

The $H[\infty]$ -wellposed Cauchy problem for Schroedinger type equations

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THE H_∞ -WELLPOSED CAUCHY PROBLEM FOR SCHRÖDINGER TYPE EQUATIONS

By

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§ 1. Introduction.

We study the Cauchy problem for a Schrödinger type operator

$$L = L(t, x, D_t, D_x) = D_t + \frac{1}{2} \sum_{j=1}^n (D_j - a_j(t, x))^2 + c(t, x),$$

where $a_j(t, x), c(t, x) \in C_i^k([0, T]; \mathcal{B}^\infty(R^n))$, ($T > 0$), $a_j(t, x) = a_j^R(t, x) + i a_j^I(t, x)$ ($a_j^R(t, x)$ and $a_j^I(t, x)$ are real valued functions) for $j=1, \dots, n$ and $D_t = -i\partial/\partial t$, $D_j = -i\partial/\partial x_j$. Here $\mathcal{B}^\infty(R^n)$ denotes the set of C^∞ -functions whose derivatives of any order are all bounded in R^n and $g(t, x) \in C_i^k([0, T]; X)$ ($k=0, 1, 2, \dots$) means that the mapping: $[0, T] \ni t \rightarrow g(t) \in X$ is k -times continuously differentiable in the topology of X .

In this paper we give a sufficient condition for the Cauchy problem

$$(1.1) \quad \begin{cases} L(t, x, D_t, D_x)u(t, x) = f(t, x), & (t, x) \in [0, T] \times R^n, \quad (T > 0), \\ u(0, x) = u_0(x), & x \in R^n \end{cases}$$

to be H_∞ -wellposed in $[0, T]$ ($T > 0$), where H_s denotes the Sobolev space of order s and $H_\infty \equiv \bigcap_{-\infty < s < \infty} H_s$.

We say that the Cauchy problem (1.1) is H_∞ -wellposed in $[0, T]$ if for any initial data $u_0 \in H_\infty$ and $f(t, x) \in C_i^k([0, T]; H_\infty)$ there exists a unique solution $u(t, x) \in C_i^k([0, T]; H_\infty)$, and for any $s \in R^1$ there exist constants $C(s, T) > 0$ and $s' \in R^1$ such that the energy inequality

$$(1.2) \quad \|u(t, \cdot)\|_{(s)} \leq C(s, T) \left\{ \|u_0\|_{(s')} + \int_0^t \|f(\tau, \cdot)\|_{(s')} d\tau \right\}$$

holds for $t \in [0, T]$. Here, $\|u(t, \cdot)\|_{(s)}$ denotes the H_s norm.

Let us briefly recall some known facts. In [3], Ichinose obtained a necessary condition of the Cauchy problem (1.1) to be H_∞ -wellposed. The sufficient conditions for the Cauchy problem (1.1) to be H_∞ -wellposed are given by Ichinose [2] and Takeuchi [6].

The following theorem, which is the main result of the present paper, gives a sufficient condition for the Cauchy problem (1.1) to be H_∞ -wellposed.

THEOREM 1.1. *Assume that the coefficients $a_j(t, x) = a_j^R(t, x) + ia_j^I(t, x)$ satisfy*

$$(1.3) \quad \begin{cases} |a_j^I(t, x)| \leq C \langle x \rangle^{-1}, \\ |D_x^\alpha a_j(t, x)| \leq C_\alpha \langle x \rangle^{-1} \end{cases}$$

for $(t, x) \in [0, T] \times R^n$ and for any multi-indices α ($|\alpha| \geq 1$) and $j=1, \dots, n$, where C and C_α are positive constants and $\langle x \rangle = (1 + |x|^2)^{1/2}$. Then the Cauchy problem for (1.1) is H_∞ -wellposed in $[0, T]$.

REMARK 1.2. If there is $\varepsilon > 0$ such that $a_j(t, x)$ satisfy

$$(1.4) \quad \begin{cases} |a_j^I(t, x)| \leq C \langle x \rangle^{-1-\varepsilon}, \\ |D_x^\alpha a_j(t, x)| \leq C_\alpha \langle x \rangle^{-1} \end{cases}$$

for $(t, x) \in [0, T] \times R^n$ and for any multi-indices α ($|\alpha| \geq 1$) and $j=1, \dots, n$, where C and C_α are positive constants, then the Cauchy problem (1.1) is L^2 -wellposed in $[0, T]$.

REMARK 1.3. In the case of $T < 0$, we can prove Theorem 1.1 in the same way.

To prove Theorem 1.1 we modify the method given in [5], Ch. 7, §3. Conjugating L by a pseudo-differential operator $K(t, x, D_x)$ with its symbol $\sigma(K)(t, x, \xi) = \exp A(t, x, \xi)$, where $A(t, x, \xi)$ is a solution of the following equation

$$(1.5) \quad \begin{cases} \left(D_t + \sum_{j=1}^n \xi_j D_j \right) A(t, x, \xi) + \frac{iM \langle \xi \rangle}{\langle x \rangle} = 0, \\ A(0, x, \xi) = 0, \end{cases}$$

we reduce L to $K^{-1} \circ L \circ K = D_t - P(t)$. Then taking a parameter $M > 0$ sufficiently large, we can make the imaginary part of $P(t)$ nonnegative in $L^2(R^n)$ and therefore can obtain an energy estimate in L^2 -sense for the operator $D_t - P(t)$.

REMARK 1.4. If (1.4) is valid, we replace $\langle x \rangle$ in (1.5) by $\langle x \rangle^{1+\varepsilon}$. Then $K(t, x, D_x)$ becomes a bounded operator from $L^2(R^n)$ to $L^2(R^n)$.

Let us sum up the contents of the paper briefly. In Section 2 we shall prove that there exists the inverse operator of K as a pseudo-differential operator. In Section 3 we shall give the expression of $P(t)$ and prove Theorem 1.1.

§ 2. Existence of inverse of e^A .

In this section we shall show the existence of inverse operator of $K = e^A(t, x, D_x)$. We can solve the solution $A(t, x, \xi)$ of (1.5) as follows

$$\begin{aligned}
 (2.1) \quad A(t, x, \xi) &= \int_0^t \frac{M \langle \xi \rangle ds}{\langle x - s\xi \rangle} = \frac{\langle \xi \rangle}{|\xi|} \int_0^{t|\xi|} \frac{M ds}{\langle x - s\omega \rangle} \\
 &= \frac{\langle \xi \rangle}{|\xi|} \int_0^{t|\xi|} \frac{M ds}{\sqrt{(s - x \cdot \omega)^2 + |x|^2 - (x \cdot \omega)^2 + 1}} \\
 &= \frac{\langle \xi \rangle}{|\xi|} \int_{-x \cdot \omega}^{t|\xi| - x \cdot \omega} \frac{M ds}{\sqrt{s^2 + |x|^2 - (x \cdot \omega)^2 + 1}} \\
 &= M \frac{\langle \xi \rangle}{|\xi|} \log \left\{ \frac{\langle x - t\xi \rangle + t|\xi| - x \cdot \omega}{\langle x \rangle - x \cdot \omega} \right\} \\
 &= M \log \left\{ \frac{\langle x - t\xi \rangle + t|\xi| - x \cdot \omega}{\langle x \rangle - x \cdot \omega} \right\} \\
 &\quad + M \left(\frac{\langle \xi \rangle}{|\xi|} - 1 \right) \log \left\{ \frac{\langle x - t\xi \rangle + t|\xi| - x \cdot \omega}{\langle x \rangle - x \cdot \omega} \right\} \\
 &\equiv A_0(t, x, \xi) + A_1(t, x, \xi).
 \end{aligned}$$

where $\omega = \xi/|\xi|$ ($\xi \neq 0$).

