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TWO MOORE SPACES ON WHICH EVERY CONTINUOUS REAL-VALUED FUNCTION IS CONSTANT

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Abstract We construct two Moore spaces on which every continuous real-valued function is constant. The first is Moore, screenable and the second, Moore separable. As corollaries we obtain two more Moore spaces on which every continuous real-valued function is constant (a Moore separable and a Moore, screenable) and having a dispersion point.

Key words: Moore, metacompact, screenable, separable, dispersion point.

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§ 1. Introduction.

Moore spaces on which every continuous real-valued function is constant are given in [1], [2], [7], [8]. The space by J.N. Younglove [8] is, in addition locally connected, complete and separable and the space in [2], by H. Brandenburg and A. Mysior, metacompact.

We construct two Moores spaces on which every continuous real-valued function is constant. The first is Moore, screenable (hence metacompact, since every developable screenable space is metacompact [4]) and the second, Moore separable. As corollaries we obtain two more Moore spaces on which every continuous real-valued function is constant (a Moore separable and a Moore, screenable) and having a dispersion point.

In order to construct these spaces, we first consider two auxiliary spaces (a Moore, screenable for the first space and a Moore separable for the second) containing two points not separated by a continuous real-valued function. Then we construct an appropriate Moore space (which is screenable in the first case or separable in the second) on which, with the help of a sequence of functions, we define a decomposition. Finally, on the quotient set we define a topology and we prove that this, in each case, is the required space.

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A space X is called (1) developable, if it has a development, i.e. a sequence $F_1, F_2, \cdots, F_n, \cdots$ of open coverings such that if K is a closed subset of X and $x \notin K$, then there exists a covering F_n such that $St(x, F_n) \cap K = \emptyset$, where $St(x, F_n)$ is the union of all sets in F_n containing x (2) metacompact, if every open covering of X has a point-finite open refinement and (3) screenable, if for every open covering F of X there exists a sequence $F_1, F_2, \cdots, F_n, \cdots$ of collections of pairwise disjoint open sets such that $\bigcup_{n=1}^{\infty} F_n$ covers X and refines F. A regular developable space is called a Moore space.

A point p of a connected space X is called a dispersion point if the space $X \setminus \{p\}$ is totally disconnected.

§2. The space X.

The following space K is a slight modification of the Heath's space [4]. The idea of "splitting" the neighbourhoods is due to A. Mysior.

We consider the set

$$K = \Big[(-1, \infty) \times [0, 1] \setminus \{(x, y) : -1 < x < 0, \ |x| > y\}\Big] \cup \{p\{.$$

Let L_1 (resp. M_1) be the set of rationals (resp. irrationals) of the intervals $[n, n+1)$, $n = 0, 2, 4, \cdots$, and L_2 (resp. M_2) be the set of rationals (resp. irrationals) of the intervals $[n, n+1)$, $n = 1, 3, 5, \cdots$.

On the set K we define the following topology: Every point $(x, y) \in K \setminus \{p\}$, $y > 0$, is isolated.

For every $(q, 0) \in L_1$ (resp. $(s, 0) \in M_1$) a basis of open neighbourhoods are the sets

$$U_n(q, 0) = \{(q, 0)\} \cup \left\{(q-y, y) : 0 < y < \frac{1}{n}\right\}$$
$$\cup \left\{(q+1-y, y) : 0 < y < \frac{1}{n}\right\}.$$
$$\left(\text{resp. } U_n(s, 0) = \{(s, 0)\} \cup \left\{(s+y, y) : 0 < y < \frac{1}{n}\right\}\right.$$
$$\left. \cup \left\{(s+1+y, y) : 0 < y < \frac{1}{n}\right\}\right),$$

$n = 1, 2, \cdots$.

