a subgroup $(E_{7(7)})_K$ of the group $E_{7(7)}$,

$$(E_{7(7)})_K = \{\alpha \in E_{7(7)} \,|\, (\alpha P, \alpha Q) = (P, Q)\}$$

is isomorphic to the group $SU(8)/\boldsymbol{Z}_2$ by the correspondence

$$\psi: SU(8) \longrightarrow (E_{7(7)})_K, \quad \psi(D)P = \chi^{-1}(D(\chi P)^t D), \; P \in \mathfrak{P}'$$

with $\mathrm{Ker}\, \psi = \boldsymbol{Z}_2 = \{E, -E\}$. Therefore the Lie algebra $(\mathfrak{e}_{7(7)})_K$ of the group $(E_{7(7)})_K$,

$$\begin{aligned}(\mathfrak{e}_7')_K = (\mathfrak{e}_{7(7)})_K &= \{\Phi \in \mathfrak{e}_7' \,|\, (\Phi P, Q) = -(P, \Phi Q)\} \\ &= \{\Phi(\phi, A, -\gamma A, 0) \,|\, \phi \in (\mathfrak{e}_6')_K, A \in \mathfrak{J}'\}\end{aligned}$$

is isomorphic to the Lie algebra $\mathfrak{su}(8)$ by the correspondence

$$\psi_*: \mathfrak{su}(8) \longrightarrow (\mathfrak{e}_7')_K,\ \psi_*(D)P=\chi^{-1}(D(\chi(P)+(\chi P)^t D),\ P\in\mathfrak{P}'.$$

If $D\in\mathfrak{su}(8)$ has the form $D=\mathrm{k}(C)+\mathrm{i}k(fA)$, $C\in\mathfrak{sp}(4)$, $A\in\mathfrak{J}'$, then ψ_* is given by

$$\psi_*(k(C)+\mathrm{i}k(fA))=\Phi(\phi_*(C), A, -\gamma A, 0).$$

2. The Lie algebra $\mathfrak{e}_8'$ and the Lie group E_8'

An exceptional Lie algebra $\mathfrak{e}_8'$ is defined as follows. In a 248 dimensional vector space $\mathfrak{e}_8'$ over $\boldsymbol{R}$:

$$\mathfrak{e}_8'=\mathfrak{e}_7'\oplus\mathfrak{P}'\oplus\mathfrak{P}'\oplus\boldsymbol{R}\oplus\boldsymbol{R}\oplus\boldsymbol{R},$$

we define the Lie bracket $[R_1, R_2]$ by

$$[(\Phi_1, P_1, Q_1, r_1, s_1, t_1), (\Phi_2, P_2, Q_2, r_2, s_2, t_2)]=(\Phi, P, Q, r, s, t)$$

where

$$\begin{cases}\Phi=[\Phi_1, \Phi_2]+P_1\times Q_2-P_2\times Q_1,\\ P=\Phi_1P_2-\Phi_2P_1+r_1P_2-r_2P_1+s_1Q_2-s_2Q_1,\\ Q=\Phi_1Q_2-\Phi_2Q_1-r_1Q_2+r_2Q_1+t_1P_2-t_2P_1,\\ r=-\dfrac{1}{8}\{P_1, Q_2\}+\dfrac{1}{8}\{P_2, Q_1\}+s_1t_2-s_2t_1,\\ s=\dfrac{1}{4}\{P_1, P_2\}+2r_1s_2-2r_2s_1,\\ t=-\dfrac{1}{4}\{Q_1, Q_2\}-2r_1t_2+2r_2t_1.\end{cases}$$

Then $\mathfrak{e}_8'$ is a simple Lie algebra of type E_8 [3], [8]. For $R\in\mathfrak{e}_8'$, the adjoint transformation ad R: $\mathfrak{e}_8'\to\mathfrak{e}_8'$, (ad R) $R_1=[R, R_1]$, $R_1\in\mathfrak{e}_8'$, will be denoted by $\Theta(R)$.

The group E_8' is defined to be the automorphism group of the Lie algebra $\mathfrak{e}_8'$:

$$E_8'=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{e}_8', \mathfrak{e}_8')\,|\,\alpha[R_1, R_2]=[\alpha R_1, \alpha R_2]\}.$$

Our purpose of this paper is to find a maximal compact subgroup of E_8' explicitly and to show that the group E_8' is connected.

The group E_8' contains a subgroup

$$E_7'=\left\{\alpha\in E_8' \;\middle|\; \begin{array}{l}\alpha(0, 0, 0, 0, 1, 0)=(0, 0, 0, 0, 1, 0)\\ \alpha(0, 0, 0, 0, 0, 1)=(0, 0, 0, 0, 0, 1)\end{array}\right\}$$

which is isomorphic to the group $E_{7(7)}$ defined in the section 1.4 by the correspondence $\alpha\in E_{7(7)}\to\alpha'\in E_7'\subset E_8'$, where

$$\alpha'(\Phi, P, Q, r, s, t)=(\alpha\Phi\alpha^{-1}, \alpha P, \alpha Q, r, s, t).$$

From now on, we identify these groups $E_{7(7)}$ and E'_7 under the above correspondence: $\alpha=\alpha'$. Therefore, elements $\gamma, \iota, \upsilon\in E_{7(7)}$ are regarded as elements $\gamma, \iota, \upsilon\in E'_7\subset E'_8$.

We define linear transformations ω, $\tilde{\iota}$ and $\tilde{\upsilon}$ of $\mathfrak{e}'_8$ respectively by

$$\omega(\Phi, P, Q, r, s, t)=(\Phi, -Q, P, -r, -t, -s),$$
$$\tilde{\iota}=\omega\iota=\iota\omega \quad\text{and}\quad \tilde{\upsilon}=\omega\upsilon=\upsilon\omega.$$

PROPOSITION 1. (1) $\omega=\exp\left(\theta\left(0, 0, 0, 0, \dfrac{\pi}{2}, -\dfrac{\pi}{2}\right)\right)\in E'_8$, $\omega^2=1$.

(2) $\tilde{\iota}, \tilde{\upsilon}\in E'_8$, $\tilde{\upsilon}=\tilde{\iota}\gamma=\gamma\tilde{\iota}$, $\tilde{\iota}^2=\tilde{\upsilon}^2=1$.

3. Connectedness of E'_8

In this section, we shall show that the group E'_8 is connected. This proof is similar to [19] Theorem 30, however we need some remarks. So we give the outline of its proof.

For $R\in\mathfrak{e}'_8$, we define a linear transformation $R\times R$ of $\mathfrak{e}'_8$ by

$$(R\times R)R_1=\Theta(R)^2R_1+\frac{1}{30}B'_8(R, R_1)R, \quad R_1\in\mathfrak{e}'_8$$

where B'_8 is the Killing form of the Lie algebra $\mathfrak{e}'_8$, and define a subspace $\mathfrak{w}'$ of $\mathfrak{e}'_8$ by

$$\mathfrak{w}'=\{R\in\mathfrak{e}'_8\,|\,R\times R=0, R\neq 0\}.$$

Since the Killing form B'_8 is calculated in [8] Theorem 27 as

$$B'_8(R_1, R_2)=\frac{5}{3}B'_7(\Phi_1, \Phi_2)+15\{Q_1, P_2\}'-15\{P_1, Q_2\}'+120r_1r_2+60t_1s_2+60s_1t_2$$

(where B'_7 is the Killing form of the Lie algebra $\mathfrak{e}'_7$) for $R_i=(\Phi_i, P_i, Q_i, r_i, s_i, t_i)\in\mathfrak{e}'_8$, $i=1, 2$, we have the following

PROPOSITION 2. *For $R=(\Phi, P, Q, r, s, t)\in\mathfrak{e}'_8$, $R\neq 0$, R belongs to $\mathfrak{w}'$ if and only if R satisfies*

(1) $2s\Phi-P\times P=0$ (2) $2t\Phi+Q\times Q=0$ (3) $2r\Phi+P\times Q=0$

(4) $\Phi P-3rP-3sQ=0$ (5) $\Phi Q+3rQ-3tP=0$ (6) $\{P, Q\}'-16(st+r^2)=0$

(7) $2(\Phi P\times Q_1+2P\times\Phi Q_1-rP\times Q_1-sQ\times Q_1)-\{P, Q_1\}'\Phi=0$