REMARK 2.1. If (1.4) is valid, we take $A(t, x, \xi)$ as

$$(2.2) \quad A(t, x, \xi) = \int_0^t \frac{M \langle \xi \rangle}{\langle x - s\xi \rangle^{1+\varepsilon}} ds.$$

Then we can see easily $|A\{\beta\}(t, x, \xi)| \leq C_{\alpha, \beta} t^{|\alpha|}$ for any multi-indices $\alpha, \beta, t \in [0, T]$ and $x, \xi \in R^n$, where $A\{\beta\}(t, x, \xi) = \partial_{\xi}^{\beta} D_x^{\beta} A(t, x, \xi)$.

LEMMA 2.2. One can find $C > 0$ such that

$$(2.3) \quad A(t, x, \xi) \leq CM(1 + \log \langle t\xi \rangle)$$

for $t \in [0, T]$ and $(x, \xi) \in R^{2n}$.

PROOF. When $\langle x \rangle \geq 2t|\xi|$, we obtain

$$\begin{aligned}
 A(t, x, \xi) &= \int_0^t \frac{M \langle \xi \rangle ds}{\langle x - s\xi \rangle} \leq \int_0^t \frac{M \sqrt{2} \langle \xi \rangle ds}{1 + |x| - t|\xi|} \\
 &\leq \int_0^t \frac{M \sqrt{2} \langle \xi \rangle}{1 + t|\xi|} ds \leq CM.
 \end{aligned}$$

When $\langle x \rangle \leq 2t|\xi|$, since

$$\frac{1}{\langle x \rangle - x \cdot \omega} = \frac{\langle x \rangle + x \cdot \omega}{\langle x \rangle^2 - (x \cdot \omega)^2} \leq 2 \langle x \rangle,$$

we have

$$\begin{aligned} A(t, x, \xi) &= M \frac{\langle \xi \rangle}{|\xi|} \log \left\{ \frac{\langle x - t\xi \rangle + t|\xi| - x \cdot \omega}{\langle x \rangle - x \cdot \omega} \right\} \\ &\leq M \frac{\langle \xi \rangle}{|\xi|} \log \left\{ \frac{2\sqrt{2}\langle t\xi \rangle + 2|x|}{\langle x \rangle - x \cdot \omega} \right\} \leq CM \log \langle t\xi \rangle. \end{aligned}$$

This completes the proof.

LEMMA 2.3. *One can find $C > 0$ and positive integer l such that*

$$(2.4) \quad \begin{cases} \exp A(t, x+y, \xi) \leq e^{CM}(1+|y|+t|\eta|)^{Ml}, \\ \exp A(t, x, \xi+\eta) \leq e^{CM}(1+|y|+t|\eta|)^{Ml} \end{cases}$$

for $|x| \geq 4t|\xi|$, $x, y, \xi, \eta \in R^n$, $t \in [0, T]$ and $M > 0$.

PROOF. When $|x| \geq 2|y|$, since $|x+y| \geq |x| - |y| \geq |x|/2 \geq 2t|\xi|$, we obtain

$$(2.5) \quad \begin{aligned} A(t, x+y, \xi) &= \int_0^t \frac{M\langle \xi \rangle}{\langle x+y-s\xi \rangle} ds \leq \int_0^t \frac{\sqrt{2}M\langle \xi \rangle}{1+|x+y|-t|\xi|} ds \\ &\leq \int_0^t \frac{\sqrt{2}M\langle \xi \rangle}{1+t|\xi|} ds \leq CM. \end{aligned}$$

When $|x| \leq 2|y|$, since $|y| \geq |x|/2 \geq 2t|\xi|$, (2.3) implies

$$(2.6) \quad A(t, x+y, \xi) \leq CM \log \langle t\xi \rangle \leq C'M \log \langle y \rangle.$$

When $|\xi| \leq |\eta|$, it follows from (2.3) that

$$(2.7) \quad A(t, x, \xi+\eta) \leq CM \log \langle t(\xi+\eta) \rangle \leq C'M \log \langle t\eta \rangle.$$

When $|\xi| \geq |\eta|$, we have

$$(2.8) \quad \begin{aligned} A(t, x, \xi+\eta) &= \int_0^t \frac{M\langle \xi+\eta \rangle}{\langle x-s(\xi+\eta) \rangle} ds \leq \int_0^t \frac{2\sqrt{2}M\langle \xi \rangle}{1+|x|-t|\xi|-t|\eta|} ds \\ &\leq \int_0^t \frac{2\sqrt{2}M\langle \xi \rangle}{1+2t|\xi|} ds \leq CM. \end{aligned}$$

This completes the proof of Lemma 2.3.

Now we put

$$(2.9) \quad f(x, \xi) = \langle x \rangle - x \cdot \left(\frac{\xi}{|\xi|} \right).$$

LEMMA 2.4. *One can find $C > 0$ such that*

$$(2.10) \quad \left(\frac{f(x', \xi + \eta)}{f(x', \xi)} \right)^{\pm 1} \leq C \langle \eta \rangle^4$$

for $|x| \leq 4t|\xi|$, $x, \xi, \eta \in R^n$ and $t \in [0, T]$, where $x' = x$ or $x - t\xi$.

PROOF. Since $1/3 \leq f(x', \xi) \leq 3$ for $|x'| \leq 1$, (2.10) is trivial for $|x'| \leq 1$. When $x' \cdot \xi \leq 0$, we have

$$(2.11) \quad \frac{f(x', \xi + \eta)}{f(x', \xi)} = \frac{\langle x' \rangle - x' \cdot ((\xi + \eta)/|\xi + \eta|)}{\langle x' \rangle - x' \cdot (\xi/|\xi|)} \leq \frac{2\langle x' \rangle}{\langle x' \rangle} \leq 2.$$

From now on we assume that $x' \cdot \xi \geq 0$ and $|x'| \geq 1$. Using the Taylor's formula, we may write

$$(2.12) \quad f(x', \xi + \eta) = f(x', \xi) + \sum_{j=1}^n f_{\xi_j}(x', \xi) \eta_j + 2 \sum_{|\alpha|=2} \int_0^1 \frac{1-\theta}{\alpha!} f^{(\alpha)}(x', \xi + \theta\eta) \eta^\alpha d\theta,$$

where $f_{\xi_j} = \partial f / \partial \xi_j$ and $f^{(\alpha)} = \partial^\alpha f$. Since $|x'| \leq 5t|\xi|$, we have

$$(2.13) \quad \begin{aligned} & \left| \sum_{|\alpha|=2} \int_0^1 \frac{1-\theta}{\alpha!} f^{(\alpha)}(x', \xi + \theta\eta) \eta^\alpha d\theta / f(x', \xi) \right| \\ & \leq \int_0^1 \frac{C|x'|}{\langle \langle x' \rangle - x' \cdot (\xi/|\xi|) \rangle} \frac{\langle \eta \rangle^2}{\langle \xi + \theta\eta \rangle^2} d\theta \\ & \leq \int_0^1 \frac{C(\langle x' \rangle + x' \cdot (\xi/|\xi|))}{\langle \langle x' \rangle^2 - (x' \cdot (\xi/|\xi|))^2 \rangle} \frac{|x'| \langle \eta \rangle^2}{\langle \xi + \theta\eta \rangle^2} d\theta \\ & \leq \int_0^1 \frac{C\langle x' \rangle^2 \langle \eta \rangle^4}{\langle \xi + \theta\eta \rangle^2 \langle \theta\eta \rangle^2} d\theta \leq \frac{C\langle x' \rangle^2 \langle \eta \rangle^4}{\langle \xi \rangle^2} \\ & \leq \frac{C\langle 5t\xi \rangle^2 \langle \eta \rangle^4}{\langle \xi \rangle^2} \leq C' \langle \eta \rangle^4 \end{aligned}$$

