Concrete Sheaf Models of Higher-Order Recursion

Cristina Matache
Balliol College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy
Trinity 2022

Abstract

This thesis studies denotational models, in the form of sheaf categories, of functional programming languages with higher-order functions and recursion. We give a general method for building such models and show how the method includes examples such as existing models of probabilistic and differentiable computation. Using our method, we build a new fully abstract sheaf model of higher-order recursion inspired by the fully abstract logical relations models of O'Hearn and Riecke. In this way, we show that our method for building sheaf models can be used both to unify existing models that have so far been studied separately and to discover new models.

The models we build are in the style of Moggi, namely, a cartesian closed category with a monad for modelling non-termination. More specifically, our general method builds sheaf categories by specifying a concrete site with a class of *admissible* monomorphisms, a concept which we define. We combine this approach with techniques from synthetic and axiomatic domain theory to obtain a lifting monad on the sheaf category and to model recursion. We then prove the models obtained in this way are computationally adequate.

Acknowledgements

First of all, I want to thank my supervisor, Sam Staton, for his guidance and patience and for sharing his ideas with me. I would also like to thank the members of the Oxford group who I have met over the years; special thanks to Ohad Kammar and Sean Moss. I am grateful to EPSRC, Balliol College, and the Clarendon Fund for funding my studies. I am also grateful to Sam Lindley for support during the final months of writing this thesis. Finally, I want to thank my parents who have supported me in every possible way.

Contents

Chapter 1

Introduction

This thesis is about denotational semantics of functional programming languages. We are interested in building models of higher-order recursive programs in the tradition of Moggi [Mog91], using cartesian closed categories (CCCs) and monads. In this setup, types are interpreted as objects in the category and programs as partial morphisms with admissible domain, since recursive programs might not terminate.

Function spaces exist in the category as objects, given formally by the cartesian closed structure, and they are used to interpret higher-order functions. Partial morphisms can be equivalently described using a lifting monad L on the category: a partial morphism $A \rightharpoonup B$ is a morphism $A \to LB$. For example, to give a partial function between sets it is enough to give a total function $A \to B \uplus \{\bot\}$.

A well-known interpretation of higher-order recursion in the form of a CCC with a monad is the $\omega\mathsf{CPO}$ model (see e.g. [Win93, Section 11.3]), of chain-complete partial orders and continuous partial functions with Scott-open domain. If we wish to combine higher-order recursion with other features, then the notion of partial map and admissible domain from the $\omega\mathsf{CPO}$ model needs to be refined. This is not straightforward because of the interaction of the new features with the higher-order recursion. Examples of such situations include:

1. probabilistic programming languages, which can encode continuous

probability distributions [VKS19, HKSY17];

2. differentiable programming and automatic differentiation [HSV20, Vák20];

3. a variation of differentiable programming where a certain degree of non-smoothness is allowed [LYRY20, LHM21].

4. A refinement of partial maps is also needed to obtain a fully abstract model of higher-order recursion [OR95, RS02]. Informally, full abstraction means that denotational equality in the model characterizes observational equality of programs, thus saying that the model is a good fit for the language in question.

The *goal* of this thesis is to show how, in all four situations described above, *higher-order recursion* can be modelled in the same way using the machinery of *concrete sheaves*. To do this, we develop a new modular method for building cartesian closed categories with a lifting monad, and show that a model of probabilistic programming, based on ω-quasi-Borel spaces [VKS19], a model of differentiable programming, based on ω-diffeological spaces [Vák20], and its variation, the $\omega\mathbf{PAP}$ spaces model [LHM21], are all instances of our method. Moreover, we use our method to build a new fully abstract model of higher-order recursion, inspired by the work of O'Hearn and Riecke [OR95, RS02] which uses logical relations [Plo73].

We note that the models (1)-(3) each have extra domain-specific structure which is not accounted for by our general method for building them, such as a probability monad in the ω-quasi-Borel spaces case. Our focus is to show that, despite these different domain-specific requirements, the structure needed to model higher-order recursion can be obtained in the same way.

This is a significant contribution because it provides a unified framework for the three existing models (1)-(3). Moreover, we show that our general method is also useful for building new models, as illustrated by our fully abstract sheaf model of higher-order recursion. We hope that in the future our method will be useful for exploring the space of denotational models for other higher-order recursive languages, such as those with computational effects [Mog91] that go beyond partiality.

We now give more details about our general method for building models. To model partiality and recursion we use techniques from axiomatic and synthetic domain theory. For example, we use the concept of dominance (e.g. [Ros86, Mul92]) to specify admissible domains of partial morphisms $A \rightharpoonup B$; (the admissible domains will be certain subobjects $A' \rightarrowtail A$). We also use an internal notion of chain and chain-completeness to replace the corresponding notions from $\omega\mathsf{CPO}$. These ingredients are explained and adapted to our setting in Chapters 2 to 4.

The core of our method for building models as categories of concrete sheaves is detailed throughout Chapters 5 to 7. In Chapter 8 we present our fully abstract sheaf model obtained using this method.

A well-known technique for obtaining denotational models of higher-order computation is via the Yoneda embedding. Informally, the Yoneda embedding is a canonical way of passing from a model of first-order types, i.e. any category $\mathbb{C}$, to a model of higher-order types, the category of presheaves on $\mathbb{C}$, which is cartesian closed.

Our models are built in the same vein, but instead of presheaves we use categories of *sheaves*, a standard concept from topos theory. Categories of sheaves refine presheaves in a canonical way by allowing certain colimits, which are generally not preserved by the Yoneda embedding, to be preserved. This refinement is useful because in our examples presheaves are not enough to model datatypes such as the natural numbers or the reals, which are interpreted as colimits.

In fact, our method gives models that are each part of a subcategory of *concrete sheaves*, which admits an elementary description in terms of sets with structure and structure preserving functions. The use of concrete presheaves and sheaves in computer science is not new, they have been used for example in [Ehr07, EX16]. The idea of using sheaf conditions to be able to model certain datatypes, like sum types, goes back to Fiore and Simpson [FS99].

Thesis outline and contributions

The material in this thesis is based on joint work with Sean Moss and Sam Staton published in the following two papers:

[MMS21] Cristina Matache, Sean Moss, and Sam Staton. Recursion and Sequentiality in Categories of Sheaves. In *FSCD*, 2021.

[MMS22] Cristina Matache, Sean K. Moss, and Sam Staton. Concrete categories and higher-order recursion: with applications including probability, differentiability, and full abstraction. In *LICS*, 2022.

Chapter 2. In this chapter we recall background material used in the rest of the thesis. We start with categories of sheaves, sites and concrete sheaves; the standard references are [MM92, Joh02]. Then we recall the treatment of partiality in a topos using the notion of dominance [Ros86] and connect it to partial maps and lifting monads. The main result of this chapter is the construction of a lifting monad from a dominance (Proposition 2.4.5 and Theorem 2.4.9).

Chapter 3. Here, we provide sufficient conditions on a cartesian closed category with a monad in order to be able to prove a fixed point theorem (Corollary 3.2.5). The development is very closely related to results in the axiomatic and synthetic domain theory literature (e.g. [FP96, LS97, Sim98, RS99]), although there are technical differences, explained in the related work section 3.3. The results in this chapter appeared in [MMS21, Section 2].

Chapter 4. In this chapter we identify sufficient conditions such that a sheaf category with a dominance can interpret call-by-value PCF (PCF_v); we refer to such an interpretation as a *normal model* (Definition 4.3.1). The most important condition is that the natural numbers object in the sheaf category is "complete", a property analogous to chain-completeness for cpos. This "completeness" allows us to interpret recursion using the fixed point theorem from Chapter 3. We then prove that the model is sound with respect to an operational semantics (Theorem 4.3.5).

The contribution in this chapter is proving that normal models satisfy the premises of the fixed point theorem, i.e. that they have the right "complete" objects. Although normal models are closely related to Simpson's natural models [Sim98], we do not know if we can deduce these completeness properties from the analogous ones for natural models because of technical reasons discussed in the related work section 7.4.

In Section 4.4, we introduce a running example of normal model, the category vSet, closely related to the category $\mathcal{H}$ of [FR01]. We show in Proposition 4.4.10 how the model of call-by-value PCF in vSet is essentially the ωCPO model. The results in this chapter appeared in [MMS21, Sections 3.1, 4, 5].

Chapter 5. In Chapters 5 and 6 we prove the main results used in our general method for building normal models. These results were published in [MMS22, Sections 5.3, 6.3, 7.1].

In Chapter 5 we introduce the notion of class of pre-admissible monos (Definition 5.1.2) and show how, given such a class in a site, we can construct a dominance in the corresponding sheaf category (Theorem 5.1.6). This result is new and generalizes a result by Mulry [Mul94] for presheaves. We then prove some results that characterize the dominance and lifting monad when the site is concrete; these results will be used in the next two chapters. We show that our running example of normal model from Chapter 4, vSet, can be seen as a category of sheaves on a concrete site with a class of pre-admissible monos.

Chapter 6. In this chapter we identify sufficient conditions such that, in a category of sheaves with a class of pre-admissible monos in its site, the natural numbers object is "complete". This completeness is the main requirement of being a normal model, introduced in Chapter 4 in order to model recursion using the fixed point theorem from Chapter 3.

The main result of the chapter is summarized in Theorem 6.2.5. Roughly, the additional conditions on the site that allow us to deduce completeness are that the site is concrete and contains the vertical natural numbers, and that the class of monos is *admissible* (Definition 6.2.1). The contributions of this chapter are introducing the notion of admissible monos and proving Theorem 6.2.5.

Chapter 7. In this chapter we summarize our recipe for building normal models of call-by-value PCF using a *concrete site* with *admissible monos* (see Theorem 7.1.1). This is the main contribution of the thesis.

We show in Theorem 7.1.3 that the normal models obtained using our recipe are *adequate*, which implies that we can prove contextual equivalence of programs by proving they have equal denotations in the model. Our adequacy result is closely related to Simpson's adequacy for natural models [Sim98]. In this setup, all types are interpreted as concrete sheaves, so the subcategory of *concrete sheaves* is also an adequate model. These result appeared in [MMS22, Section 7.2].

Another important contribution of the chapter is showing, in Section 7.2, how three existing models are an instance of the recipe from Theorem 7.1.1 for building normal models. These are the ω-quasi-Borel spaces model of probabilistic programming [VKS19], the ω-diffeological spaces model of differentiable programming [Vák20], and the ω**PAP** spaces model [LHM21]. We show that our running example vSet, which is essentially the ωCPO model, is an instance of the recipe as well.

Chapter 8. Here, we present a more involved example (see Section 8.1) of a normal model built using the recipe from Theorem 7.1.1. This is a new model of call-by-value PCF which we prove is fully abstract in Section 8.2 (see Theorem 8.2.11). The material in this chapter appears in [MMS21, Sections 6.3, 7].

Full abstraction means that denotational equality of programs coincides with contextual equivalence. This is a desirable property because the mathematical structure of the denotational model can provide simpler ways of proving contextual equivalences, which are otherwise hard to prove by operational arguments. Full abstraction ensures that such proof methods are complete. However, full abstraction is usually hard to obtain; many adequate models are not fully abstract, like the ωCPO model of PCF [Plo77].

One reason why full abstraction fails in the ωCPO model is that the model contains functions, like parallel-or, that are not definable in PCF.

The quest for fully abstract models of PCF has generated a large amount of research, for example in game semantics [AJM00, HO00]. Another earlier line of research tried to characterize the notion of "sequential" function (e.g. [Sie92, JT93, Cur93]), in order to exclude functions like parallel-or from the model. Our fully abstract model is inspired by the work of O'Hearn and Riecke [OR95, RS02, Mar00a, Str06], which uses logical relations to characterize sequentiality. We discuss the relationship with their work in Section 8.3.

To summarize, the general method for building normal models which we develop throughout Chapters 3 to 7 is used in Chapter 8 to tackle the well-established problem of building fully abstract models for PCF. By obtaining a new fully abstract model we show the wide applicability of our method, and help explain some of the ideas in the models of O'Hearn and Riecke in a principled way, by using concrete sheaves and admissible monos.

Chapter 2

Preliminaries: categories of sheaves and dominance

In this chapter we present well-known definitions and results which will be used in the rest of the thesis. We assume familiarity with category theory, for example with notions like cartesian closed categories, presheaves, the Yoneda lemma, monads.

In Section 2.1, we start by defining categories of sheaves (i.e. Grothendieck toposes) which will be the basis of all the denotational models in the thesis. In Section 2.2 we discuss subobjects, a generalization of the notion of subset from the category Set to a Grothendieck topos, and use this in Section 2.4 to generalize partial functions to partial maps. Partial maps will be used to model possibly non-terminating programs.

An important point of this chapter is to introduce the notion of dominance (Definition 2.3.2) in a Grothendieck topos as a means of specifying possible domains of partial maps. Dominances are then connected to monads (Theorem 2.4.9), which will later allow us to build models in the style of Moggi [Mog91], using a cartesian closed category with a monad.

Most of the results in this chapter are well-known, although we had to prove ourselves some facts about Δ-subobject, and the lifting monad being pointed. We also provide our own proof of Proposition 2.4.5 whose statement appears in [Mul92].

2.1 Sites and Sheaves

In this section we recall background material about sites and categories of sheaves. There are other useful facts about categories of sheaves which we use later but omit here for reasons of brevity; the standard references are [Joh02, MM92].

Definition 2.1.1. A *site* $(\mathbb{C}, J)$ is a small category $\mathbb{C}$ with a coverage J. A *coverage* consists of, for every object $c \in \mathbb{C}$, a set $J(c)$ of *sets* of morphisms with codomain c, $\{f_i : c_i \to c\}_{i \in I}$. We call $\{f_i : c_i \to c\}_{i \in I}$ in $J(c)$ a *covering family* of c, or say that it covers c. A coverage must satisfy the following axiom:

(C) For every map $h : d \to c$ in $\mathbb{C}$, if $\{f_i : c_i \to c\}_{i \in I}$ is in $J(c)$, then there is a covering family $\{g_j : d_j \to d\}_{j \in I'}$ of d such that every $h \circ g_j$ factors through some f_i.

Remark 2.1.2. This definition of coverage is the minimal one ([Joh02, A2.1.9]), and there can be several coverages on $\mathbb{C}$ giving rise to the same category of sheaves (Definition 2.1.4). It can be useful to add saturation conditions to the coverage to make calculation easier:

(M) J contains $\{\mathrm{id}_c : c \to c\}$ for all $c \in \mathbb{C}$;

(L) If $\{f_i : c_i \to c\}_{i \in I} \in J(c)$ and $\{g_{ij} : b_{ij} \to c_i\}_{j \in I'_i} \in J(c_i)$ for $i \in I$ then $\{f_i \circ g_{ij} : b_{ij} \to c\}_{i \in I, j \in I'_i} \in J(c)$.

All the coverages that we consider in later chapters will satisfy the (M) and (L) axioms.

Definition 2.1.3. Given a site $(\mathbb{C}, J)$, a covering family $\{f_i : c_i \to c\}_{i \in I} \in J(c)$, and a presheaf $F \in \mathsf{PSh}(\mathbb{C})$, a *matching family* is a set

$$\{s_i \in F(c_i)\}_{i \in I}$$

such that for all $i, j \in I$, $d \in \mathbb{C}$, $g : d \to c_i$, and $h : d \to c_j$ with

$$f_i \circ g = f_j \circ h$$

we have

$$F(g)(s_i) = F(h)(s_j).$$

Definition 2.1.4. A sheaf on a site $(\mathbb{C}, J)$ is a presheaf $F \in \mathsf{PSh}(\mathbb{C})$ such that for every covering family $\{f_i : c_i \to c\}_{i \in I}$ and every matching family $\{s_i \in F(c_i)\}_{i \in I}$ there is a unique *amalgamation* $s \in F(c)$ such that:

$$F(f_i)(s) = s_i \text{ for all } i \in I.$$

Denote by $\mathsf{Sh}(\mathbb{C}, J)$ the full subcategory of $\mathsf{PSh}(\mathbb{C})$ whose objects are J-sheaves. We define a Grothendieck topos to be a category of sheaves $\mathsf{Sh}(\mathbb{C}, J)$.

Proposition 2.1.5 (e.g. [Joh02, A4.1.8],[MM92, III.5, Theorem 1])**.** *The embedding* $i : \mathsf{Sh}(\mathbb{C}, J) \to \mathsf{PSh}(\mathbb{C})$ *has a left adjoint*

$$a : \mathsf{PSh}(\mathbb{C}) \to \mathsf{Sh}(\mathbb{C}, J)$$

called the associated sheaf functor, or sheafification functor, which preserves finite limits.

A coverage is defined to be *subcanonical* if all representable functors $y(c) = \mathbb{C}(-, c)$ are already sheaves. In this case the sheafification functor a leaves them unchanged. Many coverages however are not subcanonical, for example the coverage used in Chapter 8 for a fully abstract model of $\mathsf{PCF_v}$. In this case, we will use instead the sheafified representables $ay(c)$.

We recall some more useful properties of the category of sheaves $\mathsf{Sh}(\mathbb{C}, J)$ from [MM92, III.6]:

- $\mathsf{Sh}(\mathbb{C}, J)$ has all small limits and they are computed as in $\mathsf{PSh}(\mathbb{C})$.
- $\mathsf{Sh}(\mathbb{C}, J)$ has all small colimits and they are computed by first computing the colimit in presheaves followed by applying the sheafification functor a.
- Exponentials are computed like in presheaves. Moreover, sheaves are an exponential ideal: if F and P are presheaves, and F is also a sheaf, then the exponential F^P computed in the category of presheaves is a sheaf.

2.1.1 Concrete sites and sheaves

We recall the definitions of concrete site and concrete sheaf from [Dub79, BH11]. Both concrete presheaves and concrete sheaves have received applications in computer science before, for example in [Ehr07, EX16].

Definition 2.1.6. A *concrete site* (Definition 2.1.1) is a site $(\mathbb{C}, J)$ with a terminal object $\star$ such that the maps

$$\mathbb{C}(a, b) \to \mathsf{Set}\big(\mathbb{C}(\star, a), \mathbb{C}(\star, b)\big)$$

are all injective, and for every covering family $\{f_i : c_i \to c \mid i \in I\} \in J(c)$, the map

$$\coprod_{i \in I} \mathbb{C}(\star, c_i) \to \mathbb{C}(\star, c)$$

is surjective. We will sometimes refer to the first condition by saying $\mathbb{C}$ is well-pointed.

In a concrete site $(\mathbb{C}, J)$, for any object c in $\mathbb{C}$ we define:

$$|c| = \mathbb{C}(\star, c)$$

and refer to $|c|$ as the points of c. Since every map in $\mathbb{C}$ is determined by what it does on points, we can identify a morphism $f : c \to d$ with the induced function $|f| : |c| \to |d|$. Thus

$$|-| : \mathbb{C} \to \mathsf{Set}$$

is a faithful functor, but not necessarily full.

Given a presheaf F on $\mathbb{C}$, we also define:

$$|F| = F(\star).$$

Remark 2.1.7. Given a concrete site $(\mathbb{C}, J)$, the sheafified representables $ay(c)$ can be calculated as follows. Notice that the presheaf

$$\mathsf{Set}(|-|, |c|) : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$$

is a sheaf. This is because thanks to the matching condition and the fact that $\mathbb{C}$ is well-pointed, we can always glue a matching family $\{|d_i| \to |c|\}_{i\in I}$, for a cover $\{d_i \to d\}_{i\in I}$ of d, to a function $|d| \to |c|$.

The representable $\mathbb{C}(-, c)$ embeds into the sheaf $\mathsf{Set}(|-|, |c|)$ thanks to concreteness. Then $ay(c)$ is the smallest subfunctor of $\mathsf{Set}(|-|, |c|)$ containing $\mathbb{C}(-, c)$ and closed under amalgamation. This fact can be deduced using the isomorphism of homsets given by the adjunction $a \dashv i$.

If the coverage satisfies axioms (M) and (L), then it is enough to close under amalgamations in just one step:

$$ay(c)(d) \cong \big\{\phi \in \mathsf{Set}(|d|, |c|) \,\big|\, \exists\{f_i : d_i \to d\} \in J(d). \\ \forall i.\phi \circ |f_i| \in \mathsf{im}\big(y(c)(d_i) \hookrightarrow \mathsf{Set}(|d_i|, |c|)\big)\big\}.$$

Definition 2.1.8. Let $(\mathbb{C}, J)$ be a concrete site. A *concrete presheaf* is a presheaf F on $\mathbb{C}$ such that, for every $c \in \mathbb{C}$, the function

$$\langle F(x : \star \to c)\rangle_{x\in|c|} : F(c) \to \mathsf{Set}(|c|, F(\star))$$

is injective. Denote by $\mathsf{ConcPSh}(\mathbb{C})$ the category of concrete presheaves on $\mathbb{C}$, which is a full subcategory of $\mathsf{PSh}(\mathbb{C})$.

A *concrete sheaf* on the site $(\mathbb{C}, J)$ is a concrete presheaf that is also a sheaf for J. Denote by $\mathsf{Conc}(\mathbb{C}, J)$ the category of concrete sheaves on $(\mathbb{C}, J)$.

One advantage of working with concrete presheaves is that they have an explicit description: we can think of F as being the set $|F|$, together with a set of functions $|c| \to |F|$ for each $c \in \mathbb{C}$. Often we will think of the set $F(c)$ as a *relation* on the set $|F|$ with arity $|c|$.

If G and F are both presheaves and F is concrete, then a natural transformation $\alpha : G \to F$ is determined by the function $\alpha_\star : |G| \to |F|$. Thus, concrete presheaves form a well-pointed category.

From Remark 2.1.7 we can see that a sheafified representable $ay(c)$ is still concrete. From here we can deduce that the sheafification functor a preserves concreteness.

The following proposition shows that concrete sheaves are a well-behaved subcategory of sheaves:

Proposition 2.1.9 (e.g. [Joh02, C2.2], [BH11])**.** *Let* $(\mathbb{C}, J)$ *be a concrete site. The full inclusion* $\mathsf{Conc}(\mathbb{C}, J) \to \mathsf{Sh}(\mathbb{C}, J)$ *preserves all limits, exponentials, and coproducts, and has a left adjoint, so* $\mathsf{Conc}(\mathbb{C}, J)$ *is a reflective subcategory.*

2.1.2 Grothendieck topologies

It is common to define a category of sheaves using a Grothendieck topology T on a small category $\mathbb{C}$, as in [MM92, III.2], rather than using a coverage (Definition 2.1.1). A Grothendieck topology contains *sieves*, where a sieve on an object c is a family of morphisms with codomain c closed under precomposition. Given a sieve S on c and a morphism $f : d \to c$, define:

$$f^*(S) = \{h \mid \mathsf{cod}(h) = d \text{ and } (f \circ h) \in S\}.$$

Then $f^*(S)$ is a sieve on d, sometimes called the pullback of S along f.

Definition 2.1.10 (e.g. [MM92, III.2])**.** A Grothendieck topology T on a small category $\mathbb{C}$ consists of, for each object $c \in \mathbb{C}$, a collection $T(c)$ of sieves on c satisfying the following axioms:

(C′) If S is in $T(c)$, then for any map $h : d \to c$, the pullback sieve $h^*(S)$ is in $T(d)$.

(M′) The maximal sieve $\{f \mid \mathsf{cod}(f) = c\}$ is in $T(c)$.

(L′) If S is in $T(c)$, and R is any sieve on c such that for any map $h : d \to c$ in S the sieve $h^*(R)$ is in $T(d)$, then R is in $T(c)$.

The notions of matching family for a sieve, amalgamation of a matching family, and sheaf on $(\mathbb{C}, T)$ are defined analogously to the corresponding notions for covering families (Definitions 2.1.3 and 2.1.4).

It is a standard result that every coverage J on $\mathbb{C}$ has a corresponding Grothendieck topology T that gives rise to the same sheaves. The construction of T is detailed in the proof of [Joh02, Proposition C.2.1.9]. In the case when J satisfies the (M) and (L) axioms this construction can be simplified as follows.

Proposition 2.1.11. *Let $(\mathbb{C}, J)$ be a small site where the coverage J satisfies axioms (M) and (L) (Remark 2.1.2). For every object c in $\mathbb{C}$, let $T(c)$ contain all the sieves on c that contain some covering family from $J(c)$.*

Then T is a Grothendieck topology on $\mathbb{C}$ which determines the same sheaves as J.

Proof. To show T satisfies axioms (C′) and (M′) use the corresponding axioms (C) and (M) for coverages. To show axiom (L′) holds consider a sieve $S \in T(c)$ and let $\{f_i : c_i \to c\}_{i \in I}$ be the cover from $J(c)$ that is included in S. Then because $f_i^*(R) \in T(c_i)$, for each f_i there must be a cover $\{g_{ij} : d_{ij} \to c_i\}_{i \in I, j \in I'_i}$ from $J(c_i)$ which is included in $f_i^*(R)$. By axiom (L) for coverages we know that the following family is in $J(c)$:

$$\{f_i \circ g_{ij} : d_{ij} \to c\}_{i \in I, j \in I'_i}.$$

But this family is also included in R so $R \in T(c)$.

Consider a sheaf X for T and consider a cover $\{f_i\}_{i \in I}$ in $J(c)$. By closing this cover under precomposition with any morphism, we obtain a sieve S in $T(c)$. To show that X satisfies the sheaf condition for $\{f_i\}_{i \in I}$ it is enough to use the fact that it satisfies the sheaf condition for S. Hence X is a sheaf for J as well.

Consider a sheaf X for J and a sieve S in $T(c)$. Then S contains a cover $\{f_i : c_i \to c\}_{i \in I}$ from $J(c)$. Consider a matching family $\{x_h\}_{h \in S}$ for S, where each x_h is in $X(\mathsf{dom}(h))$. This gives a matching family $\{x_{f_i}\}_{i \in I}$ for the cover $\{f_i : c_i \to c\}_{i \in I}$, which has a unique amalgamation $x \in X(c)$.

Now consider a map $h : d \to c$ in S. We need to show that:

$$X(h)(x) = x_h.$$

This would mean that x is also an amalgamation for $\{x_h\}_{h \in S}$, and must be the unique one, thus concluding the proof that X satisfies the sheaf condition for S.

Let $\{g_j : d_j \to d\}_{j \in I'}$ be the cover in $J(d)$ whose composite with h factors through $\{f_i\}_{i \in I}$, as guaranteed by axiom (C) for coverages. Then for each g_j there is an $i_j \in I$ such that g_j factors as:

$$h \circ g_j = f_{i_j} \circ z_j \tag{2.1}$$

where $z_j : d_j \to c_{i_j}$. From eq. (2.1) we can see that $\{X(z_j)(x_{f_{i_j}})\}_{j \in I'}$ is a matching family for the cover $\{g_j\}_{j \in I'}$ of d.

Because x is an amalgamation for $\{x_{f_i}\}_{i \in I}$ we get that for all $j \in I'$:

$$X(z_j)(x_{f_{i_j}}) = X(f_{i_j} \circ z_j)(x) = X(h \circ g_j)(x).$$

So $X(h)(x)$ is an amalgamation for the matching family $\{X(z_j)(x_{f_{i_j}})\}_{j \in I'}$, and because X satisfies the sheaf condition for $\{g_j\}_{j \in I'}$, this must be the unique amalgamation.

But because $\{x_{h'}\}_{h' \in S}$ is a matching family for S, we know for all $j \in I'$:

$$X(z_j)(x_{f_{i_j}}) = X(g_j)(x_h).$$

So x_h is an amalgamation for the matching family $\{X(z_j)(x_{f_{i_j}})\}_{j \in I'}$ as well, and therefore $X(h)(x) = x$. This concludes the proof that X is a sheaf for T. □

In this thesis we chose to define sheaves in terms of coverages because coverages are easier to specify individually than Grothendieck topologies, and because it is easier to check that a functor satisfies the sheaf condition for a coverage. Nevertheless, in the proofs of Proposition 4.4.7, Lemma 5.1.5 and Theorem 5.1.6 we will use sieves and Grothendieck topologies generated by a coverage via Proposition 2.1.11.

2.2 Subobjects

Throughout the thesis, we will use the standard notion of subobject, see for example [Bor94, Chapter 4] or [Joh02, Section A.1.3]. Subobjects generalize the intuition of subsets from Set to an arbitrary category.

Definition 2.2.1. In any category, two monomorphisms $f : B \rightarrowtail A$ and $g : C \rightarrowtail A$ are equivalent if there exists an isomorphism $e : B \to C$ such that

$$g \circ e = f.$$

A *subobject* of A is an equivalence class of monomorphisms with codomain A.

A monomorphism $f : B \rightarrowtail A$ is smaller than a monomorphism $g : C \rightarrowtail A$ when there exists a map $i : B \to C$ such that $g \circ i = f$, that is, f factors through g.

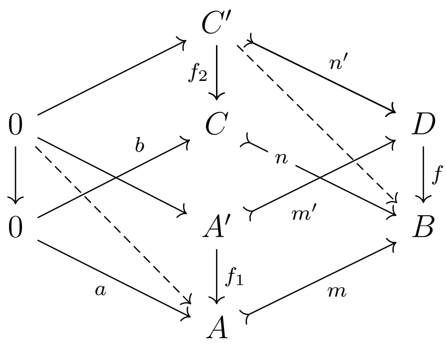
Throughout the thesis, we will use standard facts about subobjects which hold in a Grothendieck topos. Using the order on monomorphisms from Definition 2.2.1, we obtain a partial order on the set $\mathsf{Sub}(A)$ of subobjects of A. The union of a family of subobjects of A is defined to be their supremum in $\mathsf{Sub}(A)$, and their intersection is defined to be their infimum.

