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著者	Sakai Kazuhiro, Satoh Yuji
journal or publication title	Physics letters. B
volume	661
number	2-3
page range	216-219
year	2008-03
権利	(C) 2008 Elsevier B.V
URL	http://hdl.handle.net/2241/98662

doi: 10.1016/j.physletb.2008.02.015

UTHEP-540
hep-th/0703177

On the origin of the dressing phase in $\mathcal{N} = 4$ Super Yang–Mills

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Abstract

We derive the phase factor proposed by Beisert, Eden and Staudacher for the S-matrix of planar $\mathcal{N} = 4$ Super Yang–Mills, from the all-loop Bethe ansatz equations without the dressing factor. We identify a configuration of the Bethe roots, from which the closed integral formula of the phase factor is reproduced in the thermodynamic limit. This suggests that our configuration describes the "physical vacuum" in the sense that the dressing phase is nothing but the effective phase for the scattering of fundamental excitations above this vacuum, providing an interesting clue to the physical origin of the dressing phase.

March 2007

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Integrability has become of increasing importance in the study of $\mathcal{N}=4$ Super Yang–Mills (SYM) and of the dual superstrings in $AdS_5 \times S^5$. The spectral problem of the dilatation operator at one loop was identified with that of a conventional integrable spin-chain [1, 2], which can be systematically solved by using Bethe ansatz. Integrability beyond one loop has also been extensively studied and, in particular, the all-loop Bethe ansatz equations were postulated [3]. Note, however, that the spin-chain picture does not fully apply at higher loops due to several new features yet unknown in the field of integrable models, such as length fluctuation. Nevertheless, conventional integrability revives by converting the picture into a particle model, at least in the limit of infinite length of operators, or the large-spin limit [4–6]. Asymptotic particle states were realized in terms of SYM operators [7]. It is expected that they exhibit the factorized scattering property and thus all the multi-particle scattering processes are governed by the elementary two-particle S-matrix. This S-matrix was determined up to an overall scalar factor by purely algebraic consideration of the centrally extended $\mathfrak{su}(2|2)$ symmetry [8] and further algebraic aspects have been investigated [9–11].

As is expected from the AdS/CFT correspondence, this S-matrix with a pair of the $\mathfrak{su}(2|2)$ symmetries also emerges on the string theory side [11–14]. The choice of the gauge breaks the conformal invariance in two dimensions and one obtains a massive worldsheet theory, where S-matrix is naturally defined as the scattering of elementary excitations. As the symmetry completely constrains the form of the matrix, what is left to be determined is again the overall scalar factor.

The determination of the scalar factor, as a function of two momenta and the coupling, is important in two aspects: Firstly, it is the last missing element for the systematic construction of the spectrum of the scaling dimension/energy on the Yang–Mills/string side. Secondly, identification of the scalar factors on both sides serves as a strong quantitative check of the AdS/CFT correspondence.

The form of the scalar factor was first studied on the string side, based on the data of classical string spectrum [15]. Succeedingly $1/\sqrt{\lambda}$ corrections were analyzed [16–18] and an all-order form was postulated [19]. This form was shown [18,19] to be consistent with the constraint from the crossing symmetry [20]. On the other hand, the form of scalar factor was rather obscure on the Yang–Mills side, since it stays trivial up to three loops. However, it turned out to deviate from the unity at four loops [21, 22]. Meanwhile, Beisert, Eden and Staudacher managed to construct a closed integral formula [23] consistent with the above four-loop result as well as a sort of analytic continuation of the proposal on the string side [19]. The integral formula is highly intricate, but does not seem totally hopeless to handle analytically. We refer to a reference [24] for recent investigations.

In this short article, we present the derivation of the integral formula purely within the framework of quantum integrable models. Our result is of conceptual importance, since it would indicate that even the scalar factor has no room to consult model-specific degrees of freedom. The integrable structure together with the $\mathfrak{su}(2|2)$ symmetry would completely determine the S-matrix without knowing which side of the AdS/CFT correspondence we are looking at.

Before getting into our computation, we would like to remind the reader of the derivation of the Zamolodchikovs' S-matrix [25]. This S-matrix describes the scattering of elementary particles of the principal chiral field model. It was originally determined by imposing three conditions: unitarity, associativity (Yang–Baxter equation), and crossing symmetry. The first two determine the form of R-matrix, while the last constrains the overall scalar factor up to the CDD ambiguity. The S-matrix can also be derived by direct computations [26–28]. In this case, the starting point is bare Bethe ansatz equations derived from the R-matrix. The physical S-matrix is realized as the scattering matrix of excitations over the non-trivial physical vacuum state, which is built on the bare vacuum state by acting with bare Bethe roots filling up the Dirac sea. Scattering of fundamental excitations above the Dirac sea acquires a phase shift due to the interaction with those background Bethe roots. The phase shift then turns into the scalar factor in front of the bare R-matrix, giving the Zamolodchikovs' S-matrix.