For every $(r, 0) \in L_2$ (resp. $(t, 0) \in M_2$) a basis of open neighbourhoods are the sets

$$U_n(r, 0)=\{(r, 0)\}\cup\left\{(r+y, y): 0<y<\frac{1}{n}\right\}$$

$$\cup\left\{(r+1+y, y): 0<y<\frac{1}{n}\right\},$$

$$\left(\text{resp. } U_n(t, 0)=\{(t, 0)\}\cup\left\{(t-y, y): 0<y<\frac{1}{n}\right\}\right.$$

$$\left.\cup\left\{(t+1-y, y): 0<y<\frac{1}{n}\right\}\right),$$

$n=1, 2, \cdots$.

For the point p a basis of open neighbourhoods are the sets

$$U_n(p)=\{p\}\cup\{(x, y): x>n\}, \qquad n=1, 2, \cdots.$$

It can be easily proved that K is Moore, screenable not completely regular.

Let K^+, K^- be two disjoint copies of K and let $[0, 1)^+$, $[0, 1)^-$ be the copies of the interval $[0, 1)$ in K^+, K^-, respectively. We attach K^+ to K^- identifying each point of $[0, 1)^+$ with its corresponding point of $[0, 1)^-$. We set $[0, 1)^+=[0, 1)^-=[0, 1)$ and we consider the space

$$X=(K^+\setminus[0, 1)^+)\cup[0, 1)\cup(K^-\setminus[0, 1)^-).$$

It is easy to prove that X is regular, first countable, containing two points a, b (the copies of p in K^+, K^-, respectively) not separated by a continuous real-valued function of X.

Let $x\in X$ and $U_n(x)$, $n=1, 2, \cdots$, be a countable local basis of x. It is obvious that the collection $F_n=\{U_n(x): x\in X\}$, $n=1, 2, \cdots$, is a development for X and hence X is a Moore space.

Let L_1^+, L_1^- (resp. M_1^+, M_1^-) and L_2^+, L_2^- (resp. M_2^+, M_2^-) be the copies of L_1 (resp. M_1) and L_2 (resp. M_2) in K^+, K^-, respectively.

We set

$$P=(L_1^+\setminus\{0^+\})\cup\{0\}\cup(L_1^-\setminus\{0^-\})$$

$$R=M_1^+\cup M_1^-$$

$$Q=L_2^+\cup L_2^-$$

$$T=M_2^+\cup M_2^-$$

and we observe that P, R, Q, T are pairwise disjoint sets and that if p, p' (resp. r, r', q, q' and t, t') are distinct points of F (resp. of R, Q and T) then for every n, m, $U_n(p)\cap U_m(p')=\emptyset$ (resp. $U_n(r)\cap U_m(r')=\emptyset$, $U_n(q)\cap U_m(q')=\emptyset$ and $U_n(t)\cap U_m(t')=\emptyset$). Based on this, it is easy to prove that X is screenable.

§ 3. The space (Z, τ).

The set of isolated points of X has cardinality c. Let I be an index set having the same cardinality and let $X^{(i)}$, $i \in I$ be disjoint copies of X and $a^{(i)}$, $b^{(i)} \in X^{(i)}$ be points corresponding to $a, b \in X$, respectively. Let Y be the disjoint union (i.e. topological sum) of $X^{(i)}$, $i \in I$ and let D be the dense subset of isolated points of Y. Obviously, $|D| = c$.

Set $A = \{a^{(i)} : i \in I\}$ and on the quotient set $Z = Y/A$ we define a topology τ as follows: For every point $x^{(i)} \in X^{(i)}$, $x^{(i)} \neq a^{(i)}$, a basis of open neighbourhoods is $B(x^{(i)})$, where $B(x)$ is the basis of x in X. For the point A of Z a basis of open neighbourhoods are the sets

$$O_n(A) = \{A\} \cup \bigcup V_n(a^{(i)}), \qquad n = 1, 2, \cdots.$$

where $V_n(a^{(i)})$ is the copy of $U_n(a) \setminus \{a\}$ in $X^{(i)}$.

Observe that this topology is regular, first countable, strictly weaker than the quotient topology on Z and that the subspace $(X^{(i)} \setminus \{a^{(i)}\}) \cup \{A\}$ is homeomorphic to $X^{(i)}$, for every $i \in I$.