In a Grothendieck topos, we can calculate the intersection of two subobjects by taking their pullback. Two subobjects are defined to be disjoint if their pullback is the initial object 0; in this case, their union is obtained by taking their coproduct. Notice that the pullback of a subobject with an arbitrary map is also a subobject.

We will use standard facts about the union and intersection of subobjects, such as the fact that taking union of subobjects commutes with taking pullback. We give the proof for one of these facts below, the rest have a similar flavour.

Lemma 2.2.2. *In a Grothendieck topos, the respective pullbacks of disjoint subobjects along an arbitrary map are again disjoint subobjects.*

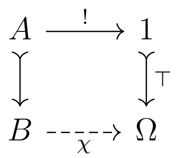
Proof. Consider subobjects $m : A \rightarrowtail B$ and $n : C \rightarrowtail B$ which are disjoint, meaning that their pullback is 0. Let m' be the pullback of m along f and n' the pullback of n.



The pullback of f_1 along a, and that of f_2 along b must both be 0. This means that the diagonal square consisting of 0, C', A and B is also a pullback, so the top square is a pullback as well. Therefore, m' and n' are disjoint. □

Recall the following definition from e.g. [MM92, Chapter I]:

Definition 2.2.3. In a category with finite limits, a subobject classifier is a monomorphism $\top : 1 \rightarrowtail \Omega$, such that for every monomorphism $A \rightarrowtail B$, there is a unique map $\chi : B \to \Omega$ that makes the following square a pullback:



Proposition 2.2.4 (e.g [MM92, Chapter 3, Section 7])**.** *Every Grothendieck topos has a subobject classifier.*

The subobject classifier can be understood as generalizing the notion of "truth values" from Set, its Set analogue being the set with two elements, true and false. Similarly, the classifying map χ corresponds to the characteristic function of the subset $A \subseteq B$. Moreover, $\top : 1 \to \Omega$ can be thought of as a "universal monomorphism", such that all other subobjects are a pullback of it, in a unique way.

2.3 Dominance

In the following chapters (e.g. Chapters 4 to 8), we will be interested in studying only a certain class of monomorphisms from our topos. For this, we refine the notion of subobject classifier to that of dominance. Since a dominance will be a subobject of the subobject classifier, using the dominance will intuitively mean restricting the available set of truth values. In this section we continue working in a Grothendieck topos $\mathcal{E}$.

Definition 2.3.1. Consider a fixed object Δ and a fixed morphism $\top : 1 \to \Delta$. We say a subobject $m : A' \rightarrowtail A$ is a Δ-*subobject*, or is *classified by* Δ, if it is a pullback of $\top : 1 \to \Delta$ along some map $A \to \Delta$.

We will use Mulry's [Mul92, Mul94] definition of dominance (which he calls a partial truth value object) and follow his development in this and the next section. An equivalent definition is given by Rosolini [Ros86]. A more general definition is given in [FP96].

Definition 2.3.2. In a topos Grothendieck $\mathcal{E}$, a *dominance* is a subobject of the subobject classifier $\delta : \Delta \rightarrowtail \Omega$ such that:

1. $\mathsf{true} \in \Delta$, i.e. $\top : 1 \to \Omega$ factors through Δ.
2. Δ-subobjectsare closed under composition: if $A \rightarrowtail B$ and $B \rightarrowtail C$ are both classified by Δ, then so is $A \rightarrowtail C$.

We will often consider a dominance such that $0 \rightarrowtail 1$ is a Δ-subobject. This means that there is a map $\bot : 1 \rightarrowtail \Delta$ which classifies $0 \rightarrowtail 1$.

Proposition 2.3.3. *For every Δ-subobject $m : A' \rightarrowtail A$ there is a* unique *map $\chi : A \to \Delta$ such that m is the pullback of $\top : 1 \to \Delta$ along χ.*

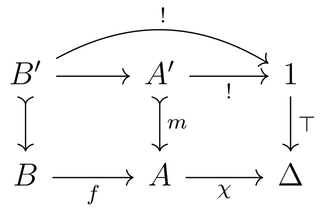
Proof. Suppose that there is another such map χ'. Both $\delta \circ \chi$ and $\delta \circ \chi' : A \to \Omega$ are classifying maps for m, so by definition of the subobject classifier they must be equal. Because δ is mono, χ must be equal to χ'. □

Example 2.3.4. Ω is a trivial example of a dominance.

Below are some useful facts about Δ-subobjects.

Lemma 2.3.5. *Δ-subobjects are stable under pullback.*

Proof. Let $m : A' \rightarrowtail A$ be a Δ-subobject with classifying map χ. If we pull back m along another map $f : B \to A$, we obtain another Δ-subobject $B' \rightarrowtail B$ with classifying map $\chi \circ f$, like in the diagram below:



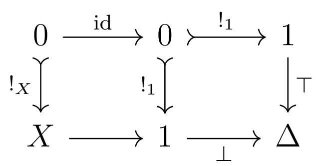
$\square$

Lemma 2.3.6. *Δ-subobjects are closed under finite intersections, including maximal subobjects.*

Proof. For an identity subobject $\mathrm{id}_A : A \rightarrowtail A$, the classifying map that makes it a Δ-subobject is $A \xrightarrow{!} 1 \xrightarrow{\top} \Delta$. The intersection of two Δ-subobjects is their pullback. Since Δ-subobjects are closed under pullback and composition, their intersection is also a Δ-subobject. $\square$

Lemma 2.3.7. *If $!_1 : 0 \rightarrowtail 1$ is a Δ-subobject then every $!_X : 0 \rightarrowtail X$ is a Δ-subobject.*

Proof. Let $\bot : 1 \to \Delta$ be the classifying map of $!_1$. The pullback of $!_1$ along $X \to 1$ must be $!_X : 0 \rightarrowtail X$ because in a topos 0 is strict initial ([Joh02][Lemma A.1.4.1]) so any map with codomain 0 is an isomorphism. Therefore $!_X : 0 \rightarrowtail X$ is a pullback of $\top$ as well.



$\square$

Proposition 2.3.8. *If* $0 \rightarrowtail 1$ *is a* Δ*-subobject, then coproduct inclusions are* Δ*-subobjects.*

Proof. Consider the coproduct inclusion $\iota_j : A_j \to \sum_{i \in I} A_i$. We defined $\bot : 1 \to \Delta$ to be the classifying map of the empty subobject $0 \rightarrowtail 1$.

Consider the map $\alpha : \sum_{i \in I} 1 \to \Delta$ which is the co-pairing of, for every $i \in I$, $\top$ if $i = j$ and $\bot$ otherwise. Then the map $\alpha \circ (\sum_{i \in I} !) : \sum_{i \in I} A_i \to \Delta$ classifies ι_j because the following two squares are pullbacks:

$$\begin{array}{ccccc}
A_j & \xrightarrow{\;!\;} & 1 & \xrightarrow{\;\mathrm{id}\;} & 1 \\
{\scriptstyle \iota_j}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{inc}_j} & & \big\downarrow{\scriptstyle \top} \\
\sum_{i \in I} A_i & \xrightarrow[\sum_{i \in I} !]{} & \sum_{i \in I} 1 & \xrightarrow[\alpha]{} & \Delta
\end{array}$$

□

2.4 Partial maps

In this section, we generalize the intuition behind partial functions between sets. A partial function $A \rightharpoonup B$ can be specified by a subset $A' \subseteq A$, its domain, and a total function $A' \to B$. In our generalization, the subset is replaced by a subobject $A' \rightarrowtail A$.

When modelling recursive programs we might want to only allow certain subobjects as domains of partial maps. For example in the cpo model of PCF the partial maps have Scott-open domain, see for example [Fio94, Section 3.1]. Another intuition for restricting the allowed domains comes from the computable partial functions, i.e. those that can be implemented by a Turing machine. Their domain cannot be any subset but must be a *semidecidable* one, in the sense that there exists a Turing machine that accepts the inputs from the subset, but does not halt otherwise. For this purpose we will use the notion of dominance from the previous section.

The main purpose of this section is to explain the connection between the notions of dominance, partial maps, and lifting monad in a Grothendieck topos. The main results are that a dominance gives rise to a partial map classifier (Proposition 2.4.5) which is a strong monad (Theorem 2.4.9).

Definition 2.4.1 ([RR88],[CL02, Section 3.1])**.** In any category $\mathbb{C}$, a class of monos $\mathcal{M}$ is a *stable system of monos* if:

1. All the isomorphisms are in $\mathcal{M}$.

2. $\mathcal{M}$ is closed under composition.

3. The pullbacks of maps in $\mathcal{M}$ along arbitrary maps exist and are again in $\mathcal{M}$.

Definition 2.4.2. Let $\mathbb{C}$ be any category and $\mathcal{M}$ a stable system of monos. The category $\mathbf{Part}_{\mathcal{M}}(\mathbb{C})$ of $\mathcal{M}$*-partial maps in* $\mathbb{C}$ has the same objects as $\mathbb{C}$ and morphisms $A \rightharpoonup B$ are isomorphism classes of pairs $(m : A' \rightarrowtail A, f : A' \to B)$, where $m \in \mathcal{M}$ and $f \in \mathbb{C}$.

Two partial maps $(m : A' \rightarrowtail A, f : A' \to B)$ and $(m' : A'' \rightarrowtail A, f' : A'' \to B)$ are in the same isomorphism class if there exists an isomorphism $\alpha : A' \to A''$ in $\mathbb{C}$ such that:

$$m = m' \circ \alpha \qquad\qquad f = f' \circ \alpha.$$

Composition of partial maps is defined using pullback stability as for example in [CL02, Section 3.1].

Definition 2.4.3 ([CL03, Section 2.1])**.** Consider a category $\mathbb{C}$ with a stable system of monos $\mathcal{M}$. A *partial map classifier* for an object B is a monomorphism $\eta_B : B \rightarrowtail LB$ in $\mathcal{M}$ such that for every partial map $(m, f) : A \rightharpoonup B$ with $m \in \mathcal{M}$ there is a unique map $f' : A \to LB$ making the following square a pullback:

$$\begin{array}{ccc} R & \xrightarrow{f} & B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ A & \xrightarrow[f']{} & LB \end{array}$$

One can check that f' is independent of the representative we choose for the isomorphism class of the partial map (m, f).

Definition 2.4.4. In a topos $\mathcal{E}$ with a dominance Δ, let $\mathcal{M}_\Delta$ be formed of the Δ-subobjects, that is the subobjects classified by Δ.

By Lemma 2.3.6 and Lemma 2.3.5 we see that $\mathcal{M}_\Delta$ is a stable system of monos. Therefore, we have the following proposition from [Mul92, Theorem 1.7].

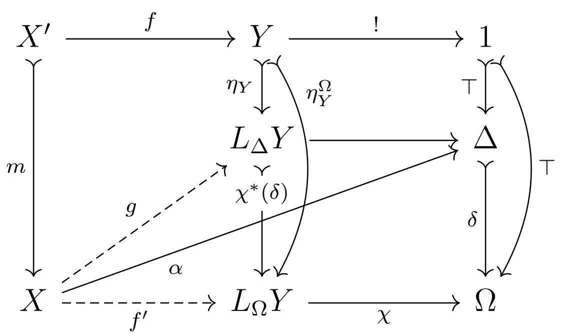
Proposition 2.4.5. *In a topos $\mathcal{E}$, for every dominance $\Delta \rightarrowtail \Omega$ there exists a corresponding partial map classifier $L_\Delta(-)$, classifying partial maps whose domain is a Δ-subobject.*

Proof. It is well-known that the subobject classifier Ω in a topos has a corresponding partial map classifier [Joh02, Proposition A2.4.7]. In this case the pullback functor between slice categories $\top^* : \mathcal{E}/\Omega \to \mathcal{E}/1 \cong \mathcal{E}$ has a right adjoint $\Pi_\top : \mathcal{E} \to \mathcal{E}/\Omega$, and the partial map classifier is defined as:

$$L_\Omega A = (\Sigma_\Omega \circ \Pi_\top)(A)$$

where $\Sigma_\Omega : \mathcal{E}/\Omega \to \mathcal{E}$ sends $f : A \to \Omega$ to A.

The same proof could be adapted for Δ, but we can also construct L_Δ explicitly as follows. For an object Y in $\mathbb{C}$ denote the partial map classifier associated to Ω by $\eta^\Omega_Y : Y \rightarrowtail L_\Omega Y$. Since Ω classifies all monos, η^Ω_Y has a classifying map χ.



By definition, Δ is a subobject of Ω, $\delta : \Delta \rightarrowtail \Omega$. Take the pullback of χ along δ, and then again along $\top : 1 \to \Delta$. The resulting map $\eta_Y : Y \to L_\Delta Y$ is our candidate partial map classifier for Δ. Note that:

$$\eta^\Omega_Y = \chi^*(\delta) \circ \eta_Y.$$

We need to consider a partial map $(m, f) : X \rightharpoonup Y$, where m is classified by Δ, and show it corresponds to a unique total map $X \to L_\Delta Y$. The mono m has a unique classifying map $\alpha : X \to \Delta$ such that:

$$m = \alpha^*(\top).$$

Also, since η_Y^Ω is a partial map classifier, there must be a unique morphism $f' : X \to L_\Omega Y$ such that:

$$m = f'^*(\eta_Y^\Omega).$$

It follows that both $\delta \circ \alpha$ and $\chi \circ f'$ classify m along $\top : 1 \to \Omega$, so they must be equal. This means that X forms a cone for the pullback $L_\Delta Y$, so there must be a comparison map $g : X \to L_\Delta Y$, with the following two properties:

$$f' = \chi^*(\delta) \circ g \qquad\qquad \alpha = \delta^*(\chi) \circ g.$$

From the first equation we can see that:

$$\eta_Y \circ f = g \circ m$$

and using the second equation and the pullback lemma, we can deduce that

$$m = g^*(\eta_Y).$$

It remains to show that g is the unique map with this property. Suppose there is another $g' : X \to L_\Delta Y$. Then m is the pullback of η_Y^Ω along $\chi^*(\delta) \circ g'$, so since f' is the unique map with this property, we have:

$$f' = \chi^*(\delta) \circ g'$$

and therefore $g = g'$. □

Remark 2.4.6. From the proof above we can see that each η_A is a Δ-subobject. Therefore, for every map $f : X \to L_\Delta A$ there is a corresponding (unique) partial map whose domain is a Δ-subobject, obtained by pulling back η_A along f.

Remark 2.4.7. A partial map $A \rightharpoonup 1$ is determined by a Δ-subobject $m : A' \rightarrowtail A$. Therefore, we see that the map $\top : 1 \rightarrowtail \Delta$ satisfies the definition of partial map classifier for 1.

As noted for example in [CL03, Section 2.1], $L_\Delta A$ being a partial map classifier for an object A is equivalent to the existence of a natural isomorphism:

$$\mathbf{Part}_{\mathcal{M}_\Delta}(\mathcal{E})(I-, A) \cong \mathcal{E}(-, L_\Delta A) \quad : \mathcal{E}^{\mathrm{op}} \to \mathsf{Set}$$

where $I : \mathcal{E} \to \mathbf{Part}_{\mathcal{M}_\Delta}(\mathcal{E})$ is the inclusion functor. Because partial maps $X \rightharpoonup 1$ correspond to monos $X' \rightarrowtail X$ classified by Δ, we see that both $L_\Delta 1$ and Δ are representing objects for the same functor $\mathbf{Part}_{\mathcal{M}_\Delta}(\mathcal{E})(I-, 1)$, so they must be isomorphic:

$$\Delta \cong L_\Delta 1.$$

Lemma 2.4.8. *L_Δ is a functor.*

Proof. Define the action of L_Δ on a map $g : A \to B$ to be the total map associated to the partial map $(\eta_A, g) : L_\Delta A \rightharpoonup B$. This is well defined because η_A is a Δ-subobject. Functoriality of L_Δ is proved using the pullback lemma and the fact that every partial map has a unique total map corresponding to it. This also shows that η is a natural transformation. □

Theorem 2.4.9. *In a topos $\mathcal{E}$ with a dominance Δ, the partial map classifier functor L_Δ is in fact a strong commutative monad on $\mathcal{E}$, such that the Kleisli category of L_Δ is equivalent to the category of $\mathcal{M}_\Delta$-partial maps in $\mathcal{E}$:*

$$\mathbf{Kl}(L_\Delta) \cong \mathbf{Part}_{\mathcal{M}_\Delta}(\mathcal{E}).$$

Proof. For a proof that L_Δ is a monad see [Mul92, Theorem 1.8]. For the strength see [CL03, Proposition 2.3]. The fact that $\mathbf{Kl}(L_\Delta) \cong \mathbf{Part}_{\mathcal{M}_\Delta}(\mathcal{E})$ is also mentioned in [CL03, Section 2.1]. □

As explained in [Mul92, CL03], the unit of the monad is given by the partial map classifier structure $\eta_A : A \rightarrowtail L_\Delta A$. The multiplication μ_A :

$L^2_\Delta A \to L_\Delta A$ is the unique total map corresponding to the partial map $(\eta_{L_\Delta A} \circ \eta_A, \mathrm{id}_A) : L^2_\Delta A \rightharpoonup A$.

The monad L_Δ satisfies the definition of *lifting monad* used by, for example, by Fiore, Plotkin and Power [FPP97, Appendix A]. This is true because L_Δ is a partial map classifier and the unit η is cartesian (meaning that the naturality squares for η are pullbacks) because of the way the action of L_Δ on morphisms is defined in Lemma 2.4.8.

Moreover, being a partial map classifier, L_Δ fits the intuition from Set that a lifting monad should add a bottom element to a set. From now on, we will refer to monads obtained from dominances using Theorem 2.4.9 as lifting monads.

Monads have been widely used in denotational semantics to build models of computational effects, starting with the seminal work of Moggi [Mog91]. Therefore, the connection between dominances, partial maps and lifting monads is crucial for us, allowing us to model the non-termination effect in the style of Moggi.

Remark 2.4.10. As remarked in [FP96] after Proposition 1.2, the underlying endofunctor of the monad L_Δ can be described as follows. Let $\Pi_\top : \mathcal{E} \to \mathcal{E}/\Delta$ be the right adjoint to the pullback functor $\top^* : \mathcal{E}/\Delta \to \mathcal{E}/1 \cong \mathcal{E}$ and $\Sigma_\Delta : \mathcal{E}/\Delta \to \mathcal{E}$ be the domain functor, then:

$$L_\Delta = \Sigma_\Delta \circ \Pi_\top.$$

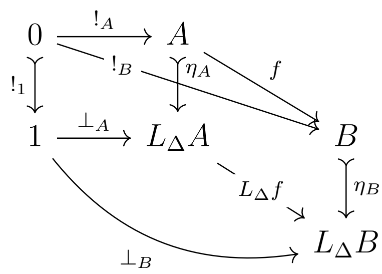
Definition 2.4.11. Let $\mathbb{1}$ be the constant 1 functor and let F be a strong monad, both on a Grothendieck topos $\mathcal{E}$. A natural transformation $\alpha : \mathbb{1} \to F$ is *strong* if for any objects A and B in $\mathcal{E}$ the following diagram commutes:

$$\begin{array}{ccc} A \times \mathbb{1}(B) \cong A \times 1 & \xrightarrow{\ !\ } & \mathbb{1}(A \times B) \cong 1 \\ \downarrow{\scriptstyle \mathrm{id}_A \times \alpha_B} & & \downarrow{\scriptstyle \alpha_{A \times B}} \\ A \times FB & \xrightarrow[\mathrm{str}_{A,B}]{} & F(A \times B) \end{array}$$

The definition above is an instance of a more general definition of strong natural transformation between strong functors (e.g. [Joh02, Section B2.1]), but this restricted definition is all we need here. In this case, the strength of $\mathbb{1}$ is given by maps into the terminal object.

Lemma 2.4.12. *If Δ classifies $0 \rightarrowtail 1$, the monad L_Δ has a point, $\bot : \mathbb{1} \to L_\Delta$, that is a strong natural transformation between the constant 1 functor and L_Δ.*

Proof. For an object A, use the assumption that $0 \rightarrowtail 1$ is a Δ-subobject to define $\bot_A$ to be the total map that corresponds to the partial map $(0 \rightarrowtail 1, 0 \to A)$.

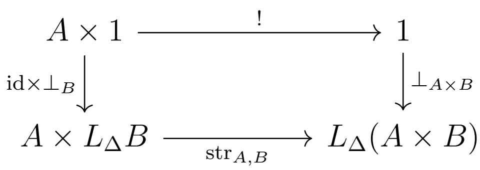


To show naturality, consider a map $f : A \to B$, we need to show that:

$$\bot_B = (L_\Delta f) \circ \bot_A.$$

By definition of $L_\Delta f$, we know that it makes the right square a pullback so both $(L_\Delta f) \circ \bot_A$ and $\bot_B$ correspond to the partial map $(!_1, !_B)$ and must therefore be equal.

To show that $\bot$ is a *strong* natural transformation (Definition 2.4.11), we need to show the following square commutes:



If we multiply the pullback square in the definition of $\bot_B$ by A it still remains a pullback, and we also know from [Joh02, Lemma A.1.4.1] that 0 is a strict initial object, therefore $A \times 0 \cong 0$. So we get the following two pullback squares:

$$\begin{array}{ccccc} 0 & \xrightarrow{!_{A\times B}} & A\times B & \xrightarrow{\mathrm{id}} & A\times B \\ {\scriptstyle !_{A\times 1}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\times\eta_B} & & \downarrow{\scriptstyle \eta_{A\times B}} \\ A\times 1 & \xrightarrow[\mathrm{id}\times\bot_B]{} & A\times L_\Delta B & \xrightarrow[\mathrm{str}_{A,B}]{} & L_\Delta(A\times B) \end{array}$$

where the right square is a pullback because the strength is the total map corresponding to the partial map $(\mathrm{id}_A \times \eta_B, \mathrm{id}_{A\times B})$ [CL03, Proposition 2.3]. Thus, $\mathrm{str}_{A,B} \circ (\mathrm{id}_A \times \bot_B)$ is the total map corresponding to the partial map $(!_{A\times 1}, !_{A\times B})$. (We have shown in Lemma 2.3.7 that every $0 \to X$ is a Δ-subobject.)

Using the pullback in the definition of $\bot_{A\times B}$, we get another two pullback squares:

$$\begin{array}{ccccc} 0 & \xrightarrow{\mathrm{id}} & 0 & \xrightarrow{!_{A\times B}} & A\times B \\ {\scriptstyle !_{A\times 1}}\downarrow & & \downarrow{\scriptstyle !_1} & & \downarrow{\scriptstyle \eta_{A\times B}} \\ A\times 1 & \xrightarrow[!]{} & 1 & \xrightarrow[\bot_{A\times B}]{} & L_\Delta(A\times B) \end{array}$$

Therefore $\bot_{A\times B}\circ!$ also corresponds to the partial map $(!_{A\times 1}, !_{A\times B})$, so we must get the equality:

$$\bot_{A\times B}\circ! = \mathrm{str}_{A,B} \circ (\mathrm{id}_A \times \bot_B).$$

□

Chapter 3

A categorical setting for recursion

In this thesis, we are interested in modelling higher-order recursive programs in the style of Moggi [Mog91] using a cartesian closed category and a monad. A monad however is not enough for modelling recursion, we also need certain maps to have fixed points in order to interpret recursively defined programs.

One example of such a model is the category $\omega\mathsf{CPO}$ of chain-complete partial orders (cpo's) and continuous functions (see e.g. [Win93, Section 11.3]), where Tarski's fixed point theorem holds. In this chapter, we construct fixed points (Corollary 3.2.5) similar to the ones constructed in Tarski's fixed point theorem, but in a more general setting.

We work in a cartesian closed category $\mathbb{C}$ with a strong monad L satisfying Assumption 3.0.1 below. Here L should be thought of as a *lifting monad* in the sense discussed in Section 2.4. We also make two further assumptions about the existence of a certain limit $\overline{\omega}$ (Assumption 3.1.1) and a certain colimit ω (Assumption 3.1.4) in $\mathbb{C}$, which intuitively are analogous to the natural numbers $\mathbb{N} + \{\infty\}$ and $\mathbb{N}$ respectively. We use ω and $\overline{\omega}$ to define analogous notions to "chain" and "chain-completeness" (Definition 3.2.2) and then prove a fixed point theorem (Theorem 3.2.3 and Corollary 3.2.5).

The assumptions in this chapter are reasonable since in Chapters 4 to 8 we deal with sheaf categories with a dominance (Definition 2.3.2). These are cartesian closed, have all small limits and colimits, and have a lifting monad induced by the dominance.

The results in this chapter appeared at FSCD 2021 [MMS21, Section 2]. Very similar results about fixed points appeared before in synthetic and axiomatic domain theory, as explained in the related work section (Section 3.3). However, we do not know whether our fixed point theorem can be deduced from these previous results because of a difference in the way ω and $\overline{\omega}$ are defined. Hence, the main contribution of this chapter is proving the fixed point theorem from Corollary 3.2.5.

Assumption 3.0.1. L is a pointed monad, i.e. there is a natural transformation $\bot : \mathbb{1} \to L$ between the constant 1 functor, $\mathbb{1}$, and L. We ask further that $\bot$ is a *strong* natural transformation (Definition 2.4.11).

3.1 The vertical natural numbers

To model recursively defined programs, we want to be able to talk about increasing chains of approximations and their suprema. For this, we consider an object ω in $\mathbb{C}$ analogous to the linear order $(0 \leq 1 \leq 2 \leq \ldots)$, the *vertical natural numbers* (Assumption 3.1.4), and an object $\overline{\omega}$ analogous to $(0 \leq 1 \leq 2 \leq \ldots \leq \infty)$, the *extended vertical natural numbers* (Assumption 3.1.1). This then allows us to think of a map $\omega \to X$ as a chain in X and $\overline{\omega} \to X$ as a chain with a supremum.

Assumption 3.1.1. Assume that the following diagram has a limit in $\mathbb{C}$, denoted by $\overline{\omega}$:

$$1 \xleftarrow{!} L1 \xleftarrow{L(!)} LL1 \xleftarrow{LL(!)} \ldots \tag{3.1}$$

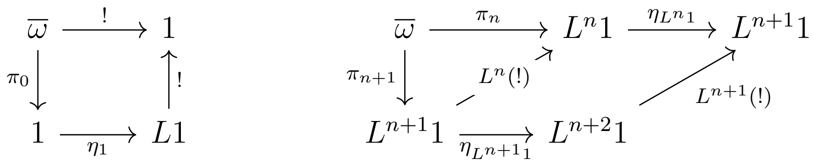
and denote the limiting cone by $(\pi_n : \overline{\omega} \to L^n 1)_{n \in \mathbb{N}}$. We refer to $\overline{\omega}$ as the *extended vertical natural numbers.*

Lemma 3.1.2. *There is another cone over diagram (3.1) with apex $\overline{\omega}$ given by:*

$$! : \overline{\omega} \to 1 = L^0 1 \qquad\qquad \overline{\omega} \xrightarrow{\pi_n} L^n 1 \xrightarrow{\eta_{L^n 1}} L^{n+1} 1.$$

Denote the morphism determined by this cone by $\mathsf{succ}_{\overline{\omega}} : \overline{\omega} \to \overline{\omega}$*.*

Proof. To show we have a cone, there are two diagrams that need to commute:



The first one commutes because 1 is terminal. The second one commutes because $(\pi_n : \overline{\omega} \to L^n(1))_{n\in\mathbb{N}}$ forms a cone and η is natural. $\square$

Lemma 3.1.3. *Another cone for diagram (3.1) with apex* 1 *is given by:*

$$\mathrm{id} : 1 \to 1 = L^0 1 \qquad\qquad 1 \xrightarrow{\eta_{L^n 1}\circ \ldots \circ \eta_1} L^{n+1}1$$

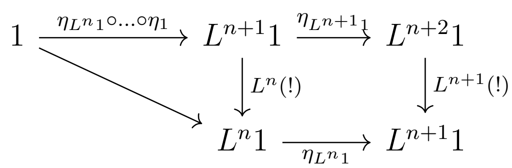
and determines a morphism $\infty : 1 \to \overline{\omega}$*. Furthermore, it is the case that:*

$$\mathsf{succ}_{\overline{\omega}} \circ \infty = \infty.$$

Proof. To show we have a cone, we can see immediately that:

$$1 \xrightarrow{\eta_1} L1 \xrightarrow{!} 1 = \mathrm{id}_1$$

and for all $n \in \mathbb{N}$ the following diagram commutes:



because η is natural and the left triangle was checked in the previous step.