In what follows we consider an analog of this derivation. Along this line, possibilities of deriving the scalar (dressing) factor for planar $\mathcal{N}=4$ Super Yang–Mills have been discussed in a recent work [29]. We refer to a work [30] for a somewhat similar approach.

Our starting point is the all-loop Bethe ansatz equations [3] without the dressing factor. Most generally the Bethe ansatz equations consist of seven sets of equations. For our purpose, we set the number of Bethe roots as[1]

$$(K_1, \ldots, K_7) = (2M, M, 0, K_4, 0, M, 2M). \tag{1}$$

Bethe roots $u_{3,k}\, u_{5,k}$ as well as equations for them are absent in this case. Throughout this article we consider configurations of Bethe roots symmetric with respect to the

[1]This set of occupation numbers is not allowed at one loop, but in the present case it is consistent with the bound from the consistency of the nested Bethe ansatz $K_2 \le K_1 + K_3 \le K_4 \ge K_5 + K_7 \ge K_6$, as long as $K_4 \ge 2M$ is satisfied [10]. (We would like to thank A. Rej, M. Staudacher and S. Zieme for discussions making us clarify this point.) The numbers of Bethe roots are determined so that the corresponding state is neutral under the pair of $su(2|2)$ symmetries. See for details [34].

interchange of the two $\mathfrak{su}(2|2)$ sectors: distribution of roots $u_{1,k}, u_{2,k}$ is just the same as that of $u_{7,k}, u_{6,k}$, respectively. We thus omit the equations for $u_{1,k}, u_{2,k}$ below. After all, we are left with the following reduced sets of equations

$$\left(\frac{x_{4,k}^+}{x_{4,k}^-}\right)^L = \prod_{j\neq k}^{K_4} \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i} \prod_{j=1}^{2M} \frac{1-g^2/x_{4,k}^- x_{1,j}}{1-g^2/x_{4,k}^+ x_{1,j}} \prod_{j=1}^{2M} \frac{1-g^2/x_{4,k}^- x_{7,j}}{1-g^2/x_{4,k}^+ x_{7,j}}\,, \tag{2}$$

$$1 = \prod_{j\neq 1}^{M} \frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i} \prod_{j=1}^{2M} \frac{u_{6,k}-u_{7,j}+i/2}{u_{6,k}-u_{7,j}-i/2}\,, \tag{3}$$

$$1 = \prod_{j=1}^{M} \frac{u_{7,k}-u_{6,j}+i/2}{u_{7,k}-u_{6,j}-i/2} \prod_{j=1}^{K_4} \frac{1-g^2/x_{7,k}\, x_{4,j}^+}{1-g^2/x_{7,k}\, x_{4,j}^-}\,. \tag{4}$$

It turns out that these equations describe, among others, a generalization of the anti-ferromagnetic state of the $\mathfrak{su}(2)$ Heisenberg spin-chain. We consider the case where both M and K_4 are of order L, which will be sent to infinity in the thermodynamic limit. We follow the standard parametrization that rapidity variables x, u are related by

$$x^{\pm}(u) = x(u \pm \tfrac{i}{2}), \quad x(u) = \frac{u}{2}\left(1+\sqrt{1-4g^2/u^2}\right), \tag{5}$$

and

$$g = \frac{\sqrt{\lambda}}{4\pi} \tag{6}$$

is the normalized coupling constant.

We first recall that neighboring roots $u_{6,k}$ and $u_{7,j}$ attract each other and may form bound states called stacks [31]. Here we consider a particular type of stacks studied in [29] that every bosonic root $u_{6,k}$ is combined with a 2-string of fermionic roots $u_{7,k}$. The center of the 2-string coincides with the bosonic root up to $\mathcal{O}(\frac{1}{L})$ correction. With appropriate ordering of Bethe roots, one can express the present formation of stacks as

$$u_{7,2k-1} \approx u_{6,k} + \tfrac{i}{2}, \quad u_{7,2k} \approx u_{6,k} - \tfrac{i}{2}, \quad \text{for} \quad k = 1, \dots, M, \tag{7}$$