and

$$(2.14) \quad \begin{aligned} \frac{f_{\xi_j}(x', \xi)}{f(x', \xi)} &= \frac{((x' \cdot \xi)\xi_j - x'_j |\xi|^2) (\langle x' \rangle + (x' \cdot \xi)/|\xi|)}{|\xi|^3 \langle \langle x' \rangle^2 - ((x' \cdot \xi)/|\xi|)^2 \rangle} \\ &= \frac{((x' \cdot \omega)\omega_j - x'_j) (\langle x' \rangle + x' \cdot \omega)}{|\xi| (|x'|^2 + 1 - (x' \cdot \omega)^2)} \\ &\leq \frac{2((x' \cdot \omega)\omega_j - x'_j) \langle x' \rangle}{|\xi| \{ (|x'| + x' \cdot \omega)(|x'| - x' \cdot \omega) + 1 \}} \\ &\leq \frac{2\langle x' \rangle |h_j(x', \omega)|}{|\xi| \{ (|x'| + x' \cdot \omega)(|x'| - x' \cdot \omega) + 1 \}} \end{aligned}$$

where $\omega_j = \xi_j/|\xi|$ and $h_j(x', \omega) = (x' \cdot \omega)\omega_j - x'_j$. Since $h_j(\omega, \omega) = 0$, we have $|x'| |h_j(\omega, \omega) - h_j(x', \omega)| = 0$. From the mean value theorem it follows that

$$\begin{aligned} h_j(x', \omega) &= h_j(x', \omega) - h_j(|x'| \omega, \omega) \\ &= \sum_{k=1}^n h_{jx'_k}(x' + \theta(x' - |x'| \omega), \omega)(x'_k - |x'| \omega_k). \end{aligned}$$

Since $h_j(x', \omega)$ is a homogeneous functions of degree 1 with respect to x' , there exists a constant $c > 0$ such that $|h_{jx'_k}(x' + \theta(x' - |x'| \omega), \omega)| \leq c$. Hence we obtain

$$\begin{aligned} |h_j(x', \omega)|^2 &\leq c^2 |x' - \omega| |x'|^2 = c^2 (|x'|^2 - 2(x' \cdot \omega) |x'| + |\omega|^2 |x'|^2) \\ &= 2c^2 |x'| (|x'| - (x' \cdot \omega)). \end{aligned}$$

Consequently we have

$$(2.15) \quad |h_j(x', \omega)| \leq \sqrt{2c} |x'|^{1/2} (|x'| - (x' \cdot \omega))^{1/2}.$$

Noting that $1 \leq |x'| \leq 5t|\xi|$, from (2.14) and (2.15) we have

$$(2.16) \quad \frac{f_{\xi_j}(x', \xi)}{f(x', \xi)} \leq \frac{2\sqrt{2c}(|x'| (|x'| - x' \cdot \omega))^{1/2} \langle x' \rangle}{|\xi| \{|x'| (|x'| - (x' \cdot \omega)) + 1\}} \leq C.$$

By (2.11), (2.12), (2.13) and (2.16), we obtain

$$(2.17) \quad \frac{f(x', \xi + \eta)}{f(x', \xi)} \leq C \langle \eta \rangle^4.$$

Moreover if we put $\xi + \eta = z$, it follows from (2.17) that

$$(2.18) \quad \frac{f(x', \xi)}{f(x', \xi + \eta)} = \frac{f(x', z - \eta)}{f(x', z)} \leq C \langle -\eta \rangle^4 = C \langle \eta \rangle^4.$$

This proves Lemma 2.4.

LEMMA 2.5. *One can find $C > 0$ such that*

$$(2.19) \quad \left(\frac{f(x+y, \xi)}{f(x, \xi)} \right)^{\pm 1} \leq C \langle y \rangle^3$$

for $x, y, \xi \in R^n$.

PROOF. (2.19) is trivial for $|x| \leq 1$. When $x \cdot \xi \leq 0$, we have

$$\begin{aligned} (2.20) \quad \frac{f(x+y, \xi)}{f(x, \xi)} &= \frac{\langle x+y \rangle - (x+y) \cdot (\xi/|\xi|)}{\langle x \rangle - x \cdot (\xi/|\xi|)} \\ &\leq \frac{2\langle x+y \rangle}{\langle x \rangle} \leq 2^2 \langle y \rangle. \end{aligned}$$

From now on we assume that $x \cdot \xi \geq 0$ and $|x| \geq 1$. Using Taylor's formula, we may write

$$(2.21) \quad f(x+y, \xi) = f(x, \xi) + \sum_{j=1}^n f_{x_j}(x, \xi) y_j + 2 \sum_{|\alpha|=2} \int_0^1 \frac{1-\theta}{\alpha!} (iy)^\alpha f_{(\alpha)}(x+\theta y, \xi) d\theta,$$

where $f_{x_j} = \partial f / \partial x_j$ and $f_{(\alpha)} = D_x^\alpha f$. It follows that

$$(2.22) \quad \left| \sum_{|\alpha|=2} \int_0^1 \frac{1-\theta}{\alpha!} (iy)^\alpha f_{(\alpha)}(x+\theta y, \xi) d\theta / f(x, \xi) \right| \\ \leq \int_0^1 \frac{C|y|^2}{\langle x+\theta y \rangle \langle x \rangle - x \cdot (\xi/|\xi|)} d\theta \\ \leq \int_0^1 \frac{C\langle x \rangle \langle y \rangle^2}{\langle x+\theta y \rangle} d\theta \leq \int_0^1 \frac{C\langle x \rangle \langle y \rangle^3}{\langle x+\theta y \rangle \langle \theta y \rangle} d\theta \leq C' \langle y \rangle^3,$$

and

$$(2.23) \quad \frac{f_{x_j}(x, \xi)}{f(x, \xi)} = \frac{x_j - \omega_j \langle x \rangle}{\langle x \rangle f(x, \xi)} = \frac{(\langle x \rangle + (x \cdot \omega))(x_j - \omega_j \langle x \rangle)}{\langle x \rangle (|x|^2 - (x \cdot \omega)^2 + 1)} \\ \leq \frac{2|x - \omega \langle x \rangle|}{|x|^2 - (x \cdot \omega)^2 + 1} \leq \frac{2(|x - \omega||x| + |\omega||x| - \omega \langle x \rangle)}{(|x| + x \cdot \omega)(|x| - x \cdot \omega) + 1} \\ \leq \frac{2p(x, \omega)}{|x|(|x| - x \cdot \omega) + 1} + \frac{2}{\langle x \rangle + |x|}$$

where $\omega_j = \xi_j / |\xi|$ and $p(x, \omega) = |x - \omega||x|$. Since

$$p(x, \omega)^2 = |x - \omega||x|^2 = |x|^2 - 2|x| x \cdot \omega + |x|^2 \\ = 2(|x| - x \cdot \omega)|x|,$$

we have

$$(2.24) \quad p(x, \omega) = \sqrt{2} (|x| - x \cdot \omega) |x|^{1/2}.$$

It follows from (2.23) and (2.24) that

$$(2.25) \quad \frac{f_{x_j}(x, \xi)}{f(x, \xi)} \leq \frac{2\sqrt{2}(|x|(|x| - x \cdot \omega))^{1/2}}{|x|(|x| - x \cdot \omega) + 1} + \frac{2}{\langle x \rangle + |x|} \leq C,$$

By (2.20), (2.21), (2.22) and (2.25), we have

$$(2.26) \quad \frac{f(x+y, \xi)}{f(x, \xi)} \leq C \langle y \rangle^3.$$

Moreover if we put $x+y=z$, it follows from (2.26) that

$$(2.27) \quad \frac{f(x, \xi)}{f(x+y, \xi)} = \frac{f(z-y, \xi)}{f(z, \xi)} \leq C \langle -y \rangle = C \langle y \rangle^3.$$

This completes the proof of Lemma 2.5.