Finally, to show $\mathsf{succ}_{\overline{\omega}} \circ \infty = \infty$, we use uniqueness of the comparison map ∞. We can see immediately that:

$$1 \xrightarrow{\infty} \overline{\omega} \xrightarrow{\mathsf{succ}_{\overline{\omega}}} \overline{\omega} \xrightarrow{\pi_0} 1 = \mathrm{id}_1.$$

The next step is to show that the following square commutes:

$$
\begin{array}{ccccc}
\overline{\omega} & & \xrightarrow{\mathsf{succ}_{\overline{\omega}}} & & \overline{\omega} \\
\uparrow{\scriptstyle\infty} & \searrow^{\pi_n} & & & \downarrow{\scriptstyle\pi_{n+1}} \\
 & & L^n 1 & & \\
 & \nearrow & & \searrow^{\eta_{L^n 1}} & \\
1 & & \xrightarrow[\eta_{L^n 1}\circ \ldots \circ \eta_1]{} & & L^{n+1}1
\end{array}
$$

The top-right triangle commutes because $\mathsf{succ}_{\overline{\omega}}$ is a comparison map, and the top-left triangle because ∞ is one, so we are done. □

Assumption 3.1.4. Assume the following diagram has a colimit ω in $\mathbb{C}$:

$$1 \xrightarrow{\perp_1} L1 \xrightarrow{L(\perp_1)} LL1 \xrightarrow{LL(\perp_1)} \ldots \tag{3.2}$$

and denote the colimiting cocone by $(\iota_n : L^n 1 \to \omega)_{n\in\mathbb{N}}$. We refer to ω as the *vertical natural numbers.*

Lemma 3.1.5. *There is another cocone for diagram (3.2) given by*

$$L^n 1 \xrightarrow{\eta_{L^n 1}} L^{n+1} 1 \xrightarrow{\iota_{n+1}} \omega$$

which determines a comparison map $\mathsf{succ}_\omega : \omega \to \omega$*.*

Proof. To show we have a cocone, we need to show the following commutes:

$$
\begin{array}{ccc}
L^n 1 & \xrightarrow{L^n(\perp_1)} & L^{n+1} 1 \\
{\scriptstyle\eta_{L^n 1}}\downarrow & & \downarrow{\scriptstyle\eta_{L^{n+1} 1}} \\
L^{n+1} 1 & \xrightarrow{L^{n+1}(\perp_1)} & L^{n+2} 1 \\
 & {}_{\iota_{n+1}}\searrow \quad \swarrow_{\iota_{n+2}} & \\
 & \omega &
\end{array}
$$

The top square commutes by naturality of η, and the bottom triangle commutes because $(\iota_n : L^n 1 \to \omega)_{n\in\mathbb{N}}$ forms a cocone. □

Lemma 3.1.6. *For a fixed $n \in \mathbb{N}$, consider the following family of maps indexed by m:*

$$L^m 1 \xrightarrow{L^m(\bot_1)} \dots \xrightarrow{L^{n-1}(\bot_1)} L^n 1 \text{ for } m < n \qquad\qquad L^n 1 \xrightarrow{\text{id}} L^n 1$$

$$L^m 1 \xrightarrow{L^{m-1}(!)} \dots \xrightarrow{L^n(!)} L^n 1 \text{ for } m > n.$$

This family forms a cocone for diagram (3.2) with apex $L^n 1$. Denote by $f_n : \omega \to L^n 1$ the comparison map given by the universal property of the colimit ω.

The family of maps $(f_n : \omega \to L^n 1)_{n\in\mathbb{N}}$ forms a cone for diagram (3.1) with apex ω. Denote by $i : \omega \to \overline{\omega}$ the comparison map given by the universal property of the limit $\overline{\omega}$.

Then i satisfies the following equation:

$$i \circ (\mathsf{succ}_\omega : \omega \to \omega) = (\mathsf{succ}_{\overline{\omega}} : \overline{\omega} \to \overline{\omega}) \circ i.$$

Proof. We first show that the family of maps $(L^m 1 \to L^n 1)_{m\in\mathbb{N}}$ defined above indeed forms a cocone for diagram (3.2). If $m < n$ it is clear that the triangles in the cocone commute. Moreover, $1 \xrightarrow{\bot_1} L1 \xrightarrow{!} 1 = \text{id}_1$; by applying L^m to this equation for $m \geq n$, we see that the triangle at step m commutes. (We can also deduce that $\bot_1$ is monic because it has a retraction.)

Next, we show the family $(f_n : \omega \to L^n 1)_{n\in\mathbb{N}}$ forms a cone for diagram (3.1), that is, for all $n \in \mathbb{N}$:

$$f_n = L^n(!) \circ f_{n+1}.$$

By uniqueness of f_n, it is enough to show for $m < n$:

$$L^{n-1}(\bot_1) \circ \dots \circ L^m(\bot_1) = L^n(!) \circ f_{n+1} \circ \iota_m$$

which is true because $f_{n+1} \circ \iota_m = L^n(\bot_1) \circ L^{n-1}(\bot_1) \circ \dots \circ L^m(\bot_1)$ and

$! \circ \bot_1 = \mathrm{id}_1$. For $m = n$, it is enough to show:

$$\mathrm{id}_{L^n 1} = L^n(!) \circ f_{n+1} \circ \iota_n$$

which is proved similarly. And for $m > n$, show:

$$L^n(!) \circ \ldots \circ L^{m-1}(!) = L^n(!) \circ f_{n+1} \circ \iota_m$$

which is true because the right-hand side equals $L^{n+1}(!) \circ \ldots \circ L^{m-1}(!)$. Therefore, the family $(f_n : \omega \to L^n 1)_{n \in \mathbb{N}}$ forms a cone and we get a comparison map $i : \omega \to \overline{\omega}$.

To show

$$i \circ (\mathsf{succ}_\omega : \omega \to \omega) = (\mathsf{succ}_{\overline{\omega}} : \overline{\omega} \to \overline{\omega}) \circ i$$

using the universal properties of ω and $\overline{\omega}$ it is enough to show for all $n, m \in \mathbb{N}$:

$$\pi_m \circ i \circ \mathsf{succ}_\omega \circ \iota_n = \pi_m \circ \mathsf{succ}_{\overline{\omega}} \circ i \circ \iota_n \quad : L^n 1 \to L^m 1.$$

If $m = 0$, then the two sides must be equal because 1 is terminal.

If $m > 0$, because $\mathsf{succ}_{\overline{\omega}}$, succ_ω and i are comparison maps, we can rewrite the left-hand side as:

$$\mathrm{LHS} = (\pi_m \circ i) \circ (\mathsf{succ}_\omega \circ \iota_n) = f_m \circ (\iota_{n+1} \circ \eta_{L^n 1})$$

and the right-hand side as:

$$\begin{aligned} \mathrm{RHS} = (\pi_m \circ \mathsf{succ}_{\overline{\omega}}) \circ i \circ \iota_n &= (\eta_{L^{m-1} 1} \circ \pi_{m-1}) \circ i \circ \iota_n \\ &= \eta_{L^{m-1} 1} \circ f_{m-1} \circ \iota_n. \end{aligned}$$

Since f_m and f_{m-1} are comparison maps out of the colimit ω we get:

$$\mathrm{LHS} = \begin{cases} L^m(!) \circ \ldots \circ L^n(!) \circ \eta_{L^n 1} & \text{if } n \geq m \\ \mathrm{id}_{L^n 1} \circ \eta_{L^n 1} & \text{if } n + 1 = m \\ L^{m-1}(\bot_1) \circ \ldots \circ L^{n+1}(\bot_1) \circ \eta_{L^n 1} & \text{if } n + 1 < m \end{cases}$$

$$\text{RHS} = \begin{cases} \eta_{L^{m-1}1} \circ L^{m-1}(!) \circ \ldots \circ L^{n-1}(!) & \text{if } n \geq m \\ \eta_{L^{m-1}1} \circ \mathrm{id}_{L^{m-1}1} & \text{if } n+1 = m \\ \eta_{L^{m-1}1} \circ L^{m-2}(\bot_1) \circ \ldots \circ L^{n}(\bot_1) & \text{if } n+1 < m \end{cases}$$

and we can see that in each case the two sides are equal because η is natural.

□

Example 3.1.7. Consider the category Set with the lifting monad L which adds a bottom element to each set: $L(X) = X \cup \{\bot\}$ and $L(f)$ preserves $\bot$ and otherwise acts like f. We can regard $L^n 1$ as the set $\{0, 1, \ldots, n\}$. The maps $\bot_1 : 1 \to L1$ and $\eta_{L^n 1} : L^n 1 \to L^{n+1} 1$ are defined as $\bot_1(0) = 0$ and $\eta_{L^n 1}(k) = k + 1$.

Because $\overline{\omega}$ is the limit of $1 \xleftarrow{!} L1 \xleftarrow{L(!)} LL1 \xleftarrow{LL(!)} \ldots$, it can be regarded as $\mathbb{N} \cup \{\infty\}$, where n is represented by the eventually constant tuple $(0, 1, 2, \ldots, n, n, \ldots)$, and ∞ is represented by the always increasing tuple $(0, 1, 2, \ldots)$. Similarly, ω is the colimit of $1 \xrightarrow{\bot_1} L1 \xrightarrow{L(\bot_1)} LL1 \xrightarrow{LL(\bot_1)} \ldots$ so it is a (quotiented) union and can thus be thought of as $\mathbb{N}$.

The successor maps perform the obvious successor operation on $\mathbb{N}$ and $\mathbb{N} \cup \{\infty\}$ respectively. The comparison map $i : \omega \to \overline{\omega}$ is the inclusion $\mathbb{N} \hookrightarrow \mathbb{N} \cup \{\infty\}$.

In the rest of the thesis we will not be using this example of ω and $\overline{\omega}$ in Set, but instead we will use their description in presheaves on a concrete site extensively (see Section 6.3.1). The Set example provides a good intuition for the situation in presheaves because, as explained in Remark 6.3.1, the underlying sets of ω and $\overline{\omega}$ are still $\mathbb{N}$ and $\mathbb{N} \cup \{\infty\}$ respectively.

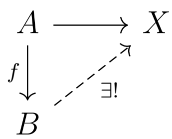
Another example worth mentioning, but which we will not use, is $\omega\mathsf{CPO}$, the category of chain-complete partial orders and continuous maps. In this case, the lifting monad has the same underlying structure as in the Set example, but ω and $\overline{\omega}$ are both $\mathbb{N} \cup \{\infty\}$ with the usual order on natural numbers.

3.2 A fixed point theorem

Recall that Tarski's fixed point theorem (e.g. [Win93, Theorem 5.11]), which applies to cpo's with bottom and continuous functions, constructs a least fixed point as the least upper bound of an ω-chain of approximations. We will use a similar strategy to construct fixed points in our more general setting.

Informally, we think of a map $\omega \to X$ as a chain valued in X. In order to express that the chain $\omega \to X$ has a least upper bound, we will ask that it has a unique extension to a map $\overline{\omega} \to X$. We do this using the notion of orthogonality:

Definition 3.2.1 ([AR94, Definition 1.32])**.** An object $X \in \mathbb{C}$ is *right-orthogonal* to a morphism $f : A \to B$ if every map $A \to X$ factors uniquely through f.

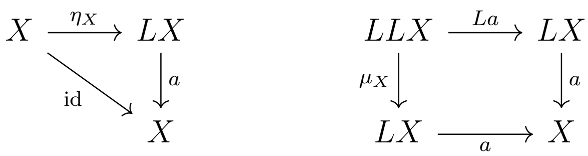


Intuitively, if an object X is right-orthogonal to the comparison map $i : \omega \to \overline{\omega}$, then all the chains $\omega \to X$ have a least upper bound. It is usual in synthetic domain theory to restrict to a subcategory of *complete* objects that admit fixed points (e.g. [FP96, LS97, RS99, vOS00]). We define completeness as follows:

Definition 3.2.2. An object $X \in \mathbb{C}$ is *L-complete* if it is right-orthogonal to $(i \times \mathrm{id}_A) : \omega \times A \to \overline{\omega} \times A$ for every $A \in \mathbb{C}$. Moreover, X is *well-complete* if LX is L-complete.

Because we need to compute fixed points of terms *in context*, it is not enough to ask for orthogonality with respect to $i : \omega \to \overline{\omega}$, instead we ask for the slightly stronger condition above. We will obtain a parametrised fixed point theorem for well-complete objects X that are L-algebras. The requirement that X is an L-algebra is analogous to the requirement from Tarski's fixed point theorem for cpo's that X has a bottom element.

Recall that an L-algebra consists of an object $X \in \mathbb{C}$ and a map $a : LX \to X$ such that the following diagrams commute:



Theorem 3.2.3. *Let $X \in \mathbb{C}$ be well-complete and an L-algebra. For any map $g : \Gamma \times X \to X$, we can construct a fixed point $\phi_g : \Gamma \to X$ such that, for any $\rho : Y \to \Gamma \in \mathbb{C}$:*

$$\phi_g(\rho) = g(\rho, \phi_g(\rho)).$$

In addition to the notion of L-completeness from Definition 3.2.2, in Section 4.1.1 we will introduce a stronger notion (Definition 4.1.1), in order to give a different characterization of well-completeness in Proposition 4.1.5. This characterization will be useful when proving that PCF types are interpreted as complete objects, in Proposition 4.3.4.

Before proving Theorem 3.2.3 above we first prove the following lemma:

Lemma 3.2.4. *Consider $X \in \mathbb{C}$ such that LX is L-complete. Then for any map $f : \Gamma \times LX \to LX$ we can construct a fixed point $\xi : \Gamma \to LX$ such that, for any $\rho : Y \to \Gamma \in \mathbb{C}$:*

$$f(\rho, \xi(\rho)) = \xi(\rho).$$

Proof. The strategy is to define a map $\mathsf{ap}_\omega : \Gamma \times \omega \to LX$ which is a chain of finite approximations of a fixed point for f. Then we extend this map to $\overline{\omega}$ and show that the value at $\infty : 1 \to \overline{\omega}$ gives a fixed point.

Constructing $\mathsf{ap}_\omega : \Gamma \times \omega \to LX$. First define a family of maps $\mathsf{ap}_n : \Gamma \times L^n 1 \to LX$ using generalized elements:

$$\begin{aligned} \mathsf{ap}_0(\rho, !) &= \bot_X(!) & \mathsf{ap}_0 &= \bot_X \circ \pi_2 : \Gamma \times 1 \to LX \\ \mathsf{bp}_n(\rho, i) &= f(\rho, \mathsf{ap}_n(\rho, i)) & \mathsf{bp}_n &= f \circ \langle \pi_1, \mathsf{ap}_n \rangle : \Gamma \times L^n 1 \to LX \\ \mathsf{ap}_{n+1}(\rho, i) &= \mathsf{bp}_n^*(\rho, i) \end{aligned}$$

where bp_n^* is the Γ-indexed Kleisli extension of $\mathsf{bp}_n : \Gamma \times L^n 1 \to LX$:

$$\Gamma \times L^{n+1}1 \xrightarrow{\mathrm{str}_{\Gamma, L^n 1}} L(\Gamma \times L^n 1) \xrightarrow{L\mathsf{bp}_n} L^2 X \xrightarrow{\mu_X} LX.$$

The functor $\Gamma \times (-)$ preserves colimits because it is a left adjoint, so $\Gamma \times \omega$ is the colimit of the diagram:

$$\Gamma \times 1 \xrightarrow{\mathrm{id}_\Gamma \times \bot_1} \Gamma \times L1 \xrightarrow{\mathrm{id}_\Gamma \times L(\bot_1)} \Gamma \times LL1 \xrightarrow{\mathrm{id}_\Gamma \times LL(\bot_1)} \dots. \tag{3.3}$$

Using the universal property of this colimit, we show that the family $(\mathsf{ap}_n)_n$ defines a map $\mathsf{ap}_\omega : \Gamma \times \omega \to LX$. For this we show by induction:

$$\mathsf{ap}_n = \mathsf{ap}_{n+1} \circ (\mathrm{id}_\Gamma \times L^n(\bot_1)).$$

For $n = 0$ this becomes:

$$\begin{aligned} \bot_X \circ ! &= \mu \circ L\mathsf{bp}_0 \circ \mathrm{str} \circ (\mathrm{id}_\Gamma \times \bot_1) \\ &= \mu \circ L\mathsf{bp}_0 \circ \bot_{\Gamma \times 1} \circ ! && \bot \text{ is a strong natural transformation} \end{aligned}$$

and the two sides are equal because $\bot : \mathbb{1} \to L$ is natural.

For $n > 0$, the right hand side becomes:

$$\begin{aligned} \mathsf{ap}_{n+1} \circ (\mathrm{id}_\Gamma \times L^n(\bot_1)) &= \mathsf{bp}_n^* \circ (\mathrm{id}_\Gamma \times L^n(\bot_1)) \\ &= (\mathsf{bp}_n \circ (\mathrm{id}_\Gamma \times L^{n-1}(\bot_1)))^* \\ &\qquad \text{by naturality of strength} \\ &= (f \circ \langle \pi_1, \mathsf{ap}_n \rangle \circ (\mathrm{id}_\Gamma \times L^{n-1}(\bot_1)))^* \\ &= (f \circ \langle \pi_1, \mathsf{ap}_{n-1} \rangle)^* \\ &\qquad \text{by induction hypothesis} \\ &= \mathsf{bp}_{n-1}^* \\ &= \mathsf{ap}_n. \end{aligned}$$

Applying f gives the next element in the chain ap_ω. This intuition is captured by the following equation:

$$\mathsf{ap}_\omega \circ (\mathrm{id}_\Gamma \times \mathsf{succ}_\omega) = f \circ \langle \pi_1, \mathsf{ap}_\omega \rangle \quad : \Gamma \times \omega \to LX. \tag{3.4}$$

The family of maps $(\mathsf{ap}_{n+1} \circ (\mathrm{id}_\Gamma \times \eta_{L^n 1}) : \Gamma \times L^n 1 \to LX)_{n \in \mathbb{N}}$ forms a cocone for diagram (3.3). This can be proved using naturality of η and the fact that $(\mathsf{ap}_n)_{n\in\mathbb{N}}$ forms a cocone for the same diagram. The comparison map for the above cocone is $\mathsf{ap}_\omega \circ (\mathrm{id}_\Gamma \times \mathsf{succ}_\omega)$. To prove that the required triangles commute, we use the fact that ap_ω is the comparison map of the cocone $(\mathsf{ap}_n)_{n\in\mathbb{N}}$, and that $(\mathrm{id}_\Gamma \times \mathsf{succ}_\omega)$ is the comparison map of the cocone $(\mathrm{id}_\Gamma \times (\iota_{n+1} \circ \eta_{L^n 1}))_{n\in\mathbb{N}}$.

Similarly, the family $(f \circ \langle \pi_1, \mathsf{ap}_n \rangle)_{n\in\mathbb{N}}$ is a cocone for diagram (3.3) because $(\mathsf{ap}_n)_{n\in\mathbb{N}}$ is, and has comparison map $f \circ \langle \pi_1, \mathsf{ap}_\omega \rangle$. Therefore to prove Equation (3.4) it is enough to show for all $n \in \mathbb{N}$:

$$\mathsf{ap}_{n+1} \circ (\mathrm{id}_\Gamma \times \eta_{L^n 1}) = f \circ \langle \pi_1, \mathsf{ap}_n \rangle.$$

Notice that the right hand side is the definition of bp_n. The left hand side can be unfolded by definition as:

$$\begin{aligned}
\mathsf{ap}_{n+1} \circ (\mathrm{id}_\Gamma \times \eta_{L^n 1}) &= \mu_X \circ L\mathsf{bp}_n \circ \mathrm{str}_{\Gamma, L^n 1} \circ (\mathrm{id}_\Gamma \times \eta_{L^n 1}) \\
&= \mu_X \circ L\mathsf{bp}_n \circ \eta_{\Gamma \times L^n 1} \qquad \text{by a strength equation} \\
&= \mu_X \circ \eta_X \circ \mathsf{bp}_n \qquad \text{by naturality of } \eta \\
&= \mathsf{bp}_n \qquad \text{by a monad equation.}
\end{aligned}$$

Constructing the fixed point $\xi : \Gamma \to LX$. Because LX is L-complete, ap_ω has a unique extension $\mathsf{ap}_{\overline{\omega}} : \Gamma \times \overline{\omega} \to LX$ along $(\mathrm{id}_\Gamma \times i) : \Gamma \times \omega \to \Gamma \times \overline{\omega}$.

We can show that $\mathsf{ap}_{\overline{\omega}} \circ (\mathrm{id}_\Gamma \times \mathsf{succ}_{\overline{\omega}})$ is an extension of $\mathsf{ap}_\omega \circ (\mathrm{id}_\Gamma \times \mathsf{succ}_\omega)$ using the fact that i commutes with succ (Lemma 3.1.6). Similarly, $f \circ \langle \pi_1, \mathsf{ap}_{\overline{\omega}} \rangle$ is an extension of $f \circ \langle \pi_1, \mathsf{ap}_\omega \rangle$. Using Equation (3.4) we see that these two extensions must be equal so:

$$\mathsf{ap}_{\overline{\omega}} \circ (\mathrm{id}_\Gamma \times \mathsf{succ}_{\overline{\omega}}) = f \circ \langle \pi_1, \mathsf{ap}_{\overline{\omega}} \rangle \quad : \Gamma \times \overline{\omega} \to LX. \tag{3.5}$$

Finally, define the candidate fixed point of f, $\xi : \Gamma \to LX$, to be:

$$\xi(\rho) = \mathsf{ap}_{\overline{\omega}}(\rho, \infty)$$

where ρ is any map into Γ and $\infty : 1 \to \overline{\omega}$ from Lemma 3.1.3. Now we show that ξ indeed has the fixed point property:

$$\begin{aligned}
\xi(\rho) &= \mathsf{ap}_{\overline{\omega}}(\rho, \infty) \\
&= \mathsf{ap}_{\overline{\omega}}(\rho, \mathsf{succ}_{\overline{\omega}}(\infty)) && \text{by Lemma 3.1.3} \\
&= f(\rho, \mathsf{ap}_{\overline{\omega}}(\rho, \infty)) && \text{by Equation (3.5)} \\
&= f(\rho, \xi(\rho)).
\end{aligned}$$

□

Using Lemma 3.2.4 we can finally prove the fixed point theorem, Theorem 3.2.3.

Proof of Theorem 3.2.3. Denote the algebra structure of X by $\alpha : LX \to X$, which we use to construct the following map:

$$\Gamma \times LX \xrightarrow{\mathrm{id}_\Gamma \times \alpha} \Gamma \times X \xrightarrow{g} X \xrightarrow{\eta_X} LX.$$

For the map above, we can construct a fixed point $\xi : \Gamma \to LX$ using Lemma 3.2.4. The candidate fixed point for g is $\alpha \circ \xi : \Gamma \to X$. The following calculation shows that it is indeed a fixed point:

$$\begin{aligned}
g \circ \langle \mathrm{id}_\Gamma, \alpha \circ \xi \rangle &= \alpha \circ \big(\eta_X \circ g \circ (\mathrm{id}_\Gamma \times \alpha)\big) \circ \langle \mathrm{id}_\Gamma, \xi \rangle \\
&\qquad \text{by the first algebra equation of } \alpha \\
&= \alpha \circ \xi \qquad \text{because } \xi \text{ is a fixed point.}
\end{aligned}$$

□

Using the fixed point constructed in Theorem 3.2.3 we can obtain fixed points suitable for interpreting recursion in a call-by-value language:

Corollary 3.2.5. *Consider objects* Γ*,* A*,* B *in* $\mathbb{C}$ *such that* LB^A *is a well-complete object. For a morphism* $m : \Gamma \times LB^A \times A \to LB$ *we can construct a fixed point* $\mathsf{rec}_m : \Gamma \to LB^A$ *such that, for any* $\langle \rho, a \rangle : Y \to \Gamma \times A \in \mathbb{C}$*:*

$$\mathsf{rec}_m(\rho)(a) = m(\rho, \mathsf{rec}_m(\rho), a).$$

Proof. The object LB^A has an algebra structure given by currying the following morphism:

$$L(LB^A) \times A \xrightarrow{\mathrm{str}_{A,LB^A}} L(LB^A \times A) \xrightarrow{L\mathsf{ev}_{A,B}} LLB \xrightarrow{\mu_B} LB.$$

Currying m we get a map of type $\Gamma \times LB^A \to LB^A$ for which we can apply Theorem 3.2.3 to obtain the fixed point rec_m. □

3.3 Related work

The treatment of recursion described in this chapter originates in the axiomatic domain theory literature [FP96, Fio96, FPP97, Fio97]. The ideas of constructing the vertical natural numbers objects ω and $\overline{\omega}$, and of using orthogonality to require that chains have least upper bounds are not new.

Perhaps the closest work is [FP96]. There however, ω and $\overline{\omega}$ are defined as the initial algebra and final coalgebra of the functor L respectively, rather than being defined as a colimit and limit. When we made our definitions however we were not familiar with these developments in axiomatic domain theory.

We do not know whether the assumptions we made in this chapter imply that ω is the initial algebra, but this fact is not necessary for proving the fixed point theorem (Corollary 3.2.5) that we use for the denotational interpretation of call-by-value PCF in Section 4.3. Nevertheless, we conjecture that in all the examples we consider (Sections 4.4 and 7.2 and Chapter 8) ω is the initial algebra.

The same ideas appear in the synthetic domain theory literature. Longley and Simpson [LS97, Section 5] (and others [RS99, vOS00]) also define ω and $\overline{\omega}$ as initial algebra and final coalgebra, and use the notion of complete object and dominance. However, their results are in the context of realizability toposes rather than sheaf categories, which we study in the following chapters.

Chapter 4

Normal models of $\mathsf{PCF_v}$

The aim of this chapter is to identify sufficient conditions such that a Grothendieck topos has enough structure to interpret call-by-value PCF ($\mathsf{PCF_v}$). We will call such a category a *normal model* (Definition 4.3.1).

In Section 4.1, we prove some technical results about completeness, as defined in Definition 3.2.2. In Section 4.2 we define $\mathsf{PCF_v}$ and its operational semantics. In Section 4.3 we define normal models and, using the completeness results from Section 4.1, we show that normal models can interpret $\mathsf{PCF_v}$; Theorem 4.3.5 shows that this interpretation is sound. Finally, in Section 4.4 we introduce our first example of normal model, the category vSet of presheaves on the vertical natural numbers (Definition 4.4.1), which will be a running example. In Proposition 4.4.10 we show that the vSet model is essentially the traditional $\omega\mathsf{CPO}$ model of call-by-value PCF.

In Chapters 5 to 7 we will develop a recipe for building adequate normal models and show that the vSet model is an instance of this recipe. Other examples of the same recipe, that go beyond $\omega\mathsf{CPO}$, will be presented in Section 7.2 and in Chapter 8.

The results in this chapter were published at FSCD 2021 [MMS21, Section 3.1, 4, 5]. The notion of normal model is closely related to Simpson's notion of natural model [Sim98], and the interpretation of $\mathsf{PCF_v}$ in a normal model is the usual interpretation in a cartesian closed category with a monad (see e.g [Mog91]). Nevertheless, the contribution of this chapter is to prove that normal models satisfy the premises of the fixed point theorem from Corollary 3.2.5. This is done using the results about completeness from Section 4.1, and we do not know if it can be deduced from the work of Simpson [Sim98], because of the different way in which completeness is defined there; this issue is discussed in Section 7.4. The notion of normal model will be useful in the next three chapters, as a target for our recipe of building sound and adequate models.

In this chapter we work in a Grothendieck topos $\mathcal{E}$ with a dominance Δ (Definition 2.3.2) which classifies the subobject $0 \rightarrowtail 1$. Let L_Δ be the lifting monad associated to the dominance Δ. Then by Lemma 2.4.12, L_Δ is a pointed monad. Denote by ω_Δ and $\overline{\omega}_\Delta$ the vertical natural numbers objects (defined in Section 3.1) obtained from the lifting monad L_Δ. Notice that ω_Δ and $\overline{\omega}_\Delta$ exist because $\mathcal{E}$ has all small limits and colimits. Hence all the assumptions about $\mathcal{E}$ from Chapter 3, needed for fixed points, are satisfied.

4.1 Consequences of the dominance being a complete object

In order to interpret $\mathsf{PCF_v}$ one condition we will ask for is that Δ is an L_Δ-complete object (Definition 3.2.2). Roughly speaking, this will ensure that there are enough complete objects in $\mathcal{E}$ to interpret types. In this section, we prove some technical consequences of Δ being complete, which will be used in Section 4.3.

4.1.1 A strengthening of completeness

Recall that in Section 3.2 we defined an object X to be L_Δ-complete if it is right-orthogonal (Definition 3.2.1) to all maps of the form $\omega_\Delta \times A \to \overline{\omega}_\Delta \times A$. Intuitively, this means that ω-chains valued in X and parameterised by A have a least upper bound.

Now, we consider a strengthening of the L_Δ-completeness condition, which roughly says that *partial* maps $\omega_\Delta \times A \rightharpoonup X$ can be extended uniquely to partial maps $\overline{\omega}_\Delta \times A \rightharpoonup X$, rather than just total maps. The reason for introducing this new definition is to characterize L_Δ-completeness for objects of the form $L_\Delta X$, in Proposition 4.1.5. This characterization will be used when proving that $\mathsf{PCF_v}$ types are interpreted as complete objects (Proposition 4.3.4).

Definition 4.1.1. Let $\mathcal{O}_\Delta$ be the class of maps in $\mathcal{E}$ which are pullbacks of maps of the form $i\times\mathrm{id}_A : \omega_\Delta\times A \to \overline{\omega}_\Delta\times A$ along Δ-subobjects of $\overline{\omega}_\Delta\times A$ (Definition 2.3.1). Write $\mathcal{O}_\Delta^\perp$ for the class of objects right-orthogonal (Definition 3.2.1) to every map in $\mathcal{O}_\Delta$.

The following are useful facts about $\mathcal{O}_\Delta$; we sketch the proofs:

- $\mathcal{O}_\Delta$ is closed under pullback along Δ-subobjects.

  Proof sketch. This is because the composition of Δ-subobjects is a Δ-subobject. □

- $\mathcal{O}_\Delta$ is closed under the operations $(-) \times \mathrm{id}_A$.

  Proof sketch. We can use the fact that Δ-subobjects are closed under these operations. □

The orthogonality class $\mathcal{O}_\Delta^\perp$ seen as a full subcategory of $\mathcal{E}$ has the following standard properties:

- $\mathcal{O}_\Delta^\perp$ is contained in the class of L_Δ-complete objects.