where we let $\approx$ denote equality up to $\mathcal{O}(\frac{1}{L})$ correction. After substituting (7), (2) read

$$\begin{aligned}\left(\frac{x_{4,k}^+}{x_{4,k}^-}\right)^L \approx \prod_{j\neq k}^{K_4} \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i} \prod_{j=1}^{M} \frac{1-g^2/x_{4,k}^- x_{2,j}^+}{1-g^2/x_{4,k}^+ x_{2,j}^+} \prod_{j=1}^{M} \frac{1-g^2/x_{4,k}^- x_{2,j}^-}{1-g^2/x_{4,k}^+ x_{2,j}^-} \\ \times \prod_{j=1}^{M} \frac{1-g^2/x_{4,k}^- x_{6,j}^+}{1-g^2/x_{4,k}^+ x_{6,j}^+} \prod_{j=1}^{M} \frac{1-g^2/x_{4,k}^- x_{6,j}^-}{1-g^2/x_{4,k}^+ x_{6,j}^-}\,,\end{aligned} \tag{8}$$

while (3) are identically satisfied[2]. Equations (4) split up into two kinds, namely, for $u_{7,2k-1}$ and for $u_{7,2k}$. Multiplying the former by the latter for the same k, we obtain a set of center equations

$$1 \approx \prod_{j\neq k}^{M} \frac{u_{6,k}-u_{6,j}+i}{u_{6,k}-u_{6,j}-i} \prod_{j=1}^{K_4} \frac{1-g^2/x_{6,k}^+\,x_{4,j}^+}{1-g^2/x_{6,k}^+\,x_{4,j}^-}\,\frac{1-g^2/x_{6,k}^-\,x_{4,j}^+}{1-g^2/x_{6,k}^-\,x_{4,j}^-}\,. \tag{9}$$

Let us next consider the thermodynamic limit $L\to\infty$ of these effective equations. We are looking for a solution analogous to the anti-ferromagnetic state of spin-chains, for which all the Bethe roots sit along the real axis with consecutive mode numbers. For such a solution we may well assume the distribution of roots to be symmetric under $u\mapsto -u$. For the sake of simplicity we set $K_4 = L/2$, which may be the possible maximal number for the real roots $u_{4,k}$. The number of stacks M should also be fixed[3], but in the following discussion we merely need it to be macroscopic. By taking logarithm, differentiating with respect to the spectral parameter u and performing Fourier transform successively, (8), (9) give rise to the following set of integral equations

$$J_0(2gt) = e^{|t|}\hat\rho_4(t) + \hat\rho_4(t) - 4g^2 t\int_0^\infty dt'\hat K_1(2gt,2gt')\,[\hat\rho_2(t')+\hat\rho_6(t')]\,, \tag{10}$$

$$0 = -e^{|t|}\hat\rho_6(t) + \hat\rho_6(t) - 4g^2 t\int_0^\infty dt'\hat K_0(2gt,2gt')\hat\rho_4(t')\,. \tag{11}$$

The computation is almost parallel with that in [29], where Fourier transform of the density function is defined by

$$\hat\rho(t) = e^{-|t|/2}\int_{-\infty}^{\infty} e^{itu}\rho(u)du\,, \tag{12}$$

and the integration kernels are given by

$$\hat K_0(t,t') = \frac{tJ_1(t)J_0(t')-t'J_0(t)J_1(t')}{t^2-t'^2}\,,\quad \hat K_1(t,t') = \frac{t'J_1(t)J_0(t')-tJ_0(t)J_1(t')}{t^2-t'^2}\,. \tag{13}$$

$J_k(u)$ is the Bessel function of the first kind.

Note that the first term of the r.h.s. of equations (10), (11) comes from the growth of the mode number along the real axis, while the second term comes from the scattering

[2]There appear seemingly indeterminate factors $0/0$ at the leading order of the large L approximation. This indeterminateness is resolved if one takes account of the $1/L$ correction. The requirement for the correction is that $u_{7,2k-1}-u_{6,k} = i/2+\epsilon_k$ and $u_{7,2k}-u_{6,k} = -i/2-\epsilon_k$, with $\epsilon_k = \mathcal{O}(1/L)$. Such adjustments are possible stack by stack.

[3]M may be fixed so that the vacuum is maximally neutral with respect to the global symmetry. For $K_4 = 2M = L/2$, global charges listed in [3] vanish (except for the scaling dimension).

of the Bethe roots of the same flavor. Observe that the relative signs of these two terms are different in (10) and (11). This is due to the fact that $u_{4,k}$ correspond to excitations in the compact $\mathfrak{so}(6)$ sector, while $u_{6,k}$ correspond to the non-compact $\mathfrak{so}(4,2)$ sector.