(8) $2(\Phi Q\times P_1+2Q\times\Phi P_1+rQ\times P_1-tP\times P_1)-\{Q, P_1\}'\Phi=0$

(9) $8((P\times Q_1)Q-stQ_1-r^2Q_1-\Phi^2Q_1+2r\Phi Q_1)+5\{P, Q_1\}'Q-2\{Q, Q_1\}'P=0$

(10) $8((Q\times P_1)P+stP_1+r^2P_1+\Phi^2P_1+2r\Phi P_1)+5\{Q, P_1\}'P-2\{P, P_1\}'Q=0$

(11) $18((ad\Phi)^2\Phi_1+Q\times\Phi_1P-P\times\Phi_1Q)+B'_7(\Phi, \Phi_1)\Phi=0$

(12) $18(\Phi_1\Phi P-2\Phi\Phi_1P-r\Phi_1P-s\Phi_1Q)+B'_7(\Phi, \Phi_1)P=0$

(13) $18(\Phi_1\Phi Q-2\Phi\Phi_1Q+r\Phi_1Q-t\Phi_1P)+B'_7(\Phi, \Phi_1)Q=0$

for any $\Phi_1\in\mathfrak{e}_7'$, P_1, $Q_1\in\mathfrak{P}'$.

For a while, we use the following notations briefly.

$$\begin{aligned}
&\dot{1}=(0,0,1,0)\in\mathfrak{P}', &&\underline{1}=(0,0,0,1)\in\mathfrak{P}',\\
&\bar{s}=(0,0,0,0,s,0)\in\mathfrak{e}_8', &&\underline{t}=(0,0,0,0,0,t)\in\mathfrak{e}_8',\\
&\bar{P}=(0,P,0,0,0,0)\in\mathfrak{e}_8', &&\underline{Q}=(0,0,Q,0,0,0)\in\mathfrak{e}_8',\\
&\underline{\mathfrak{P}}'=\{\underline{P}\in\mathfrak{e}_8'\,|\,P\in\mathfrak{P}'\}, &&\underline{\boldsymbol{R}}=\{\underline{t}\in\mathfrak{e}_8'\,|\,t\in\mathbf{R}\}.
\end{aligned}$$

THEOREM 3. *The group E_8' acts transitively on $\mathfrak{w}'$ (which is connected) and the isotropy subgroup $(E_8')_{\underline{1}}$ of E_8' at $\underline{1}\in\mathfrak{w}'$ is* $(\exp(\underline{\mathfrak{P}}')\exp(\underline{\boldsymbol{R}}))E_7'$. *Therefore we have the homeomorphism*

$$E_8'/(\exp(\underline{\mathfrak{P}}')\exp(\underline{\boldsymbol{R}}))E_7'\simeq\mathfrak{w}'.$$

In particular, the group E_8' is connected.

PROOF. Obviously the group E_8' acts on $\mathfrak{w}'$. Since $\underline{1}\in\mathfrak{w}'$, in order to prove the transitivity of E_8', it suffices to show that any element $R\in\mathfrak{w}'$ can be transformed to $\underline{1}$ by a certain element $\alpha\in(E_8')_0$ (which denotes the identity component of E_8').

Case (1) $R=(\Phi, P, Q, r, s, t)$, $t>0$. In this case, from (2), (5), (6) of Proposition 2, we have

$$\Phi=-\frac{1}{2t}Q\times Q,\quad P=\frac{r}{t}Q-\frac{1}{6t^2}(Q\times Q)Q,\quad s=-\frac{r^2}{t}+\frac{1}{96t^3}\{Q,(Q\times Q)Q\}.$$

Put

$$P_1=\frac{\log t}{2(\sqrt{t}-t)}Q,\quad s_1=\frac{r\log t}{t-1},\quad r_1=-\frac{\log t}{2}.$$

Then, for $\Theta=\Theta(0,P_1,0,r_1,s_1,0)\in\mathrm{ad}\,\mathfrak{e}_8'$, we have $(\exp\Theta)\underline{1}=R$ (about its calculation, see [19] Theorem 30). So R is transformed to $\underline{1}$ by $\exp(-\Theta)\in(E_8')_0$.

Case (1′) $\Theta=(\Phi, P, Q, r, s, t)$, $t<0$. Similarly as the case (1), we see that R can be transformed to $-\underline{1}$ by $(E_8')_0$. Furthermore $-\underline{1}$ can be transformed to $\underline{1}$ by $(E_8')_0$. In fact, for $\Theta=\Theta(0,0,1,0,0,0)\in\mathrm{ad}\,\mathfrak{e}_8'$,

$$(\exp\Theta)(0,0,\dot{1},0,0,0)=\left(0,0,\dot{1},0,0,\frac{1}{4}\right)\underset{(1)}{\longrightarrow}\underline{1},$$

$$(\exp(-\Theta))(0,0,\dot{1},0,0,0)=\left(0,0,\dot{1},0,0,-\frac{1}{4}\right)\underset{(1')}{\longrightarrow}-\underline{1}.$$

This shows that $-\underline{1}$ can be transformed to $\underline{1}$ by $(E_8')_0$.

Case (2) $R=(\Phi, P, Q, r, s, t)$, $s>0$. Similarly as the case (1), R can be transformed to $\bar{1}$ and $\bar{1}$ is transformed to $-\underline{1}$ by $\omega=\exp(\Theta(0,0,0,0,\frac{\pi}{2},-\frac{\pi}{2})\in(E_8')_0$: $\omega\bar{1}=-\underline{1}$. Furthermore, $-\underline{1}$ is transformed to $\underline{1}$ as was seen in the case (1′).

Case (2′) $R=(\Phi, P, Q, r, s, t)$, $s<0$. Similarly as the case (2), R can be transformed to

$-\bar{1}$ and then to $\underline{1}$ by ω.

Case (3) $R=(\Phi, P, Q, r, 0, 0)$, $r\neq 0$. In this case, from (2), (5), (6) of Proposition 2, we have $Q\times Q=0$, $\phi Q=-3rQ$, $\{P, Q\}'=16r^2$. Now, for $\Theta=\Theta(0, Q, 0, 0, 0, 0)\in \mathrm{ad}\,\mathfrak{e}_8'$, we have

$$(\exp\Theta)R=(\Phi, P+2rQ, Q, r, -4r^2, 0),\ -4r^2\neq 0.$$

So we can reduce to the case (2′).

Case (4) $R=(\Phi, P, Q, 0, 0, 0)$, $Q\neq 0$. Choose $P_1\in\mathfrak{P}'$ such that $\{P_1, Q\}'\neq 0$. Then for $\Theta=\Theta(0, P_1, 0, 0, 0, 0)\in\mathrm{ad}\,\mathfrak{e}_8'$, we have

$$(\exp\Theta)R=\left(*, *, *, -\frac{1}{8}\{P_1, Q\}', *, *\right).$$

So we can reduce to the case (3).

Case (5) $R=(\Phi, P, Q, 0, 0, 0)$, $P\neq 0$. This is similar to the case (4).

Case (6) $R=(\Phi, 0, 0, 0, 0, 0)$, $\Phi\neq 0$. In this case, from (10) of Proposition 2, we have $\Phi^2=0$. Now, choose $P_1\in\mathfrak{P}'$ such that $\Phi P_1\neq 0$. Then for $\Theta=\Theta(0, P_1, 0, 0, 0, 0)\in\mathrm{ad}\,\mathfrak{e}_8'$, we have

$$(\exp\Theta)=\left(\Phi, -\Phi P_1, 0, 0, -\frac{1}{8}\{\Phi P_1, P_1\}', 0\right).$$

So we can reduce to the case (5).