  Proof sketch. This is because the identity subobject $\mathrm{id} : \overline{\omega}\times A \to \overline{\omega}\times A$ is always a Δ-subobject. □

- $\mathcal{O}_\Delta^\perp$ is closed under limits in $\mathcal{E}$ [AR94, Observation 1.34].
- $\mathcal{O}_\Delta^\perp$ is a reflective subcategory of $\mathcal{E}$ (defined in e.g. [Joh02, A1.1.1]).
- $\mathcal{O}_\Delta^\perp$ is an exponential ideal (see e.g. [Joh02, A1.5.10] for a definition).

We give proofs for the last two facts above:

Lemma 4.1.2. *$\mathcal{O}_\Delta^\perp$ is a reflective subcategory of $\mathcal{E}$.*

Proof. From [AR94, Observation 1.36] we know that every *small* orthogonality class, that is, objects orthogonal to a small class of maps, is a reflective subcategory of the Grothendieck topos $\mathcal{E}$.

Every Grothendieck topos $\mathcal{E}$ admits a set of objects $\mathcal{S}$ which generates $\mathcal{E}$ under colimits, by Giraud's theorem e.g. [Joh02, C2.2.8]. In the case of a sheaf topos, we can take this set to be the sheafified representables $ay(c)$. The class of maps $\mathcal{O}_\Delta$ is not small, but the subset of $\mathcal{O}_\Delta$ consisting of pullbacks of maps of the form $i \times \mathrm{id}_{ay(c)}$, where c is an object in the site of $\mathcal{E}$, is small. We can deduce that $\mathcal{O}_\Delta^\perp$ is equivalently the class of objects right orthogonal to this small subset of $\mathcal{O}_\Delta$, using the fact that any object in $\mathcal{E}$ is a colimit of sheafified representables and colimits are preserved by products and pullbacks. Thus, we can apply Observation 1.36 from [AR94] to $\mathcal{O}_\Delta^\perp$. □

Lemma 4.1.3. *$\mathcal{O}_\Delta^\perp$ is an exponential ideal.*

Proof. Let $A \in \mathcal{O}_\Delta^\perp$. We need to prove that for any object B in $\mathcal{E}$, the exponential $B \Rightarrow A$ is in $\mathcal{O}_\Delta^\perp$.

Let $x : X' \to X$ be the pullback of a Δ-subobject $X \rightarrowtail \overline{\omega}_\Delta \times C$ along $\omega_\Delta \times C \to \overline{\omega}_\Delta \times C$, and consider a map $f : X' \to (B \Rightarrow A)$. Let $f' : X' \times B \to A$ be the uncurrying of f.

Then $x \times \mathrm{id}_B : X' \times B \to X \times B$ is in $\mathcal{O}_\Delta$ because $\mathcal{O}_\Delta$ is closed under $(-) \times \mathrm{id}_B$. Since $A \in \mathcal{O}_\Delta^\perp$, f' has a unique extension $\overline{f'} : X \times B \to A$. Then the currying of $\overline{f'}$, $\overline{f} : X \to (B \Rightarrow A)$, is the unique extension of f. □

4.1.2 Some completeness results

We now establish some consequences of the dominance Δ being an L_Δ-complete object. First, we show that for any object X, $L_\Delta X$ being L_Δ-complete (i.e X being well-complete) is equivalent to X being in $\mathcal{O}_\Delta^\perp$ (Proposition 4.1.5). Then we show that the initial object is in $\mathcal{O}_\Delta^\perp$ (Proposition 4.1.6) and give a necessary and sufficient condition for $\mathcal{O}_\Delta^\perp$ to be closed under coproducts (Proposition 4.1.9). These propositions will be used in Section 4.3

to show that we can interpret recursion in a Grothendieck topos with a dominance using the fixed point constructed in Corollary 3.2.5.

Lemma 4.1.4. *Suppose* Δ *is* L_Δ*-complete. Then* Δ *is in* $\mathcal{O}_\Delta^\perp$.

Proof. Consider a Δ-subobject $\varphi : X' \rightarrowtail \overline{\omega}_\Delta \times A$. Because Δ-subobjects are stable under pullback (Lemma 2.3.5), $\alpha : X \rightarrowtail \omega_\Delta \times A$ in the pullback diagram below is also a Δ-subobject:

$$\begin{array}{ccc} X & \overset{\alpha}{\rightarrowtail} & \omega_\Delta \times A \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle i\times \mathrm{id}_A} \\ X' & \underset{\varphi}{\rightarrowtail} & \overline{\omega}_\Delta \times A \end{array}$$

We need to show that Δ is right orthogonal to x so consider a map $f : X \to \Delta$. We will show that f has a unique extension $f' : X' \to \Delta$.

From Remark 2.4.7 we know that Δ satisfies the definition of a partial map classifier for 1. So f corresponds to a partial map $(f^*(\top) : Y \rightarrowtail X, ! : Y \to 1)$ which makes the following diagram a pullback:

$$\begin{array}{ccc} Y & \overset{!}{\longrightarrow} & 1 \\ {\scriptstyle f^*(\top)}\downarrow & & \downarrow{\scriptstyle \top} \\ X & \underset{f}{\longrightarrow} & \Delta \cong L_\Delta 1 \end{array}$$

The aim is to find another Δ-subobject $Y' \rightarrowtail X'$ which corresponds to a map f' extending f.

Δ-subobjects are closed under composition by definition so $\alpha \circ f^*(\top)$ is also a Δ-subobject, and has a classifying map $\beta : \omega_\Delta \times A \to \Delta$. Because Δ is right orthogonal to $i \times \mathrm{id}_A : \omega_\Delta \times A \to \overline{\omega}_\Delta \times A$, β has a unique extension $\overline{\beta}$. The pullback of $\top$ along $\overline{\beta}$ is again a Δ-subobject $\overline{\beta}^*(\top) : Y' \rightarrowtail \overline{\omega}_\Delta \times A$. All these maps are depicted in Figure 4.1.

Because $\alpha \circ f^*(\top)$ is the pullback of $\top$ along β, and $\overline{\beta}^*(\top)$ the pullback of $\top$ along $\overline{\beta}$ and:

$$\beta = \overline{\beta} \circ (i \times \mathrm{id}_A)$$

then by the pullback lemma, $\alpha \circ f^*(\top)$ is also the pullback of $\overline{\beta}^*(\top)$ along

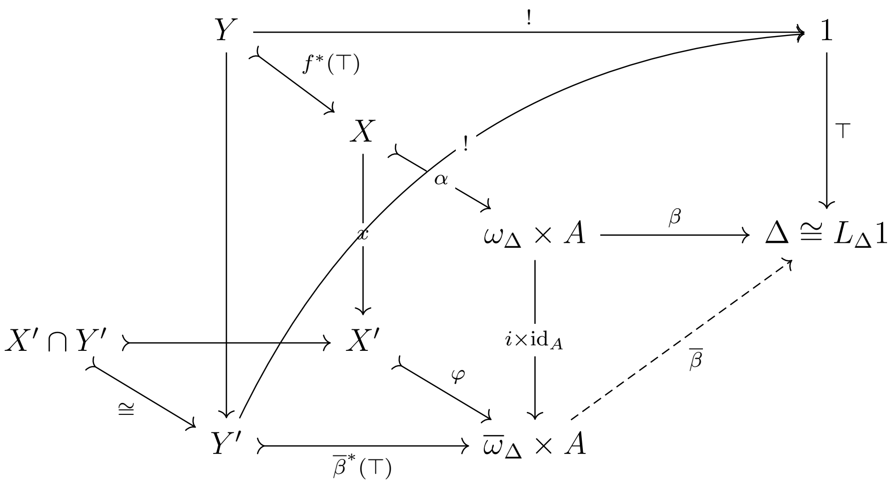
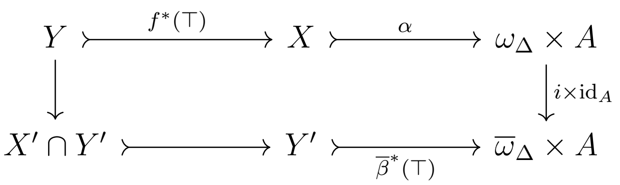


Figure 4.1: Showing Δ is in $\mathcal{O}_\Delta^\perp$.

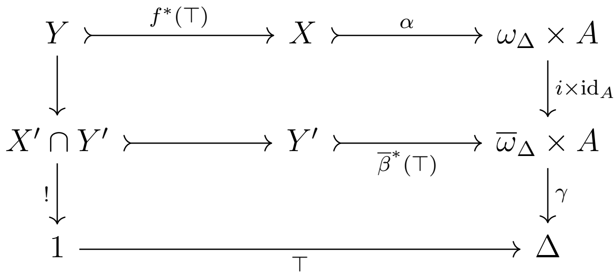
$(i \times \mathrm{id}_A)$.

The next step is to show that Y' actually factors through X', giving the Δ-subobject we want.

Taking the intersection of the subobjects $\alpha \circ f^*(\top)$ and α, via pullback, gives again $\alpha \circ f^*(\top)$. Because they are the pullback of $\overline{\beta}^*(\top)$ and φ respectively, along $(i \times \mathrm{id}_A)$, we must get the same result if we first take the pullback of $\overline{\beta}^*(\top)$ and φ, denoted by $X' \cap Y'$, and then pull back the result along $(i \times \mathrm{id}_A)$. Therefore the following square is also a pullback:



Δ-subobjects are stable under pullback and closed under composition so $X' \cap Y' \rightarrowtail Y' \rightarrowtail \overline{\omega}_\Delta \times A$ is also a Δ-subobject. Let its classifying map be $\gamma : \overline{\omega}_\Delta \times A \to \Delta$. By the pullback lemma the outer square in the following diagram is a pullback:



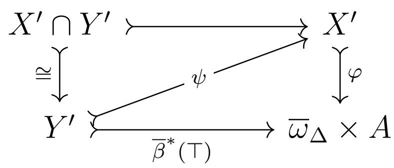
But we assumed that the unique classifying map of $\alpha \circ f^*(\top)$ is β, therefore:

$$\beta = \gamma \circ (i \times \mathrm{id}_A).$$

Because Δ is complete, $\overline{\beta}$ is the unique extension of β, therefore:

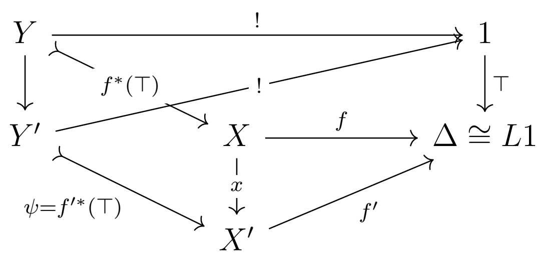
$$\overline{\beta} = \gamma.$$

This means that the subobjects that $\overline{\beta}$ and γ classify must be isomorphic, so $X' \cap Y' \rightarrowtail Y'$ is an isomorphism and we have a Δ-subobject $\psi : Y' \rightarrowtail X'$:



Define the candidate extension of $f : X \to \Delta$, $f' : X' \to \Delta$, to be the classifying map of ψ, or equivalently the total map associated to the partial map $(\psi, ! : Y' \to 1)$. So $\psi = f'^*(\top)$.

Consider the following diagram. We need to show that $f = f' \circ x$.



By looking at Figure 4.1 we can see that $f^*(\top)$ is the pullback of $f'^*(\top)$ along x. Therefore, by the pullback lemma, $f^*(\top)$ is the pullback of $\top$ along $f' \circ x$. But the unique classifying map of $f^*(\top)$ is f, therefore:

$$f = f' \circ x.$$

Finally, we need to show that f' is unique. Consider another $f'' : X' \to \Delta$ such that:

$$f = f'' \circ x.$$

Let the Δ-subobject $f''^*(\top) : Y'' \rightarrowtail X'$ be the pullback of $\top$ along f''. Using the pullback lemma, we see that $f^*(\top) : Y \rightarrowtail X$ must be the pullback of $f''^*(\top)$ along x.

Therefore $\varphi \circ f''^*(\top) : Y'' \rightarrowtail X' \rightarrowtail \overline{\omega}_\Delta \times A$ pulled back along $i \times \mathrm{id}_A$ gives $\alpha \circ f^*(\top) : Y \rightarrowtail X \rightarrowtail \omega_\Delta \times A$. The map $\varphi \circ f''^*(\top) : Y'' \rightarrowtail X' \rightarrowtail \overline{\omega}_\Delta \times A$ is a Δ-subobject, so let its classifying map be $\rho : \overline{\omega}_\Delta \times A \to \Delta$. Because the classifying map of $\alpha \circ f^*(\top)$ is β we get that:

$$\beta = \rho \circ (i \times \mathrm{id}_A).$$

But the unique extension of β is $\overline{\beta}$ so:

$$\overline{\beta} = \rho$$

and the Δ-subobjects that they classify must be isomorphic:

$$\overline{\beta}^*(\top) = \varphi \circ \psi \cong \varphi \circ f''^*(\top).$$

The map φ is mono so the following Δ-subobjects are also isomorphic:

$$\psi = f'^*(\top) \cong f''^*(\top)$$

which means they must have the same classifying map and therefore:

$$f' = f''. \qquad \square$$

Proposition 4.1.5. *Assume that* Δ *is* L_Δ*-complete. Then for any* A *in* $\mathcal{E}$ *the following three statements are equivalent:*

1. A *is in* $\mathcal{O}_\Delta^\perp$*;*

2. $L_\Delta A$ *is* L_Δ*-complete;*

3. $L_\Delta A$ *is in* $\mathcal{O}_\Delta^\perp$*.*

Notice that the fact that Item 1 implies Item 3 means that $\mathcal{O}_\Delta^\perp$ *is closed under* L_Δ*.*

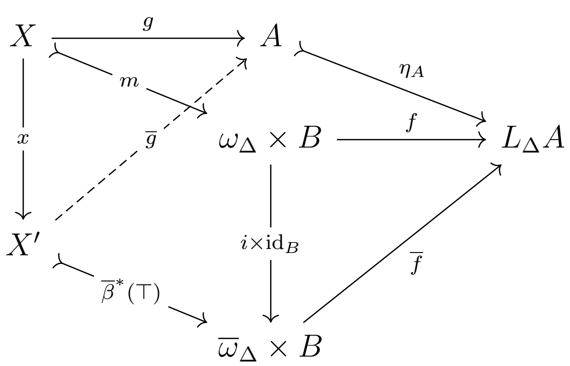
Proof. We first show that Item 1 is equivalent to Item 2.

Item 1 implies Item 2. Assume that $A \in \mathcal{O}_\Delta^\perp$. To show that $L_\Delta A$ is L_Δ-complete, consider a map $f : \omega_\Delta \times B \to L_\Delta A$ and show it has a unique extension to $\overline{\omega}_\Delta \times B$.

The total map $f : \omega_\Delta \times B \to L_\Delta A$ corresponds to a partial map $(m : X \rightarrowtail \omega_\Delta \times B, g : X \to A)$ with m a Δ-subobject. Therefore, m has a classifying map $\beta : \omega_\Delta \times B \to \Delta$. Because Δ is L_Δ-complete, β has a unique extension $\overline{\beta} : \overline{\omega}_\Delta \times B \to \Delta$ such that:

$$\beta = \overline{\beta} \circ (i \times \mathrm{id}_B).$$

Let $\overline{\beta}^*(\top)$ be the Δ-subobject classified by $\overline{\beta}$. We can deduce by the pullback lemma that m is the pullback of $\overline{\beta}^*(\top)$ along $(i \times \mathrm{id}_B)$. The whole construction appears in the diagram below.



Because $A \in \mathcal{O}_\Delta^\perp$, A is orthogonal to x, so g has a unique extension $\overline{g}$ such that:

$$g = \overline{g} \circ x.$$

The pair $(\overline{\beta}^*(\top), \overline{g})$ is a partial map which determines a total map $\overline{f} : \overline{\omega}_\Delta \times B \to L_\Delta A$, such that $\overline{\beta}^*(\top)$ is the pullback of η_A along $\overline{f}$.

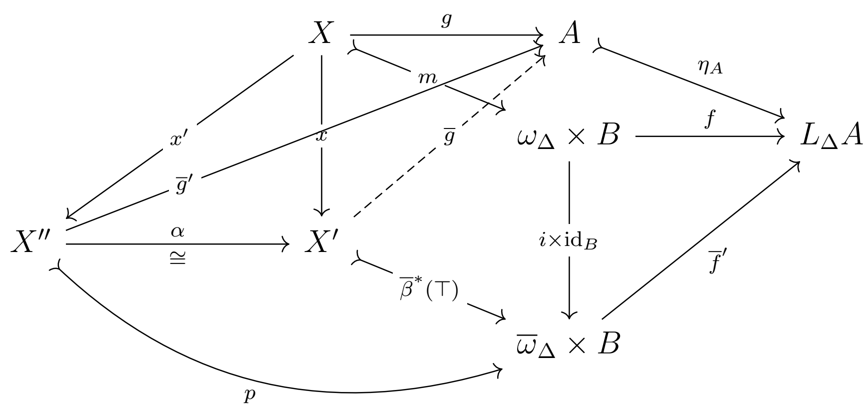
We already know that m is the pullback of $\overline{\beta}^*(\top)$ along $(i \times \mathrm{id}_B)$, so by the pullback lemma m is the pullback of η_A along $\overline{f} \circ (i \times \mathrm{id}_B)$. So $\overline{f} \circ (i \times \mathrm{id}_B)$ represents the partial map (m, g). But we assumed that the unique total map corresponding to the partial map (m, g) is f, therefore:

$$f = \overline{f} \circ (i \times \mathrm{id}_B).$$

It now remains to show that $\overline{f}$ is the unique map with this property. Assume there is another $\overline{f}'$ such that:

$$f = \overline{f}' \circ (i \times \mathrm{id}_B).$$

This determines a partial map $(p : X'' \rightarrowtail \overline{\omega}_\Delta \times B, \overline{g}' : X'' \to A)$, where p is a Δ-subobject, as in the diagram below. By the pullback lemma, m is the pullback of p along $i \times \mathrm{id}_B$, where the other side of the pullback is x'.



Let $\beta_p : \overline{\omega}_\Delta \times B \to \Delta$ be the classifying map of p. Again by the pullback lemma, m is the pullback of $\top$ along $\beta_p \circ (i \times \mathrm{id}_B)$, so by uniqueness of β, the classifying map of m, we have:

$$\overline{\beta} = \beta_p.$$

This means that the Δ-subobjects $\overline{\beta}^*(\top)$ and p are isomorphic. So there exists an iso $\alpha : X'' \to X'$ such that $\overline{\beta}^*(\top) \circ \alpha = p$, and also:

$$x = \alpha \circ x',$$

which means that

$$g = \overline{g} \circ \alpha \circ x'.$$

We know that A is orthogonal to x', therefore g has a unique extension along x', which we have shown to be $\overline{g} \circ \alpha$. We can also show by a diagram chase that:

$$\eta_A \circ g = \eta_A \circ \overline{g}' \circ x'.$$

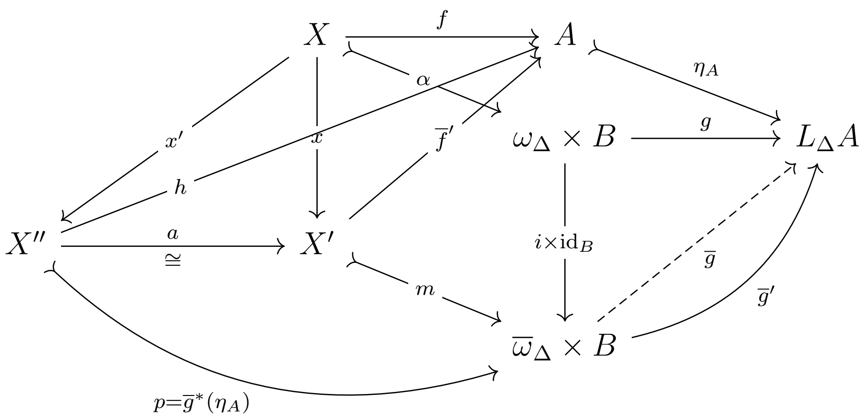
Because η_A is mono by the definition of partial map classifier we get that:

$$\overline{g}' = \overline{g} \circ \alpha.$$

We have shown that the partial maps $(p, \overline{g}')$ and $(\overline{\beta}^*(\top), \overline{g})$ are in fact equal. Therefore they must correspond to the same total map, so:

$$\overline{f}' = \overline{f}.$$

Item 2 implies Item 1. Assume that $L_\Delta A$ is L_Δ-complete. To show $A \in \mathcal{O}_\Delta^\perp$, consider a Δ-subobject $m : X' \rightarrowtail \overline{\omega}_\Delta \times B$ and denote its pullback along $i \times \mathrm{id}_B$ by $x : X \to X'$ and $\alpha : X \rightarrowtail \omega_\Delta \times B$. We need to show that A is right orthogonal to x, so consider a map $f : X \to A$ which we will show can be extended uniquely to X'. The situation is depicted in the diagram below.



The pair (α, f) is a partial map, so it must correspond to a total map $g : \omega_\Delta \times B \to L_\Delta A$. Because $L_\Delta A$ is complete, g has a unique extension $\overline{g}$ such that:

$$g = \overline{g} \circ (i \times \mathrm{id}_B).$$

Let (p, h) be the partial map corresponding to $\overline{g}$. We will show that the Δ-subobjects p and m are in fact isomorphic.

Using the pullback lemma we can see that α is the pullback of p along $i \times \mathrm{id}_B$. Denote by $x' : X \to X''$ the other leg of the pullback. The map p is a Δ-subobject so it has a classifying map $\gamma : \overline{\omega}_\Delta \times B \to \Delta$. Therefore $\gamma \circ (i \times \mathrm{id}_B)$ must be the classifying map of the α.

Similarly, let β, be the classifying map of m, then $\beta \circ (i \times \mathrm{id}_B)$ classifies α so we get:

$$\gamma \circ (i \times \mathrm{id}_B) = \beta \circ (i \times \mathrm{id}_B).$$

We know that Δ is complete, so the map $\gamma \circ (i \times \mathrm{id}_B) : \omega_\Delta \times B \to L_\Delta A$ has a unique extension, therefore:

$$\gamma = \beta.$$

This means that m and p are isomorphic Δ-subobjects, so there exists an iso $a : X'' \to X'$ such that $p = m \circ a$, and therefore $x = a \circ x'$.

Let the candidate extension of f be:

$$\overline{f} = a^{-1} \circ h.$$

and we need to show $f = \overline{f} \circ x$. This reduces to showing:

$$f = h \circ x'.$$

By a diagram chase we can see that:

$$\eta_A \circ f = \eta_A \circ h \circ x'$$

and this is sufficient because η_A is monic.

Now we need to show that $\overline{f} = h \circ a^{-1}$ is the unique extension of f along x. Consider another $\overline{f}'$ such that:

$$f = \overline{f}' \circ x.$$

The pair $(m, \overline{f}')$ is a partial map, so it corresponds to a total map $\overline{g}' : \overline{\omega}_\Delta \times B \to L_\Delta A$. Therefore, α is the pullback of η_A along $\overline{g}' \circ (i \times \mathrm{id}_B)$, which means that:

$$g = \overline{g}' \circ (i \times \mathrm{id}_B).$$

But the unique extension of g is $\overline{g}$, so $\overline{g} = \overline{g}'$. This means that the partial maps they correspond to, (p, h) and $(m, \overline{f}')$, are equal. We already know that a is an iso between p and m (and m is mono), so we obtain:

$$\overline{f} = h \circ a^{-1} = \overline{f}'.$$

Item 1 implies Item 3. Assume that $A \in \mathcal{O}_\Delta^\perp$. We need to prove that $L_\Delta A \in \mathcal{O}_\Delta^\perp$. For this consider a Δ-subobject $q : Z' \rightarrowtail \overline{\omega}_\Delta \times B$, and $z : Z \to Z'$ its pullback along $i \times \mathrm{id}_B$. To show $L_\Delta A$ is right orthogonal to z consider a map $f : Z \to L_\Delta A$ and show it has a unique extension to Z'.

From Lemma 4.1.4 we know that $\Delta \in \mathcal{O}_\Delta^\perp$, so we know Δ is right orthogonal to z. Using this fact the proof proceeds in the same way as the proof that Item 1 implies Item 2 ($A \in \mathcal{O}_\Delta^\perp$ implies $L_\Delta A$ is complete). The role of $i \times \mathrm{id}_B$ from that proof is now played by z, and orthogonality of Δ to $i \times \mathrm{id}_B$ is replaced by orthogonality to z.

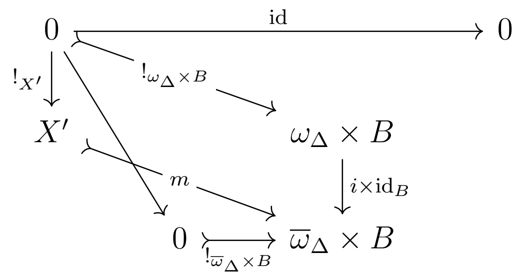
Item 3 implies Item 2. We need to prove that $L_\Delta A \in \mathcal{O}_\Delta^\perp$ implies $L_\Delta A$ is L_Δ-complete. This is immediate because the top subobjects are Δ-subobjects.

□

Proposition 4.1.6. *Assume that Δ is L_Δ-complete. Then 0, the initial object, is in $\mathcal{O}_\Delta^\perp$.*

Proof. Let $m : X' \rightarrowtail \overline{\omega}_\Delta \times B$ be a Δ-subobject and consider $x : X \to X'$ the pullback of $i \times \mathrm{id}_B : \omega_\Delta \times B \to \overline{\omega}_\Delta \times B$ along m. Consider a map $f : X \to 0$. We need to show f can be extended uniquely to X'.

Since we are working in a topos, we know that 0 is a strict initial object ([Joh02, Lemma A.1.4.1]), so any map with codomain 0 is an isomorphism. Therefore X is isomorphic to 0, and it is enough to show that id_0 can be extended to a map $X' \to 0$. The situation is illustrated in the following diagram.



It is enough to show that X' is isomorphic to 0. Consider the Δ-subobject $!_{\overline{\omega}_\Delta \times B} : 0 \rightarrowtail \overline{\omega}_\Delta \times B$. We know both $!_{\overline{\omega}_\Delta \times B}$ and $!_{\omega_\Delta \times B}$ are Δ-subobjects from Lemma 2.3.7. The map $!_{\omega_\Delta \times B}$ must be the pullback of $!_{\overline{\omega}_\Delta \times B}$ along $i \times \mathrm{id}_B$ because 0 is strict initial.

Using the pullback lemma, we can see that the classifying map of $!_{\omega_\Delta \times B}$ is equal to $i \times \mathrm{id}_B$ composed with either the classifying map if m or the classifying map of $!_{\overline{\omega}_\Delta \times B}$. Because Δ is L_Δ-complete, it means that m and $!_{\overline{\omega}_\Delta \times B}$ have the same classifying map, so they are isomorphic subobjects. Therefore X' is isomorphic to 0.

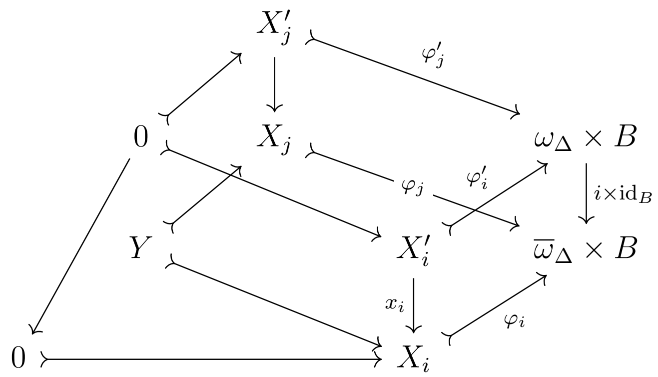
□

Before we can prove the last consequence of completeness of Δ, which is about coproducts (Proposition 4.1.9), we need to prove the following two lemmas:

Lemma 4.1.7. *Assume that Δ is L_Δ-complete. A J-indexed family of subobjects of $\overline{\omega}_\Delta \times B$ is disjoint (meaning that their pullback is the initial object 0) if their pullback along $i \times \mathrm{id}_B : \omega_\Delta \times B \to \overline{\omega}_\Delta \times B$ is a disjoint family.*

Proof. Consider a family of Δ-subobjects $(\varphi_j : X_j \rightarrowtail \overline{\omega}_\Delta \times B)_{j\in J}$ and their pullback along $i \times \mathrm{id}_B$, $(\varphi'_j : X'_j \rightarrowtail \omega_\Delta \times B)_{j\in J}$. It is enough to show that any two subobjects X_i and X_j are disjoint. Let their pullback be Y.

We know that X'_i and X'_j are disjoint so their pullback is 0. Now we want to show that the Δ-subobject $Y \rightarrowtail X_i$ is isomorphic to $0 \rightarrowtail X_i$.



We know that $0 \rightarrowtail X_i$ and $0 \rightarrowtail X'_i$ are Δ-subobjects from Lemma 2.3.7. Let $\alpha_i : X_i \to \Delta$ be the classifying map of $0 \rightarrowtail X_i$. By the pullback lemma, $\alpha_i \circ x_i$ is the classifying map of $0 \rightarrowtail X'_i$.