LEMMA 2.6. One can find $C > 0$ and a positive integer l such that

$$(2.28) \quad \exp \{-\Lambda(t, x, \xi + \eta) + \Lambda(t, x+y, \xi)\} \leq e^{CM} (|y| + t|\eta| + 1)^{Ml}$$

for $x, y, \xi, \eta \in R^n$, $t \in [0, T]$ and $M > 0$.

PROOF. When $|x| \geq 4t|\xi|$, we have (2.28) from Lemma 2.3. When $|x| \leq 4t|\xi|$, it follows from (2.1) that

$$\begin{aligned}
& \exp \{-A(t, x, \xi + \eta) + A(t, x + y, \xi)\} \\
&= \left\{ \frac{\langle x \rangle - x \cdot ((\xi + \eta) / |\xi + \eta|)}{\langle x - t(\xi + \eta) \rangle + t|\xi + \eta| - x \cdot ((\xi + \eta) / |\xi + \eta|)} \times \frac{\langle x + y - t\xi \rangle + t|\xi| - (x + y) \cdot (\xi / |\xi|)}{\langle x + y \rangle - (x + y) \cdot (\xi / |\xi|)} \right\}^M \\
& \quad \times \exp \{A_1(t, x, \xi + \eta) + A_1(t, x + y, \xi)\} \\
&= \left\{ \frac{\langle x \rangle - x \cdot ((\xi + \eta) / |\xi + \eta|)}{\langle x \rangle - x \cdot (\xi / |\xi|)} \times \frac{\langle x - t\xi \rangle - (x - t\xi) \cdot (\xi / |\xi|)}{\langle x - t\xi \rangle - (x - t\xi) \cdot ((\xi + \eta) / |\xi + \eta|)} \right. \\
& \quad \times \frac{\langle x - t\xi \rangle - (x - t\xi) \cdot ((\xi + \eta) / |\xi + \eta|)}{\langle x - t(\xi + \eta) \rangle - (x - t(\xi + \eta)) \cdot ((\xi + \eta) / |\xi + \eta|)} \times \frac{\langle x \rangle - x \cdot (\xi / |\xi|)}{\langle x + y \rangle - (x + y) \cdot (\xi / |\xi|)} \\
& \quad \left. \times \frac{\langle x + y - t\xi \rangle - (x + y - t\xi) \cdot (\xi / |\xi|)}{\langle x - t\xi \rangle - (x - t\xi) \cdot (\xi / |\xi|)} \right\}^M \times \exp \{-A_1(t, x, \xi + \eta) + A_1(t, x + y, \xi)\} \\
&= \left\{ \frac{f(x, \xi + \eta)}{f(x, \xi)} \times \frac{f(x - t\xi, \xi)}{f(x - t\xi, \xi + \eta)} \times \frac{f(x - t\xi, \xi + \eta)}{f(x - t(\xi + \eta), \xi + \eta)} \times \frac{f(x, \xi)}{f(x + y, \xi)} \right. \\
& \quad \left. \times \frac{f(x + y - t\xi, \xi)}{f(x - t\xi, \xi)} \right\}^M \times \exp \{-A_1(t, x, \xi + \eta) + A_1(t, x + y, \xi)\}.
\end{aligned}$$

Because $A_1(t, x, \xi)$ is bounded when $|\xi| \geq 1$, (2.28) can be obtained by using Lemmas 2.4 and 2.5.

LEMMA 2.7. For any multi-indices α, β ($|\alpha + \beta| \geq 1$), we have

$$(2.29) \quad |A_{\{\beta\}}^{\{\alpha\}}(t, x, \xi)| \leq C_{\alpha, \beta} t^{|\alpha|}$$

for $x, \xi \in R^n$ and $t \in [0, T]$.

PROOF. For any multi-indices α, β ($|\alpha + \beta| \geq 1$) we can estimate

$$\left| \partial_{\xi}^{\alpha} D_x^{\beta} \left(\frac{\langle \xi \rangle}{\langle x - s\xi \rangle} \right) \right| \leq \begin{cases} C_{\alpha, \beta} \frac{s^{|\alpha| - 1}}{\langle x - s\xi \rangle^{|\alpha + \beta|}} + C'_{\alpha, \beta} \frac{\langle \xi \rangle s^{|\alpha|}}{\langle x - s\xi \rangle^{|\alpha + \beta| + 1}} & \text{for } |\alpha| \geq 1, \\ C''_{\alpha, \beta} \frac{\langle \xi \rangle}{\langle x - s\xi \rangle^{|\beta| + 1}} & \text{for } |\alpha| = 0 \end{cases}$$

Therefore we have

$$|A_{\{\beta\}}^{\{\alpha\}}(t, x, \xi)| = \left| \partial_{\xi}^{\alpha} D_x^{\beta} \int_0^t \frac{M \langle \xi \rangle ds}{\langle x - s\xi \rangle} \right| \leq C_{\alpha, \beta} t^{|\alpha|}.$$

This proves Lemma 2.7.

Let $\sigma(K)(t, x, D_x)$ and $\sigma(\tilde{K})(t, x, D_x)$ be pseudo-differential operators with its symbols $\sigma(K)(t, x, \xi) = \exp(A(t, x, \xi))$ and $\sigma(\tilde{K})(t, x, \xi) = \exp(-A(t, x, \xi))$ respectively. Then the symbol of the product of $\tilde{K}(t, x, D_x)$ and $K(t, x, D_x)$ is

given by

$$(2.30) \quad \sigma(\tilde{K} \circ K)(t, x, \xi) = 1 - \sigma(R)(t, x, \xi),$$

where

$$(2.31) \quad \sigma(R)(t, x, \xi) = \sum_{|\gamma|=1} \int_0^1 O_s - \iint e^{-iy \cdot \eta} \{D_\xi^\gamma A(t, x, \xi + \eta)\} e^{-A(t, x, \xi + \eta)} \\ \times \{\partial_x^\gamma A(t, x + \theta y, \xi)\} e^{A(t, x + \theta y, \xi)} dy d\eta d\theta \quad (d\eta = (2\pi)^{-n} d\eta).$$

Here an oscillatory integral of a symbol $a(x, \xi)$ means

$$O_s - \iint e^{-iy \cdot \eta} a(y, \eta) dy d\eta = \lim_{\varepsilon \rightarrow 0} \iint e^{-iy \cdot \eta} \chi(\varepsilon y, \varepsilon \eta) a(y, \eta) dy d\eta$$

for $\chi \in \mathcal{S}$ in R^{2n} such that $\chi(0, 0) = 1$.

