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COMPLETE RIEMANNIAN MANIFOLD MINIMALLY IMMERSED IN A UNIT SPHERE $S^{n+p}(1)$

By

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1. Innroduction.

Let M^n be an n-dimensional Riemannian manifold which is minimally immersed in a unit sphere $S^{n+p}(1)$ of dimension $n+p$. If M^n is compact, then many authors studied them and obtained many beautiful results (for examples [1], [3], [4], [5] and [6]). In this paper, we make use of Yau's *maximum principle* to extend these results to complete manifolds with Ricci curvature bounded from below.

2. Preliminaries.

Let M^n be an n-dimensional Riemannian manifold which is minimally immersed in a unit sphere $S^{n+p}(1)$ of dimension $n+p$. Then the *second fundamental form* h of the immersion is given by $h(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y$ and it satisfies $h(X, Y)=h(Y, X)$, where $\tilde{\nabla}$ and ∇ denote the covariant differentiation on $S^{n+p}(1)$ and M^n respectively, X and Y are vector fields on M^n. We choose a local field of orthonormal frames $e_1, \cdots, e_n, \cdots, e_{n+p}$ in $S^{n+p}(1)$ such that, restricted to M^n, the vector $e_1, \cdots, e_n$ are tangent to M^n. We use the following convention on the range of indices unless otherwise stated: $A, B, C, \cdots=1, 2, \cdots, n+p$; $i, j, k, \cdots=1, 2, \cdots, n$; $\alpha, \beta, \cdots=n+1, \cdots, n+p$. And we agree that repeated indices under a summation sign without indication are summed over the respective range. With respect to the frame field of $S^{n+p}(1)$ chosen above, let $\tilde{\omega}_1, \cdots, \tilde{\omega}_{n+p}$ be the dual frames. Then structure equations of $S^{n+p}(1)$ are given by

$$d\tilde{\omega}_A=\sum\tilde{\omega}_{AB}\wedge\tilde{\omega}_B, \quad \tilde{\omega}_{AB}+\tilde{\omega}_{BA}=0, \tag{2.1}$$

$$d\tilde{\omega}_{AB}=\sum\tilde{\omega}_{AC}\wedge\tilde{\omega}_{CB}-\tilde{\omega}_A\wedge\tilde{\omega}_B. \tag{2.2}$$

Restricting these forms to M^n, we have the structure equations of the immersion:

$$\omega_\alpha=0, \tag{2.3}$$

$$\omega_{i\alpha}=\sum h_{ij}^\alpha\omega_j, \quad h_{ij}^\alpha=h_{ji}^\alpha, \tag{2.4}$$

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(2.5) $$d\omega_i=\sum\omega_{ij}\wedge\omega_j,\quad \omega_{ij}+\omega_{ji}=0,$$

(2.6) $$d\omega_{ij}=\sum\omega_{ik}\wedge\omega_{kj}-\Omega_{ij},\quad \Omega_{ij}=\frac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l,$$

(2.7) $$R_{ijkl}=(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum(h_{ik}^\alpha h_{jl}^\alpha-h_{il}^\alpha h_{jk}^\alpha),$$

(2.8) $$d\omega_{\alpha\beta}=\sum\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}-\Omega_{\alpha\beta},\quad \Omega_{\alpha\beta}=\frac{1}{2}\sum R_{\alpha\beta ij}\omega_i\wedge\omega_j,$$

(2.9) $$R_{\alpha\beta ij}=\sum(h_{ik}^\alpha h_{kj}^\beta-h_{jk}^\alpha h_{ki}^\beta).$$

Then, the *second fundamental form* h can be written as

(2.10) $$h(e_i, e_j)=\sum h_{ij}^\alpha e_\alpha.$$

If we define h_{ijk}^α by

(2.11) $$\sum h_{ijk}^\alpha\omega_k=dh_{ij}^\alpha+\sum h_{ik}^\alpha\omega_{kj}+\sum h_{kj}^\alpha\omega_{ki}+\sum h_{ij}^\beta\omega_{\beta\alpha},$$

then, from (2.2), (2.3) and (2.4), we have $h_{ijk}^\alpha=h_{ikj}^\alpha$.