Let β_j be the classifying map of φ'_j. Then by the pullback lemma $\beta_j \circ \varphi'_i$ classifies $0 \rightarrowtail X'_i$. Because Δ is L_Δ-complete, β_j has an extension $\overline{\beta}_j$ which classifies φ_j, so we have the equality:

$$\begin{aligned}\alpha_i \circ x_i &= \overline{\beta}_j \circ (i \times \mathrm{id}_B) \circ \varphi'_i \\ &= \overline{\beta}_j \circ \varphi_i \circ x_i.\end{aligned}$$

From Lemma 4.1.4, Δ is right-orthogonal to x_i, so it must be the case that:

$$\alpha_i = \overline{\beta}_j \circ \varphi_i.$$

This means that $Y \rightarrowtail X_i$ and $0 \rightarrowtail X_i$ have the same classifying map, so they must be isomorphic subobjects. Hence, φ_i and φ_j are disjoint.

□

Lemma 4.1.8. *Assume that Δ is L_Δ-complete, and that $\coprod_{j \in J} 1$ is in $\mathcal{O}_\Delta^\perp$. A J-indexed join of disjoint Δ-subobjects of $\overline{\omega}_\Delta \times A$ is a Δ-subobject if and only if the join of their pullbacks along $i \times \mathrm{id}_A : \omega_\Delta \times A \to \overline{\omega}_\Delta \times A$ is a Δ-subobject.*

Proof. Let $(f_j : X_j \rightarrowtail \overline{\omega}_\Delta \times A)_{j \in J}$ be a family of disjoint Δ-subobjects. Let $(f'_j : X'_j \rightarrowtail \omega_\Delta \times A)_{j \in J}$ be their pullback along $i \times \mathrm{id}_A : \omega_\Delta \times A \to \overline{\omega}_\Delta \times A$.

The left to right implication is true because taking the join of $(f_j : X_j \rightarrowtail \overline{\omega}_\Delta \times A)_{j \in J}$ then pulling back along $i \times \mathrm{id}_A$ gives the same subobject as pulling back and then taking the join (see Section 2.2).

For the right to left implication, assume that the join of $(f'_j : X'_j \rightarrowtail \omega_\Delta \times A)_{j \in J}$ is a Δ-subobject. By Lemma 2.2.2, $(f'_j : X'_j \rightarrowtail \omega_\Delta \times A)_{j \in J}$ is a disjoint family of subobjects, so its join is the co-pairing:

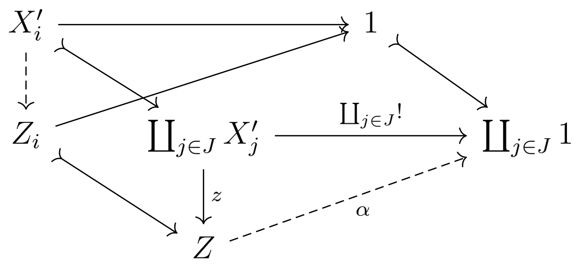
$$[f'_j]_{j \in J} : \coprod_{j \in J} X'_j \rightarrowtail \omega_\Delta \times A.$$

Because Δ is L_Δ-complete, there is a Δ-subobject $\varphi : Z \rightarrowtail \overline{\omega}_\Delta \times A$ which pulls back to $[f'_j]_{j \in J}$, as in the following diagram:

$$\begin{array}{ccc} \coprod_{j \in J} X'_j & \overset{[f'_j]_{j \in J}}{\rightarrowtail} & \omega_\Delta \times A \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle i \times \mathrm{id}_A} \\ Z & \underset{\varphi}{\rightarrowtail} & \overline{\omega}_\Delta \times A \end{array}$$

The strategy is to show that the join $[f_j]_{j \in J} : \coprod_{j \in J} X_j \rightarrowtail \overline{\omega}_\Delta \times A$ is an isomorphic subobject to φ and hence a Δ-subobject.

We know that $\coprod_{j \in J} 1$ is in $\mathcal{O}_\Delta^\perp$, so $\coprod_{j \in J} 1$ is right-orthogonal to z. Therefore, the map $\coprod_{j \in J} ! : \coprod_{j \in J} X'_j \to \coprod_{j \in J} 1$ has a unique extension α to z:



From [Joh02], we know that a Grothendieck topos is *infinitary extensive*, meaning that for any family of commutative squares:

$$\begin{array}{ccc} X_j & \longrightarrow & B \\ \big\downarrow & & \big\downarrow f \\ A_j & \longrightarrow & \coprod_{i \in I} A_i \end{array}$$

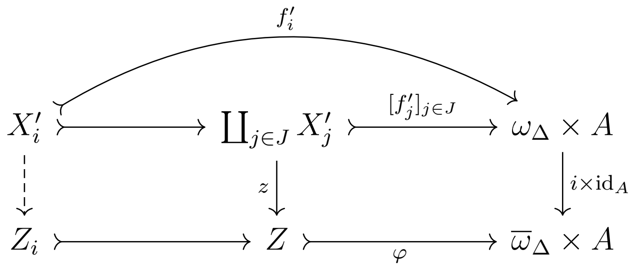
where f is fixed and each $A_j \to \coprod_{i \in I} A_i$ is the respective coproduct inclusion, B is the coproduct $\coprod_{i \in I} X_i$ if and only if all the squares are pullbacks. It also follows that coproduct inclusions are disjoint.

We now use the fact that our category is infinitary extensive: pulling back α along the coproduct inclusions $1 \rightarrowtail \coprod_{j \in J} 1$ we get a coproduct decomposition of Z:

$$Z \cong \coprod_{j \in J} Z_j.$$

The map $X'_i \to Z_i$ is the comparison map that comes from the universal property of Z_i as a pullback, so the left square starting at X'_i commutes. The top square starting at X'_i is also a pullback so by the pullback lemma the left square is a pullback as well.

We now have the following two pullback squares:



The map $\varphi \circ \mathrm{inc}_i : Z_i \rightarrowtail Z \rightarrowtail \overline{\omega}_\Delta \times A$ is a Δ-subobject because coproduct inclusions are Δ-subobjects (Proposition 2.3.8) and it pulls back along $i \times \mathrm{id}_A$ to f'_i. But $f_i : X_i \rightarrowtail \overline{\omega}_\Delta \times A$ also pulls back to f'_i, so because Δ is L_Δ-complete, f_i and $\varphi \circ \mathrm{inc}_i$ are isomorphic subobjects.

The isomorphism holds for any i in J, so the co-pairings:

$$[f_j]_{j\in J} \qquad \text{and} \qquad [\varphi \circ \mathrm{inc}_j]_{j\in J} = \varphi$$

are isomorphic subobjects as well, hence the join $[f_j]_{j\in J}$ is a Δ-subobject.

□

Now we give a sufficient condition for the orthogonality class $\mathcal{O}_\Delta^\perp$ to be closed under I-indexed coproducts. We will use the proposition below in the proof of Proposition 4.3.4 to show that sum types in $\mathsf{PCF_v}$ are interpreted as complete objects, by instantiating the coproduct $\coprod_J 1$ to be the natural numbers $\mathsf{Nat}_\mathcal{E} \cong \coprod_\mathbb{N} 1$.

Proposition 4.1.9. *Assume that Δ is L_Δ-complete. Then $\mathcal{O}_\Delta^\perp$ is closed under I-indexed coproducts if and only if the coproduct of the terminal object $\coprod_J 1$ is in $\mathcal{O}_\Delta^\perp$ for some J with $|I| \leq |J|$.*

Proof. The left to right direction is immediate because we can choose J to be I, and 1 is in $\mathcal{O}_\Delta^\perp$ because it is the terminal object.

For the right to left direction, consider a coproduct $\coprod_{i\in I} A_i$ where each A_i is in $\mathcal{O}_\Delta^\perp$. Consider a Δ-subobject $m : X' \rightarrowtail \overline{\omega}_\Delta \times B$ and let the pullback of $i \times \mathrm{id}_B : \omega_\Delta \times B \to \overline{\omega}_\Delta \times B$ along m be $x : X \to X'$. Consider a map

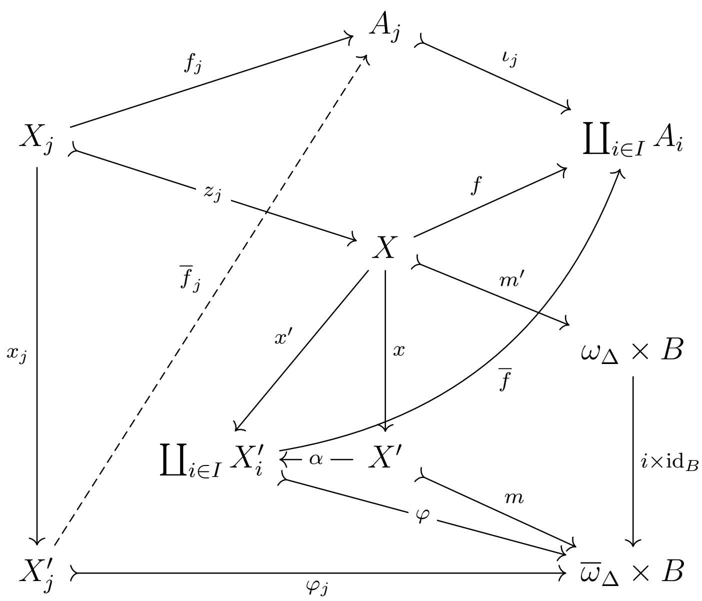


Figure 4.2: Diagram for the proof of Proposition 4.1.9.

$f : X \to \coprod_{i\in I} A_i$. We need to show that f has a unique extension to X'. The situtation is depicted in Figure 4.2.

Consider one of the coproduct inclusions $\iota_j : A_j \rightarrowtail \coprod_{i\in I} A_i$. We know from Proposition 2.3.8 that coproduct inclusions are Δ-subobjects. Therefore z_j, the pullback of ι_j along f, is also a Δ-subobject.

Δ-subobjects are closed under composition so $m' \circ z_j$ is a Δ-subobject. Therefore, it has a classifying map $\beta_j : \omega_\Delta \times B \to \Delta$. Because Δ is L_Δ-complete, β_j has a unique extension $\overline{\beta}_j : \overline{\omega}_\Delta \times B \to \Delta$. Let $\varphi_j : X'_j \rightarrowtail \overline{\omega}_\Delta \times B$ be the Δ-subobject classified by $\overline{\beta}_j$.

It follows by the pullback lemma that $m' \circ z_j$ is the pullback of φ_j along $i \times \mathrm{id}_B$. Because A_j is in $\mathcal{O}_\Delta^\perp$ it means that A_j is right-orthogonal to x_j, so f_j has a unique extension $\overline{f}_j : X'_j \to A_j$.

From [Joh02], we know that the topos $\mathcal{E}$ we are working in is infinitary extensive. Therefore, X is the coproduct $\coprod_{i\in I} X_i$ and f is $\coprod_{i\in I} f_i$. Moreover, we know that the coproduct inclusions ι_j are all pairwise disjoint.

The aim is to show that X' is actually the coproduct $\coprod_{i\in I} X'_i$ and that $\overline{f} = \coprod_{i\in I} \overline{f}_i : X' \to \coprod_{i\in I} A_i$ is an extension of f.

Notice that all the φ_j's are pairwise disjoint because they pull back to a pairwise disjoint family of subobjects, $(m' \circ z_i)_{i \in I}$ (Lemma 4.1.7). Therefore, their union:

$$\varphi = [\varphi_i]_{i \in I} : \coprod_{i \in I} X'_i \to \overline{\omega}_\Delta \times B$$

is still a monomorphism. Now we can apply Lemma 4.1.8 to deduce that ϕ is a Δ-subobject, because the join of $(m' \circ z_i)_{i \in I}$ is m' which is a Δ-subobject.

Let $x' = \coprod_{i \in I} x_i : X \to \coprod_{i \in I} X'_i$. Then (x', m') is the pullback of ϕ and $i \times \mathrm{id}_B$, because taking pullback of a family of subobjects and then taking the union is the same as taking the union followed by pullback (see Section 2.2).

We know that both φ and m are Δ-subobjects and their pullback along $i \times \mathrm{id}_B$ is m'. Because Δ is L_Δ-complete, they must have the same classifying map and so they are isomorphic subobjects. Let $\alpha : X' \to \coprod_{i \in I} X'_i$ be the isomorphism such that:

$$m = \varphi \circ \alpha \qquad\qquad \alpha \circ x = x'.$$

Because $f = \coprod_{i \in I} f_i$, $\overline{f} = \coprod_{i \in I} \overline{f}_i$ and $x' = \coprod_{i \in I} x_i$, and because each $\overline{f}_i$ extends f_i along x_i we get that:

$$f = \overline{f} \circ x' = \overline{f} \circ \alpha \circ x.$$

Therefore $\overline{f} \circ \alpha$ extends f along x. It remains to show this is the unique extension.

Assume there is another $g : X' :\to \coprod_{i \in I} A_i$ such that $f = g \circ x$. We can pull back g along ι_j to get another coproduct decomposition of $X' \cong \coprod_{i \in I} Y_i$, because a Grothendieck topos is infinitary extensive [Joh02], as shown in Figure 4.3.

Ignoring the isomorphisms $\coprod_{i \in I} X'_i \cong X' \cong \coprod_{i \in I} Y_i$ we need to show that $g = \overline{f}$. We know that:

$$g = \coprod_{i \in I} g_i \qquad\qquad \overline{f} = \coprod_{i \in I} \overline{f}_i$$

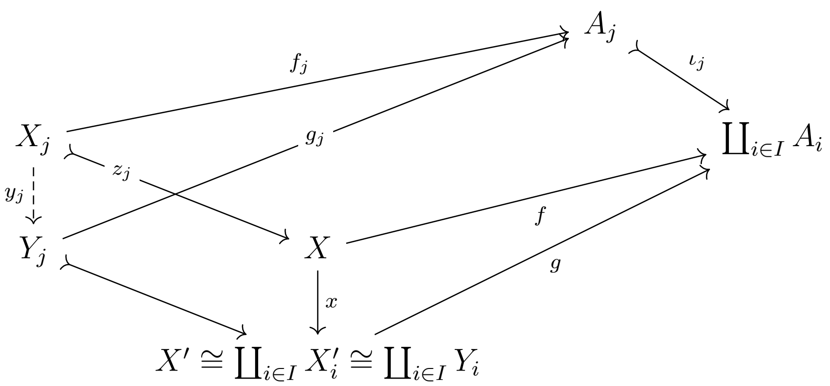


Figure 4.3: Diagram for the proof of Proposition 4.1.9.

so it is enough to show that for each j, Y_j is isomorphic to X'_j and $g_j = \overline{f}_j$ (up to isomorphism).

Let $y_j : X_j \to Y_j$ be the comparison map that comes from the universal property of Y_j as the pullback of g and ι_j. Then by the pullback lemma, we see that the inclusion map $Y_j \rightarrowtail X'$ pulls back to z_j, along x.

The inclusion map $X'_j \rightarrowtail X'$ also pulls back to z_j along x, because φ_j pulls back along $i \times \mathrm{id}_B$ to $m' \circ z_j$, and m pulls back to m'. Both inclusion maps $Y_j \rightarrowtail X'$ and $X'_j \rightarrowtail X'$ are Δ-subobjects by Proposition 2.3.8.

By assumption Δ is L_Δ-complete, so by Lemma 4.1.4 Δ is in $\mathcal{O}^\perp_\Delta$, therefore it is right-orthogonal to x. Therefore $Y_j \rightarrowtail X'$ and $X'_j \rightarrowtail X'$ are isomorphic subobjects, so $Y_j \cong X'_j$.

By a diagram chase:

$$\iota_j \circ f_j = \iota_j \circ g_j \circ y_j$$

so because ι_j is mono, we have that $f_j = g_j \circ y_j$. Therefore, g_j and $\overline{f}_j$ both extend f_j along the same map (up to isomorphism), so they must be equal because f_j has a unique extension.

□

4.2 $\mathsf{PCF_v}$: a higher-order language with recursion

Traditionally, PCF [Plo77] is a call-by-name lambda-calculus with one or two base types and a fixed point combinator for each type. PCF has been used extensively as a basis for studying denotational semantics of programming languages. This section introduces $\mathsf{PCF_v}$, a (fine-grain) call-by-value version of PCF, whose models will be studied in the rest of the thesis.

4.2.1 Typing rules and operational semantics

$\mathsf{PCF_v}$ has as base types unit, empty and natural numbers; it also has binary products and sums, and function types. The grammar of $\mathsf{PCF_v}$ types is defined as follows:

$$\text{Types: } \tau ::= \mathsf{0} \mid \mathsf{1} \mid \mathsf{nat} \mid \tau + \tau \mid \tau \times \tau \mid \tau \to \tau$$

The ground types are $\mathsf{0}$, $\mathsf{1}$ and nat.

The calculus is fine-grain call-by-value [LPT03], meaning there is a syntactic distinction between values and computations.

$$\begin{aligned} \text{Values: } v, w &::= x \mid \star \mid \mathsf{inl}\, v \mid \mathsf{inr}\, v \mid (v, v) \mid \underline{0} \mid \mathsf{S}(v) \mid \lambda x.\, t \mid \mathsf{rec}\, f\, x.\, t \\ \text{Computations: } t &::= \mathsf{return}\, v \mid \mathsf{case}\, v\, \mathsf{of}\, \{\mathsf{inl}\, x \to t,\ \mathsf{inr}\, y \to t'\} \mid \pi_1 v \mid \pi_2 v \\ &\mid v\ w \mid \mathsf{case}\, v\ \mathsf{of}\ \{\underline{0}{\to}t,\ \mathsf{S}(x){\to}t'\} \mid \mathsf{let}\, x = t\, \mathsf{in}\, t' \end{aligned}$$

Each type comes with the usual constructors (values) and destructors (computations). The value $\mathsf{S}(v)$ is thought of as successor of the natural number v. In addition, we can define recursive functions ($\mathsf{rec}\, x\, f.\, t$) which should be thought of as a definition $f(x) = t$ where both f and x can appear free in t. Values can be embedded into computations using $\mathsf{return}\, v$ and computations can be sequenced using let.

There are two typing judgements, $\vdash^{\mathsf{v}}$ for values, and $\vdash^{\mathsf{c}}$ for computations, shown in Figure 4.4. The big-step operational semantics is defined as a relation $\Downarrow$ between closed computations and values; it is the least relation

$$\frac{}{\Gamma \vdash^{\mathsf{v}} \star : 1} \qquad \frac{\Gamma \vdash^{\mathsf{v}} v : \tau}{\Gamma \vdash^{\mathsf{v}} \mathsf{inl}\, v : \tau + \tau'} \qquad \frac{\Gamma \vdash^{\mathsf{v}} v : \tau'}{\Gamma \vdash^{\mathsf{v}} \mathsf{inr}\, v : \tau + \tau'}$$

$$\frac{}{\Gamma, x : \tau, \Gamma' \vdash^{\mathsf{v}} x : \tau} \qquad \frac{}{\Gamma \vdash^{\mathsf{v}} \underline{0} : \mathsf{nat}} \qquad \frac{\Gamma \vdash^{\mathsf{v}} v : \mathsf{nat}}{\Gamma \vdash^{\mathsf{v}} \mathsf{S}(v) : \mathsf{nat}}$$

$$\frac{\Gamma,\, x : \tau \vdash^{\mathsf{c}} t : \tau'}{\Gamma \vdash^{\mathsf{v}} \lambda x.\, t : \tau \to \tau'} \qquad \frac{\Gamma,\, f : \tau \to \tau',\, x : \tau \vdash^{\mathsf{c}} t : \tau'}{\Gamma \vdash^{\mathsf{v}} \mathsf{rec}\, f\, x.\, t : \tau \to \tau'}$$

$$\frac{\Gamma \vdash^{\mathsf{v}} v : \tau \qquad \Gamma \vdash^{\mathsf{v}} v' : \tau'}{\Gamma \vdash^{\mathsf{v}} (v, v') : \tau \times \tau'} \qquad \frac{\Gamma \vdash^{\mathsf{v}} v : \tau \times \tau'}{\Gamma \vdash^{\mathsf{c}} \pi_1 v : \tau} \qquad \frac{\Gamma \vdash^{\mathsf{v}} v : \tau \times \tau'}{\Gamma \vdash^{\mathsf{c}} \pi_2 v : \tau'}$$

$$\frac{\Gamma \vdash^{\mathsf{v}} v : \tau + \tau' \qquad \Gamma, x : \tau \vdash^{\mathsf{c}} t : \sigma \qquad \Gamma, y : \tau' \vdash^{\mathsf{c}} t' : \sigma}{\Gamma \vdash^{\mathsf{c}} \mathsf{case}\, v\, \mathsf{of}\, \{\mathsf{inl}\, x \to t,\ \mathsf{inr}\, y \to t'\} : \sigma}$$

$$\frac{\Gamma \vdash^{\mathsf{v}} v : 0}{\Gamma \vdash^{\mathsf{c}} \mathsf{case}\, v\, \mathsf{of}\ \{\} : \tau} \qquad \frac{\Gamma \vdash^{\mathsf{v}} v : \tau \to \tau' \qquad \Gamma \vdash^{\mathsf{v}} w : \tau}{\Gamma \vdash^{\mathsf{c}} v\ w : \tau'}$$

$$\frac{\Gamma \vdash^{\mathsf{v}} v : \mathsf{nat} \qquad \Gamma \vdash^{\mathsf{c}} t : \tau \qquad \Gamma, x : \mathsf{nat} \vdash^{\mathsf{c}} t' : \tau}{\Gamma \vdash^{\mathsf{c}} \mathsf{case}\, v\ \mathsf{of}\ \{\underline{0} \to t,\ \mathsf{S}(x) \to t'\} : \tau}$$

$$\frac{\Gamma \vdash^{\mathsf{v}} v : \tau}{\Gamma \vdash^{\mathsf{c}} \mathsf{return}\, v : \tau} \qquad \frac{\Gamma \vdash^{\mathsf{c}} t : \tau \qquad \Gamma, x : \tau \vdash^{\mathsf{c}} t' : \tau'}{\Gamma \vdash^{\mathsf{c}} \mathsf{let}\, x = t\, \mathsf{in}\, t' : \tau'}$$

Figure 4.4: Typing rules for $\mathsf{PCF_v}$.

closed under the rules in Figure 4.5.

We now sketch an extension of $\mathsf{PCF_v}$ with new type and term constants. Denote by α, β the existing ground types $\mathsf{1}$, $\mathsf{0}$, nat and the type constants we wish to add, which will also be treated as ground types.

To be able to extend the operational semantics, assume that each new type constant α comes equipped with a set Val_α of values of type α. The existing ground types already have such sets of values:

$$Val_{\mathsf{1}} \cong 1 \qquad\qquad Val_{\mathsf{0}} = \emptyset \qquad\qquad Val_{\mathsf{nat}} \cong \mathbb{N}.$$

For each element u in Val_α, we add to the calculus a term constant u of type α. To add term constants of type $(\alpha_1 \times \ldots \times \alpha_n) \to \beta$, assume that each

$$\frac{}{\mathsf{return}\, v \Downarrow v} \qquad \frac{}{\pi_1(v, v') \Downarrow v} \qquad \frac{}{\pi_2(v, v') \Downarrow v'}$$

$$\frac{t[v/x] \Downarrow w}{\mathsf{case}\,\mathsf{inl}\, v\,\mathsf{of}\,\{\mathsf{inl}\, x \to t,\ \mathsf{inr}\, y \to t'\} \Downarrow w}$$

$$\frac{t'[v/x] \Downarrow w}{\mathsf{case}\,\mathsf{inr}\, v\,\mathsf{of}\,\{\mathsf{inl}\, x \to t,\ \mathsf{inr}\, y \to t'\} \Downarrow w}$$

$$\frac{t[(\mathsf{rec}\, f\, x.\, t)/f,\ v/x] \Downarrow w}{(\mathsf{rec}\, f\, x.\, t)\ v \Downarrow w} \qquad \frac{t[v/x] \Downarrow w}{(\lambda x.\, t)\ v \Downarrow w} \qquad \frac{t \Downarrow v \qquad t'[v/x] \Downarrow w}{\mathsf{let}\, x = t\, \mathsf{in}\, t' \Downarrow w}$$

$$\frac{t \Downarrow w}{\mathsf{case}\, \underline{0}\ \mathsf{of}\ \{\underline{0} \to t,\ \mathsf{S}(x) \to t'\} \Downarrow w} \qquad \frac{t'[v/x] \Downarrow w}{\mathsf{case}\, \mathsf{S}(v)\ \mathsf{of}\ \{\underline{0} \to t,\ \mathsf{S}(x) \to t'\} \Downarrow w}$$

Figure 4.5: Operational semantics of $\mathsf{PCF_v}$.

$$\tau ::= \ldots \mid \alpha$$
$$v, w ::= \ldots \mid u \mid f$$
where $u \in \mathit{Val}_\alpha$, and f represents the new term constants

$$\frac{}{\Gamma \vdash^{\mathsf{v}} u : \alpha} \qquad \frac{}{\Gamma \vdash^{\mathsf{v}} f : (\alpha_1 \times \ldots \times \alpha_n) \to \beta} \qquad \frac{}{f\ v \Downarrow w}\ \text{if } f(v) = w \in \mathit{Val}_\beta$$

Figure 4.6: Extension of $\mathsf{PCF_v}$ with new type and term constants.

such term constant $f : (\alpha_1 \times \ldots \times \alpha_n) \to \beta$ is associated to a partial function $f : (\mathit{Val}_{\alpha_1} \times \ldots \times \mathit{Val}_{\alpha_n}) \rightharpoonup \mathit{Val}_\beta$.

We add all elements of Val_α and the term constants $f : (\alpha_1 \times \ldots \times \alpha_n) \to \beta$ as values, and implement an operational semantics based on the underlying function of each term constant. The additions to $\mathsf{PCF_v}$ are summarised in Figure 4.6:

In Section 7.2, we study sheaf models for languages that are richer than $\mathsf{PCF_v}$, like the ω-quasi-Borel spaces model for a probabilistic language [VKS19], or the ω-diffeological spaces model for differentiable programming [Vák20]. Our aim in the thesis is to show that these models use the same recipe for modelling $\mathsf{PCF_v}$ with type and term constants (even though the language modelled in each case is even richer), which is why we add such constants to our calculus. Below are some examples of new constants we might want to add.

Example 4.2.1. For all the examples in Section 7.2 we can add a type real with $Val_{\mathsf{real}} \cong \mathbb{R}$. In each case, and for each $n \in \mathbb{N}$, there would be a term constant $f : (\underbrace{\mathsf{real} \times \ldots \times \mathsf{real}}_{n \text{ times}}) \to \mathsf{real}$ corresponding to:

- each measurable function $f : \mathbb{R}^n \to \mathbb{R}$, for the ω-quasi-Borel spaces example;
- each smooth function $f : \mathbb{R}^n \to \mathbb{R}$, for the ω-diffeological spaces example; for example $\mathsf{sin} : \mathbb{R} \to \mathbb{R}$;
- each PAP [LYRY20] function $f : \mathbb{R}^n \to \mathbb{R}$, for the ω**PAP** example [LHM21]; for example a piecewise smooth function $f : \mathbb{R} \to \mathbb{R}$ like:

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{otherwise.} \end{cases}$$

4.2.2 Contextual equivalence

Using the operational semantics, we can define the usual notion of contextual equivalence (e.g. [Pit11, Section 5.6]). Intuitively, a context is a term with a hole; we define the grammar of contexts in Figure 4.7.

Because of our distinction between values and computations, the grammar of contexts is made up of four different syntactic classes depending on whether the hole in the contexts needs to be filled in with a value or a computation, and on whether the filled in context produces a value or a computation. For example, we denote by $C_{\mathrm{v}}^{\mathrm{c}}$ a context which takes in a value and produces a computation.