Thus the transitivity of $(E_8')_0$ on $\mathfrak{w}'$ is proved. Hence $\mathfrak{w}'=(E_8')_0\underline{1}$, so $\mathfrak{w}'$ is connected. Next, the isotropy subgroup $(E_8')_{\underline{1}}$ of E_8' at $\underline{1}$ is the semi-direct product of subgroups exp $(\underline{\mathfrak{P}}')$ exp $(\underline{\boldsymbol{R}})$ and E_7' of the group $(E_8')_{\underline{1}}$: $(E_8')_{\underline{1}}=(\exp(\underline{\mathfrak{P}}')\exp(\underline{\boldsymbol{R}}))E_7'$ (about its proof, see [9], [18]). Thus we have the homeomorphism $E_8'/(\exp(\underline{\mathfrak{P}}')\exp(\underline{\boldsymbol{R}}))E_7'\simeq\mathfrak{w}'$. The space $\mathfrak{w}'$ and the group $(\exp(\underline{\mathfrak{P}}')\exp(\underline{\boldsymbol{R}}))E_7'$ are connected, so the group E_8' is also connected.

4. The positive definite inner product (R_1, R_2) *in* $\mathfrak{e}_8'$

We define an inner product (R_1, R_2) in $\mathfrak{e}_8'$ by

$$(R_1, R_2)=-\frac{1}{15}B_8'(\tilde{\upsilon}R_1, R_2)$$

(the coefficient $\frac{1}{15}$ is not essential).

PROPOSITION 4. *The inner product* (R_1, R_2) *in* $\mathfrak{e}_8'$ *is positive definite.*

PROOF.

$$(R_1, R_2)=-\frac{1}{15}B_8'(\tilde{\upsilon}R_1, R_2)$$

$$=-\frac{1}{15}B_8'(\tilde{\upsilon}(\Phi_1, P_1, Q_1, r_1, s_1, t_1), (\Phi_2, P_2, Q_2, r_2, s_2, t_2))$$

$$=-\frac{1}{15}B_8'((\upsilon\Phi_1\upsilon^{-1}, -\upsilon Q_1, \upsilon P_1, -r_1, -s_1, -t_1), (\Phi_2, P_2, Q_2, r_2, s_2, t_2))$$

$$=-\frac{1}{15}\Big(\frac{5}{3}B_7'(\upsilon\Phi_1\upsilon^{-1}, \Phi_2)+15\{\upsilon P_1, P_2\}'-15\{-\upsilon Q_1, Q_2\}'$$

$$+120(-r_1)r_2+60(-s_1)s_2+60(-t_1)t_2\Big)$$

$$=\frac{1}{9}B_7'(\upsilon\Phi_1\upsilon, \Phi_2)+(P_1, P_2)+(Q_1, Q_2)+8r_1r_2+4s_1s_2+4t_1t_2.$$

So it is sufficient to show that B_7' $(\upsilon\Phi_1\upsilon, \Phi_2)$ is positive definite. From [8] Theorem 27, this is calculated as follows.

$$\begin{aligned}B_7'(\upsilon\Phi_1\upsilon, \Phi_2)&=B_7'(\upsilon\Phi(\phi_1, A_1, B_1, \nu_1)\upsilon, \Phi(\phi_2, A_2, B_2, \nu_2))\\&=B_7'(\Phi(-\gamma\phi_1'\gamma, \gamma B_1, \gamma A_1, \nu_1), \Phi(\phi_2, A_2, B_2, \nu_2))\\&=\frac{3}{2}B_6'(-\gamma\phi_1'\gamma, \phi_2)+36(\gamma B_1, B_2)'+36(\gamma A_1, A_2)'+24\nu_1\nu_2\\&=-\frac{3}{2}B_6'(\gamma\phi_1'\gamma, \phi_2)+36(B_1, B_2)+36(A_1, A_2)+24\nu_1\nu_2\end{aligned}$$

where B_6' is the Killing form of the Lie algebra $\mathfrak{e}_6'$. So it is sufficient to show that B_6' $(\gamma\phi_1'\gamma, \phi_2)$ is positive definite. From [8] Theorem 27, we have

$$\begin{aligned}B_6'(\gamma\phi_1'\gamma, \phi_2)&=B_6'(\gamma(\delta_1+\tilde{T}_1)'\gamma, \delta_2+\tilde{T}_2)=B_6'(\gamma(\delta_1-\tilde{T}_1)\gamma, \delta_2+\tilde{T}_2)\\&=B_6'(\gamma\delta_1\gamma-\gamma\tilde{T}_1, \delta_2+\tilde{T}_2)=\frac{4}{3}B_4'(\gamma\delta_1\gamma, \delta_2)-12(\gamma T_1, T_2)'\\&=\frac{4}{3}B_4'(\gamma\delta_1\gamma, \phi_2)-12(T_1, T_2)\end{aligned}$$

where B_4' is the Killing form of the Lie algebra $\mathfrak{f}_4'=\{\delta\in\mathfrak{e}_6'\,|\,\delta E=0\}$, $\delta_i\in\mathfrak{f}_4'$, $T_i\in\mathfrak{J}_0'=\{T\in\mathfrak{J}'\,|\,\mathrm{tr}\,(T)=0\}$ and $\tilde{T}\in\mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}')$ is defined by $\tilde{T}X=T\circ X$. So it is sufficient to show that B_4' $(\gamma\delta_1\gamma, \delta_2)$ is negative definite. Since the Lie algebra $\mathfrak{f}_4'$ is simple, $\mathfrak{f}_4'$ is generated by $\mathfrak{J}'$: $\mathfrak{f}_4'=\{\Sigma_i[A_i, B_i]\,|\,A_i, B_i\in\mathfrak{J}'\}$. We define an inner product $(\delta_1, \delta_2)_4$ in $\mathfrak{f}_4'$ by

$$(\delta, \sum_i[A_i, B_i])_4=\sum_i(\gamma\delta\gamma B_i, A_i)', \quad \delta\in\mathfrak{f}_4', A_i, B_i\in\mathfrak{J}'.$$

Then this inner product is well-defined ([8] Proposition 2) and positive definite. In fact, under the notations

$$E_i=\begin{pmatrix} \delta_{i1} & 0 & 0 \\ 0 & \delta_{i2} & 0 \\ 0 & 0 & \delta_{i3} \end{pmatrix},\quad F_i(x)=\begin{pmatrix} 0 & \delta_{i3}x & \delta_{i2}\bar{x} \\ \delta_{i3}\bar{x} & 0 & \delta_{i1}x \\ \delta_{i2}x & \delta_{i1}\bar{x} & 0 \end{pmatrix}\in\mathfrak{J}',\ x\in\mathfrak{C}',\ i=1,2,3$$

(δ_{ij} is the Kronecker's delta), we can easily verify that

$$\sqrt{2}[\tilde{E}_1, \tilde{F}_2(e_i)],\ \sqrt{2}[\tilde{E}_1, \tilde{F}_3(e_i)],\ \sqrt{2}[\tilde{E}_3, \tilde{F}_1(e_i)],\ i=0, 1, 2, \cdots, 7$$

$$\frac{1}{\sqrt{2}}[\tilde{F}_1(e_i), \tilde{F}_1(e_j)],\ 0\leqq i<j\leqq 7$$

(where $\{e_0=1, e_1=\mathrm{i}, e_2=\mathrm{j}, e_3=\mathrm{k}, e_4=e', e_5=\mathrm{i}e', e_6=\mathrm{j}e', e_7=\mathrm{k}e'\}$ ($\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\}$ is the canonical basis of $\boldsymbol{H}$)) is an orthonormal basis with respect to the inner product $(\delta_1, \delta_2)_4$. Hence $(\delta_1, \delta_2)_4$ is positive definite. Now, $B_4'(\gamma\delta_1\gamma, \delta_2)$ is again calculated in [8] Proposition 27 (cf. §8, §11) as

$$B_4'(\gamma\delta_1\gamma, \sum_i [A_i, B_i])=-9\sum_i(\gamma\delta_1\gamma B_i, A_i)'.$$

Therefore $B_4'(\gamma\delta_1\gamma, \delta_2)$ is negative definite. Thus we see that the inner product (R_1, R_2) is positive definite.