Obviously (Z, τ) is Moore screenable.

§ 4. The space $(S_\infty/L, \tau*)$.

We consider a copy Z_0 of Z and let A_0, B_0 be the copies of the point A and of the set $B = \{b^{(i)} : i \in I\}$, in Z_0, respectively.

Let Y_k, $k = 1, 2, \cdots$, be disjoint copies of Y and let A_k, B_k be the copies of A, B, in Y_k, respectively.

We attach the space Y_1 to Z_0 replacing each point $b_0^{(i)}$ of B_0 by its corresponding point $a_1^{(i)}$ of A_1.

We set $S_1 = (Z_0 \setminus B_0) \cup Y_1$.

By induction (replacing each point $b_{k-1}^{(i)}$ of B_{k-1} by its corresponding point $a_k^{(i)}$ of A_k) we construct the space $S_k = (S_{k-1} \setminus B_{k-1}) \cup Y_k$, $k = 2, 3, \cdots$.

Finally, we consider the space

$$S_\infty = \bigcup_{k=1}^{\infty} S_k .$$

It can be easily proved that S_∞ is Moore, screenable and that every continuous real-valued function of S_∞ is constant on $\{A, a_k^{(i)} : k = 1, 2, \cdots, i \in I\}$.

Observe that the basis of open neighbourhoods of each point $a_k^{(i)} \in A_k$, $k = 1, 2, \cdots$, has the form

$$O_n(a_k^{(i)}) = V_n(a_k^{(i)}) \cup U_n(a_k^{(i)}), \qquad n = 1, 2, \cdots,$$

where $V_n(a_k^{(i)})$ is the deleted neighbourhood of $b_{k-1}^{(i)}$ in S_{k-1} and $U_n(a_k^{(i)})$ is the neighbourhood of $a_k^{(i)}$ in Y_k.

Let $D_0, D_1, D_2, \cdots, D_k, \cdots$, be the sets of isolated points of $Z_0, Y_1, Y_2, \cdots, Y_k, \cdots$, respectively.

Since the sets A_k, D_{k-2}, $k=2, 3, \cdots$ have the same cardinality there exists an one-to-one function f_k of A_k onto D_{k-2}.

Let L be the decomposition of S_∞ consisting of the points A_0, $a_1^{(i)}$, $i\in I$, the pairs $(a_k^{(i)}, f_k(a_k^{(i)}))$, $k=2, 3, \cdots$, and the points of the sets

$$P_k=\{p_k^{(i)}: p\in P,\ k=0, 1, 2, \cdots, i\in I\}$$

$$R_k=\{r_k^{(i)}: r\in R,\ k=0, 1, 2, \cdots, i\in I\}$$

$$Q_k=\{q_k^{(i)}: q\in Q,\ k=0, 1, 2, \cdots, i\in I\}$$

$$T_k=\{t_k^{(i)}: t\in T,\ k=0, 1, 2, \cdots, i\in I\}$$

where again P_0, R_0, Q_0, T_0 are the corresponding copies for $k=0$, in Z_0.

On the quotient set S_∞/L we define a topology $\tau*$ as follows:

If $s\in S_\infty/L$ and $s=(a_k^{(i)}, f_k(a_k^{(i)}))$ we set

$$E_n^0(s)=\{f_k(a_k^{(i)})\}\cup V_n(a_k^{(i)})\cup U_n(a_k^{(i)})$$

and we consider the set

$$E_n^1(s)=E_n^0(s)\cup M_n^{k+1}(s)\cup N_n^{k+2}(s)\,.$$

where,

$$M_n^{k+1}(s)=\bigcup\{O_n(a_{k+1}^{(i)}): f_{k+1}(a_{k+1}^{(i)})\in V_n(a_k^{(i)})\}$$

and

$$N_n^{k+2}(s)=\bigcup\{O_n(a_{k+2}^{(i)}): f_{k+2}(a_{k+2}^{(i)})\in U_n(a_k^{(i)})\}.$$

By induction, we consider the set

$$E_n^{m+1}(s)=E_n^m(s)\cup\bigcup\{O_n(a_{k+m+1}^{(i)}): f_{k+m+1}(a_{k+m+1}^{(i)})\in M_n^{k+m}\}$$
$$\cup\bigcup\{O_n(a_{k+m+2}^{(i)}): f_{k+m+2}(a_{k+m+2}^{(i)})\in N_n^{k+m+1}\}$$

and we set $E_n(s)=\bigcup_{m=0}^{\infty} E_n^m(s)$.