Definition 4.2.2. Two computations $\Gamma \vdash^{\mathbf{c}} t : \tau$ and $\Gamma \vdash^{\mathbf{c}} t' : \tau$ are contextually equivalent, written as

$$\Gamma \vdash^{\mathbf{c}} t \simeq t' : \tau,$$

$$
\begin{aligned}
C_{\mathrm{v}}^{\mathrm{c}} ::= {} & \mathsf{return}\, C_{\mathrm{v}}^{\mathrm{v}} \mid \mathsf{case}\, C_{\mathrm{v}}^{\mathrm{v}}\, \mathsf{of}\, \{\mathsf{inl}\, x \to t,\ \mathsf{inr}\, y \to t'\} \mid \\
& \mathsf{case}\, v\, \mathsf{of}\, \{\mathsf{inl}\, x \to C_{\mathrm{v}}^{\mathrm{c}},\ \mathsf{inr}\, y \to t'\} \mid \mathsf{case}\, v\, \mathsf{of}\, \{\mathsf{inl}\, x \to t,\ \mathsf{inr}\, y \to C_{\mathrm{v}}^{\mathrm{c}}\} \mid \\
& \pi_1 C_{\mathrm{v}}^{\mathrm{v}} \mid \pi_2 C_{\mathrm{v}}^{\mathrm{v}} \mid C_{\mathrm{v}}^{\mathrm{v}}\ w \mid v\ C_{\mathrm{v}}^{\mathrm{v}} \mid \mathsf{case}\, C_{\mathrm{v}}^{\mathrm{v}}\ \mathsf{of}\ \{\underline{\mathsf{0}} \to t,\ \mathsf{S}(x) \to t'\} \mid \\
& \mathsf{case}\, v\ \mathsf{of}\ \{\underline{\mathsf{0}} \to C_{\mathrm{v}}^{\mathrm{c}},\ \mathsf{S}(x) \to t'\} \mid \mathsf{case}\, v\ \mathsf{of}\ \{\underline{\mathsf{0}} \to t,\ \mathsf{S}(x) \to C_{\mathrm{v}}^{\mathrm{c}}\} \mid \\
& \mathsf{let}\, x = C_{\mathrm{v}}^{\mathrm{c}}\, \mathsf{in}\, t' \mid \mathsf{let}\, x = t\, \mathsf{in}\, C_{\mathrm{v}}^{\mathrm{c}}
\end{aligned}
$$

$$
\begin{aligned}
C_{\mathrm{c}}^{\mathrm{c}} ::= {} & \Box \mid \mathsf{return}\, C_{\mathrm{c}}^{\mathrm{v}} \mid \mathsf{case}\, C_{\mathrm{c}}^{\mathrm{v}}\, \mathsf{of}\, \{\mathsf{inl}\, x \to t,\ \mathsf{inr}\, y \to t'\} \mid \\
& \mathsf{case}\, v\, \mathsf{of}\, \{\mathsf{inl}\, x \to C_{\mathrm{c}}^{\mathrm{c}},\ \mathsf{inr}\, y \to t'\} \mid \mathsf{case}\, v\, \mathsf{of}\, \{\mathsf{inl}\, x \to t,\ \mathsf{inr}\, y \to C_{\mathrm{c}}^{\mathrm{c}}\} \mid \\
& \pi_1 C_{\mathrm{c}}^{\mathrm{v}} \mid \pi_2 C_{\mathrm{c}}^{\mathrm{v}} \mid C_{\mathrm{c}}^{\mathrm{v}}\ w \mid v\ C_{\mathrm{c}}^{\mathrm{v}} \mid \mathsf{case}\, C_{\mathrm{c}}^{\mathrm{v}}\ \mathsf{of}\ \{\underline{\mathsf{0}} \to t,\ \mathsf{S}(x) \to t'\} \mid \\
& \mathsf{case}\, v\ \mathsf{of}\ \{\underline{\mathsf{0}} \to C_{\mathrm{c}}^{\mathrm{c}},\ \mathsf{S}(x) \to t'\} \mid \mathsf{case}\, v\ \mathsf{of}\ \{\underline{\mathsf{0}} \to t,\ \mathsf{S}(x) \to C_{\mathrm{c}}^{\mathrm{c}}\} \mid \\
& \mathsf{let}\, x = C_{\mathrm{c}}^{\mathrm{c}}\, \mathsf{in}\, t' \mid \mathsf{let}\, x = t\, \mathsf{in}\, C_{\mathrm{c}}^{\mathrm{c}}
\end{aligned}
$$

$$
C_{\mathrm{v}}^{\mathrm{v}} ::= \Box \mid \mathsf{inl}\, C_{\mathrm{v}}^{\mathrm{v}} \mid \mathsf{inr}\, C_{\mathrm{v}}^{\mathrm{v}} \mid (C_{\mathrm{v}}^{\mathrm{v}}, v) \mid (v, C_{\mathrm{v}}^{\mathrm{v}}) \mid \mathsf{S}(C_{\mathrm{v}}^{\mathrm{v}}) \mid \lambda x.\, C_{\mathrm{v}}^{\mathrm{c}} \mid \mathsf{rec}\, f\, x.\, C_{\mathrm{v}}^{\mathrm{c}}
$$

$$
C_{\mathrm{c}}^{\mathrm{v}} ::= \mathsf{inl}\, C_{\mathrm{c}}^{\mathrm{v}} \mid \mathsf{inr}\, C_{\mathrm{c}}^{\mathrm{v}} \mid (C_{\mathrm{c}}^{\mathrm{v}}, v) \mid (v, C_{\mathrm{c}}^{\mathrm{v}}) \mid \mathsf{S}(C_{\mathrm{c}}^{\mathrm{v}}) \mid \lambda x.\, C_{\mathrm{c}}^{\mathrm{c}} \mid \mathsf{rec}\, f\, x.\, C_{\mathrm{c}}^{\mathrm{c}}
$$

Figure 4.7: Grammar of $\mathsf{PCF}_{\mathsf{v}}$ contexts.

if for any context $C^{\mathrm{c}}_{\mathrm{c}}$ such that $C^{\mathrm{c}}_{\mathrm{c}}[t]$ and $C^{\mathrm{c}}_{\mathrm{c}}[t']$ are well-typed closed computations of ground type (in this case $\mathsf{0}$, $\mathsf{1}$ or nat or one of the type constants):

$$C^{\mathrm{c}}_{\mathrm{c}}[t] \Downarrow v \quad \text{if and only if} \quad C^{\mathrm{c}}_{\mathrm{c}}[t'] \Downarrow v.$$

The definition for values is analogous, using contexts of the form $C^{\mathrm{c}}_{\mathrm{v}}$.

4.3 Denotational semantics of $\mathsf{PCF_v}$

We now describe the denotational semantics of $\mathsf{PCF_v}$ with type constants in a Grothendieck topos. This denotational semantics is not new, it follows the same pattern as Moggi's [Mog91] interpretation of the computational lambda-calculus in a cartesian closed category with a monad. The notion of normal model of $\mathsf{PCF_v}$ defined below is very closely related to the notion of natural model of synthetic domain theory studied by Simpson [Sim98]; the connection is discussed in Section 7.4.

Definition 4.3.1. A *normal model* of $\mathsf{PCF_v}$ with type constants is a Grothendieck topos $\mathcal{E}$ (Definition 2.1.4) with a dominance Δ (Definition 2.3.2), which classifies $0 \rightarrowtail 1$, such that:

- $L_\Delta(\mathsf{Nat}_\mathcal{E})$ is L_Δ-complete (Definition 3.2.2), where $\mathsf{Nat}_\mathcal{E} = \coprod_\mathbb{N} 1$.

- For each type constant α, there is an object A_α in $\mathcal{E}$ such that there is a mapping from Val_α to the set of points $1 \to A_\alpha$ of A_α. Moreover, $L_\Delta(A_\alpha)$ is L_Δ-complete.

- For each term constant $f : (\alpha_1 \times \ldots \times \alpha_n) \to \beta$, there is a morphism

  $$\phi_f : (A_{\alpha_1} \times \ldots \times A_{\alpha_n}) \to L_\Delta(A_\beta)$$

  which agrees with $f : (\mathit{Val}_{\alpha_1} \times \ldots \times \mathit{Val}_{\alpha_n}) \rightharpoonup \mathit{Val}_\beta$ on points. This means that:

  - if $f(v) = u$, then

    $$1 \xrightarrow{p_v} (A_{\alpha_1} \times \ldots \times A_{\alpha_n}) \xrightarrow{\phi_f} L_\Delta(A_\beta) = 1 \xrightarrow{p_u} A_\beta \xrightarrow{\eta} L_\Delta(A_\beta)$$

– if $f(v)$ is undefined, then

$$1 \xrightarrow{p_v} (A_{\alpha_1} \times \ldots \times A_{\alpha_n}) \xrightarrow{\phi_f} L_\Delta(A_\beta) = 1 \xrightarrow{\perp_{A_\beta}} L_\Delta(A_\beta),$$

where $\perp$ is the point of L_Δ constructed in Lemma 2.4.12.

Remark 4.3.2. Notice that for the existing ground types of $\mathsf{PCF_v}$ ($\mathsf{1}$, $\mathsf{0}$, nat) we always have a corresponding object A_α in $\mathcal{E}$. For example, A_{nat} is $\coprod_{\mathbb{N}} 1$.

Lemma 4.3.3. *In a normal model, the dominance Δ is L_Δ-complete.*

Proof. As we know from Remark 2.4.7, $\Delta \cong L_\Delta 1$. $L_\Delta 1$ is a retract of $L_\Delta(\mathsf{Nat}_{\mathcal{E}})$ because:

$$\left(L_\Delta 1 \xrightarrow{L_\Delta \mathrm{inc}_0} L_\Delta \left(\coprod_{\mathbb{N}} 1 \right) \xrightarrow{L_\Delta !} L_\Delta 1 \right) = \mathrm{id}_{L_\Delta 1}.$$

Given a map $f : \omega_\Delta \times A \to L_\Delta 1$, the composite $(L_\Delta \mathrm{inc}_0) \circ f$ has a unique extension $h : \overline{\omega}_\Delta \times A \to L_\Delta(\mathsf{Nat}_{\mathcal{E}})$, because $L_\Delta(\mathsf{Nat}_{\mathcal{E}})$ is L_Δ-complete. Hence, we obtain a unique extension of f:

$$(L_\Delta !) \circ h.$$

□

A normal model $\mathcal{E}$ has the cartesian closed structure and the lifting monad L_Δ needed to interpret $\mathsf{PCF_v}$ types. The interpretation of types, including type constants, is:

$$[\![\mathsf{nat}]\!] = \coprod_{\mathbb{N}} 1 \qquad [\![\mathsf{0}]\!] = 0 \qquad [\![\mathsf{1}]\!] = 1 \qquad [\![\alpha]\!] = A_\alpha \qquad [\![\tau + \tau']\!] = [\![\tau]\!] + [\![\tau']\!]$$

$$[\![\tau \times \tau']\!] = [\![\tau]\!] \times [\![\tau']\!] \qquad\qquad [\![\tau \to \tau']\!] = \big([\![\tau]\!] \Rightarrow L_\Delta[\![\tau']\!]\big).$$

A typing context $\Gamma = x_1 : \tau_1, \ldots, x_n : \tau_n$ is interpreted using product in $\mathcal{E}$:

$$[\![\Gamma]\!] = [\![\tau_1]\!] \times \ldots \times [\![\tau_n]\!].$$

In Figure 7.1, we will spell out the interpretation of types explicitly for a special class of normal models.

A value $\Gamma \vdash^{\mathsf{v}} v : \tau$ is interpreted as a morphism:

$$[\![v]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

in $\mathcal{E}$, and a computation $\Gamma \vdash^{\mathsf{c}} t : \tau$ as a morphism

$$[\![t]\!] : [\![\Gamma]\!] \to L_\Delta[\![\tau]\!].$$

A term constant $\Gamma \vdash^{\mathsf{v}} u : \alpha$, $u \in \mathit{Val}_\alpha$, where α is one of the type constants, is interpreted by the corresponding point of A_α which is part of the definition of normal model:

$$[\![\Gamma \vdash^{\mathsf{v}} u : \alpha]\!] = [\![\Gamma]\!] \xrightarrow{!} 1 \xrightarrow{p_u} A_\alpha.$$

Similarly, a term constant $\Gamma \vdash^{\mathsf{v}} f : (\alpha_1 \times \ldots \times \alpha_n) \to \beta$ is interpreted as:

$$[\![\Gamma \vdash^{\mathsf{v}} f : (\alpha_1 \times \ldots \times \alpha_n) \to \beta]\!] = \\ [\![\Gamma]\!] \xrightarrow{!} 1 \xrightarrow{\mathsf{curry}(\phi_f)} \big((A_{\alpha_1} \times \ldots \times A_{\alpha_n}) \Rightarrow L_\Delta(A_\beta)\big).$$

For the other values and computations, the interpretation uses the structure of the category $\mathcal{E}$ and the strong monad L_Δ in a standard way (e.g. [Mog91]). Figures 4.8 and 4.9 contain this interpretation.

The only interesting case is the interpretation of fixed points which uses Corollary 3.2.5. To apply this result we need to show that for every type $\tau_1 \to \tau_2$, $L_\Delta\big([\![\tau_1]\!] \Rightarrow (L_\Delta[\![\tau_2]\!])\big)$ is L_Δ-complete. We prove the following more general statement. In this proof we use the orthogonality class $\mathcal{O}_\Delta^\perp$ to strengthen the induction hypothesis.

Proposition 4.3.4. *For any* $\mathsf{PCF_v}$ *type* τ, $L_\Delta[\![\tau]\!]$ *is* L_Δ*-complete.*

Proof. We prove by induction on τ that $[\![\tau]\!] \in \mathcal{O}_\Delta^\perp$. Because in a normal model Δ is L_Δ-complete we have:

- $0 \in \mathcal{O}_\Delta^\perp$ by Proposition 4.1.6.

$$\llbracket \Gamma \vdash^{\mathsf{v}} \star : \mathsf{1} \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{!} 1$$

$$\llbracket \Gamma, x : \tau, \Gamma' \vdash^{\mathsf{v}} x : \tau \rrbracket = \llbracket \Gamma \rrbracket \times \llbracket \tau \rrbracket \times \llbracket \Gamma' \rrbracket \xrightarrow{\pi_x} \llbracket \tau \rrbracket$$

$$\llbracket \Gamma \vdash^{\mathsf{v}} \mathsf{inl}\, v : \tau + \tau' \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\llbracket v \rrbracket} \llbracket \tau \rrbracket \xrightarrow{\mathsf{inj}_0} \llbracket \tau \rrbracket + \llbracket \tau' \rrbracket$$

$$\llbracket \Gamma \vdash^{\mathsf{v}} \mathsf{inr}\, v : \tau + \tau' \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\llbracket v \rrbracket} \llbracket \tau' \rrbracket \xrightarrow{\mathsf{inj}_1} \llbracket \tau \rrbracket + \llbracket \tau' \rrbracket$$

$$\llbracket \Gamma \vdash^{\mathsf{v}} (v, v') : \tau \times \tau' \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket v \rrbracket, \llbracket v' \rrbracket \rangle} \llbracket \tau \rrbracket \times \llbracket \tau' \rrbracket$$

$$\llbracket \Gamma \vdash^{\mathsf{v}} \underline{0} : \mathsf{nat} \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\mathsf{inj}_0} \coprod_{\mathbb{N}} 1$$

$$\llbracket \Gamma \vdash^{\mathsf{v}} \mathsf{S}(v) : \mathsf{nat} \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\llbracket v \rrbracket} \coprod_{\mathbb{N}} 1 \xrightarrow{[\mathsf{inj}_{n+1}]_{n \in \mathbb{N}}} \coprod_{\mathbb{N}} 1$$

$$\llbracket \Gamma \vdash^{\mathsf{v}} \lambda x.\, t : \tau \to \tau' \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\mathsf{curry}(\llbracket t \rrbracket)} (\llbracket \tau \rrbracket \Rightarrow L_\Delta \llbracket \tau' \rrbracket)$$

$$\llbracket \Gamma \vdash^{\mathsf{v}} \mathsf{rec}\, f\, x.\, t : \tau \to \tau' \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{\mathsf{rec}_{\llbracket t \rrbracket}} (\llbracket \tau \rrbracket \Rightarrow L_\Delta \llbracket \tau' \rrbracket)$$

$$\text{where } \llbracket t \rrbracket : \llbracket \Gamma \rrbracket \times (\llbracket \tau \rrbracket \Rightarrow L_\Delta \llbracket \tau' \rrbracket) \times \llbracket \tau \rrbracket \to L_\Delta \llbracket \tau' \rrbracket$$

$$\llbracket \Gamma \vdash^{\mathsf{v}} u : \alpha \rrbracket = \llbracket \Gamma \rrbracket \xrightarrow{!} 1 \xrightarrow{p_u} A_\alpha$$

$$\llbracket \Gamma \vdash^{\mathsf{v}} f : (\alpha_1 \times \ldots \times \alpha_n) \to \beta \rrbracket =$$

$$\llbracket \Gamma \rrbracket \xrightarrow{!} 1 \xrightarrow{\mathsf{curry}(\phi_f)} \big((A_{\alpha_1} \times \ldots \times A_{\alpha_n}) \Rightarrow L_\Delta(A_\beta) \big)$$

Figure 4.8: Interpretation of $\mathsf{PCF_v}$ values in a normal model.

$$[\![\Gamma \vdash^{\mathbf{c}} \pi_1 v : \tau]\!] = [\![\Gamma]\!] \xrightarrow{[\![v]\!]} [\![\tau]\!] \times [\![\tau']\!] \xrightarrow{\pi_1} [\![\tau]\!] \xrightarrow{\eta_{[\![\tau]\!]}} L_\Delta[\![\tau]\!]$$

$$[\![\Gamma \vdash^{\mathbf{c}} \pi_2 v : \tau']\!] = [\![\Gamma]\!] \xrightarrow{[\![v]\!]} [\![\tau]\!] \times [\![\tau']\!] \xrightarrow{\pi_2} [\![\tau']\!] \xrightarrow{\eta_{[\![\tau']\!]}} L_\Delta[\![\tau']\!]$$

$$[\![\Gamma \vdash^{\mathbf{c}} \mathsf{case}\, v\, \mathsf{of}\, \{\mathsf{inl}\, x \to t,\ \mathsf{inr}\, y \to t'\} : \sigma]\!]$$

$$= [\![\Gamma]\!] \xrightarrow{\langle \mathrm{id}_{[\![\Gamma]\!]}, [\![v]\!]\rangle} [\![\Gamma]\!] \times ([\![\tau]\!] + [\![\tau']\!]) \cong ([\![\Gamma]\!] + [\![\tau]\!]) \times ([\![\Gamma]\!] + [\![\tau']\!]) \xrightarrow{\langle [\![t]\!], [\![t']\!]\rangle} L_\Delta[\![\sigma]\!]$$

$$[\![\Gamma \vdash^{\mathbf{c}} \mathsf{case}\, v\, \mathsf{of}\ \{\} : \tau]\!] = [\![\Gamma]\!] \xrightarrow{[\![v]\!]} 0 \xrightarrow{!} L_\Delta[\![\tau]\!]$$

$$[\![\Gamma \vdash^{\mathbf{c}} v\ w : \tau']\!] = [\![\Gamma]\!] \xrightarrow{\langle \mathrm{id}_{[\![\Gamma]\!]}, [\![w]\!]\rangle} [\![\Gamma]\!] \times [\![\tau]\!] \xrightarrow{\mathsf{uncurry}([\![v]\!])} L_\Delta[\![\tau']\!]$$

$$[\![\Gamma \vdash^{\mathbf{c}} \mathsf{case}\, v\ \mathsf{of}\ \{\underline{0} \to t,\ \mathsf{S}(x) \to t'\} : \tau]\!]$$

$$= [\![\Gamma]\!] \xrightarrow{\langle \mathrm{id}_{[\![\Gamma]\!]}, [\![v]\!]\rangle} [\![\Gamma]\!] \times (1 + \coprod_{n \geq 1, n \in \mathbb{N}} 1) \cong [\![\Gamma]\!] + ([\![\Gamma]\!] \times \coprod_{\mathbb{N}} 1) \xrightarrow{[[\![t]\!], [\![t']\!]]} L_\Delta[\![\tau]\!]$$

$$[\![\Gamma \vdash^{\mathbf{c}} \mathsf{return}\, v : \tau]\!] = [\![\Gamma]\!] \xrightarrow{[\![v]\!]} [\![\tau]\!] \xrightarrow{\eta_{[\![\tau]\!]}} L_\Delta[\![\tau]\!]$$

$$[\![\Gamma \vdash^{\mathbf{c}} \mathsf{let}\, x = t\, \mathsf{in}\, t' : \tau']\!]$$

$$= [\![\Gamma]\!] \xrightarrow{\langle \mathrm{id}_{[\![\Gamma]\!]}, [\![t]\!]\rangle} [\![\Gamma]\!] \times L_\Delta[\![\tau]\!] \xrightarrow{\mathsf{str}} L_\Delta([\![\Gamma]\!] \times [\![\tau]\!]) \xrightarrow{[\![t']\!]} L_\Delta[\![\tau']\!]$$

Figure 4.9: Interpretation of $\mathsf{PCF_v}$ computations in a normal model.

- $L_\Delta(\mathsf{Nat}_\mathcal{E})$ being L_Δ-complete implies that $\mathsf{Nat}_\mathcal{E} \in \mathcal{O}_\Delta^\perp$ by Proposition 4.1.5.

For the type constants, we assumed in the definition of normal model that $L_\Delta(\llbracket\alpha\rrbracket) = L_\Delta(A_\alpha)$ is an L_Δ-complete object. For the unit type, $1 \in \mathcal{O}_\Delta^\perp$ because 1 is the terminal object. Therefore, we have proved the base cases of the induction.

If $\llbracket\tau_1\rrbracket, \llbracket\tau_2\rrbracket \in \mathcal{O}_\Delta^\perp$, then we can use Proposition 4.1.9 and $\coprod_\mathbb{N} 1 \in \mathcal{O}_\Delta^\perp$ to deduce that $\llbracket\tau_1 + \tau_2\rrbracket \in \mathcal{O}_\Delta^\perp$. Also, $\llbracket\tau_1 \times \tau_2\rrbracket \in \mathcal{O}_\Delta^\perp$ because $\mathcal{O}_\Delta^\perp$ is closed under limits, as we noted at the beginning of Section 4.1.

To show $(L_\Delta\llbracket\tau_2\rrbracket)^{\llbracket\tau_1\rrbracket} \in \mathcal{O}_\Delta^\perp$, note that $\llbracket\tau_2\rrbracket \in \mathcal{O}_\Delta^\perp$ implies by Proposition 4.1.5 that $L_\Delta\llbracket\tau_2\rrbracket \in \mathcal{O}_\Delta^\perp$. We can then use the fact that $\mathcal{O}_\Delta^\perp$ is an exponential ideal (Lemma 4.1.3).

Finally, by Proposition 4.1.5, $\llbracket\tau\rrbracket \in \mathcal{O}_\Delta^\perp$ implies that $L_\Delta\llbracket\tau\rrbracket$ is L_Δ-complete. □

To connect the interpretation of $\mathsf{PCF_v}$ in a normal model with the operational semantics from Section 4.2.1, we prove the following soundness theorem. We will later prove adequacy (Theorem 7.1.3) for a more restricted class of models rather than for all normal models, which nevertheless covers all our examples of models (from Sections 4.4 and 7.2 and Chapter 8).

Theorem 4.3.5 (Soundness)**.** *For any closed computation t of type τ, if t reduces to a value v, $t \Downarrow v$, then the denotation of t is:*

$$\llbracket t\rrbracket = \eta_{\llbracket\tau\rrbracket} \circ \llbracket v\rrbracket \qquad : 1 \to L_\Delta\llbracket\tau\rrbracket.$$

Proof sketch. The proof is by induction on the rules for the reduction relation $\Downarrow$, following the usual strategy from $\omega\mathsf{CPO}$ (e.g [Win93, Lemma 11.11]). All the cases work because the interpretation of $\mathsf{PCF_v}$ is defined compositionally using the categorical structure.

We spell out the case of the term constants $f : (\alpha_1 \times \ldots \times \alpha_n) \to \beta$. Recall that the reduction rule is

$$\frac{}{f\ v \Downarrow w} \text{ if } f(v) = w \in \mathit{Val}_\beta.$$

The interpretation of $(f\ v)$ is:

$$[\![f\ v]\!] = 1 \xrightarrow{p_v} (A_{\alpha_1} \times \ldots \times A_{\alpha_n}) \xrightarrow{\phi_f} L_\Delta(A_\beta)$$

and we already know from the definition of normal model that this must be interpretation of $\eta_{[\![\beta]\!]} \circ [\![w]\!]$ as well. $\square$

4.4 Example: presheaves on the vertical natural numbers

In this section we introduce the category vSet, a first example of a normal model of $\mathsf{PCF_v}$ with ground types 1, 0, nat, that is, without any type constants. The category vSet will be a running example for our recipe of constructing normal models:

- In Section 5.3, we will show that vSet is equivalent to a category of sheaves on a concrete site, and that the dominance $\Delta_{\mathbb{V}}$ we define here (Definition 4.4.5) can also be constructed from a class of pre-admissible monos in the site, via Theorem 5.1.6.
- In Section 6.2, we show that this class of monos is in fact an example of a class of *admissible* monos. This allows us to show Proposition 6.2.6: that the lifted natural numbers in vSet, $L_{\mathbb{V}}(\mathsf{Nat}_{\mathbb{V}}) \cong L_{\mathbb{V}}\big(\coprod_{\mathbb{N}} 1\big)$, is an $L_{\mathbb{V}}$-complete object, via Theorem 6.2.5.
- In Section 7.1, we will show that vSet is an adequate model of $\mathsf{PCF_v}$ via Theorem 7.1.3.

The vSet model is also the simplest example of normal model we discuss. The other examples include ω-quasi-Borel spaces and ω-diffeological spaces, in Section 7.2, and a fully abstract model of $\mathsf{PCF_v}$ without type constants in Chapter 8.

4.4.1 Defining the category vSet

The category vSet is almost the same as the category $\mathcal{H}$ from [FR97, FR01], which was considered as a model of synthetic domain theory, except we omit their coverage which we do not need.

Consider the poset of vertical natural numbers with a top element:

$$\mathrm{V} = \{0 \leq 1 \leq \ldots \leq n \leq \ldots \leq \infty\}.$$

Let $\mathbb{V}$ be the category with one object V, such that $\mathbb{V}$ is a full subcategory of ωCPO. This means that the maps in $\mathbb{V}$ are the continuous endomorphisms of V, in the ωCPO sense. So we can think of a map $e : \mathrm{V} \to \mathrm{V}$ as a monotone increasing chain valued in $\mathbb{N} \cup \{\infty\}$, with a least upper bound.

Definition 4.4.1. vSet is the category PSh($\mathbb{V}$) of presheaves on $\mathbb{V}$.

We can understand an object of PSh($\mathbb{V}$) as a right-action of the monoid of endomorphisms of V.

For a presheaf $X \in$ vSet, let its set of global elements, which we think of as the set of its *points*, be denoted by:

$$|X| = \mathsf{vSet}(1, X).$$

We can also describe $|X|$ as those elements $x \in X(\mathrm{V})$ such that for any $e \in \mathbb{V}(\mathrm{V}, \mathrm{V})$:

$$X(e)(x) = x.$$

Every element $s \in X(\mathrm{V})$ can be thought of as an *abstract chain* with a supremum, which gives an actual chain of points of X with a supremum:

$$X(c_0)(s),\ X(c_1)(s),\ \ldots,\ X(c_\infty)(s),$$

where

$$c_n = \lambda x.\, n \quad : \mathrm{V} \to \mathrm{V}$$

is intuitively the constant n chain. Notice that each $X(c_n)(s)$ is invariant under the action of any $X(e)$ because $c_n \circ e = c_n$.

The category $\mathbb{V}$ cannot be part of a concrete site in the sense of Definition 2.1.6 because it does not have a terminal object. Nevertheless, given the discussion above we can define a notion of concrete presheaf on $\mathbb{V}$ analogous to the one from Definition 2.1.8.

Definition 4.4.2. A *concrete presheaf* X *on* $\mathbb{V}$ is one for which the function:

$$\begin{aligned} X(\mathrm{V}) &\to \mathsf{Set}(\mathbb{N} \cup \{\infty\}, |X|) \\ s &\mapsto \lambda n.\, X(c_n)(s) \end{aligned}$$

is injective.

Remark 4.4.3. A concrete presheaf $X \in \mathsf{vSet}$ (in the sense of Definition 4.4.2) is a set $|X|$ together with a set $X(\mathrm{V})$ of actual chains with a sup valued in $|X|$. By functoriality, $X(\mathrm{V})$ must be closed under precomposition with any $e \in \mathbb{V}(\mathrm{V}, \mathrm{V})$, and must contain all constant chains. A morphism of concrete presheaves is a function between their sets of points that preserves all chains.

Remark 4.4.4. The category $\omega\mathsf{CPO}$ embeds fully and faithfully into vSet. The embedding sends an ωcpo D to the presheaf X such that $X(\mathrm{V})$ is the set of ω-chains in D together with their least upper bound. Notice that the image of an ωcpo D is a concrete presheaf (in the sense of Definition 4.4.2) whose set of points is D. A continuous function between ωcpo's $f : D \to E$ is mapped to the same function between the sets of points D and E. This embedding was already noted in [FR01].

4.4.2 Dominance in vSet

For vSet to be a normal model according to Definition 4.3.1 we need to define a dominance $\Delta_{\mathbb{V}}$. Let $L_{\mathbb{V}}$ be the resulting lifting monad according to Theorem 2.4.9.

Consider the idempotent $r_1 : \mathrm{V} \to \mathrm{V}$ such that:

$$\begin{aligned} r_1(0) &= 0 \\ r_1(x) &= 1 \quad \text{if } x \geq 1. \end{aligned}$$

Definition 4.4.5. Define $\Delta_{\mathbb{V}}$ in vSet to be the splitting of the idempotent map $y(r_1) : y(\mathrm{V}) \to y(\mathrm{V})$.

Remark 4.4.6. To see what $\Delta_{\mathbb{V}}$ looks like more concretely, notice the following equalities:

$$\begin{gathered} \mathbb{V}(\mathrm{V},\mathrm{V}) \xrightarrow{r_1 \circ (-)} \Delta_{\mathbb{V}}(\mathrm{V}) \xrightarrow{\subseteq} \mathbb{V}(\mathrm{V},\mathrm{V}) = \big(y(r_1)\big)_{\mathrm{V}} \\ \Delta_{\mathbb{V}}(\mathrm{V}) \xrightarrow{\subseteq} \mathbb{V}(\mathrm{V},\mathrm{V}) \xrightarrow{r_1 \circ (-)} \Delta_{\mathbb{V}}(\mathrm{V}) = \mathrm{id}_{\Delta_{\mathbb{V}}(\mathrm{V})}. \end{gathered}$$

This means that we can think of $\Delta_{\mathbb{V}}(\mathrm{V})$ as containing those continuous endomorphisms of V that have image $\{0, 1\}$; or equivalently, as containing the monotone binary sequences $\mathbb{N} \to \{0, 1\}$.