5. The subgroup $(E_8')_K$ of E_8' and its Lie algebra $(\mathfrak{e}_8')_K$

We define a subgroup $(E_8')_K$ of the group E_8' by

$$\begin{aligned}(E_8')_K&=\{\alpha\in E_8' \mid \tilde{\upsilon}\alpha=\alpha\tilde{\upsilon}\}\\ &=\{\alpha\in E_8' \mid (\alpha R_1, \alpha R_2)=(R_1, R_2)\}.\end{aligned}$$

Our present purpose is to show that the group $(E_8')_K$ is isomorphic to the semispinor group $Ss(16)$. First, we shall show that the Lie algebra $(\mathfrak{e}_8')_K$ of the group $(E_8')_K$,

$$\begin{aligned}(\mathfrak{e}_8')_K&=\{\Theta\in\mathrm{ad}\,\mathfrak{e}_8' \mid \tilde{\upsilon}\Theta=\Theta\tilde{\upsilon}\}\\ &=\{\Theta\in\mathrm{ad}\,\mathfrak{e}_8' \mid (\Theta R_1, R_2)=-(R_1, \Theta R_2)\}\\ &=\{\Theta(\Phi, \upsilon P, P, 0, s, -s)\in\mathrm{ad}\,\mathfrak{e}_8' \mid \Phi\in\mathfrak{e}_7', \upsilon\Phi=\Phi\upsilon, P\in\mathfrak{P}'\}\end{aligned}$$

is isomorphic to the Lie algebra $\mathfrak{so}(16)$. For this purpose, we give some preliminaries (cf. §1).

LEMMA 5. *For $X_1, X_2\in\mathfrak{J}'$, we have*

$$\phi_*[fX_1, fX_2]=2(X_1\vee\gamma X_2-X_2\vee\gamma X_1).$$

PROOF. First note that $[fX_1, fX_2]\in\mathfrak{sp}(4)$. Now, for $X\in\mathfrak{J}'$,

$$f(2(X_1\vee\gamma X_2)X)$$

$$=f\left((\gamma X_2, X)'X_1+\frac{1}{3}(X_1, \gamma X_2)'X-4\gamma X_2\times(X_1\times X)\right)$$

$$=(X_2, X)fX_1+\frac{1}{3}(X_1, X_2)fX-4fX_2\circ f(\gamma(X_1\times X))+(X_2, \gamma(X_1\times X))E$$

$$=(X_2, X)fX_1+\frac{1}{3}(X_1, X_2)fX-4fX_2\circ(fX_1\circ fX)+(X_1, X)fX_2+(X_2, X_1, X)'E.$$

Hence we have

$$\begin{aligned}f(2(X_1\vee\gamma X_2-X_2\vee\gamma X_1)X)&=-4fX_2\circ(fX_1\circ fX)+4fX_1\circ(fX_2\circ fX)\\&=[fX_1, fX_2]fX-fX[fX_1, fX_2]=f((\phi_*[fX_1, fX_2])X).\end{aligned}$$

Since f is a momomorphism, we have the required formula.

PROPOSITION 6. *For $S, S_1, S_2\in\mathfrak{S}(8, \boldsymbol{C})$, we have*

(1) $\upsilon\chi^{-1}S=-\chi^{-1}\mathrm{i}S$.

(2) $\operatorname{tr}(S_1\bar{S}_2-S_2\bar{S}_1)=4\mathrm{i}\{\chi^{-1}S_1, \chi^{-1}S_2\}'$.

(3) $\psi_*\left(S_1\bar{S}_2-S_2\bar{S}_1-\frac{1}{8}\operatorname{tr}(S_1\bar{S}_2-S_2\bar{S}_1)E\right)=4(\upsilon\chi^{-1}S_1\times\chi^{-1}S_2-\upsilon\chi^{-1}S_2\times\chi^{-1}S_1)$.

PROOF. (1) Put $\chi^{-1}S=P=(X, Y, \xi, \eta)$. Then we have

$$\begin{aligned}\chi\upsilon\chi^{-1}S&=\chi\upsilon P=\chi\upsilon(X, Y, \xi, \eta)=\chi(\gamma Y, -\gamma X, \eta, -\xi)\\&=\left(k\left(f(\gamma Y)-\frac{\eta}{2}E\right)+\mathrm{i}k\left(f(\gamma(-\gamma X))-\frac{-\xi}{2}E\right)\right)J\\&=-\mathrm{i}\left(k\left(fX-\frac{\xi}{2}E\right)+\mathrm{i}k\left(f(\gamma Y)-\frac{\eta}{2}E\right)\right)J\\&=-\mathrm{i}\chi(X, Y, \xi, \eta)=-\mathrm{i}\chi P=-\mathrm{i}S.\end{aligned}$$

(2), (3) Put $\chi^{-1}S_i=P_i=(X_i, Y_i, \xi_i, \eta_i)$, $i=1, 2$. First note that $S_1\bar{S}_2-S_2\bar{S}_1-\frac{1}{8}\operatorname{tr}(S_1\bar{S}_2-S_2\bar{S}_1)E\in\mathfrak{su}(8)$. Now,

$$\begin{aligned}S_1\bar{S}_2&=\chi P_1\overline{\chi P_2}=\chi(X_1, Y_1, \xi_1, \eta_1)\overline{\chi(X_2, Y_2, \xi_2, \eta_2)}\\&=\left(k\left(fX_1-\frac{\xi_1}{2}E\right)+\mathrm{i}k\left(f(\gamma Y_1)-\frac{\eta_1}{2}E\right)\right)J\,\overline{\left(k\left(fX_2-\frac{\xi_2}{2}E\right)+\mathrm{i}k\left(f(\gamma Y_2)-\frac{\eta_2}{2}E\right)\right)J}\\&=\left(k\left(fX_1-\frac{\xi_1}{2}E\right)+\mathrm{i}k\left(f(\gamma Y_1)-\frac{\eta_1}{2}E\right)\right)\left(k\left(fX_2-\frac{\xi_2}{2}E\right)-\mathrm{i}k\left(f(\gamma Y_2)-\frac{\eta_2}{2}E\right)\right)J^2\\&=-k\left(\left(fX_1-\frac{\xi_1}{2}E\right)\left(fX_2-\frac{\xi_2}{2}E\right)+\left(f(\gamma Y_1)-\frac{\eta_1}{2}E\right)\left(f(\gamma Y_2)-\frac{\eta_2}{2}E\right)\right)\\&\quad-ik\left(\left(f(\gamma Y_1)-\frac{\eta_1}{2}E\right)\left(fX_2-\frac{\xi_2}{2}E\right)-\left(fX_1-\frac{\xi_1}{2}E\right)\left(f(\gamma Y_2)-\frac{\eta_2}{2}E\right)\right).\end{aligned}$$

Hence we have

$$\begin{aligned}S_1\bar{S}_2-S_2\bar{S}_1 &= -k(fX_1fX_2-fX_2fX_1+f(\gamma Y_1)f(\gamma Y_2)-f(\gamma Y_2)f(\gamma Y_1))\\ &\quad -\mathrm{i}k(f(\gamma Y_1)fX_2-f(\gamma Y_2)fX_1-fX_1f(\gamma Y_2)+fX_2f(\gamma Y_1)-\eta_1 fX_2+\eta_2 fX_1\\ &\quad +\xi_1 f(\gamma Y_2)-\xi_2 f(\gamma Y_1))+\frac{\mathrm{i}}{2}(\xi_1\eta_2-\xi_2\eta_1)E\\ &= k(-[fX_1,fX_2]-[f(\gamma Y_1),f(\gamma Y_2)])\\ &\quad +\mathrm{i}k(f(2\gamma X_1\times Y_2-2\gamma X_2\times Y_1+\eta_1 X_2-\eta_2 X_1-\xi_1\gamma Y_2+\xi_2\gamma Y_1))\\ &\quad +\frac{\mathrm{i}}{2}((X_1,\ Y_2)'-(X_2,\ Y_1)'+\xi_1\eta_2-\xi_2\eta_1))E\\ &\underset{\text{put}}{=\!=} k(C)+\mathrm{i}k(fA)+\frac{\mathrm{i}}{2}\{P_1,P_2\}'E,\quad C\in\mathfrak{sp}(4),\ A\in\mathfrak{J}'.\end{aligned}$$

Take the traces of the above, then we have

$$\mathrm{tr}\,(S_1\bar{S}_2-S_2\bar{S}_1)=4\mathrm{i}\{P_1,P_2\}'=4\mathrm{i}\,\{\chi^{-1}S_1,\chi^{-1}S_2\}'.$$