LEMMA 2.8. Assume that $A(t, x, \xi)$ satisfies (2.28) and (2.29). Let $R_\pm(t, x, D_x)$ be a pseudo-differential operators with its symbol $r_\pm(t, x, \xi)$ satisfying

$$(2.32) \quad |r_\pm\{\beta\}(t, x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m_\pm} \langle x \rangle^{k_\pm} e^{\pm A(t, x, \xi)}$$

for $t \in [0, T]$ and $x, \xi \in R^n$, where m_\pm and k_\pm are real numbers. Then $q(t, x, \xi) = \sigma(R_- \circ R_+)(t, x, \xi)$ satisfies

$$|q\{\beta\}(t, x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m_- + m_+} \langle x \rangle^{k_- + k_+}$$

for $t \in [0, T]$ and $x, \xi \in R^n$.

PROOF. $q(t, x, \xi)$ is written by

$$q(t, x, \xi) = O_s - \iint e^{-iy \cdot \eta} r_-(t, x, \xi + \eta) r_+(t, x + y, \xi) dy d\eta.$$

Noting that (2.32), (2.28), (2.29) and $e^{-iy \cdot \eta} = \langle y \rangle^{-2m} \langle D_\eta \rangle^{2m} e^{-iy \cdot \eta} = \langle \eta \rangle^{-2m} \langle D_y \rangle^{2m} e^{-iy \cdot \eta}$ are valid, we get by use of integration by parts

$$|q\{\beta\}(t, x, \xi)| \\ \leq \sum_{\substack{\alpha' \leq \alpha \\ \beta' \leq \beta}} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \iint |\langle y \rangle^{-2m} \langle D_\eta \rangle^{2m} \\ \times (\langle \eta \rangle^{-2m} \langle D_y \rangle^{2m} (r_-)\{\alpha'\}(t, x, \xi + \eta) (r_+)\{\beta - \beta'\}(t, x + y, \xi)) dy d\eta \\ \leq C_{\alpha, \beta, m} \langle \xi \rangle^{m_- + m_+} \langle x \rangle^{k_- + k_+} \iint \langle y \rangle^{-2m + k_+} \langle \eta \rangle^{-2m + m_-} \\ \times \exp \{-A(t, x, \xi + \eta) + A(t, x + y, \xi)\} dy d\eta \\ \leq C'_{\alpha, \beta, m} \langle \xi \rangle^{m_- + m_+} \langle x \rangle^{k_- + k_+} \iint \langle y \rangle^{-2m + k_+ + Ml} \langle \eta \rangle^{-2m + m_- + Ml} dy d\eta.$$

Taking $m = \max([\frac{k_+ + Ml + n}{2} + 1], [\frac{m_- + Ml + n}{2} + 1])$, we get Lemma 2.8.

Here $[s]$ denotes the largest integer not greater than s .

It follows from Lemma 2.7 and Lemma 2.8 that $\sigma(R)(t, x, \xi)$ given in (2.31) satisfies $|\sigma(R)(t, x, \xi)| \leq t C_{\alpha, \beta}$ for $t \in [0, T]$ and $x, \xi \in R^n$. So we get the inverse operator of K by the following Lemma 2.9.

LEMMA 2.9 (Ichinose [2], Lemma 2). *If T_0 ($0 < T_0 \leq T$) is small, for $t \in [0, T_0]$ the inverse operator $(I-R)^{-1}(t, x, D_x)$ of $(I-R)(t, x, D_x)$ exists as the continuous map from H_s to H_s space ($s \in R^+$) and $\sigma((I-R)^{-1})(t, x, \xi)$ belongs to $S_{0,0}^s$ uniformly in $t \in [0, T_0]$. Moreover the inverse operator $K^{-1}(t, x, D_x)$ of $K(t, x, D_x)$ is given by*

$$(2.33) \quad K^{-1}(t, x, D_x) = (I-R)^{-1}(t, x, D_x) \circ \tilde{K}(t, x, D_x).$$

§ 3. Proof of Theorem.

We put $u(t, x) = Kv(t, x)$, where $\sigma(K)(t, x, \xi) = \exp(A(t, x, \xi))$. Then noting that $A(t, x, \xi)$ satisfies (1.5) we have

$$(3.1) \quad \begin{aligned} Lu(t, x) &= L \circ Kv(t, x) \\ &= K \circ \left(D_t - \frac{1}{2} \Delta \right) v(t, x) \\ &\quad + \int e^{ix \cdot \xi} \left(D_t A(t, x, \xi) - \sum_{j=1}^n a_j(t, x) \xi_j + \sum_{j=1}^n \xi_j D_j A \right) e^A v(t, \xi) d\xi \\ &\quad + \int e^{ix \cdot \xi} \left\{ \frac{1}{2} \sum_{j=1}^n ((D_j A)^2 + (D_j a_j) - 2a_j D_j A + a_j^2) - \frac{1}{2} \Delta A + c(t, x) \right\} \\ &\quad \times e^A v(t, \xi) d\xi \\ &= K \circ \left(D_t - \frac{1}{2} \Delta \right) v(t, x) \\ &\quad - K \circ \int e^{ix \cdot \xi} \left(\frac{iM \langle \xi \rangle}{\langle x \rangle} + \sum_{j=1}^n a_j \xi_j \right) v(t, \xi) d\xi + \sum_{j=1}^n K_j v(t, x) = f(t, x), \end{aligned}$$

where $\Delta = \sum_{j=1}^n \partial^2 / \partial x_j^2$,

$$(3.2) \quad \begin{aligned} \sigma(K_1)(t, x, \xi) &= \left\{ \frac{1}{2} \sum_{j=1}^n ((D_j A(t, x, \xi))^2 + (D_j a_j(t, x)) - 2a_j D_j A + a_j^2) - \frac{1}{2} \Delta A + c(t, x) \right\} e^A, \end{aligned}$$

and

$$(3.3) \quad \begin{aligned} \sigma(K_2)(t, x, \xi) &= - \sum_{|\gamma|=1} \int_0^1 O_s - \iint e^{-iy \cdot \eta} \{ e^{A(t, x, \xi + \eta)} \}_{(\gamma)} \\ &\quad \times \left\{ \sum_{j=1}^n a_j(t, x + \theta y) \xi_j + \frac{iM \langle \xi \rangle}{\langle x + \theta y \rangle} \right\}_{(\gamma)} dy d\eta d\theta. \end{aligned}$$

LEMMA 3.1. One can find $C_{\alpha, \beta, M}$ and $C'_{\alpha, \beta, M}$ such that

$$(3.4) \quad \begin{cases} |\sigma(K_1)\{\beta\}(t, x, \xi)| \leq C_{\alpha, \beta, M} e^{A(t, x, \xi)}, \\ |\sigma(K_2)\{\beta\}(t, x, \xi)| \leq t C'_{\alpha, \beta, M} \langle \xi \rangle \langle x \rangle^{-1} e^{A(t, x, \xi)} \end{cases}$$

for $t \in [0, T]$ and $x, \xi \in R^n$.

