TSUKUBA J. MATH.
Vol. 15 No. 1 (1991), 215-217

A NOTE ON STRONG LOCALLY DIVIDED DOMAINS

By

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Abstract. A uniform proof is given for the following five assertions. Let R be an integral domain such each overring of R is a pseudovaluation domain (resp., divided domain; resp., going-down domain; resp., locally pseudovaluation domain; resp., locally divided domain). Then R/P has the same property, for each prime ideal P of R. The assertion for pseudovaluation domains was proved recently by Okabe-Yoshida by other methods.

1. Introduction and summary.

Let R be a pseudovaluation domain (for short, PVD), in the sense of [7]; that is, (R, M) is a quasilocal (commutative integral) domain with a canonically associated valuation overring V such that M is the maximal ideal of V. In [7, Proposition 2.7], Hedstrom-Houston showed that each overring of R (including R itself) is a PVD if and only if V is the integral closure of R. Recently, Okabe-Yoshida have given other equivalent characterizations [8, Theorem 2.4], and have shown that the property in question is stable under passage to factor domains [8, Proposition 2.5]. In this note, we use uniform methods to prove [8, Proposition 2.5] and four new related results.

Following [8], if $\boldsymbol{P}$ is a property of domains, we may say that a domain R is *strong* $\boldsymbol{P}$ in case each overring of R satisfies $\boldsymbol{P}$. With this terminology, [8, Proposition 2.5] asserts that if R is a strong PVD, then so is R/P, for each prime ideal P of R. This is recovered in Theorem 2.2(a) below. In Theorem 2.2(b), (c), we prove the analogue for divided domains and going-down domains. (Recall from [1] and [3] that a domain R is called *divided* in case $PR_P=P$ for each prime ideal P of R. Each PVD is a divided domain [4, page 560]; and each divided domain is a going-down domain [3, Proposition 2.1], in the sense of [2].) In Theorem 2.2(d), (e), we establish the analogues for locally pseudovaluation domains (or LPVD's, in the sense of [6]) and locally divided domains (in the sense of [5]).

Received January 9, 1990.

Any unexplained material is standard, as in the cited articles.

2. Results.

We begin by isolating two assertions that will be needed in the proof of parts (d) and (e) of Theorem 2.2.

LEMMA 2.1. *Let P be a prime ideal of an LPVD (resp., a locally divided domain) R. Then R/P is an LPVD (resp., a locally divided domain).*

PROOF. Put $S=(R/P)_{Q/P}$, where $P\subset Q$ are prime ideais of R. It suffices to show that S is a PVD (resp., divided domain). Since S is canonically isomorphic to R_Q/PR_Q and R_Q is a PVD (resp., divided domain), it suffices to observe that the class of PVD's (resp., divided domains) is stable under passage to factor domains. For this, one may appeal to [4, Lemma 4.5(i)] (resp., [3, Lemma 2.2(c)]). The proof is complete.

Theorem 2.2(a) recovers [8, Proposition 2.5]. In view of the many inequivalent usages of "strong" in the literature, we prefer to state this result without the "strong" terminology.

THEOREM 2.2. *Let P be a prime ideal of a domain R. Then:*

(a) *If each overring of R is a PVD, then each overring of R/P is a PVD.*

(b) *If each overring of R is a divided domain, then each overring of R/P is a divided domain.*

(c) *If each overring of R is a going-down domain, then each overring of R/P is a going-down domain.*

(d) *If each overring of R is an LPVD, then each overring of R/P is an LPVD.*

(e) *If each overring of R is a locally divided domain, then each overring of R/P is a locally divided domain.*

PROOF. Consider an overring E of R/P. Thus $R/P\subset E\subset R_P/RP_P$. Put $T=R+PR_P$. One has a canonical isomorphism $T/PR_P\cong R/P$. Thus, by a standard homomorphism theorem, $E\cong D/PR_P$ where D is a domain satisfying $T\subset D\subset R_P$. In particular, D is an overring of R. By hypothesis, D is a PVD (resp., divided; resp., a going-down domain; resp., an LPVD; resp., locally divided). It suffices to observe that the class of PVD's (resp., divided domains; resp., going-down domains; resp., LPVD's; resp., locally divided domains) is stable under passage to factor domains. For this, one may appeal to [4, Lemma

4.5(i)], [3, Lemma 2.2(c) and Remark 2.11], and Lemma 2.1. The proof is complete.

REMARK 2.3. (a) Perhaps the simplest example of a domain R satisfying the conditions in Theorem 2.2 is $F+XK[[X]]$, where $F\subset K$ is an algebraic field extension.

(b) The absence of an assertion about "locally going-down domains" in Theorem 2.2 is explained by the fact that a domain R is a going-down domain if and only if R_P is a going-down domain for each prime ideal P of R [3, page 357]. In particular, a domain satisfying any of the conditions in Theorem 2.2 must be a going-down domain. However, not all going-down domains (or PVD's) satisfy those conditions: cf. [2, Theorem 4.2(ii)].

(c) The unified proof in Theorem 2.2 becomes especially simple in part (a) and (b). (Recall that (a) recovered the motivating result, [8, Proposition 2.5].) Indeed, in these cases, $T=R$ since R is divided, and so the proof amounts just to the isomorphism $E\cong D/P$ and an appeal to the stability results ([4, Lemma 4.5(i)], [3, Lemma 2.2(c)]).

(d) The interested reader may find more complicated proofs of Theorem 2.2 by using pullbacks and various results in the literature on seminormality, i-domains, and Prüfer domains. One such proof uses [6, Theorem 2.9]. In this regard, we close by noting that [6, Corollary 2.10] provides some characterizations of strong PVD's, and thus may be viewed as a companion for [8, Theorem 2.4].

References

[1] T. Akiba, A note on AV-domains, Bull. Kyoto Univ. Education Ser. B, **31** (1967), 1-3.

[2] D. E. Dobbs, On going down for simple overrings, II, Comm. Algebra, **1** (1974), 439-458.

[3] D. E. Dobbs, Divided rings and going-down, Pac. J. Math., **67** (1976), 353-363.

[4] D. E. Dobbs, Coherence, ascent of going-down, and pseudo-valuation domains, Houston J. Math., **4** (1978), 551-567.

[5] D. E. Dobbs, On locally divided integral domains and CPI-overrings, Internat. J. Math. & Math. Sci., **4** (1981), 119-135.

[6] D. E. Dobbs and M. Fontana, Locally pseudo-valuation domains, Annali Mat. Pura Appl., **134** (1983), 147-168.

[7] J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, II, Houston J. Math., **4** (1978), 199-207.

[8] A. Okabe and K. Yoshida, Note on strong pseudo-valuation domains, Bull. Fac. Sci. Ibaraki Univ. Ser. A, **21** (1989), 9-12.

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