By eliminating $\hat{\rho}_2, \hat{\rho}_6$, one obtains a single integral equation for $\hat{\rho}_4(t)$

$$J_0(2gt) = (e^{|t|}+1)\hat{\rho}_4(t) + 4g^2 t \int_0^\infty dt' \hat{K}_d(2gt, 2gt')\hat{\rho}_4(t')\,, \tag{14}$$

where the integration kernel reads

$$\hat{K}_d(t,t') = 8g^2 \int_0^\infty dt'' \hat{K}_1(t, 2gt'') \frac{t''}{e^{t''}-1} \hat{K}_0(2gt'', t')\,. \tag{15}$$

We obtain the very kernel describing the dressing phase proposed in [23].

The meaning of the result is as follows. Let us first see what state we have at one loop by taking the limit $g \to 0$ in our starting equations (2)–(4). One immediately sees that equations (3) and (4) decouple from (2). In fact, they reduce to trivial equations via a duality transformation and we are left only with the simple Bethe ansatz equations for $u_{4,k}$ describing the $\mathfrak{su}(2)$ Heisenberg spin-chain. In particular, our solution with maximally filling real $u_{4,k}$ corresponds to the anti-ferromagnetic state of the $\mathfrak{su}(2)$ chain. In fact, (14) has the form of the continuous all-loop Bethe ansatz equation for the $\mathfrak{su}(2)$ anti-ferromagnetic state [32,33], plus the integral term expressing the contribution from the background stacks.

Next, let us consider the $\mathfrak{su}(2)$ anti-ferromagnetic state using the all-loop Bethe ansatz equations with the dressing factor of Beisert, Eden and Staudacher [23] instead of introducing our background stacks. In the thermodynamic limit, the same equation (14) appears [29], but now the integral term comes from the dressing factor. This shows that introduction of the background stacks is equivalent to that of the dressing factor. In other words, the dressing phase is nothing but the effective phase due to the existence of our background stacks, which provides an interesting clue to the physical origin of the dressing factor.

In this letter, we have focused on the $\mathfrak{su}(2)$ anti-ferromagnetic state. Other states are also described similarly to the hole excitations above the anti-ferromagnetic vacuum of the $\mathfrak{su}(2)$ Heisenberg spin-chain [34].

Now, one can argue that there are two equivalent formulations also for planar $\mathcal{N}=4$ Super Yang–Mills, as discussed in the introduction: one could start either from physical Bethe ansatz equations with a trivial reference state, or from bare Bethe ansatz equations with a non-trivial reference state. The former is derived from the physical

S-matrix involving the dressing factor, which is analogous to the Bethe ansatz formulation of quantum sigma-models [25]. The latter is derived from a bare R-matrix without the dressing factor, which is analogous to the lattice (spin-chain) realization of particle models.

Our result indicates the possibility of the latter. In this formulation, the fundamental R-matrix can be determined by purely algebraic consideration [8, 9], while the physical S-matrix is dynamically generated as the scattering matrix of fundamental excitations over the Fermi surface. The dressing phase is then regarded as the effective phase over the Fermi surface, or the "physical vacuum" with the stacks. To pursue this program, there are still many questions open for further investigations. One has to examine, for example, what are the allowed excitations, how a small number of excitations is described, and how each Yang–Mills field is realized in terms of Bethe roots. We leave detailed analysis for future publication [34].

Note added: After the submission of this article we were informed by A. Rej, M. Staudacher and S. Zieme that they were aware of similar results, which were later presented in the revised version of [29]. Despite the formal resemblance, our approach is different from theirs in several ways, in particular conceptually, and the open questions in [29] are resolved in ours [34].

Acknowledgments

We would like to thank H-Y. Chen, N. Dorey, V. Kazakov, I. Klebanov, T. Klose, C. Kristjansen, M. Martins, A. Rej, D. Serban, M. Staudacher, A. Volovich, K. Yoshida and S. Zieme for useful comments and discussions. We are especially grateful to M. Shiroishi for collaborative discussions at the early stage of this project. K.S. is very grateful to the Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, the Department of Applied Mathematics and Theoretical Physics at University of Cambridge, and the String Theory Group at National Taiwan University for their warm hospitality. Research of K.S. is supported by the Keio Gijuku Academic Development Funds.

References

[1] J. A. Minahan and K. Zarembo, JHEP **0303** (2003) 013 [arXiv:hep-th/0212208].