PROOF. By (2.28) and (2.29), the first estimate of (3.4) can be shown by simple computation. Noting that (2.28), (2.29) and $e^{-iy \cdot \eta} = \langle y \rangle^{-2m} \langle D_\eta \rangle^{2m} e^{-iy \cdot \eta} = \langle \eta \rangle^{-2m} \langle D_y \rangle^{2m} e^{-iy \cdot \eta}$ are valid, we get by use of integration by parts

$$\begin{aligned} |\sigma(K_2)\{\beta\}(t, x, \xi)| &= \left| \sum_{\substack{\alpha' \leq \alpha \\ \beta' \leq \beta}} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \sum_{l=1}^{\infty} \int_0^1 O_S - \iint e^{-iy \cdot \eta} \{e^{A(t, x, \xi + \eta)}\}_{\{\beta'\}^{\alpha'}} \right. \\ &\quad \times \left. \left\{ \sum_{j=1}^n a_j(t, x + \theta y) \xi_j + \frac{iM \langle \xi \rangle}{\langle x + \theta y \rangle} \right\}_{(\gamma + \beta - \beta')}^{(\alpha - \alpha')} dy d\eta d\theta \right| \\ &\leq t C_{\alpha, \beta, m} \frac{\langle \xi \rangle}{\langle x \rangle} e^{A(t, x, \xi)} \iint \langle y \rangle^{-2m+1} \langle \eta \rangle^{-2m} \\ &\quad \times \exp \{A(t, x, \xi + \eta) - A(t, x, \xi)\} dy d\eta \\ &\leq t C_{\alpha, \beta, m} \frac{\langle \xi \rangle}{\langle x \rangle} e^{A(t, x, \xi)} \iint \langle y \rangle^{-2m+1} \langle \eta \rangle^{-2m+Ml} dy d\eta, \end{aligned}$$

where l is a positive integer given in Lemma 2.6. Taking $m = \max([\frac{n+1}{2} + 1], [\frac{n+Ml}{2} + 1])$, we get the second estimate of (3.4).

Therefore we can transform the Cauchy problem (1.1) to the following problem

$$(3.5) \quad \begin{cases} (D_t - P(t))v(t, x) = \tilde{f}(t, x), & (t, x) \in [0, T] \times R^n, \quad (T > 0), \\ v(0, x) = v_0(x) \quad (= u_0(x)), & x \in R^n \end{cases}$$

where $\tilde{f}(t, x) = K^{-1}f(t, x)$ and

$$(3.6) \quad P(t, x, D_x) = \frac{1}{2} \Delta + \tilde{a}(t, x, D_x) - \sum_{j=1}^2 K^{-1} \circ K_j(t, x, D_x).$$

Here

$$(3.7) \quad \sigma(\tilde{a})(t, x, \xi) = \frac{iM \langle \xi \rangle}{\langle x \rangle} + \sum_{j=1}^n a_j(t, x) \xi_j.$$

Then it follows from Lemma 2.7, Lemma 2.9 and Lemma 3.1 that $\sigma(K^{-1} \circ K_1) \times (t, x, \xi)$ and $\sigma(K^{-1} \circ K_2)(t, x, \xi)$ satisfy

$$(3.8) \quad \begin{cases} |\sigma(K^{-1} \circ K_1)\{\beta\}(t, x, \xi)| \leq C_{\alpha, \beta, M}, \\ |\sigma(K^{-1} \circ K_2)\{\beta\}(t, x, \xi)| \leq t C'_{\alpha, \beta, M} \langle \xi \rangle \langle x \rangle^{-1} \end{cases}$$

for $t \in [0, T_0]$ and $x, \xi \in R^n$.

THEOREM 3.2. *Suppose (1.3) is valid. Then there is $0 < T_1 \leq T$ such that for any $v_0 \in H_{s+2}$ and $\tilde{f}(t, x) \in C^0([0, T_1]; H_{s+2})$ there exists a unique solution $v(t, x)$ of (3.5) which belongs to $C^1([0, T_1]; H_s) \cap C^0([0, T_1]; H_{s+2})$ and moreover for any $s \in R^1$ there exists a constant $C(s, T) > 0$ such that*

$$(3.9) \quad \|v(t, \cdot)\|_{(s+j)} \leq C(s, T) \left\{ \|v_0\|_{(s+j)} + \int_0^t \|\tilde{f}(\tau, \cdot)\|_{(s+j)} d\tau \right\}$$

for $t \in [0, T_1]$, $j=0, 1, 2$.

The proof of this theorem is the same way as that of Theorem 4.1 in [1].

Following the idea of Kumano-go [4] we introduce the series $\{\zeta_\nu(\xi)\}_{\nu=1}^\infty$ as

$$(3.10) \quad \zeta_\nu(\xi) = \left(\nu \sin \frac{\xi_1}{\nu}, \dots, \nu \sin \frac{\xi_n}{\nu} \right)$$

and define $P_\nu(t) = p_\nu(t, x, D_x)$ as

$$(3.11) \quad p_\nu(t, x, \xi) = p(t, x, \zeta_\nu(\xi)).$$

We consider the following Cauchy problem

$$(3.12) \quad \begin{cases} L_\nu v_\nu = D_t v_\nu - P_\nu(t) v_\nu = \tilde{f}(t) & (t \in [0, T]), \\ v_\nu|_{t=0} = u_0. \end{cases}$$

We define the series of weight functions $\{\lambda_\nu(\xi)\}_{\nu=1}^\infty$ as

$$(3.13) \quad \lambda_\nu(\xi) = \langle \zeta_\nu(\xi) \rangle = \left\{ 1 + \sum_{j=1}^n \left(\nu \sin \frac{\xi_j}{\nu} \right)^2 \right\}^{1/2}.$$

Then $\{\lambda_\nu(\xi)\}_{\nu=1}^\infty$ satisfies

$$(3.14) \quad \begin{cases} \text{i) } 1 \leq \lambda_\nu(\xi) \leq \min(\langle \xi \rangle, \sqrt{1+n\nu^2}), \\ \text{ii) } |\partial_\xi^\alpha \lambda_\nu(\xi)| \leq A_\alpha \lambda_\nu(\xi)^{1-|\alpha|}, \\ \text{iii) } \lambda_\nu(\xi) \longrightarrow \langle \xi \rangle \quad (\nu \longrightarrow \infty) \text{ on } R_\xi^n, \end{cases}$$

(uniform convergence in a compact set).

Denote by $S_{\lambda_\nu, \rho, \delta}^m$ ($0 \leq \delta \leq \rho \leq 1$, $\delta < 1$) the set of symbols $q(x, \xi) \in C^\infty(R^{2n})$ satisfying

$$(3.15) \quad |q^{(\alpha)}(x, \xi)| \leq C_{\alpha\beta} \lambda_\nu(\xi)^{m-\rho|\alpha|+\delta|\beta|}$$

for any multi-index α, β and $S_{\rho, \delta}^m = S_{\langle \xi \rangle, \rho, \delta}^m$. Then we get the following lemma (Kumanogo [4], Ch. 7, Lemma 3.3).

LEMMA 3.3. For $q(x, \xi) \in S_{\rho, \delta}^m$ ($0 \leq \delta \leq \rho \leq 1$, $\delta < 1$), we put $q_\nu(x, \xi) = q(x, \zeta_\nu(\xi))$. Then $q_\nu(x, \xi) \in S_{\lambda_\nu, \rho, \delta}^m$ ($0 \leq \delta \leq \rho \leq 1$, $\delta < 1$), and for any α, β there is constant $A_{\alpha, \beta}$ which is independent of ν and q such that

$$(3.16) \quad \begin{cases} |q_{\nu\{\beta\}}(x, \xi)| \leq (A_{\alpha, \beta} |q|_{\alpha+\beta}^{(m)}) \lambda_\nu(\xi)^{m-\rho|\alpha|+\delta|\beta|}, \\ q_\nu(x, \xi) \longrightarrow q(x, \xi) \quad (\text{uniformly}) \quad (\nu \rightarrow \infty) \text{ in } R_x^n \times K_\xi \end{cases}$$

where K_ξ is an arbitrary compact set of R_ξ^n and $|q|_l^{(0)} = \max_{|\alpha+\beta| \leq l} \sup_{x, \xi \in R^n \times \{|q\{\beta\}}(x, \xi)|}$.