Let K_N be the square of the length of curvature tensor of the *normal bundle*, that is,

(2.12) $$K_N=\sum(\sum_k(h_{ik}^\alpha h_{kj}^\beta-h_{jk}^\alpha h_{ki}^\beta))^2$$

Setting

(2.13) $$L_N=\sum(\sum_{ij}h_{ij}^\alpha h_{ij}^\beta)^2.$$

In this paper, we used the notations in [2].

LEMMA 1 ([4] or [6]). *If M^n is an n-dimensional Riemannian manifold minimally immersed in a unit sphere $S^{n+p}(1)$ of dimension $n+p$. Then, Simons' equation*

(2.14) $$\frac{1}{2}\Delta\|h\|^2=\sum(h_{ijk}^\alpha)^2+\sum(h_{ij}^\alpha h_{kl}^\alpha R_{lijk}+h_{ij}^\alpha h_{il}^\alpha R_{lkjk})-\frac{1}{2}K_N$$
$$=\sum(h_{jik}^\alpha)^2-K_N-L_N+n\|h\|^2$$

holds good and

(2.15) $$\|h\|^2=n(n-1)-R,$$

where $\|h\|$ denotes the length of the second fundamental form h such that $\|h\|^2=\sum(h_{ij}^\alpha)^2$ and R is the scalar curvature of M^n.

LEMMA 2 ([7]). *Let M^n be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function bounded from above on M^n, then for all $\varepsilon>0$, there exists a point x in M^n at which*

(2.16) $$\sup f-\varepsilon<f(x),$$

(2.17) $$\|\nabla f\|<\varepsilon,$$

(2.18) $$\Delta f<\varepsilon.$$

3. Main results.

THEOREM 1. *Let M^n be an n-dimensional complete Riemannian manifold minimally immersed in a unit sphere $S^{n+p}(1)$ of dimension $n+p$ with Ricci curvature bounded from below. Then either M^n is totally geodesic, in this case, $M^n=S^n(1)$ holds locally or* $\inf R\leqq n(n-1)-n/(2-1/p)$.

PROOF. According to Lemma 1, we have

(3.1) $$\frac{1}{2}\Delta\|h\|^2=\sum(h_{ijk}^{\alpha})^2-K_N-L_N+n\|h\|^2,$$

Because

(3.2) $$\sum_{ij}(\sum_k(h_{ik}^{\alpha}h_{kj}^{\beta}-h_{jk}^{\alpha}h_{ki}^{\beta}))^2\leqq 2\sum_{ij}(h_{ij}^{\alpha})^2\sum_{ij}(h_{ij}^{\beta})^2,$$

we get

(3.3) $$\begin{aligned}K_N=\sum(\sum_k(h_{ik}^{\alpha}h_{kj}^{\beta}-h_{jk}^{\alpha}h_{ki}^{\beta}))^2&\leqq 2\sum(\sum_{ij}(h_{ij}^{\alpha})^2\sum_{ij}(h_{ij}^{\beta})^2)\\&=2\|h\|^4-2\sum(\sum_{ij}(h_{ij}^{\alpha})^2)^2.\end{aligned}$$

(3.1) and (3.3) imply

(3.4) $$\frac{1}{2}\Delta\|h\|^2\geqq\|h\|^2\{n-(2-1/p)\|h\|^2\}.\quad\text{(see [1] or [4])}$$

If $\inf R>n(n-1)-n/(2-1/p)$, from Lemma 1, we have

(3.5) $$\|h\|^2=n(n-1)-R$$

Hence, $\|h\|^2$ is bounded. We define $f=\|h\|^2$, $F=(f+a)^{1/2}$ (where $a>0$ is any positive constant number). F is bounded because $\|h\|^2$ is bounded.

$$dF=\frac{1}{2}(f+a)^{-1/2}df,$$

$$\begin{aligned}\Delta F&=\frac{1}{2}\left\{-\frac{1}{2}(f+a)^{-3/2}\|df\|^2+(f+a)^{-1/2}\Delta f\right\}\\&=\frac{1}{2}\{-2\|dF\|^2+\Delta f\}(f+a)^{-1/2}\\&=\frac{1}{2F}\{-2\|dF\|^2+\Delta f\}.\end{aligned}$$