Therefore we have also

$$S_1\bar{S}_2-S_2\bar{S}_1-\frac{1}{8}\,\mathrm{tr}\,(S_1\bar{S}_2-S_2\bar{S}_1)E=k(C)+\mathrm{i}k(fA).$$

On the other hand

$$\upsilon P_1\times P_2=\begin{pmatrix}\gamma Y_1\\ -\gamma X_1\\ \eta_1\\ -\xi_1\end{pmatrix}\times\begin{pmatrix}X_2\\ Y_2\\ \xi_2\\ \eta_2\end{pmatrix}=\Phi\begin{bmatrix}-\dfrac{1}{2}(\gamma Y_1\vee Y_2+X_2\vee(-\gamma X_1))\\ -\dfrac{1}{4}(2(-\gamma X_1)\times Y_2-\eta_1 X_2-\xi_2\gamma Y_1)\\ \dfrac{1}{4}(2\gamma Y_1\times X_2-(-\xi_1)Y_2-\eta_2(-\gamma X_1))\\ \dfrac{1}{8}((\gamma Y_1,\ Y_2)'+(X_2,\ -\gamma X_1)'-3(\eta_1\eta_2+\xi_2(-\xi_1))\end{bmatrix}$$

Thus we have

$$\upsilon P_1\times P_2-\upsilon P_2\times P_1=\Phi\begin{bmatrix}\dfrac{1}{2}(-X_1\vee\gamma X_2+X_2\vee\gamma X_1-\gamma Y_1\vee Y_2+\gamma Y_2\vee Y_1)\\ \dfrac{1}{4}(2\gamma X_1\times Y_2-2\gamma X_2\times Y_1+\eta_1 X_2-\eta_2 X_1+\xi_2\gamma Y_1-\xi_1\gamma Y_2)\\ \dfrac{1}{4}(2X_2\times\gamma Y_1-2X_1\times\gamma Y_2+\eta_2\gamma X_1-\eta_1\gamma X_2+\xi_1 Y_2-\xi_2 Y_1)\\ 0\end{bmatrix}$$

$$=\Phi\left(\frac{1}{4}\phi_*C, \frac{1}{4}A, -\frac{1}{4}\gamma A, 0\right) \quad \text{(Lemma 5)}$$

$$=\frac{1}{4}\psi_*(k(C)+ik(fA))=\frac{1}{4}\psi_*(S_1\bar{S}_2-S_2\bar{S}_1-\frac{1}{8}\operatorname{tr}(S_1\bar{S}_2-S_2\bar{S}_1)E).$$

We define an algebraic homomorphism l: $M(8, \boldsymbol{C})\to M(16, \boldsymbol{R})$ by

$$l(x_{kj}+iy_{kj})=\begin{pmatrix} x_{kj} & -y_{kj} \\ y_{kj} & x_{kj} \end{pmatrix}, \quad x_{kj}, y_{kj}\in\boldsymbol{R}.$$

Then we have

$${}^t l(X)=l(X^*), \quad Ll(X)=l(\bar{X})L, \quad Jl(X)=l(X)J \quad \text{for} \quad X\in M(8, \boldsymbol{C}).$$

PROPOSITION 7.

(1) $l(\mathfrak{u}(8))=\{B\in\mathfrak{so}(16)\,|\,BJ=JB\}$, $\quad l(\mathfrak{S}(8, C))L=\{B\in\mathfrak{so}(16)\,|\,BJ=-JB\}$.

(2) *Any element B of $(\mathfrak{e}_8')_K$ is represented by the form*

$$B=l(D')+l(S)L, \quad D'\in\mathfrak{u}(8), S\in\mathfrak{S}(8, C)$$
$$=l(D)+l(S)L+l(isE), \quad D\in\mathfrak{su}(8), S\in\mathfrak{S}(8, \boldsymbol{C}), s\in\boldsymbol{R}.$$

THEOREM 8. *The Lie algebra $(\mathfrak{e}_8')_K$ is isomorphic to the Lie algebra $\mathfrak{so}(16)$ by the correspondence ζ: $\mathfrak{so}(16)\to(\mathfrak{e}_8')_K$,*

$$\zeta(l(D)+l(S)L+l(isE))=(\psi_*D, 2\upsilon\chi^{-1}S, 2\chi^{-1}S, 0, 2s, -s)$$

where $D\in\mathfrak{so}(8)$, $S\in\mathfrak{S}(8, C)$, $s\in\boldsymbol{R}$.

PROOF. The mapping ζ is clearly bijective. We shall show that ζ is a homomorphism.

(1) $\zeta[l(D_1), l(D_2)]=\zeta l[D_1, D_2]=(\psi_*[D_1, D_2], 0, 0, 0, 0, 0)$
$=([\psi_*D_1, \psi_*D_2], 0, 0, 0, 0, 0)$
$=[(\psi_*D_1, 0, 0, 0, 0, 0), (\psi_*D_2, 0, 0, 0, 0, 0)]=[\zeta l(D_1), \zeta l(D_2)]$.

(2) $\zeta[l(D), l(S)L]=\zeta(l(DS-S\bar{D})L)=\zeta(l(DS+S{}^tD)L)$
$=(0, 2\upsilon\chi^{-1}(DS+S{}^tD), 2\chi^{-1}(DS+S{}^tD), 0, 0, 0)$. On the other hand,

$$[\zeta l(D), \zeta(l(S)L)]=[(\psi_*D, 0, 0, 0, 0, 0), (0, 2\upsilon\chi^{-1}S, 2\chi^{-1}S, 0, 0, 0)]$$
$$=(0, 2(\psi_*D)\upsilon\chi^{-1}S, 2(\psi_*D)\chi^{-1}S, 0, 0, 0).$$

Since $(\psi_*D)\upsilon=\upsilon(\psi_*D)$ and $(\psi_*D)\chi^{-1}S=\chi^{-1}(DS+S{}^tD)$, this is equal to the above.

(3) $\zeta[l(D), l(isE)]=\zeta l[D, isE]=\zeta 0=0=[(\psi_*D, 0, 0, 0, 0, 0), (0, 0, 0, 0, 2s, -2s)]$
$=[\zeta l(D), \zeta l(isE)]$.

(4) $\zeta[l(S_1)L, l(S_2)L]=\zeta l(S_1\bar{S}_2-S_2\bar{S}_1)$

$$=\zeta\left(l\left(S_1\bar{S}_2-S_2\bar{S}_1-\frac{1}{8}\operatorname{tr}(S_1\bar{S}_2-S_2\bar{S}_1)E\right)+l\left(\frac{1}{8}\operatorname{tr}(S_1\bar{S}_2-S_2\bar{S}_1)E\right)\right)$$

$$=\left(\psi_*\left(S_1\bar{S}_2-S_2\bar{S}_1-\frac{1}{8}\operatorname{tr}(S_1\bar{S}_2-S_2\bar{S}_1)E\right), 0, 0, 0,\right.$$
$$\left.-\frac{i}{4}\operatorname{tr}(S_1\bar{S}_2-S_2\bar{S}_1), \frac{i}{4}\operatorname{tr}(S_1\bar{S}_2-S_2\bar{S}_1)\right)$$
$$=(4(\upsilon\chi^{-1}S_1\times\chi^{-1}S_2-\upsilon\chi^{-1}S_2\times\chi^{-1}S_1), 0, 0, 0, \{\chi^{-1}S_1, \chi^{-1}S_2\}',$$
$$-\{\chi^{-1}S_1, \chi^{-1}S_2\}')$$

(Proposition 6 (2), (3)). On the other hand

$$[\zeta(l(S_1)L), \zeta(l(S_2)L)]$$
$$=[(0, 2\upsilon\chi^{-1}S_1, 2\chi^{-1}S_1, 0, 0, 0), (0, 2\upsilon\chi^{-1}S_2, 2\chi^{-1}S_2, 0, 0, 0)]$$
$$=\left(2\upsilon\chi^{-1}S_1\times 2\chi^{-1}S_2-2\upsilon\chi^{-1}S_2\times 2\chi^{-1}S_1, 0, 0, 0, \frac{1}{8}(-\{2\upsilon\chi^{-1}S_1, 2\chi^{-1}S_2\}'\right.$$
$$\left.+2\{\upsilon\chi^{-1}S_2, 2\chi^{-1}S_1\}'), \frac{1}{4}\{2\upsilon\chi^{-1}S_1, 2\upsilon\chi^{-1}S_2\}', -\frac{1}{4}\{2\chi^{-1}S_1, 2\chi^{-1}S_2\}'\right)$$
$$=\text{the above.}$$