We get the following lemma (Kumano-go [4], Ch. 7, Theorem 1.6).

LEMMA 3.4. $Q = q(x, D_x) \in S_{0,0}^0$ is a continuous mapping from L_2 to L_2 and there are $C > 0$ and a positive integer l such that

$$(3.17) \quad \|Qu\|_{L_2} \leq (C |q|_l^{(0)}) \|u\|_{L_2} \quad \text{for } u \in L_2(R^n).$$

PROPOSITION 3.5. Suppose (1.3) is valid. Then there is T_1 ($0 < T_1 \leq T_0$) such that for any $v_0 \in L_2$ and any $\tilde{f}(t, x) \in C^1([0, T_1]; L_2)$ and there exists a unique solution $v_\nu(t, x) \in C^1([0, T_1]; L_2)$ of (3.12) which satisfies the energy inequalities

$$(3.18) \quad \|v_\nu(t)\| \leq C_1(T_1) \left\{ \|v_0\| + \int_0^t \|\tilde{f}(\tau)\| d\tau \right\} \quad (t \in [0, T_1]),$$

$$(3.19) \quad \|A_\nu^j v_\nu(t)\| \leq C_2(T_1) \left\{ \|A_\nu^j v_0\| + \int_0^t \|A_\nu^j \tilde{f}(\tau)\| d\tau \right\} \quad (t \in [0, T_1]),$$

$$(3.20) \quad \left\| \frac{d}{dt} A_\nu^j v_\nu(t) \right\| \leq C_3(T_1) \left\{ \|A_\nu^{j+2} v_0\| + \max_{[0, T_1]} \|A_\nu^{j+2} \tilde{f}(\tau)\| \right\} \quad (t \in [0, T_1]),$$

$$(3.21) \quad \|A_\nu^j(v_\nu(t) - v_\nu(t'))\| \leq C_4(T_1) |t - t'| \left\{ \|A_\nu^{j+2} v_0\| + \max_{[0, T_1]} \|A_\nu^{j+2} \tilde{f}(\tau)\| \right\} \\ (t, t' \in [0, T_1])$$

where $C_1(T_1)$, $C_2(T_1)$, $C_3(T_1)$ and $C_4(T_1)$ are constants which are independent of ν , and $A_\nu = \lambda_\nu(D_x)$, $\|\cdot\| = \|\cdot\|_{L_2}$, $j=0, 1, 2$.

PROOF. I) If we fix ν arbitrarily, we have $p_\nu(t, x, \xi) \in \mathcal{B}^1([0, T]; \mathcal{B}^\infty(R_{x, \xi}^{2n}))$. Since $\mathcal{B}^\infty(R_{x, \xi}^{2n}) = S_{0,0}^0$, from Lemma 3.4 it follows that $P_\nu(t)$ is an L_2 -bounded operator uniformly with respect to t . Therefore there is a unique solution $v_\nu(t) \in C^1([0, T_1]; L^2)$ of the integral equation

$$(3.22) \quad v_\nu(t) = v_0 + i \int_0^t P_\nu(\tau) v_\nu(\tau) d\tau + i \int_0^t \tilde{f}(\tau) d\tau.$$

II) By straitforward computation we have

$$(3.28) \quad \langle x \rangle^{-1} \circ A_\nu = A_\nu^* \circ A_\nu + B_\nu,$$

where $\sigma(A_\nu)(\xi) = \lambda_\nu(\xi)$, $A_\nu^* = A_\nu^{1/2} \circ \langle x \rangle^{-1/2}$, $A_\nu = \langle x \rangle^{-1/2} \circ A_\nu^{1/2}$ and $\sigma(B_\nu)(x, \xi) \in S_{\nu,0}^0$ uniformly in ν . It follows from (3.12) that

$$(3.24) \quad \begin{aligned} \frac{d}{dt} \|v_\nu(t, \cdot)\|^2 &= 2 \operatorname{Re} \left(\frac{d}{dt} v_\nu, v_\nu \right) \\ &= \operatorname{Re}(i \Delta_\nu v_\nu, v_\nu) - 2M \operatorname{Re}(\langle x \rangle^{-1} \circ A_\nu v_\nu, v_\nu) \\ &\quad - \operatorname{Re}(a_\nu^I(t, x, D_x) v_\nu, v_\nu) + \operatorname{Re}(i a_\nu^R(t, x, D_x) v_\nu, v_\nu) \\ &\quad - \operatorname{Re}(i(K^{-1} \circ K_1)_\nu v_\nu, v_\nu) - \operatorname{Re}(i(K^{-1} \circ K_2)_\nu v_\nu, v_\nu) + 2 \operatorname{Re}(i \tilde{f}, v_\nu), \end{aligned}$$

where $\sigma(\Delta_\nu)(\xi) = -\sum_{j=1}^n (\nu \sin(\xi_j/\nu))^2$, $\sigma(a_\nu^I)(t, x, \xi) = \sum_{j=1}^n a_j^I(t, x) \zeta_{\nu j}(\xi)$ and $\sigma(a_\nu^R)(t, x, \xi) = \sum_{j=1}^n a_j^R(t, x) \zeta_{\nu j}(\xi)$. We put

$$J(t, x, D_x) = \langle x \rangle^{1/2} \circ A_\nu^{-1/2} \circ (K^{-1} \circ K_2)_\nu \circ A_\nu^{-1/2} \circ \langle x \rangle^{1/2}.$$

Then we have

$$(3.25) \quad |J\{\beta\}(t, x, \xi)| \leq t C_{\alpha, \beta}$$

for $t \in [0, T_0]$ and $x, \xi \in R^n$. In fact, it follows from (3.8) and Lemma 3.3 that $W_\nu = (K^{-1} \circ K_2)_\nu$ satisfies

$$(3.26) \quad |W_\nu\{\beta\}(t, x, \xi)| \leq t C_{\alpha, \beta, M} \lambda_\nu(\xi) \langle x \rangle^{-1}$$

for $t \in [0, T_1]$, $x, \xi \in R^n$ and $\nu = 1, 2, \dots$. Moreover we can express

$$\begin{aligned} J(t, x, \xi) &= O_S - \iint e^{-i\tilde{y}^3 \cdot \tilde{\eta}^3} \langle x + \tilde{y}^1 \rangle^{1/2} \lambda_\nu^{-1/2}(\xi + \eta^1) W_\nu(t, x + \tilde{y}^2, \xi + \eta^2) \\ &\quad \times \lambda_\nu^{-1/2}(\xi + \eta^3) \langle x + \tilde{y}^3 \rangle^{1/2} d y^1 d y^2 d y^3 d \eta^1 d \eta^2 d \eta^3, \end{aligned}$$

where $\tilde{y}^3 \cdot \tilde{\eta}^3 = y^1 \cdot \eta^1 + y^2 \cdot \eta^2 + y^3 \cdot \eta^3$ and $\tilde{y}^j = y^1 + \dots + y^j$ ($j=1, 2, 3$). By virtue of (3.26) we get (3.25) in the same way as the proof of Lemma 3.1. By (3.25), Lemma 3.4, and the Schwartz' inequality, we have

$$(3.27) \quad \begin{aligned} \operatorname{Re}((K^{-1} \circ K_2)_\nu v_\nu, v_\nu) &= \operatorname{Re}(J(x, D) \langle x \rangle^{-1/2} A_\nu^{1/2} v_\nu, \langle x \rangle^{-1/2} A_\nu^{1/2} v_\nu) \\ &\leq \|J(x, D) \langle x \rangle^{-1/2} A_\nu^{1/2} v_\nu\| \|\langle x \rangle^{-1/2} A_\nu^{1/2} v_\nu\| \\ &\leq t C_M |J\{i^0\}| \|\langle x \rangle^{-1/2} A_\nu^{1/2} v_\nu\|^2. \end{aligned}$$