A basis of open neighbourhoods for the point $s=(a_k^{(i)}, f(a_k^{(i)}))$ are the sets $E_n(s)$, $n=1, 2, \cdots$.

Similarly, we define the open bases $E_n(s)$, $n=1, 2, \cdots$, if $s=A_0$, whence we set $E_n^0(A_0)=\{A_0\}\cup V_n(a_0^{(d)})$ or, if $s=a_1^{(i)}$, $i\in I$, whence we set $E_n^0(a_1^{(i)})=V_n(a_1^{(i)})\cup U_n(a_1^{(i)})$, or if $s\in P_k\cup R_k\cup Q_k\cup T_k$, $k=0, 1, 2, \cdots$, whence we set $E_n^0(s)=U_n(s)$, where $U_n(s)$, $n=1, 2, \cdots$, is the basis of s in S_∞.

It can be easily proved that the space $(S_\infty/L, \tau*)$ is regular, first countable and that the topology $\tau*$ is strictly weaker than the quotient topology on S_∞/L.

PROPOSITION. *The space* $(S_\infty/L, \tau*)$ *is Moore, screenable, on which every continuous real-valued function is constant.*

PROOF. Since the collection $F_n=\{E_n(s): s\in S_\infty/L\}$, $n=1, 2, \cdots$, is a development, it follows that S_∞/L is a Moore space.

To prove that S_∞/L is screenable observe that for every $k=0, 1, 2, \cdots$, the sets P_k, R_k, Q_k and T_k, are pairwise disjoint and that if $p_k^{(i)}$, $p_k^{(j)}$ (resp. $r_k^{(i)}$, $r_k^{(j)}$, $q_k^{(i)}$, $q_k^{(j)}$ and $t_k^{(i)}$, $t_k^{(j)}$) are distinct points of P_k (resp. R_k, Q_k and T_k) then for every n, m, $E_n(p_k^{(i)})\cap E_m(p_k^{(j)})=\emptyset$ (resp. $E_n(r_k^{(i)})\cap E_m(r_k^{(j)})=\emptyset$, $E_n(q_k^{(i)})\cap E_m(q_k^{(j)})=\emptyset$ and $E_n(t_k^{(i)})\cap E_m(t_k^{(j)})=\emptyset$). Based on this it is easy to prove that S_∞/L is screenable.

Finally, since every continuous real-valued function of S^∞/L is constant on the dense subset $\{(a_k^{(i)}, f_k(a_k^{(i)})): k=1, 2, \cdots, i\in I\}$, it follows that every continuous real-valued function of S_∞/L is constant.

REMARK 1. Based on the above we can easily construct a Moore separable space on which every continuous real-valued function is constant (see, also, [8]): Let K be the set

$$\{(x, y): x, y\in Q, x, y>0\}\cup\{(r, 0): r\geqq 0, r\in R\}\cup\{p\}$$

(Q, R denote the rationals and the reals, respectively). On K we define the following topology: Every point (x, y), $x, y\in Q$, $y>0$ is isolated. For every point $(r, 0)$, $r\geqq 0$ a basis of open neighbourhoods are the sets

$$U_n(r, 0)=\{(r, 0)\}\cup\left\{(t, s)\in K: t>r, (t-r)^2+\left(s-\frac{1}{n}\right)^2<\frac{1}{n^2}\right\}$$
$$\cup\left\{(t, s)\in K: t<r+1, (t-r-1)^2+\left(s-\frac{1}{n}\right)^2<\frac{1}{n^2}\right\}$$

$n=1, 2, \cdots$. For the point p, a basis of open neighbourhoods are the sets

$$U_n(p)=\{p\}\cup\{(t, s)\in K: t>n\}, \qquad n=1, 2, \cdots .$$

The space K (which is called splitted Niemytzki's space) is Moore, separable not completely regular and it is due to A. Mysior.