(5) $\zeta[l(\mathrm{i}sE), l(S)L]=\zeta(2l(\mathrm{i}sS)L)$
$$=(0, 4\upsilon\chi^{-1}(\mathrm{i}sS), 4\chi^{-1}(\mathrm{i}sS), 0, 0, 0)$$
$$=(0, 4\chi^{-1}(sS), -4\upsilon\chi^{-1}(sS), 0, 0, 0) \quad \text{(Proposition 6(1))}$$
$$=[(0, 0, 0, 0, 2s, -2s), (0, 2\upsilon\chi^{-1}S, 2\chi^{-1}S, 0, 0, 0)]$$
$$=[\zeta l(\mathrm{i}sE), \zeta(l(S)L)].$$

(6) $\zeta[l(\mathrm{i}s_1E), l(\mathrm{i}s_2E)]=\zeta l[\mathrm{i}s_1E, \mathrm{i}s_2E]=\zeta 0=0$
$$=[(0, 0, 0, 0, 2s_1, -2s_1), (0, 0, 0, 0, 2s_2, -2s_2)]$$
$$=[\zeta l(\mathrm{i}s_1E), \zeta l(\mathrm{i}s_2E)].$$

6. The polar decomposition of E_8' and connectedness of $(E_8')_K$

To give the polar decomposition of the group E_8', we prepare

LEMMA 9. *The group E_8' is an algebraic subgroup of a general linear group $GL(248, \boldsymbol{R})=\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{e}_8', \mathfrak{e}_8')$ and satisfies the condition that $\alpha\in E_8'$ implies ${}^t\alpha\in E_8'$, where ${}^t\alpha$ is the transpose of α with respect to the inner product (R_1, R_2): $({}^t\alpha R_1, R_2)=(R_1, \alpha R_2)$.*

PROOF.

Since $({}^t\alpha R_1, R_2)=(R_1, \alpha R_2)=-\dfrac{1}{15}B_8'(\tilde{\upsilon}R_1, \alpha R_2)=-\dfrac{1}{15}B_8'(\alpha^{-1}\tilde{\upsilon}R_1, R_2)=(\tilde{\upsilon}\alpha^{-1}\tilde{\upsilon}R_1, R_2)$

for $\alpha\in E_8'$, we have

$${}^t\alpha=\tilde{\upsilon}\alpha^{-1}\tilde{\upsilon}\in E_8' \quad \text{(Proposition 1 (2)).}$$

It is obvious that the group E_8' is algebraic, because it is defined by the algebraic relation

$\alpha[R_1, R_2]=[\alpha R_1, \alpha R_2]$.

According to Chevalley's lemma ([1] Lemma 2, p.201), we have the homeomorphism

$$E_8' \simeq (E_8' \cap O(\mathfrak{e}_8')) \times \boldsymbol{R}^d = (E_8')_K \times \boldsymbol{R}^d$$

where $O(\mathfrak{e}_8') = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{e}_8', \mathfrak{e}_8') \mid (\alpha R_1, \alpha R_2) = (R_1, R_2)\}$ and d is calculated as $d = \dim E_8' - \dim \mathfrak{so}(16) = 248 - 120 = 128$. Thus we have the following

THEOREM 10. *The group E_8' is homeomorphic to the topological product of the group $(E_8')_K$ and the Euclidean space $\boldsymbol{R}^{128}$:*

$$E_8' \simeq (E_8')_K \times \boldsymbol{R}^{128}.$$

In particular, the group $(E_8')_K$ is connected.

Since $(E_8')_K$ is a connected compact simple Lie group of type D_8 (Theorem 8, 10), we see that the group $(E_8')_K$ is isomorphic to one of the following four groups

$$Spin(16), \quad Ss(16), \quad SO(16), \quad PSO(16).$$

7. Isomorphism $(E_8')_K \simeq Ss(16)$

In order to determine the group-type of the group $(E_8')_K$, consider

$$(\mathfrak{e}_8')_K = \{R \in \mathfrak{e}_8' \mid \tilde{\upsilon} R = R\}, \quad (\mathfrak{e}_8')_P = \{R \in \mathfrak{e}_8' \mid \tilde{\upsilon} R = -R\}.$$

Then we have

$$\mathfrak{e}_8' = (\mathfrak{e}_8')_K \oplus (\mathfrak{e}_8')_P$$

which is the Cartan decomposition of the Lie algebra $\mathfrak{e}_8'$ with respect to the involutive automorphism $\tilde{\upsilon}$, in particular we have $[(\mathfrak{e}_8')_K, (\mathfrak{e}_8')_P] \subset (\mathfrak{e}_8')_P$. The adjoint representation Ψ of $(\mathfrak{e}_8')_K$ to $(\mathfrak{e}_8')_P$:

$$\Psi(R)R_1 = [R, R_1], \quad R \in (\mathfrak{e}_8')_K, \ R_1 \in (\mathfrak{e}_8')_P$$

is irreducible ([4](8.5.1)). Moreover, since the Lie algebra $(\mathfrak{e}_8')_K$ is simple (Theorem 8), the complexification representation Ψ^C of Ψ to the complexification representation space $((\mathfrak{e}_8')_P)^C$ is also irreducible ([4](8.8.3)). Thus we have

LEMMA 11. *The representation of the group $(E_8')_K$ to $((\mathfrak{e}_8')_P)^C$ is irreducible.*

PROPOSITION 12. *The center $z((E_8')_K)$ of the group $(E_8')_K$ is $\{1, \tilde{\upsilon}\}$.*

PROOF. Obviously $\{1, \tilde{\upsilon}\} \subset z((E_8')_K)$. Conversely let $\alpha \in z((E_8')_K)$. The action of α to $((\mathfrak{e}_8')_P)^C$ is constant: $\alpha \mid ((\mathfrak{e}_8')_P)^C = \lambda 1$, $\lambda \in C$, from Lemma 11 and Schur's lemma. Since $\alpha \in (E_8')_K$ preserves the inner product (R_1, R_2)(which is naturally extended to the complexification Lie algebra $(\mathfrak{e}_8')^C$ of $\mathfrak{e}_8'$): $(\alpha R_1, \alpha R_2) = (R_1, R_2)$, $R_1, R_2 \in ((\mathfrak{e}_8')_P)^C$, we have $\lambda^2 = 1$. Next from the simplicity of the Lie algebra $(\mathfrak{e}_8')_K$, it is generated by $(\mathfrak{e}_8')_P$: $(\mathfrak{e}_8')_K = \{\Sigma_{i,j} [R_i, R_j] \mid R_i,$

$R_j \in (\mathfrak{e}_8')_P\}$. Therefore α is $\lambda^2 1 = 1$, i. e. the identity mapping on $(\mathfrak{e}_8')_K$. Consequently, if $\lambda = 1$ then $\alpha = 1$ and if $\lambda = -1$ then $\alpha = \tilde{\upsilon}$. Thus we have $z((E_8')_K) = \{1, \tilde{\upsilon}\}$.

From Proposition 12, we see that the group $(E_8')_K$ is not either $Spin(16)$ or $PSO(16)$. Therefore $(E_8')_K$ is isomorphic to one of $SO(16)$ and $Ss(16)$.