Since $a_j^R(t, x)$ ($j=1, \dots, n$) are real valued, we have

$$\operatorname{Re}(i a_\nu^R(t, x, D_x) v_\nu, v_\nu) \leq C \|v_\nu\|^2.$$

Putting $M_1 = \sup_{t \in [0, T], x \in R^n} \{\langle x \rangle |a^I(t, x)|\}$, we have

$$(3.28) \quad \operatorname{Re}(a_\nu^I(t, x, D_x) v_\nu, v_\nu) \leq M_1 \|\langle x \rangle^{-1/2} A_\nu^{1/2} v_\nu\|^2 + C \|v_\nu\|^2.$$

Therefore by (3.24), (3.27) and (3.28) we get

$$(3.29) \quad \begin{aligned} \frac{d}{dt} \|v_\nu(t, \cdot)\|^2 \leq & (-2M + M_1 + tC_M) \|\langle x \rangle^{-1/2} A_\nu^{1/2} v_\nu(t, \cdot)\|^2 \\ & + C_M \|v_\nu(t, \cdot)\|^2 + C \|\tilde{f}(t, \cdot)\| \|v_\nu(t, \cdot)\|. \end{aligned}$$

We take $M (> M_1/2)$ and $T_1 = \min((2M - M_1)/C_M, T_0)$. Then

$$(3.30) \quad -2M + M_1 + tC_M \leq 0$$

for $t \in [0, T_1]$. By (3.29) we get

$$(3.31) \quad \frac{d}{dt} \|v_\nu(t)\| \leq C' \{\|v_\nu(t)\| + \|\tilde{f}(t)\|\} \quad (t \in [0, T_1]; \nu = 1, 2, \dots).$$

Therefore we get (3.18). Moreover from (3.12) we have

$$(3.32) \quad \frac{d}{dt} A_\nu^j v_\nu = i(P_\nu + [A_\nu^j, P_\nu] A_\nu^{-j}) A_\nu^j v_\nu + i A_\nu^j \tilde{f},$$

and moreover

$$(3.33) \quad |\sigma([A_\nu^j, P_\nu] A_\nu^{-j})\{\beta\}(t, x, \xi)| \leq C_{\alpha, \beta, M}$$

for $t \in [0, T_1]$ and $x, \xi \in R^n$ uniformly with respect to ν . Here, $[A, B]$ denotes the commutator of operators for A and B , that is $A \circ B - B \circ A$. In fact, we have

$$\begin{aligned} & \sigma([A_\nu^j, P_\nu])(t, x, \xi) \\ &= O_s - \iint e^{-i\nu \cdot \eta} \lambda_\nu^j(\xi + \eta) P_\nu(t, x + y, \xi) dy d\eta - P_\nu(t, x, \xi) \lambda_\nu^j(\xi) \\ &= \sum_{|\gamma|=1} \int_0^1 O_s - \iint e^{-i\nu \cdot \eta} (\lambda_\nu^j)^{(\gamma)}(\xi + \eta) (P_\nu)^{(\gamma)}(t, x + \theta y, \xi) dy d\eta d\theta. \\ &= \sum_{|\gamma|=1} \iint_0^1 O_s - \iint e^{-i\nu \cdot \eta} (\lambda_\nu^j)^{(\gamma)}(\xi + \eta) \\ & \quad \times \left\{ \sigma(\tilde{a}_\nu)^{(\gamma)}(t, x + \theta y, \xi) - \sum_{j=1}^2 \sigma((K^{-1} \circ K_j)_\nu)^{(\gamma)}(t, x + \theta y, \xi) \right\} dy d\eta d\theta. \end{aligned}$$

Repeating the same argument as in the proof of (3.25), by use of (3.8), (3.16) and (3.26) we can estimate

$$(3.34) \quad |\sigma([A_\nu^j, P_\nu])\{\beta\}(t, x, \xi)| \leq C_{\alpha, \beta, M} \lambda_\nu(\xi)^j.$$

This implies (3.33). Hence by (3.32) and (3.33), we get (3.19) similarly to (3.18). On the other hand, noting

$$\frac{d}{dt} A_\nu^j v_\nu = i \{A_\nu^j \circ P_\nu \circ A_\nu^{-j-2}\} \circ A_\nu^{j+2} v_\nu + i A_\nu^j \tilde{f}$$

and $\sigma(A_\nu^j \circ P_\nu \circ A_\nu^{-j-2})(t, x, \xi) \in S_{0,0}^0$ for $t \in [0, T_1]$ (uniformly in ν), by Lemma 3.4

we have

$$(3.35) \quad \left\| \frac{d}{dt} A_v^j v_v(t) \right\| \leq C \|A_v^{j+2} v_v\| + \|A_v^j \tilde{f}\|.$$

By (3.29) and (3.19), we get (3.20). Noting $v_v(t) - v_v(t') = \int_{t'}^t (d/d\tau)v_v(\tau) d\tau$, we get (3.21) from (3.20). This completes the proof of Proposition 3.5.

From Proposition 3.5, we can prove Theorem 3.2 in the same way as Kumano-go [4], Ch. 7, Theorem 3.2.

PROOF OF THEOREM 1.1. By Theorem 3.2 we can see that there exists a solution $v(t, x) \in C^1([0, T_1]; H_\infty)$ of the Cauchy problem (3.5) and by (3.9) we get the energy inequality

$$\|v(t, \cdot)\|_{(s)} \leq C(s, T) \left\{ \|u_0\|_{(s)} + \int_0^t \|\tilde{f}(\tau, \cdot)\|_{(s)} d\tau \right\}$$

for any $s \in R^1$ and $t \in [0, T_1]$. Hence, we obtain the unique solution $u(t, x) = Kv(t, x)$ of the equation (1.1) in $[0, T_1]$ and by using Lemma 3.4 we get the energy inequality

$$(3.36) \quad \|u(t, \cdot)\|_{(s)} \leq C'(s, T) \left\{ \|u_0\|_{(s+M')} + \int_0^t \|f(\tau, \cdot)\|_{(s+2M')} d\tau \right\}$$

for any $s \in R^1$ and $t \in [0, T_1]$. We can extend the existence interval $[0, T_1]$ of the solution $u(t, x)$ to $[0, T]$ as follows. Consider the Cauchy problem

$$Lw(t, x) = f(t, x) \quad \text{on} \quad [T_1, T_2] \times R_x^n, \quad w(T_1, x) = u(T_1, x).$$

Then, we get the solution $w(t, x) \in C^1([T_1, T_2]; H_\infty)$ where $T_2 = \min(2T_1, T)$ in the same way as in the construction of $u(t, x) \in C^1([0, T_1]; H_\infty)$. Define

$$\tilde{u}(t, x) = \begin{cases} u(t, x) & \text{for } 0 \leq t \leq T_1, \\ w(t, x) & \text{for } T_1 \leq t \leq T_2. \end{cases}$$

Then $\tilde{u}(t, x)$ belongs to $C^1([0, T_2]; H_\infty)$ and satisfies (1.1) in $[0, T_2]$. Repeating this process, the solution $u(t, x)$ satisfying (1.1) in $[0, T]$ is obtained. The energy estimate (3.36) implies the uniqueness of solution of (1.1).

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