Then the corresponding space X (see § 2) is Moore separable (since its subset of isolated points is countable and dense) containing two points a, b (the copies of p in K^+, K^-, respectively) not separated by a continuous real-valued function of X. Hence, if $X^{(n)}$, $n=1, 2, \cdots$, are disjoint copies of X, then the corresponding spaces Y, Z, S_∞ (see § 3 and 4) are Moore separable and therefore S_∞/L is Moore separable on which every continuous real-valued function is constant.

COROLLARY 1. *There exists a Moore separable space on which every continuous real-valued function is constant and having a dispersion point.*

PROOF. Let Z be the Moore separable space corresponding to the space X of Remark 1. Let f be a one-to-one function of $B=\{b^{(k)}: k=1, 2, \cdots\}$ onto the countable dense subset D of isolated points of Z. If L is the decomposition of Z consisting of the points of $Z\setminus BUD$ and the pairs $(b^{(k)}, f(b^{(k)}))$, $k=1, 2, \cdots$, and if on the set Z/L we define a topology $\tau*$ in the same manner as on the set S_∞/L, then the space $(Z/L, \tau*)$ is, obviously, Moore separable on which every continuous real-valued function is constant (hence, is connected) and having the point A as a dispersion point, (since X is totally disconnected; see the remark in [3]).

COROLLARY 2. *There exists a Moore screenable space on which every continuous real-valued function is constant and having a dispersion point.*

PROOF. Let Z_k, $k=1, 2$, be disjoint copies of the space Z of § 3 and let A_k be the copy of the point A in Z_k. Let Y_∞ be the disjoint union of Z_k. We set $A_\infty=\{A_k: k=1, 2, \cdots\}$ and on the quotient set $Z_\infty=Y_\infty/A_\infty$ we define a topology as on the set $Z=Y/A$ of § 3. Let D_k, $k=1, 2, \cdots$, be the (dense) subset of isolated points of Z_k. We set $B_k=\{b_k^{(i)}: i\in I\}$ and we consider a sequence of one-to-one functions f_k, $k=1, 2, \cdots$, from B_{k+1} onto D_k. We set $B_\infty=\bigcup_{k=1}^{\infty} B_k$, $D_\infty=\bigcup_{k=1}^{\infty} D_k$ and let L be the decomposition of Z_∞ consisting of the points of $Z_\infty\setminus B_\infty\cup D_\infty$ and the pairs $(b_k^{(i)}, f_k(b_k^{(i)}))$, $k=2, 3, \cdots$, $i\in I$.

Then, defining on the quotient set Z_∞/L a topology $\tau*$ as on the set S_∞/L (in § 4), it can be proved, in a similar manner as for the space S_∞/L, that Z_∞/L is Moore, screenable on which every continuous real-valued function is constant. That A_∞ is a dispersion point, is proved as in Corollary 1.

REMARKS. A direct application of the van Douwen's method [3] on the space X either if it is the Moore, screenable of § 2, or it is the Moore separable of Remark 1, leads to a regular, not separable and nowhere first countable space. The quotient topology on S_∞/L if X is the Moore, screenable (resp. if it is the Moore separable) gives a regular, nowhere first countable, metacompact, screenable (resp. a regular, nowhere first countable, separable) space.

The quotient topology on Z/L of Corollary 1 if X is the Moore separable space of Remark 1 gives a regular, separable, nowhere first countable with a dispersion point. The quotient topology on Z_∞/L gives a regular, nowhere first countable, metacompact screenable space with a dispersion point. On each of these spaces, every continuous real-valued function is constant.

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