We shall show that 128 dimensional complex irreducible representation of the Lie algebra $\mathfrak{so}(16)$ are only two. In fact, the dimension of the complex irreducible representation corresponds to a dominant weight ω can be calculated by Weyl's demension formula (e.g. [4](7.5.9)) as follows:

ω_1	$2\omega_1$	ω_2	$2\omega_2$	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	$\cdots$
16	135	120	5304	560	1820	4368	8008	128	128	$\cdots$

where $\omega_1, \omega_2, \cdots \omega_8$ are the fundamental weights of $\mathfrak{so}(16)$. From this, we see that only ω_7 and ω_8 have 128 dimension among complex irreducible representations of $\mathfrak{so}(16)$. On the other hand, we know that the spinor group $Spin(16)$ has two 128 dimensional complex irreducible representations $\Delta_{16}^{+}, \Delta_{16}^{-}$, called the spinor representations, and these are both not representations of the group $SO(16)$(e.g. [20] Lemma 4.4.6). Now, $((\mathfrak{e}_8')_P)^C$ was a 128 dimensional complex irreducible representation space of the group $(E_8')_K$ (Lemma 11). The above arguments mean that $(E_8')_K$ is not isomorphic to $SO(16)$. That is, we have

THEOREM 13. *The group* $(E_8')_K = \{\alpha \in E_8' \mid (\alpha R_1, \alpha R_2) = (R_1, R_2)\}$ *is isomorphic to the semispinor group* $Ss(16)$: $(E_8')_K = Ss(16)$.

Thus, from Theorem 10, we have the following theorem which was our main purpose.

THEOREM 14. *The group* $E_8' = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{e}_8', \mathfrak{e}_8') \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}$ *is homeomorphic to the topological product of the semispinor group* $Ss(16)$ *and the 128 dimensional Euclidean space* $\boldsymbol{R}^{128}$:

$$E_8' \simeq Ss(16) \times \boldsymbol{R}^{128}$$

8. The subgroup $Ss(16)$ in the compact simple Lie group E_8

It is known that the simply connected compact simple Lie group E_8 has $Ss(16)$ as a subgroup of maximal rank [10]. Here we find out this subgroup $Ss(16)$ explicitly in the group E_8.

Let $\mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H}e$ denote the Cayley division algebra with the multiplication

$$(a+be)(c+de) = (ac - d\bar{b}) + (b\bar{c} + da)e$$

and $\mathfrak{C}^C = \{x + \sqrt{-1}y \mid x, y \in \mathfrak{C}\}$ its complexification. The split Cayly algebra $\mathfrak{C}'$ is naturally imbedded in $\mathfrak{C}^C$,

$$a + be' \in \mathfrak{C}' \longrightarrow a + \sqrt{-1}be \in \mathfrak{C}^C$$

and its complexification $\mathfrak{C}'+\sqrt{-1}\mathfrak{C}'$ is also $\mathfrak{C}^C$: $(\mathfrak{C}')^C=\mathfrak{C}^C$. The involutive automorphism γ: $\mathfrak{C}'\to\mathfrak{C}'$ is naturally extended to the complex linear involutive automorphism γ: $\mathfrak{C}^C\to\mathfrak{C}^C$. Let τ denote the complex conjugation on $\mathfrak{C}^C$ with respect to $\mathfrak{C}$: $\tau(a+\sqrt{-1}b)=a-\sqrt{-1}b)$, $a, b\in\mathfrak{C}$. Then $\mathfrak{C}'=\{x\in\mathfrak{C}^C\,|\,\tau x=\gamma x\}$ by the above inclusion. Similarly $\mathfrak{J}'$ and $\mathfrak{P}'$ are imbedded in $\mathfrak{J}'$ and $\mathfrak{P}^C$ as $\mathfrak{J}'=\{X\in\mathfrak{J}^C\,|\,\tau X=\gamma X\}$ and $\mathfrak{P}'=\{P\in\mathfrak{P}^C\,|\,\tau P=\gamma P\}$ respectively, where τ are the complex conjugations on $\mathfrak{J}^C$, $\mathfrak{P}^C$ with respect to $\mathfrak{J}$, $\mathfrak{P}$ respectively and finally γ is the complex extension of γ on $\mathfrak{J}^C$, $\mathfrak{P}^C$.

Let $\mathfrak{e}_8^C=\mathfrak{e}_7^C\oplus\mathfrak{P}^C\oplus\mathfrak{P}^C\oplus\boldsymbol{R}^C\oplus\boldsymbol{R}^C\oplus\boldsymbol{R}^C$ be complex Lie algebra of type E_8 constructed basing on $\mathfrak{C}^C$ ([3], [8]). The involution τ: $\mathfrak{e}_8^C\to\mathfrak{e}_8^C$ is defined by

$$\tau(\Phi, P, Q, r, s, t)=(\tau\Phi\tau, \tau P, \tau Q, \tau r, \tau s, \tau t).$$

Since $\mathfrak{e}_8^C$ is also the complexification of $\mathfrak{e}_8'$: $\mathfrak{e}_8^C=\mathfrak{e}_8'+\sqrt{-1}\,\mathfrak{e}_8'$, involutive automorphisms γ, $\tilde{\iota}$, $\tilde{\upsilon}$: $\mathfrak{e}_8'\to\mathfrak{e}_8'$ are naturally extended to involutive automorphisms γ, $\tilde{\iota}$, $\tilde{\upsilon}$: $\mathfrak{e}_8^C\to\mathfrak{e}_8^C$ respectively. Another involution ρ: $\mathfrak{e}_8^C\to\mathfrak{e}_8^C$ is defined by $\rho=\tau\gamma=\gamma\tau$:

$$\rho(\Phi, P, Q, r, s, t)=(\gamma\tau\Phi\tau\gamma, \gamma\tau P, \gamma\tau Q, \tau r, \tau s, \tau t).$$

Then $\mathfrak{e}_8'$ is naturally imbedded in $\mathfrak{e}_8^C$ as

$$\mathfrak{e}_8'=\{\theta\in\mathfrak{e}_8^C\,|\,\rho\theta=\theta\}=\{\theta\in\mathfrak{e}_8^C\,|\,\gamma\theta=\tau\theta\}.$$

In $\mathfrak{e}_8^C$, we define a positive definite Hermitian inner product $\langle R_1, R_2\rangle$ and an inner product $\langle R_1, R_2\rangle_{\tilde{\upsilon}}$ respectively by

$$\langle R_1, R_2\rangle=-\frac{1}{15}B_8(\tau\tilde{\iota}R_1, R_2),\quad \langle R_1, R_2\rangle_{\tilde{\upsilon}}=\langle\tilde{\upsilon}R_1, R_2\rangle=(\rho R_1, R_2)$$

where B_8 is the Killing form of the Lie algebra $\mathfrak{e}_8^C$. We have shown in [8] that the group

$$E_8^C=\{\alpha\in \mathrm{Iso}_C(\mathfrak{e}_8^C, \mathfrak{e}_8^C)\,|\,\alpha[R_1, R_2]=[\alpha R_1, \alpha R_2]\}$$

is a simply connected complex Lie group of type E_8 and the group

$$E_8=\{\alpha\in E_8^C\,|\,\langle\alpha R_1, \alpha R_2\rangle=\langle R_1, R_2\rangle\}$$

is a simply connected compact simple Lie group of type E_8. Now we define subgroups $(E_8^C)^\rho$, E_8^C, $\tilde{\upsilon}$ respectively by

$$(E_8^C)^\rho=\{\alpha\in E_8^C\,|\,\rho\alpha=\alpha\rho\},\ E_{8,\tilde{\upsilon}}^C=\{\alpha\in E_8^C\,|\,\langle\alpha R_1, \alpha R_2\rangle_{\tilde{\upsilon}}=\langle R_1, R_2\rangle_{\tilde{\upsilon}}\}.$$

PROPOSITION 15. *$(E_8^C)^\rho=E_{8,\tilde{\upsilon}}^C$ and it is isomorphic to the group E_8'.*

PRROF. $\langle R_1, R_2\rangle_{\tilde{\upsilon}}=\langle\tilde{\upsilon}R_1, R_2\rangle=-\frac{1}{15}B_8(\tau\tilde{\iota}\tilde{\upsilon}R_1, R_2)=-\frac{1}{15}B_8(\rho R_1, R_2)$, $R_1, R_2\in\mathfrak{e}_8^C$. Hence for $\alpha\in E_8^C$, $\langle\alpha R_1, \alpha R_2\rangle_{\tilde{\upsilon}}=\langle R_1, R_2\rangle_{\tilde{\upsilon}}$ holds if and only if $\rho\alpha=\alpha\rho$. Thus we have $(E_8^C)^\rho=E_{8,\tilde{\upsilon}}^C$. Next, since it is easy to see that the Lie algebra $(\mathfrak{e}_8^C)^\rho=\{R\in\mathfrak{e}_8^C\,|\,\rho R=R\}$ is isomorphic to the Lie algebra $\mathfrak{e}_8'$: $(\mathfrak{e}_8^C)^\rho=\mathfrak{e}_8'$ and $\mathfrak{e}_8^C$ is the complexification of $(\mathfrak{e}_8^C)^\rho$, the correspondence $\alpha\in E_8'\to\alpha^C\in(E_8^C)^\rho$(where α^C denotes the complexification of α) gives an isomorphism between E'_8 and $(E_8^C)^\rho$.