We can also regard $\Delta_{\mathbb{V}}(\mathrm{V})$ as a collection of sieves on V (see Section 2.1.2). The monotone sequence $s : \mathbb{N} \to \{0, 1\}$ which becomes 1 at position n corresponds to the sieve generated by the $\lambda y.\, y + n$ endomorphism of V. The sequence which is always 0 corresponds to the empty sieve.

Proposition 4.4.7. *The object* $\Delta_{\mathbb{V}}$ *is a dominance in* vSet *and classifies the subobject* $0 \rightarrowtail 1$.

Proof. According to Definition 2.3.2, three things are needed to show that $\Delta_{\mathbb{V}}$ is a dominance:

1. that $\Delta_{\mathbb{V}}$ is a subobject of the subobject classifier Ω in vSet;
2. that $\top : 1 \to \Omega$ factors through $\Delta_{\mathbb{V}}$;
3. and that subobjects classified by $\Delta_{\mathbb{V}}$ are closed under composition.

For Item 1, notice that the definition of Ω is:

$$\Omega(\mathrm{V}) = \{S \mid S \text{ a sieve on V}\},$$

and the action of $\Omega(e : \mathrm{V} \to \mathrm{V})$ is:

$$\Omega(e)(S) = \{f : \mathrm{V} \to \mathrm{V} \mid (e \circ f) \in S\}.$$

So we can already see that $\Delta_{\mathbb{V}}(\mathrm{V}) \subseteq \Omega(\mathrm{V})$.

To show that $\Delta_{\mathbb{V}}(\mathrm{V})$ is closed under the action of $\Omega(e)$, consider the sieve S_n generated by $\lambda y.\, y + n$. If there exists some $k \in \mathbb{N}$ such that:

$$e(\{0, \ldots, k-1\}) \subseteq \{0, \ldots, n-1\} \qquad\qquad e(k) \geq n,$$

then $\Omega(e)(S_n) = S_k \in \Delta_{\mathbb{V}}(\mathrm{V})$. Otherwise, the image of e lies entirely inside $\{0, \ldots, n-1\}$, so $\Omega(e)(S_n)$ is the empty sieve, which is in $\Delta_{\mathbb{V}}(\mathrm{V})$. If we start from the empty sieve, then $\Omega(e)(\emptyset)$ is again the empty sieve.

For Item 2, notice that $\top_{\mathrm{V}} : 1 \to \Omega(\mathrm{V})$ picks out the sieve generated by identity on V, which is also a sieve in $\Delta_{\mathbb{V}}(\mathrm{V})$.

For Item 3, consider two subobjects $m : B \rightarrowtail A$ and $n : C \rightarrowtail B$ both classified by $\Delta_{\mathbb{V}}$. Let their classifying maps be $\chi_m : A \to \Delta_{\mathbb{V}}$ and $\chi_n : B \to \Delta_{\mathbb{V}}$ respectively. Both m and n are classified by Ω as well, and since $\Delta_{\mathbb{V}}(\mathrm{V})$ is included in $\Omega(\mathrm{V})$, it must be the case that χ_m and χ_n have the same form as the classifying maps into Ω. For example, for some $x \in A(\mathrm{V})$:

$$(\chi_m)_{\mathrm{V}}(x) = \{\sigma : \mathrm{V} \to \mathrm{V} \mid A(\sigma)(x) \in B(\mathrm{V})\},$$

and similarly for χ_n.

Next, we will show that the classifying map $\chi : A \to \Omega$ of $(m \circ n) : C \rightarrowtail A$, given by:

$$(\chi)_{\mathrm{V}}(x) = \{\sigma : \mathrm{V} \to \mathrm{V} \mid A(\sigma)(x) \in C(\mathrm{V})\}$$

factors through $\Delta_{\mathbb{V}}$.

The condition $A(\sigma)(x) \in C(\mathrm{V})$ can be rephrased as: $A(\sigma)(x) \in B(\mathrm{V})$ and $A(\sigma)(x) \in C(\mathrm{V})$. So we know that each σ in $(\chi)_{\mathrm{V}}(x)$ is also part of $(\chi_m)_{\mathrm{V}}(x)$.

If $(\chi_m)_{\mathrm{V}}(x)$ is generated by $\lambda y.\, y + p_1$, then each σ in $(\chi)_{\mathrm{V}}(x)$ factors through $\lambda y.\, y + p_1$. So $(\chi)_{\mathrm{V}}(x)$ is isomorphic to:

$$\{\sigma' : \mathrm{V} \to \mathrm{V} \mid A(\sigma')\big(A(\lambda y.\, y + p_1)(x)\big) \in C(\mathrm{V})\}.$$

But $A(\lambda y.\, y + p_1)(x)$ is in $B(\mathrm{V})$, so $(\chi)_{\mathrm{V}}(x)$ is isomorphic to

$$(\chi_n)_{\mathrm{V}}\big(A(\lambda y.\, y + p_1)(x)\big).$$

If $(\chi_n)_{\mathrm{V}}\big(A(\lambda y.\, y + p_1)(x)\big)$ is generated by $\lambda y.\, y + p_2$, then it follows that $(\chi)_{\mathrm{V}}(x)$ is generated by $\lambda y.\, y + (p_1 + p_2)$, so it is part of $\Delta_{\mathbb{V}}(\mathrm{V})$.

If either $(\chi_m)_{\mathrm{V}}(x)$ or $(\chi_n)_{\mathrm{V}}\big(A(\lambda y.\, y + p_1)(x)\big)$ are the empty sieve, then $(\chi)_{\mathrm{V}}(x)$ is also the empty sieve, so it is part of $\Delta_{\mathbb{V}}(\mathrm{V})$.

$\Delta_{\mathbb{V}}$ classifies $0 \rightarrowtail 1$. The classifying map of $0 \to 1$ is the map $\chi_{\mathrm{V}} : 1 \to \Delta_{\mathbb{V}}(\mathrm{V})$ which picks out the empty sieve. It is natural because the empty sieve pulls back to the empty sieve along any endomorphism of V; and the pullback of χ along $\top$ is indeed 0. □

Remark 4.4.8. The lifting monad $L_{\mathbb{V}}$ given by the dominance $\Delta_{\mathbb{V}}$ (Definition 4.4.5) has the following explicit description:

$$(L_{\mathbb{V}}X)(\mathrm{V}) \cong \{\bot\} + \sum_{n \in \mathbb{N}} \big(X(\mathrm{V})\big).$$

An abstract chain in $(L_{\mathbb{V}}X)(\mathrm{V})$ is either the always $\bot$ chain, or is a chain from $X(\mathrm{V})$ with n $\bot$'s added at the beginning, hence the sum over $n \in \mathbb{N}$.

The action of an endomorphism $e : \mathrm{V} \to \mathrm{V}$ for $s \in \big(X(\mathrm{V})\big)_n$ is given explicitly as:

$$(L_{\mathbb{V}}X)(e)(s) = \begin{cases} \bot & \text{if } \mathsf{im}(e) \subseteq \{0, \ldots, n-1\} \\ X(e')(s) \in \big(X(\mathrm{V})\big)_k & \text{if } e(\{0, \ldots, k-1\}) \subseteq \{0, \ldots, n-1\}, \\ & e(k) \geq n,\ e'(i) = e(k+i) - n \end{cases}$$

$$(L_{\mathbb{V}}X)(e)(\bot) = \bot.$$

In case X is a concrete presheaf (in the sense of Definition 4.4.2), the action of $(L_{\mathbb{V}}X)(e)$ has a more intuitive description. An element in $\big(X(\mathrm{V})\big)_n$ is a sequence s of points from $|X|$ to which n $\perp$'s are added at the beginning; the action of e is sequence reindexing by function composition:

$$(\perp, \ldots, \perp, s) \circ e.$$

Remark 4.4.9. The poset of vertical natural numbers V is closely connected to $\overline{\omega}_{\Delta_{\mathbb{V}}}$, the vertical natural numbers defined in Assumption 3.1.1, which is an object in vSet. More precisely the Yoneda embedding of V into vSet is isomorphic to $\overline{\omega}_{\Delta_{\mathbb{V}}}$:

$$y(\mathrm{V}) \cong \overline{\omega}_{\Delta_{\mathbb{V}}}.$$

4.4.3 vSet as a normal model of $\mathsf{PCF_v}$

After defining a dominance $\Delta_{\mathbb{V}}$ in vSet, to conclude that vSet is a normal model, it remains to show that $L_{\mathbb{V}}(\mathsf{Nat}_{\mathbb{V}})$ is an $L_{\mathbb{V}}$-complete object. Here $\mathsf{Nat}_{\mathbb{V}} \cong \coprod_{\mathbb{N}} 1$ is the natural numbers in vSet. We defer this proof until Section 6.2 where it will be a consequence of Theorem 6.2.5.

The interpretation of $\mathsf{PCF_v}$ in vSet is the same as the interpretation in any normal model, described in Section 4.3. From soundness for normal models (Theorem 4.3.5) we can deduce soundness of the vSet model. In Section 7.1, we will show that vSet is also an adequate model as a consequence of Theorem 7.1.3.

The next proposition says that the model of $\mathsf{PCF_v}$ in vSet is essentially the $\omega\mathsf{CPO}$ model. Thus, even though $\omega\mathsf{CPO}$ is not a Grothendieck topos, it can be regarded as a normal model via the embedding in vSet. We will later show that normal models encompass ideas going beyond $\omega\mathsf{CPO}$, through examples such as ω-quasi-Borel spaces (Section 7.2) and a fully abstract model of $\mathsf{PCF_v}$ (Chapter 8).

Proposition 4.4.10. *There is a full and faithful functor $F : \omega\mathsf{CPO} \to \mathsf{vSet}$ which preserves products, coproducts and exponentials, and commutes with* *the lifting monads on each category:*

$$FL = L_{\mathbb{V}}F,$$

where the monad L adds a bottom element to each ωcpo. Moreover, F commutes with the interpretation of $\mathsf{PCF_v}$ *in* $\omega\mathsf{CPO}$ *and* vSet*, including the interpretation of fixed points.*

Proof. The functor F is the one defined in Remark 4.4.4. Using the fact that the image of F lands inside the subcategory of concrete presheaves on $\mathbb{V}$ (in the sense of Definition 4.4.2) and using the description of $L_{\mathbb{V}}$ for concrete presheaves (Remark 4.4.8), we can see that $L_{\mathbb{V}}$ also adds a bottom element to the underlying set $|X|$ of a concrete presheaf X. From here we can check that $FL = L_{\mathbb{V}}F$. We omit checking that F preserves products, coproducts and exponentials, which involves spelling out the categorical structure of both categories.

To show that F preserves the interpretation of $\mathsf{PCF_v}$ types, we proceed by induction on types. The only interesting case is nat. In $\omega\mathsf{CPO}$, nat is interpreted as the natural numbers $\mathbb{N}$ with the discrete ordering and by calculating the coproduct $\coprod_{\mathbb{N}} 1$ in vSet we can see that it is indeed the image of $\mathbb{N}$. For the other types, the proof goes through because F preserves the necessary categorical structure.

To prove that F preserves the interpretation of $\mathsf{PCF_v}$ values and computations, proceed by induction on the typing rules. Most cases follow because F preserves the relevant categorical structure.

The interesting case is $\Gamma \vdash^{\mathsf{v}} (\mathsf{rec}\, f\, x.\, t) : \tau \to \tau'$, which is interpreted as a fixed point of the (curried) interpretation of t. In vSet it is the fixed point constructed in Corollary 3.2.5 for the map

$$[\![t]\!]^* : [\![\Gamma]\!] \times ([\![\tau]\!] \Rightarrow L_{\mathbb{V}}[\![\tau']\!]) \to ([\![\tau]\!] \Rightarrow L_{\mathbb{V}}[\![\tau']\!]).$$

In $\omega\mathsf{CPO}$ it is the least fixed point constructed using Tarski's fixed point theorem. To show that the two fixed points agree, we will make extensive use of the fact that F is full and faithful and commutes with lifting. We will sometimes denote a map or an object in the image of F by its counterpart in $\omega\mathsf{CPO}$ and vice-versa.

First, we show a more general fact. Consider a map $h : \Gamma \times A \to A$ in $\omega\mathsf{CPO}$, where A has a bottom element $\bot_A$. This means A has an L-algebra structure $\alpha : LA \to A$ which sends $\bot$ from LA to $\bot_A$, and this remains an $L_{\mathbb{V}}$-algebra after applying F. (Notice that this is the same $L_{\mathbb{V}}$-algebra structure used in the proof of Corollary 3.2.5.) Given this algebra structure, we can construct a fixed point of $Fh : F\Gamma \times FA \to FA$ in vSet using Theorem 3.2.3; denote this fixed point by:

$$\phi : F\Gamma \to FA.$$

The corresponding map in $\omega\mathsf{CPO}$, $\phi : \Gamma \to A$, must also be a fixed point of $h : \Gamma \times A \to A$ because F is full and faithful. We will show that:

$$\text{in } \omega\mathsf{CPO}\text{, the map } \phi : \Gamma \to A \text{ is the } \textit{least} \text{ fixed point of } h : \Gamma \times A \to A, \tag{4.1}$$

and it is thus the one constructed in Tarski's fixed point theorem. This fact is enough to prove that F commutes with the interpretation of rec.

Because in vSet, $\phi : F\Gamma \to FA$ is constructed using Theorem 3.2.3, it must have the form

$$F\Gamma \xrightarrow{\xi} L_{\mathbb{V}}F(A) \xrightarrow{\alpha} FA$$

where ξ is the fixed point calculated in Lemma 3.2.4 for the map:

$$g = F\Gamma \times L_{\mathbb{V}}F(A) \xrightarrow{\mathrm{id}\times\alpha} F\Gamma \times FA \xrightarrow{Fh} FA \xrightarrow{\eta} L_{\mathbb{V}}F(A).$$

Since F is full, faithful and commutes with lifting, in $\omega\mathsf{CPO}$ the map

$$\xi : \Gamma \to LA$$

is a fixed point of:

$$g = \Gamma \times LA \xrightarrow{\mathrm{id}\times\alpha} \Gamma \times A \xrightarrow{h} A \xrightarrow{\eta} LA,$$

and

$$\Gamma \xrightarrow{\phi} A = \Gamma \xrightarrow{\xi} LA \xrightarrow{\alpha} A.$$

It would be enough to prove that $\xi : \Gamma \to LA$ is the least fixed point of $g : \Gamma \times LA \to LA$. Consider another fixed point $\phi' : \Gamma \to A$ in $\omega\mathsf{CPO}$ of $h : \Gamma \times A \to A$. Then

$$\Gamma \xrightarrow{\phi'} A \xrightarrow{\eta} LA$$

is a fixed point of g. Since ξ is the least fixed point of g we get:

$$\Gamma \xrightarrow{\xi} LA \leq \Gamma \xrightarrow{\phi'} A \xrightarrow{\eta} LA.$$

Because α is monotone:

$$\Gamma \xrightarrow{\xi} LA \xrightarrow{\alpha} A \leq \Gamma \xrightarrow{\phi'} A \xrightarrow{\eta} LA \xrightarrow{\alpha} A,$$

so we obtain $\phi \leq \phi'$ as required. This concludes the proof of fact (4.1).

It remains to prove the following fact: in $\omega\mathsf{CPO}$, the map $\xi : \Gamma \to LA$ is the *least* fixed point of $g : \Gamma \times LA \to LA$, meaning that for any $\rho \in \Gamma$

$$\xi(\rho) \in LA \text{ is the least fixed point of } g(\rho, -) : LA \to LA. \tag{4.2}$$

For the rest of the proof we fix an element $\rho \in \Gamma$, which has a corresponding map in vSet

$$\rho : 1 \to F\Gamma.$$

In the proof of Lemma 3.2.4, the map $\xi : F\Gamma \to L_{\mathbb{V}}F(A)$ is constructed in vSet by evaluating a map

$$\mathsf{ap}_{\overline{\omega}} : F\Gamma \times \overline{\omega} \to L_{\mathbb{V}}F(A)$$

at the point $\infty : 1 \to \overline{\omega}$ (defined in Lemma 3.1.3). So in vSet:

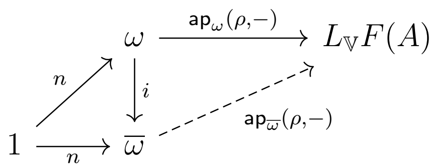
$$\xi(\rho) = \mathsf{ap}_{\overline{\omega}}(\rho, \infty) : 1 \to L_{\mathbb{V}}F(A).$$

Because in vSet: $\overline{\omega} \cong y(\mathrm{V})$ (Remark 4.4.9), by Yoneda, $\mathsf{ap}_{\overline{\omega}}(\rho, -) : \overline{\omega} \to$

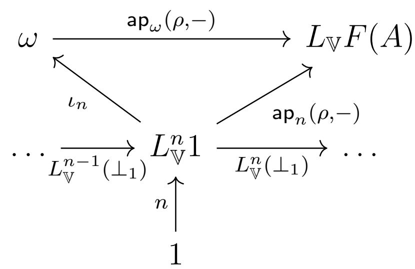
$L_{\mathbb{V}}F(A)$ corresponds to an actual chain with a least upper bound valued in the ωcpo LA. Moreover, using concreteness of $L_{\mathbb{V}}F(A)$ (in the sense of Definition 4.4.2), we can see that $\mathsf{ap}_{\overline{\omega}}(\rho, \infty)$ corresponds to the least upper bound of this chain.

So, to show that $\xi(\rho)$ is a least fixed point in $\omega\mathsf{CPO}$, it is enough to show that any other fixed point $a \in LA$ of $g(\rho, -) : LA \to LA$ is greater than every element in the chain $\mathsf{ap}_{\overline{\omega}}(\rho, -)$ (excluding the top element).

To describe the elements of the chain $\mathsf{ap}_{\overline{\omega}}(\rho, -)$, we can look at the points of $\overline{\omega}$ in vSet, which are isomorphic to $\mathbb{N} + \{\infty\}$. Recall that in the proof of Lemma 3.2.4, $\mathsf{ap}_{\overline{\omega}}(\rho, -)$ is defined as the unique extension of a map $\mathsf{ap}_{\omega}(\rho, -)$. So for each $n \in \mathbb{N}$, we get the following commuting diagram:



Recall that ω was defined as a colimit (Assumption 3.1.4), and $\mathsf{ap}_{\omega}(\rho, -)$ was defined as the comparison map into a cocone $\{\mathsf{ap}_k : L^k_{\mathbb{V}}1 \to L_{\mathbb{V}}F(A)\}_{k\in\mathbb{N}}$. Moreover, each $n : 1 \to \omega$ factors through the colimit inclusion $\iota_n : L^n_{\mathbb{V}}1 \to \omega$. Therefore, we get the following commuting diagram:



It is thus enough to show that for any $n \in \mathbb{N}$, the $\omega\mathsf{CPO}$ map corresponding to the vSet map:

$$1 \xrightarrow{n} L^n_{\mathbb{V}}1 \xrightarrow{\mathsf{ap}_n(\rho,-)} L_{\mathbb{V}}F(A)$$

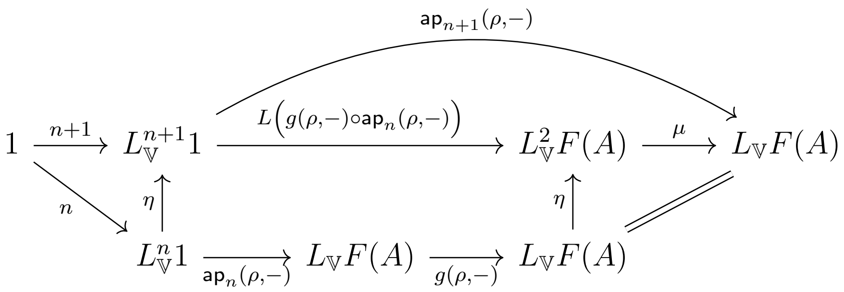
is smaller than $a \in LA$. We will prove this by induction on n, using the definition of ap_n in vSet from the proof of Lemma 3.2.4.

In the base case, from the definition of ap_0, we get the following map in vSet:

$$\bot_{FA} : 1 \to L_{\mathbb{V}} F(A),$$

which is smaller than any $a \in LA$.

For the induction step, unwind the definition of ap_{n+1} to get a commuting diagram:



We know by induction hypothesis that in ωCPO:

$$\mathsf{ap}_n(\rho, n) \leq a.$$

Because $g(\rho, -)$ is monotone:

$$g\big(\rho, \mathsf{ap}_n(\rho, n)\big) \leq g(\rho, a).$$

Because a was assumed to be a fixed point of $g(\rho, -)$ we get the required inequality:

$$\mathsf{ap}_{n+1}(\rho, n+1) = g\big(\rho, \mathsf{ap}_n(\rho, n)\big) \leq a,$$

which concludes the proof of fact (4.2).

□

Chapter 5

Building normal models: dominance from a site

In this chapter and the next, we work towards a recipe for building normal models (Definition 4.3.1) of call-by-value PCF ($\mathsf{PCF_v}$) as categories of sheaves. To do this, we identify sufficient structure on the site $(\mathbb{C}, J)$ of a sheaf category to allow us to build a dominance (Definition 2.3.2) in $\mathsf{Sh}(\mathbb{C}, J)$. The structure that we ask for is the existence of a class of pre-admissible monos (Definition 5.1.2) *in the site*; the dominance is built in Theorem 5.1.6. The main contribution of this chapter is proving Theorem 5.1.6, which generalizes a result by Mulry [Mul94] about building dominances in categories of presheaves.

In Section 5.2, under further concreteness assumptions, we prove some results about the dominance and lifting monad obtained from a class of pre-admissible monos. These results will be used in Chapters 6 and 7.

In Section 5.3, we revisit our running example of normal model, the category vSet of presheaves on the vertical natural numbers, introduced in Section 4.4. We show that the vSet model fits into our recipe of building a dominance from a class of pre-admissible monos in a site (Proposition 5.3.5), and that vSet satisfies the concreteness assumptions from Section 5.2.

The material in this chapter was published at LICS 2022 [MMS22, Sections 5.3, 6.3].

In Chapter 6, we will continue our recipe for building normal models, which will be summarized in Theorem 7.1.1. In Chapter 7 we will also prove that the normal models we obtain with our recipe are adequate.

5.1 Dominance from a class of pre-admissible monos

In this section, we explain how a class of monos $\mathcal{M}$ in a site $(\mathbb{C}, J)$ can be organized as a presheaf $\Delta_{\mathcal{M}}$ (Definition 5.1.1), and we introduce the notion of class of pre-admissible monos (Definition 5.1.2) in a site $(\mathbb{C}, J)$. The main theorem (Theorem 5.1.6) is that such a class gives rise to a dominance in the corresponding sheaf category. In Section 5.1.1, we give an explicit description of the lifting monad obtained from this dominance via Theorem 2.4.9.

Definition 5.1.1. Given a small site $(\mathbb{C}, J)$ and a stable system of monos $\mathcal{M}$ in $\mathbb{C}$ (Definition 2.4.1), define a presheaf $\Delta_{\mathcal{M}} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ as follows:

$$\Delta_{\mathcal{M}}(c) \coloneqq \mathsf{Sub}_{\mathcal{M}}(c)$$
$$\Delta_{\mathcal{M}}(f : a \to c) : \Delta_{\mathcal{M}}(c) \to \Delta_{\mathcal{M}}(a) \text{ given by pullback}$$

where $\mathsf{Sub}_{\mathcal{M}}$ contains the $\mathcal{M}$-subobjects of c, that the monos from $\mathcal{M}$ considered up to isomorphism as explained in Definition 2.2.1.

Definition 5.1.2. A stable system of monos $\mathcal{M}$ is a *class of pre-admissible monomorphisms* in $(\mathbb{C}, J)$ if the presheaf $\Delta_{\mathcal{M}}$ is a J-sheaf.

Definition 5.1.3. Define the map $\top : 1 \rightarrowtail \Delta_{\mathcal{M}}$ to be:

$$\top_c(\star) = [\mathrm{id}_c] \in \mathsf{Sub}_{\mathcal{M}}(c).$$

Naturality follows because $\Delta_{\mathcal{M}}(f)$ acts by pullback.

The aim is to show that $\Delta_{\mathcal{M}}$ is a dominance (Definition 2.3.2) in $\mathsf{Sh}(\mathbb{C}, J)$, and that $\top : 1 \rightarrowtail \Delta_{\mathcal{M}}$ is the map used to classify subobjects according to Definition 2.3.1, i.e. the map through which $\top : 1 \to \Omega_J$ factors, where

Ω_J is the subobject classifier. First we prove two lemmas that characterize the subobjects classified by $\Delta_{\mathcal{M}}$.

Lemma 5.1.4. *Let $(\mathbb{C}, J)$ be a site with $\mathcal{M}$ a class of pre-admissible monos. Given any map $\chi : ay(c) \to \Delta_{\mathcal{M}}$ from a sheafified representable $ay(c)$ (Proposition 2.1.5) into $\Delta_{\mathcal{M}}$, the pullback of $\top : 1 \rightarrowtail \Delta_{\mathcal{M}}$ along χ has the form $ay(m) : ay(c') \rightarrowtail ay(c)$ for some $m \in \Delta_{\mathcal{M}}(c)$.*

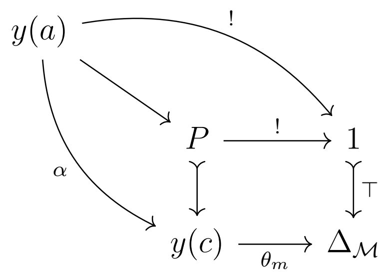
Proof. The sheafification functor $a : \mathsf{PSh}(\mathbb{C}) \to \mathsf{Sh}(\mathbb{C}, J)$ is left adjoint to inclusion, $a \dashv i$, so because $\Delta_{\mathcal{M}}$ is a sheaf:

$$\mathsf{Sh}(ay(c), \Delta_{\mathcal{M}}) \cong \mathsf{PSh}(y(c), \Delta_{\mathcal{M}}) \cong \Delta_{\mathcal{M}}(c) \cong \mathsf{Sub}_{\mathcal{M}}(c).$$

So every map $\chi : ay(c) \to \Delta_{\mathcal{M}}$ corresponds to an $\mathcal{M}$-subobject $m : c' \rightarrowtail c$.

First we show the analogous proposition for presheaves. Consider a map $\theta_m : y(c) \to \Delta_{\mathcal{M}}$ corresponding to $m : c' \rightarrowtail c \in \mathcal{M}$. Let $P \rightarrowtail y(c)$ be the pullback of $\top$ along θ_m. Then for an object a in $\mathbb{C}$:

$$P(a) \cong \mathsf{PSh}(\mathbb{C})(y(a), P) \cong \big\{\alpha : y(a) \to y(c) \;\big|\; \theta_m \circ \alpha = \top\circ!\big\}.$$



By Yoneda, α must always be θ_f for some $f \in \mathbb{C}(a, c)$. So

$$\big(\theta_m \circ y(f) = \top\circ!\big) : y(a) \to \Delta_{\mathcal{M}}$$

can be rewritten for any $b \in \mathbb{C}$ as:

$$\big(\lambda(g : b \to a).\, (f \circ g)^*(m)\big) = [\mathrm{id}_b]$$

because by Yoneda $\theta_m(f \circ g) = \Delta_{\mathcal{M}}(f \circ g)(m)$. Therefore:

$$\begin{aligned} P(a) &\cong \{ f \in \mathbb{C}(a, c) \mid \forall b. \forall g : b \to a.\ (f \circ g)^*(m) = [\mathrm{id}_b] \} \\ &\cong \{ f \in \mathbb{C}(a, c) \mid f^*(m) = [\mathrm{id}_a] \} \\ &\cong \mathbb{C}(a, c') \end{aligned}$$

So we obtain:

$$P \cong y(c').$$

Now we need to show $P \rightarrowtail y(c)$ is $y(m)$. According to the isomorphism above $\mathrm{id}_{c'}$ gets mapped to m in $P(c') \subseteq \mathbb{C}(c', c)$. So by Yoneda the inclusion $P \cong y(c') \rightarrowtail y(c)$ is given by $y(m)$.

Now to show the statement for sheaves, consider $\chi_m : ay(c) \to \Delta_{\mathcal{M}}$. This has a corresponding map $\theta_m : y(c) \to \Delta_{\mathcal{M}}$ in presheaves. Because sheafification preserves limits, we can obtain the pullback of $\top$ along χ_m by sheafifying the pullback along θ_m. And so we obtain the pullback diagram:

$$\begin{array}{ccc} ay(c') & \xrightarrow{\ !\ } & 1 \\ {\scriptstyle ay(m)} \big\downarrow & & \big\downarrow {\scriptstyle \top} \\ ay(c) & \xrightarrow[\chi_m]{} & \Delta_{\mathcal{M}} \end{array}$$

□

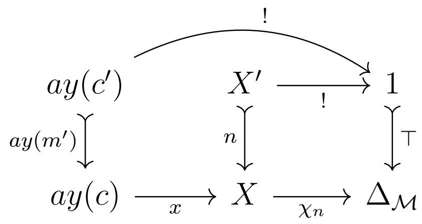
In the proof of the following lemma and of Theorem 5.1.6 we use the topology T on $\mathbb{C}$ generated by the coverage J via Proposition 2.1.11.

Lemma 5.1.5. *A subobject $n : X' \rightarrowtail X$ in $\mathsf{Sh}(\mathbb{C}, J)$ is classified by $\Delta_{\mathcal{M}}$ if and only if n is representably in $\mathcal{M}$, meaning that the pullback of n along any map from a sheafified representable $ay(c) \to X$ has the form $ay(m') : ay(c') \rightarrowtail ay(c)$ for some m' in $\mathcal{M}$.*

Proof. Recall that there is an isomorphism:

$$\mathsf{Sh}(ay(c), \Delta_{\mathcal{M}}) \cong \mathsf{Sub}_{\mathcal{M}}(c).$$

For the **left to right** direction consider $x : ay(c) \to X$.