[2] N. Beisert and M. Staudacher, Nucl. Phys. B **670** (2003) 439 [arXiv:hep-th/0307042].

[3] N. Beisert and M. Staudacher, Nucl. Phys. B **727** (2005) 1 [arXiv:hep-th/0504190].

[4] D. Berenstein, J. M. Maldacena and H. Nastase, JHEP **0204** (2002) 013 [arXiv:hep-th/0202021].

[5] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Nucl. Phys. B **636** (2002) 99 [arXiv:hep-th/0204051].

[6] S. Frolov and A. A. Tseytlin, Nucl. Phys. B **668** (2003) 77 [arXiv:hep-th/0304255].

[7] M. Staudacher, JHEP **0505** (2005) 054 [arXiv:hep-th/0412188].

[8] N. Beisert, arXiv:hep-th/0511082.

[9] N. Beisert, J. Stat. Mech. **0701** (2007) P017 [arXiv:nlin.si/0610017].

[10] M. J. Martins and C. S. Melo, Nucl. Phys. B **785** (2007) 246 [arXiv:hep-th/0703086].

[11] G. Arutyunov, S. Frolov and M. Zamaklar, JHEP **0704** (2007) 002 [arXiv:hep-th/0612229].

[12] G. Arutyunov, S. Frolov, J. Plefka and M. Zamaklar, J. Phys. A **40** (2007) 3583 [arXiv:hep-th/0609157].

[13] T. Klose, T. McLoughlin, R. Roiban and K. Zarembo, JHEP **0703**, 094 (2007) [arXiv:hep-th/0611169].

[14] T. Klose and K. Zarembo, JHEP **0702** (2007) 071 [arXiv:hep-th/0701240].

[15] G. Arutyunov, S. Frolov and M. Staudacher, JHEP **0410** (2004) 016 [arXiv:hep-th/0406256].

[16] R. Hernandez and E. Lopez, JHEP **0607** (2006) 004 [arXiv:hep-th/0603204].

[17] L. Freyhult and C. Kristjansen, Phys. Lett. B **638** (2006) 258 [arXiv:hep-th/0604069].

[18] G. Arutyunov and S. Frolov, Phys. Lett. B **639** (2006) 378 [arXiv:hep-th/0604043].

[19] N. Beisert, R. Hernandez and E. Lopez, JHEP **0611** (2006) 070 [arXiv:hep-th/0609044].

[20] R. A. Janik, Phys. Rev. D **73** (2006) 086006 [arXiv:hep-th/0603038].

[21] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower and V. A. Smirnov, Phys. Rev. D **75** (2007) 085010 [arXiv:hep-th/0610248].

[22] F. Cachazo, M. Spradlin and A. Volovich, Phys. Rev. D **75** (2007) 105011 [arXiv:hep-th/0612309].

[23] N. Beisert, B. Eden and M. Staudacher, J. Stat. Mech. **0701** (2007) P021 [arXiv:hep-th/0610251].

[24] N. Dorey, D. M. Hofman and J. Maldacena, Phys. Rev. D **76** (2007) 025011 [arXiv:hep-th/0703104].

[25] A. B. Zamolodchikov and A. B. Zamolodchikov, Annals Phys. **120** (1979) 253.

[26] V. E. Korepin, Theor. Math. Phys. **41** (1979) 953 [Teor. Mat. Fiz. **41** (1979) 169].

[27] L. D. Faddeev and L. A. Takhtajan, J. Sov. Math. **24** (1984) 241 [Zap. Nauchn. Semin. **109** (1981) 134].

[28] N. Andrei and C. Destri, Nucl. Phys. B **231** (1984) 445.

[29] A. Rej, M. Staudacher and S. Zieme, J. Stat. Mech. **0708** (2007) P08006 [arXiv:hep-th/0702151].

[30] N. Gromov and V. Kazakov, Nucl. Phys. B **780** (2007) 143 [arXiv:hep-th/0605026].

[31] N. Beisert, V. A. Kazakov, K. Sakai and K. Zarembo, JHEP **0507** (2005) 030 [arXiv:hep-th/0503200].

[32] K. Zarembo, Phys. Lett. B **634** (2006) 552 [arXiv:hep-th/0512079].

[33] A. Rej, D. Serban and M. Staudacher, JHEP **0603** (2006) 018 [arXiv:hep-th/0512077].

[34] K. Sakai and Y. Satoh, JHEP **0712** (2007) 044 [arXiv:0709.3342 [hep-th]].