THEOREM 16. *The simply connected compact simple Lie group E_8 contains a subgroup $(E_8)^\rho=\{\alpha\in E_8\,|\,\rho\alpha=\alpha\rho\}$ which is isomorphic to the semispinor group $Ss(16)$.*

PROOF.

$$\begin{aligned}(E_8)^\rho&=\{\alpha\in E_8\,|\,\langle\alpha R_1,\alpha R_2\rangle=\langle R_1,R_2\rangle,\ \rho\alpha=\alpha\rho\}\\&=\{\alpha\in (E_8^C)^\rho\,|\,\langle\alpha R_1,\alpha R_2\rangle=\langle R_1,R_2\rangle\}\\&=\{\alpha\in E'_8\,|\,(\alpha R_1,\alpha R_2)=(R_1,R_2)\}\ \text{(Proposition 16)}\\&=(E'_8)_K\cong Ss(16)\text{(Theorem 13)}.\end{aligned}$$

9. Eratta and corrections of the preceding papers about exceptional Lie groups

We have been able to realize all of connected exceptional linear Lie groups and find their maximal compact subgroups explicitely:

$G_{2(-14)}$ [2], [20], $G_{2(2)}$ [15].
$F_{4(-52)}$ [2], [20], $F_{4(-20)}$ [2], [14], $F_{4(4)}$ [16].
$E_{6(-78)}$ [17], $E_{6(2)}$ [12], $E_{6(-14)}$ [12], $E_{6(-26)}$ [2], $E_{6(6)}$ [11].
$E_{7(-133)}$ [5], [7], $E_{7(-25)}$ [6], [7], $E_{7(-5)}$ [13], $E_{7(7)}$ [18].
$E_{8(-248)}$ [8], $E_{8(-24)}$ [9], $E_{8(8)}$ [this paper].

Here we point out some of their errata and correct them.

[5] p. 384, *l.* 12, [6] p. 10, *l.* 16. In front of "$\mathfrak{J}^C\oplus\mathfrak{J}^C\oplus C$" insert "$\mathfrak{e}_6^C\oplus$".
[5] p. 384, *l.* 13. Upon "$2X\times Z-\eta W-\xi Y$" insert "$X\vee W+Z\vee Y$".
[8] p. 761, *l.* 6. For "$-9(\delta A, B)$" read "$9(\delta A, B)$" or"$-9(\delta B, A)$".
[8] p. 761, *l.* 6. For "$-12(\phi A, B)$" read "$12(\phi A, B)$".
[9] p. 70, *l.* 1−6. Omit and replace with "$\mathfrak{T}=\{R\in\mathfrak{e}_{8,1}\,|\,R\times R=0,\ R\neq 0\}$ where $R\times R\in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{e}_{8,1},\mathfrak{e}_{8,1})$ is defined by $(R\times R)R_1=(\mathrm{ad}\,R^2R_1+\frac{1}{30}B_{8,1}(R,R_1)R$ for $R_1\in\mathfrak{e}_{8,1}$ ($B_{8,1}$ denotes the Killing form of the Lie algebra $\mathfrak{e}_{8,1}$)".
[9] p. 70, *l.* 10-12. Omit Proposition 10 and replace with "Proposition 10, $\mathfrak{T}$ is connected". Added in [9]. The group $E_{8,1}$ can be also defined by $E_{8,1}=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{e}_{8,1},\mathfrak{e}_{8,1})\,|\,\alpha[R_1,R_2]=[\alpha R_1,\alpha R_2]\}$ (which is connected (see [9] Theorem 16).
[17] p. 461, *l.* 2. For "$\boldsymbol{Z}_2=\{1,-1\}$" read "$\boldsymbol{Z}_2=\{1,\sigma\}$".
[18] p. 60, *l.* 2. Instead of χ use χ of Remark.

References

[1] Chevalley, C., Theory of Lie groups I, Princeton univ. Press, 1946.
[2] Freudenthal, H., Oktaven, Ausnahmegruppen und Oktavengeometrie, Math. Inst. Rijksuniv. te Utrecht, 1951.
[3] Freudenthal, H., Beziehungen der E_7 und E_8 zur Oktavenebene I, II, VIII, IX, X, XI, Nedel. Akad. Wetensch. Proc. Ser. A. **57**=Indag. Math., **16** (1954) 218–230, 363–368, Ser. A. **62**=**21** (1959) 447–465, 466-474, Ser. A. **66**=**25** (1963), 457–471, 472–487.
[4] Goto, M. and Grosshans, F. D., Semisimple Lie algebras, Dekker, 1978.

[5] Imai, T. and Yokota, I., Simply connected compact simple Lie group $E_{7(-133)}$ of type E_7, J. Math., Kyoto Univ., **21** (1981) 383–395.
[6] Imai, T. and Yokota, I., Non-compact simple Lie group $E_{7(-25)}$ of type E_7, J. Fac. Sci., Shinshu Univ., **15** (1980) 1–18.
[7] Imai, T. and Yokota, I., Another definitions of exceptional simple Lie groups of type $E_{7(-25)}$ and $E_{7(-133)}$, J. Fac. Sci., Shinshu Univ., **15** (1980) 47–52.
[8] Imai, T. and Yokota, I., Simply connected compact simple Lie group $E_{8(-248)}$ of type E_8, J. Math., Kyoto Univ., **21** (1981) 471–762.
[9] Imai, T. and Yokota, I., Non-compact simple Lie group $E_{8(-24)}$ of type E_8, J. Fac. Sci., Shinshu Univ., **15** (1980) 53–76.
[10] Ishitoya, K. and Toda, H., On the cohomology of irreducible symmetric spaces of exceptional type, J. Math., Kyoto Univ., **17** (1977) 225-243.
[11] Shukuzawa, O. and Yokota, I., Non-compact simple Lie group $E_{6(6)}$ of type E_6, J. Fac. Sci., Shinshu Univ., **14** (1979) 1–13.
[12] Shukuzawa, O. and Yokota, I., Non-compact simple Lie groups $E_{6(-14)}$ and $E_{6(2)}$ of type E_6, J. Fac. Sci., Shinshu Univ., **14** (1979) 15–28.
[13] Yasukura, O. and Yokota, I., Subgroup $(SU(2)\times Spin(12))/\boldsymbol{Z}_2$ of compact simple Lie group E_7 and non-compact simple Lie group $E_{7,\sigma}$ of type $E_{7(-5)}$, Hiroshima Math. J. **12** (1981) 59–76.
[14] Yokota, I., On a non compact simple Lie group $F_{4,1}$ of type F_4, J. Fac. Sci., Shinshu Univ., **10** (1975) 71–80.
[15] Yokota, I., Non-compact simple Lie group $G_2$of type G_2, J. Fac. Sci., Shinshu Univ., **12** (1977) 45–52.
[16] Yokota, I., Non-compact simple Lie group $F_{4,2}$ of type F_4, J. Fac. Sci., Shinshu Univ., **12** (1977) 53–64.
[17] Yokota, I., Simply connected compact simple Lie group $E_{6(-78)}$ of type E_6 and its involutive automorphisms, J. Math., Kyoto Univ., **20** (1980) 447–573.
[18] Yokota, I., Subgroup $SU(8)/\boldsymbol{Z}_2$ of compact simple Lie group E_7 and non-compact simple Lie group $E_{7(7)}$ of type E_7, Math. J. Okayama Univ., **24** (1982) 53–71.
[19] Yokota, I. and Imai, T., and Yasukura, O., On the homogeneous space E_8/E_7, J. Math., Kyoto Univ., **23** (1983) 467–473.
[20] Yokota, I., Groups and Representations (in Japanese), Shôkabo, 1973.

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