Hence, $F\Delta F=-\|dF\|^2+\frac{1}{2}\Delta f$, namely,

$$\frac{1}{2}\Delta f=F\Delta F+\|dF\|^2. \tag{3.6}$$

Applying the Lemma 2 to F, we have for all $\varepsilon>0$, there exists a point x in M^n such that at x

$$\|dF(x)\|<\varepsilon. \tag{3.7}$$

$$\Delta F(x)<\varepsilon, \tag{3.8}$$

$$F(x)>\sup F-\varepsilon. \tag{3.9}$$

(3.6), (3.7) and (3.8) imply

$$\frac{1}{2}\Delta f<\varepsilon^2+F\varepsilon=\varepsilon(\varepsilon+F) \quad (\text{from } F>0). \tag{3.10}$$

We take the sequence $\{\varepsilon_m\}$ such that $\varepsilon_m\to 0$ $(m\to\infty)$ and for all m, there exists a point x_m in M^n such that (3.7), (3.8) and (3.9) hold good. Hence, $\varepsilon_m(\varepsilon_m+F(x_m))\to 0$ $(m\to\infty)$ (because F is bounded).

On the other hand, from (3.9),

$$F(x_m)>\sup F-\varepsilon_m.$$

Because F is bounded, $\{F(x_m)\}$ is a bounded sequence, we get

$$F(x_m)\longrightarrow F_0 \text{ (if necessary, we can choose subsequence)}.$$

Hence,

$$F_0\geqq\sup F.$$

According to the properties of supremum, we have

$$F_0=\sup F. \tag{3.11}$$

From the definition of F, we get

$$f(x_m)\longrightarrow f_0=\sup f_0 \quad (\text{from } F_0=\sup F) \tag{3.12}$$

From (3.4) and (3.10), we obtain

$$f[n-(2-1/p)f]\leqq\frac{1}{2}\Delta f<\varepsilon^2+\varepsilon F,$$

$$f(x_m)[n-(2-1/p)f(x_m)]<\varepsilon_m^2+\varepsilon_m F(x_m)\leqq\varepsilon_m^2+\varepsilon_m F_0,$$

Let $m\to\infty$, we have $\varepsilon_m\to 0$, $f(x_m)\to f_0$. Hence,

$$f_0[n-(2-1/p)f_0]\leqq 0$$

1) If $f_0=0$, we have $f=\|h\|^2\equiv 0$, hence M^n is totally geodesic, from [4], we know $M^n=S^n(1)$ holds locally.

2) If $f_0>0$, we have

$$n-(2-1/p)f_0\leqq 0,$$

$$f_0\geqq n/(2-1/p),$$

that is, $\sup\|h\|^2\geqq n/(2-1/p)$. From (2.15),

$$\inf R\leqq n(n-1)-n/(2-1/p)$$

This completes the proof of Theorem 1.

THEOREM 2. *Let M^n be an n-dimensional complete Riemannian manifold with Ricci curvature bounded from below which is minimally immersed in a unit sphere $S^{n+p}(1)$ of dimension $n+p$. If $K_N=0$, then, either M^n is totally geodesic and $M^n=S^n(1)$ holds locally or* $\inf R\leqq n(n-2)$.

PROOF. Because $K_N=0$ if and only if, for any α, β,

$$\sum_{ij}(\sum_k(h^\alpha_{ik}h^\beta_{kj}-h^\alpha_{jk}h^\beta_{ki}))^2=0,$$

(3.13)
$$\begin{aligned}\frac{1}{2}\Delta\|h\|^2&\geqq -L_N+n\|h\|^2\\&\geqq n\|h\|^2-\|h\|^4\\&=\|h\|^2(n-\|h\|^2)\quad\text{(see [4])}\end{aligned}$$

Hence, using the same arguments as Theorem 1, we have $\|h\|^2=0$ or $\sup\|h\|^2\geqq n$. Thus, from Lemma 1, we know either M^n is totally geodesic and $M^n=S^n(1)$ holds locally or $\inf R\leqq n(n-2)$.

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