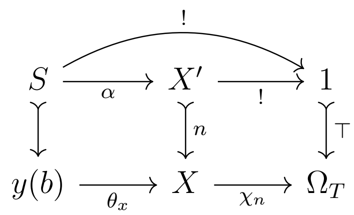
By Lemma 5.1.4, the pullback of $\top$ along $\chi_n \circ x$ is a representable. Using the pullback lemma we can deduce that is also the pullback of n along x.

For the **right to left** direction, we show that the classifying map of n from sheaves $\chi_n : X \to \Omega_T$ factors through $\Delta_{\mathcal{M}}$. (Where Ω_T is the subobject classifier from sheaves, and is the same as Ω_J which we used before.)

For any $b \in \mathbb{C}$ and any $x \in X(b)$, $(\chi_n)_b(x)$ is defined to be the T-closed sieve [MM92, Chapter III.7]:

$$(\chi_n)_b(x) = S = \big\{ f : b' \to b \bigm| X(f)(x) \in X'(b') \big\}.$$

Regarding the sieve S as a subobject of a representable, this means that S is the pullback of $\top : 1 \to \Omega_T$ along $\chi_n \circ \theta_x$:



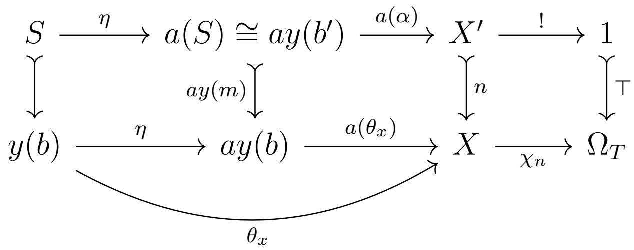
because $(\chi_n \circ \theta_x)_{b'}(f : b' \to b) = f^*(S)$. For this to be the maximal sieve we need $f \in S$. (See Section 2.1.2 for a definition of sieves and $f^*(S)$.)

The map α is defined as follows:

$$\alpha_{b'}(f : b' \to b) = X(f)(x) \in X'(b').$$

It is natural by functoriality of X, and it makes the left square commute because by Yoneda $(\theta_x)_{b'}(f) = X(f)(x)$. Thus, by the pullback lemma, the left square is a pullback too.

If we sheafify the previous diagram we obtain:



where the middle square is a pullback because sheafification preserves limits, and $a(S) \rightarrowtail ay(b)$ is by assumption $ay(m) : ay(b') \rightarrowtail ay(b)$ for some $m : b' \rightarrowtail b \in \mathcal{M}$. The map η is the unit of the adjunction $a \dashv i$, and therefore $\eta \circ a(\theta_x) = \theta_x$.

We now show that the sieve S is generated by m, so it is part of $\Delta_{\mathcal{M}}(b)$. For this it is enough to show that $S \rightarrowtail y(b)$ is equal to $ay(m) : ay(b') \to ay(b)$ by showing they have the same classifying map into Ω_T.

By the pullback lemma, the leftmost square is a pullback (it commutes because $ay(m)$ is the sheafification of $S \rightarrowtail y(b)$). So the classifying map of $S \rightarrowtail y(b)$ is $(\chi_n \circ a(\theta_x) \circ \eta)$. The map $ay(m)$, which is the sheafification of $y(m)$, has the unique classifying map $(\chi_n \circ a(\theta_x))$. Because $y(m)$ has a classifying map and a preserves limits, this must be $(\chi_n \circ a(\theta_x) \circ \eta)$. $\square$

The following theorem constructs a dominance from a class of pre-admissible monos. It generalizes [Mul94, Theorem 2.6] which only applies to categories of presheaves.

Theorem 5.1.6. *Let $(\mathbb{C}, J)$ be a site with $\mathcal{M}$ a class of pre-admissible monos. Then the object $\Delta_{\mathcal{M}}$ is a dominance in $\mathsf{Sh}(\mathbb{C}, J)$, and $\top : 1 \to \Delta_{\mathcal{M}}$ (Definition 5.1.3) is the factoring of the map $\top : 1 \to \Omega_J$ through $\Delta_{\mathcal{M}}$.*

Proof. According to the definition of dominance (Definition 2.3.2) there are three steps in the proof:

$\Delta_{\mathcal{M}}$ is a subobject of Ω_J. First we need to show $\Delta_{\mathcal{M}}$ is a subobject of Ω_T, the subobject classifier in $\mathsf{Sh}(\mathbb{C}, J) \cong \mathsf{Sh}(\mathbb{C}, T)$. Recall that we denote by T

the topology on $\mathbb{C}$ generated by the coverage J via Proposition 2.1.11. The subobject classifier Ω_T, which is the same as Ω_J, is [MM92, Chapter III.7]:

$$\Omega_T(c) = \{S \mid S \text{ is a } T\text{-closed sieve on } c\}$$

where $\Omega_T(f : d \to c)$ takes the pullback $f^*(S)$ of a sieve S on c (see Section 2.1.2). Being a T-closed sieve means that for any $f : d \to c$ in $\mathbb{C}$:

$$\text{if } f^*(S) \in T(c), \text{ then } f \in S.$$

Since $\Delta_{\mathcal{M}}(c) = \mathsf{Sub}_{\mathcal{M}}(c)$, we need to show that the sieve S generated by each $m : c' \rightarrowtail c \in \mathcal{M}$ is T-closed. Consider $f : d \to c$ in $\mathbb{C}$, and $f^*(S) \in T(c)$.

For all $g \in f^*(S)$, $f \circ g$ factors through m. Let:

$$m' = \big(f^*(m) : d' \rightarrowtail d\big) \in \Delta_{\mathcal{M}}(d).$$

By the universal property of the pullback, every $g \in f^*(S)$ factors through m'.

Notice that m' is an amalgamation for the following matching family for the sieve $f^*(S)$:

$$\big(\Delta_{\mathcal{M}}(g)(m') = g^*(m')\big)_{g \in f^*(S)}.$$

But each g factors through m', so the matching family is in fact:

$$\big(\mathrm{id}_{\mathrm{dom}(g)}\big)_{g \in f^*(S)}$$

and has amalgamation id_d. We assumed $f^*(S) \in T(c)$ so there can be only one amalgamation for this matching family, therefore:

$$m' = \mathrm{id}_d.$$

The map m' was defined to be the pullback of m along f, so it must be the case that f factors through m. Thus $f \in S$.

To show $\Delta_{\mathcal{M}}$ is a subobject of Ω_T, we also need to show that $\Delta_{\mathcal{M}}$ is closed under the action of $\Omega_T(f : d \to c)$. We have already shown that every g in

$f^*(S)$ factors through $m' : d' \rightarrowtail d$. For every $g' : d'' \to d'$:

$$f \circ m' \circ g' = m \circ m^*(f) \circ g'$$

so $(f \circ m' \circ g') \in S$, and therefore $(m' \circ g') \in f^*(S)$. Thus $f^*(S)$ is the sieve generated by $m' \in \mathcal{M}$, so $f^*(S)$ is in $\Delta_{\mathcal{M}}(d)$.

$\top : 1 \to \Omega_T$ factors through $\Delta_{\mathcal{M}}$. In sheaves, $\top_c : 1 \to \Omega_T(c)$ picks out the maximal sieve on c (generated by id_c), so it factors through the map $\top : 1 \to \Delta_{\mathcal{M}}$ (Definition 5.1.3).

Subobjects classified by $\Delta_{\mathcal{M}}$ are closed under composition. Consider two maps $X'' \rightarrowtail X'$ and $X' \rightarrowtail X$ classified by $\Delta_{\mathcal{M}}$. In Lemma 5.1.5, we have shown they are both representably in $\mathcal{M}$. By the pullback lemma, their composition is also representably in $\mathcal{M}$, so by it is classified by $\Delta_{\mathcal{M}}$. □

5.1.1 Lifting from a class of pre-admissible monos

Definition 5.1.7. Given a site $(\mathbb{C}, J)$ with a class of pre-admissible monos $\mathcal{M}$, denote by $L_{\mathcal{M}}$ the lifting monad on $\mathsf{Sh}(\mathbb{C}, J)$ obtained from the dominance $\Delta_{\mathcal{M}}$ using Theorem 2.4.9.

We now give an explicit description of the monad $L_{\mathcal{M}}$ in terms of $\mathcal{M}$ as follows. Instantiate the formula for $L_{\mathcal{M}}$ from Remark 2.4.10:

$$L_{\mathcal{M}} = \Sigma_{\Delta_{\mathcal{M}}} \circ \Pi_{\top}$$

where $\Pi_{\top} : \mathsf{Sh}(\mathbb{C}, J) \to \mathsf{Sh}(\mathbb{C}, J)/\Delta_{\mathcal{M}}$ is left adjoint to the pullback functor $\top^*$ and $\Sigma_{\Delta_{\mathcal{M}}}$ is the domain functor. We can then prove the following proposition:

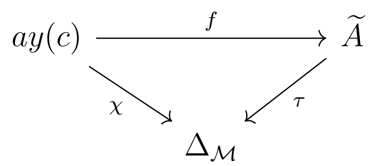
Proposition 5.1.8. *For a sheaf A, the lifting monad $L_{\mathcal{M}}$ can be described as:*

$$L_{\mathcal{M}}(A)(c) = \sum_{(m : d \rightarrowtail c) \in \mathsf{Sub}_{\mathcal{M}}(c)} A(d).$$

Proof.

$$
\begin{aligned}
L_{\mathcal{M}}(A)(c) &\cong (\Sigma_{\Delta_{\mathcal{M}}} \circ \Pi_{\top})(A)(c) \\
&\cong \mathsf{PSh}\big(y(c), (\Sigma_{\Delta_{\mathcal{M}}} \circ \Pi_{\top})(A)\big) && \text{by Yoneda} \\
&\cong \mathsf{Sh}\big(ay(c), (\Sigma_{\Delta_{\mathcal{M}}} \circ \Pi_{\top})(A)\big) && \text{by going across } a \dashv i \\
&\cong \sum_{\chi : ay(c) \to \Delta_{\mathcal{M}}} \mathsf{Sh}(\mathbb{C}, J)/\Delta_{\mathcal{M}}\big(\chi, \tau\big)
\end{aligned}
$$

where $\tau : \widetilde{A} \to \Delta_{\mathcal{M}}$ is $\Pi_{\top}(A)$, and the last isomorphism holds because for every $f : ay(c) \to \widetilde{A}$, we can choose $\chi = \tau \circ f$ to make the triangle commute:



Continuing the chain of isomorphisms:

$$
\begin{aligned}
L_{\mathcal{M}}(A)(c) &\cong \sum_{\chi : ay(c) \to \Delta_{\mathcal{M}}} \mathsf{Sh}(\mathbb{C}, J)/\Delta_{\mathcal{M}}\big(\chi, \tau\big) \\
&\cong \sum_{\chi : ay(c) \to \Delta_{\mathcal{M}}} \mathsf{Sh}\big(\top^*(\chi), A\big) && \text{by going across } \top^* \dashv \Pi_{\top} \\
&\cong \sum_{(m : d \rightarrowtail c) \in \Delta_{\mathcal{M}}(c)} \mathsf{Sh}\big(ay(d), A\big) && \text{by Lemma 5.1.4} \\
&\cong \sum_{(m : d \rightarrowtail c) \in \Delta_{\mathcal{M}}(c)} \mathsf{PSh}\big(y(d), A\big) && \text{by going across } a \dashv i \\
&\cong \sum_{(m : d \rightarrowtail c) \in \mathsf{Sub}_{\mathcal{M}}(c)} A(d) && \text{by Yoneda.}
\end{aligned}
$$

We have used the result from Lemma 5.1.4 that says that the pullback of $\top$ along χ is a representable morphism $ay(m) : ay(d) \to ay(c)$.

□

Remark 5.1.9. For any map $f : c' \to c$ in the site, the action of

$$L_{\mathcal{M}}(A)(f : c' \to c) : \left(\sum_{(m:d \rightarrowtail c) \in \mathsf{Sub}_{\mathcal{M}}(c)} A(d) \right) \to \left(\sum_{(m':d' \rightarrowtail c') \in \mathsf{Sub}_{\mathcal{M}}(c')} A(d') \right)$$

can be described as follows:

$$(a)_{m:d \rightarrowtail c} \mapsto \big(A(f')(a)\big)_{m':d' \rightarrowtail c'}$$

where the following diagram is a pullback

$$\begin{array}{ccc} d' & \overset{m'}{\rightarrowtail} & c' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ d & \underset{m}{\rightarrowtail} & c \end{array}$$

For any natural transformation $\alpha : A \to B$ between sheaves, the action of:

$$\big(L_{\mathcal{M}}(\alpha : A \to B)\big)_c : \left(\sum_{(m:d \rightarrowtail c) \in \mathsf{Sub}_{\mathcal{M}}(c)} A(d) \right) \to \left(\sum_{(m:d \rightarrowtail c) \in \mathsf{Sub}_{\mathcal{M}}(c)} B(d) \right)$$

is

$$(a)_{m:d \rightarrowtail c} \mapsto \big(\alpha_d(a)\big)_{m:d \rightarrowtail c}.$$

5.2 Consequences of concreteness

In this section we make further assumptions about the class of pre-admissible monos $\mathcal{M}$ and the dominance $\Delta_{\mathcal{M}}$ that we obtain from it via Theorem 5.1.6. Most importantly, we assume that $\Delta_{\mathcal{M}}$ is a *concrete* sheaf. Recall from Section 2.1.1 that a concrete sheaf X on a concrete site can be regarded as a set $|X|$ together with a set of functions into $|X|$, and a map $\alpha : Y \to X$ into a concrete sheaf is determined by the function $\alpha_\star : |Y| \to |X|$.

Assumption 5.2.1. We make the following concreteness assumptions:

- $(\mathbb{C}, J)$ is a concrete site (Definition 2.1.6),

- with an initial object covered by the empty set: $\emptyset \in J(0)$.
- $\mathcal{M}$ is a class of pre-admissible monos in $(\mathbb{C}, J)$,
- such that for all $c \in \mathcal{M}$, $0 \to c$ is in $\mathcal{M}$.
- $\Delta_{\mathcal{M}}$ is a *concrete* sheaf.

These are reasonable assumptions because all the models we consider in Section 7.2 and Chapter 8, as well as the vSet example (Sections 4.4 and 5.3), satisfy them.

Using Assumption 5.2.1, we now prove some useful facts about the dominance $\Delta_{\mathcal{M}}$ (Lemma 5.2.2) and the monad $L_{\mathcal{M}}$ (Proposition 5.2.5 and Proposition 5.2.6), and give an explicit description of the natural numbers object in $\mathsf{Sh}(\mathbb{C}, J)$ (Remark 5.2.7). These fact will be used later in Chapters 6 and 7.

Lemma 5.2.2. *Under the conditions in Assumption 5.2.1,* $\Delta_{\mathcal{M}}(\star)$ *has exactly two elements:*

$$\Delta_{\mathcal{M}}(\star) = \{[\mathrm{id}_\star], [0 \to \star]\}.$$

Proof. Because $\mathcal{M}$ is a stable system of monos, $[\mathrm{id}_\star]$ must be part of $\Delta_{\mathcal{M}}(\star)$; $[0 \to \star]$ is part of $\Delta_{\mathcal{M}}(\star)$ by assumption.

Now we show that $\Delta_{\mathcal{M}}(\star)$ has at most two elements. Consider a subobject $d \rightarrowtail \star \in \mathcal{M}$. If d has a point, then:

$$\star \to d \rightarrowtail \star = \mathrm{id}_\star$$

so $d \rightarrowtail \star$ must be an isomorphism.

If d has no points, then because $\Delta_{\mathcal{M}}$ is concrete we have:

$$\Delta_{\mathcal{M}}(d) \subseteq \big([\emptyset \to \Delta_{\mathcal{M}}(\star)] \cong 1\big).$$

We require that both $[\mathrm{id}_d]$ and $[0 \to d]$ are in $\Delta_{\mathcal{M}}(d)$, so it must be the case that $d \cong 0$. Therefore $\star$ has exactly two subobjects, $\star \to \star$ and $0 \to \star$. □

The next lemma shows that $\mathcal{M}$-subobjects are determined by their points:

Lemma 5.2.3. *Under the concreteness conditions in Assumption 5.2.1:*

1. *Each $\mathcal{M}$-subobject $(m : d \rightarrowtail c)$ of each $c \in \mathbb{C}$ is determined by the subset $|m| \subseteq |c|$ of points of c that factorize through it.*

2. *The order relation $m \leq m'$ between $\mathcal{M}$-subobjects is reflected by the inclusion relation $|m| \subseteq |m'|$. In other words, to show $m \leq m'$ it is enough to show that the points of m factor through m'.*

Proof. We prove each of the statements in turn:

1. Consider two $\mathcal{M}$-subobjects $m : d \rightarrowtail c$ and $m' : d' \rightarrowtail c$ such that a point $p : \star \to c$ factors through m if and only if it factors through m'. We need to show that $m = m'$.

   Because $\Delta_{\mathcal{M}}$ is concrete, the map:

   $$\alpha_c : \Delta_{\mathcal{M}}(c) \to \big[\mathbb{C}(\star, c) \to \Delta_{\mathcal{M}}(\star)\big]$$

   mapping each m to $\lambda(p : \star \to c).\, p^*(m)$ is injective. So it is enough to show that for all p:

   $$p^*(m) = p^*(m').$$

   From Lemma 5.2.2, we see that $p^*(m) : d'' \to \star$ can either be $\mathrm{id}_\star$ or $0 \to \star$. The first case is equivalent to p factoring through m, and the second to p not factoring. A similar reasoning applies to $p^*(m')$. Thus, by assumption, $p^*(m) = p^*(m')$.

2. Consider $\mathcal{M}$-subobjects $m : d \rightarrowtail c$ and $m' : d' \rightarrowtail c$ such that the points of m factor through m'. We need to show that m factors through m', which is equivalent to the pullback $m^*(m')$ being id_d.

   $$\begin{array}{ccccc} d'' & \longrightarrow & \cdot & \rightarrowtail & d' \\ \downarrow & & \downarrow {\scriptstyle m^*(m')} & & \downarrow {\scriptstyle m'} \\ \star & \xrightarrow{p} & d & \overset{}{\underset{m}{\rightarrowtail}} & c \end{array}$$

   Consider a point $p : \star \to d$ of m (a point of m because $m \circ p$ factors through m). By assumption $m \circ p$ must factor through m'. So taking the pullback of $m^*(m')$ along p we must get $d'' \to \star = \mathrm{id}_\star$.

This means that $m^*(m') \in \Delta_{\mathcal{M}}(d)$ must be sent by α_d to the function $\lambda(p : \star \to d).\, \mathrm{id}_\star$. But id_d is sent to the same function, and because $\Delta_{\mathcal{M}}$ is concrete, α is injective, so indeed:

$$m^*(m') = \mathrm{id}_d.$$

□

Because we assume that the initial object in $(\mathbb{C}, J)$ is covered by the empty set, then for a sheaf A:

$$A(0) \cong 1.$$

Remark 5.2.4. We can then calculate the underlying set of a sheaf $L_{\mathcal{M}}(A)$ using the formula from Proposition 5.1.8:

$$L_{\mathcal{M}}(A)(\star) \cong \sum_{(d \rightarrowtail \star) \in \mathsf{Sub}_{\mathcal{M}}(\star)} A(d) \cong A(\star) + A(0) \cong A(\star) + 1.$$

Proposition 5.2.5. *Under Assumption 5.2.1, the lifting monad $L_{\mathcal{M}}$ induced by the dominance $\Delta_{\mathcal{M}}$ preserves concreteness.*

Proof. Let A be a concrete sheaf in $\mathsf{Sh}(\mathbb{C}, J)$. We have shown in Proposition 5.1.8 that:

$$\begin{aligned} (L_{\mathcal{M}}A)(\star) &= A(\star) + 1 \\ (L_{\mathcal{M}}A)(c) &= \sum_{(m : d \rightarrowtail c) \in \Delta_{\mathcal{M}}(d)} A(d). \end{aligned}$$

We need to show that the map:

$$\alpha_c : (L_{\mathcal{M}}A)(c) \to \big[\mathbb{C}(\star, c) \to A(\star) + 1\big]$$

sending a_m (where $m : d \rightarrowtail c$) to $\lambda p.\, (L_{\mathcal{M}}A)(p)(a_m)$ is injective.

For any $a_m \in (L_{\mathcal{M}}A)(c)$ and any $p : \star \to c$, we see from Remark 5.1.9

that the action of $(L_{\mathcal{M}}A)(p)(a_m)$ is:

$$(L_{\mathcal{M}}A)(p)(a_m) = \begin{cases} A(p')(a_m) & \text{if } p \text{ factors through } m \text{ via } p' \\ 1 & \text{otherwise.} \end{cases}$$

Consider a_m and $a_{m'}$ in $(L_{\mathcal{M}}A)(c)$ such that $\alpha_c(a_m) = \alpha_c(a_{m'})$. From the formula above it follows that exactly the same points $p : \star \to c$ must factor through both m and m'. Therefore from Lemma 5.2.3 we have that $m = m'$.

This means that for all $p' : \star \to d$:

$$A(p')(a_m) = A(p')(a_{m'})$$

so because we assumed A is concrete we have $a_m = a_{m'}$. □

Proposition 5.2.6. *Let $(\mathbb{C}, J)$ be a concrete site with an initial object covered by the empty set, and a class of pre-admissible monos $\mathcal{M}$, such that every $0 \to c$ is in $\mathcal{M}$. (These are the conditions in Assumption 5.2.1 without the requirement that $\Delta_{\mathcal{M}}$ is concrete.) Then the dominance $\Delta_{\mathcal{M}}$ classifies $0 \to 1 \in \mathsf{Sh}(\mathbb{C}, J)$. Hence the monad $L_{\mathcal{M}}$ is pointed.*

Proof. Because the initial object in $\mathbb{C}$ is covered by the empty set, for any sheaf A, including the initial one, $A(0) \cong 1$. For other objects $c \in \mathbb{C}$, $0(c) = \emptyset$.

We can construct a classifying map $\chi : 1 \to \Delta_{\mathcal{M}}$ for $0 \to 1$ as follows:

$$\chi_c(*) = [0 \to c] \in \Delta_{\mathcal{M}}(c).$$

Naturality follows because $\Delta_{\mathcal{M}}$ acts by pullback.

The pullback of χ with $\top : 1 \to \Delta_{\mathcal{M}}$ can be calculated componentwise in Set. Since $\top_c(*) = [\mathrm{id}_c]$, the pullback is empty except when $c = 0$, in which case it has one element, so the pullback is indeed the initial object 0 from $\mathsf{Sh}(\mathbb{C}, J)$.

We can now deduce from Lemma 2.4.12 that the monad $L_{\mathcal{M}}$ is pointed. □

Remark 5.2.7. Under Assumption 5.2.1, and assuming the coverage J satisfies axioms (M) and (L) (introduced in Remark 2.1.2), we can calculate the value of $\mathsf{Nat} \cong \coprod_{\mathbb{N}} 1$ in $\mathsf{Sh}(\mathbb{C}, J)$ explicitly, for any object c in the site $\mathbb{C}$. The assumption on the coverage is reasonable because all examples in Section 7.2 and Chapter 8 satisfy it.

The coproduct $\coprod_{\mathbb{N}} 1$ in sheaves is obtained by sheafifying the coproduct $\coprod_{\mathbb{N}}^{\mathsf{PSh}} 1$ in presheaves. In presheaves

$$\left(\coprod_{\mathbb{N}}^{\mathsf{PSh}} 1\right)(c) \cong \mathbb{N}$$

including at 0, because it is calculated pointwise.

We assumed the coverage J is such that 0 is covered by the empty set. Therefore, sheafifying with respect to J can be split into two stages:

- sheafifying with respect to a coverage J_0, where $J_0(0) = \{\emptyset\}$, and for $c \neq 0$, $J_0(c) = \big\{\{\mathrm{id}_c\}\big\}$;
- followed by sheafifying with respect to J.

Denote by $\left(\coprod_{\mathbb{N}}^{\mathsf{PSh}} 1\right)_{J_0}$ the result of sheafifying with respect to J_0. Then:

$$\left(\coprod_{\mathbb{N}}^{\mathsf{PSh}} 1\right)_{J_0}(c) \cong \mathbb{N}$$

for $c \neq 0$, as was the case before sheafifying. At 0 however:

$$\left(\coprod_{\mathbb{N}}^{\mathsf{PSh}} 1\right)_{J_0}(0) \cong 1$$

because the empty cover of 0 gives only one matching family, so the sheaf must contain exactly one amalgamation.

Notice that $\left(\coprod_{\mathbb{N}}^{\mathsf{PSh}} 1\right)_{J_0}$ is concrete, because $\left(\coprod_{\mathbb{N}}^{\mathsf{PSh}} 1\right)_{J_0}(c)$ contains exactly the constant functions $|c| \to \mathbb{N}$. Therefore, it is a subpresheaf of the following concrete sheaf:

$$\left(\coprod_{\mathbb{N}}^{\mathsf{PSh}} 1\right)_{J_0} \hookrightarrow \left[|{-}| \to \mathbb{N}\right].$$

The presheaf $\left(\coprod_{\mathbb{N}}^{\mathsf{PSh}} 1\right)_{J_0}$ is also separated for J because all its elements are constant functions.

Therefore, to sheafify $\left(\coprod_{\mathbb{N}}^{\mathsf{PSh}} 1\right)_{J_0}$ with respect to J it suffices to add the missing amalgamations from $\left[|{-}| \to \mathbb{N}\right]$, obtaining:

$$\mathsf{Nat}(c) = \{f : |c| \to \mathbb{N} \mid f \text{ locally constant on a cover of } c\}.$$

It is enough to add amalgamations in one step because the coverage J satisfies axiom (L).

5.3 Example: presheaves on the vertical natural numbers – revisited

The category $\mathsf{vSet} = \mathsf{PSh}(\mathbb{V})$ defined in Section 4.4, which is our running example of normal model (Definition 4.3.1), has another description as a category of sheaves on a concrete site with a class of pre-admissible monos, more precisely, as a category that satisfies Assumption 5.2.1.

Definition 5.3.1. Let $\mathbb{V}_0$ be a three object category: one object is the vertical natural numbers V and the others are a terminal and initial object, such that $\mathbb{V}_0$ is a full subcategory of $\omega\mathsf{CPO}$. This means that there is a map $n : \star \to \mathrm{V}$ for each $n \in \mathbb{N} \cup \{\infty\}$.

Definition 5.3.2. Define a coverage $J_{\mathbb{V}_0}$ on $\mathbb{V}_0$ as follows:

$$J_{\mathbb{V}_0}(\mathrm{V}) = \left\{\{\mathrm{id}_\mathrm{V}\}\right\} \qquad J_{\mathbb{V}_0}(\star) = \left\{\{\mathrm{id}_\star\}\right\} \qquad J_{\mathbb{V}_0}(0) = \left\{\emptyset, \{\mathrm{id}_0\}\right\}.$$

Unwinding the definitions, we can see that $(J_{\mathbb{V}_0}, \mathbb{V}_0)$ forms a concrete site that satisfies axioms (M) and (L) (Remark 2.1.2).

Proposition 5.3.3. *There is an equivalence* $\overline{(-)} : \mathsf{PSh}(\mathbb{V}) \to \mathsf{Sh}(\mathbb{V}_0, J_{\mathbb{V}_0})$, *given by:*

$$\overline{X}(\mathrm{V}) = X(\mathrm{V}) \qquad \overline{X}(0) \cong 1 \qquad \overline{X}(\star) = \mathsf{PSh}(\mathbb{V})(1, X),$$

where $\overline{X}$ *has the obvious action on maps, and* $\overline{(\alpha : X \to Y)}$ *is defined as:*

$$\begin{aligned} \overline{\alpha}_{\mathrm{V}} &= \alpha_{\mathrm{V}} \\ \overline{\alpha}_0 &= \mathrm{id}_1 \\ \overline{\alpha}_\star(\beta : 1 \to X) &= \alpha \circ \beta. \end{aligned}$$

Proof. We will show that $\overline{(-)}$ is full, faithful and essentially surjective. Consider $\overline{\alpha} = \overline{\beta}$. Then $\alpha_{\mathrm{V}} = \beta_{\mathrm{V}}$, so $\alpha = \beta$, and thus $\overline{(-)}$ is faithful.

For fullness, consider $\gamma : \overline{X} \to \overline{Y}$, and choose $\alpha : X \to Y$ such that $\alpha_{\mathrm{V}} = \gamma_{\mathrm{V}}$. Then α is natural and $\overline{\alpha} = \gamma$.

To show $\overline{(-)}$ is essentially surjective, consider $A \in \mathsf{Sh}(\mathbb{V}_0, J_{\mathbb{V}_0})$, and define $X \in \mathsf{PSh}(\mathbb{V})$ as:

$$X(\mathrm{V}) = A(\mathrm{V}) \qquad X(e : \mathrm{V} \to \mathrm{V}) = A(e).$$

We will show $\overline{X} \cong A$. The only interesting part is showing $\overline{X}(\star) \cong A(\star)$.

Notice that:

$$\overline{X}(\star) = \mathsf{PSh}(\mathbb{V})(1, X) \cong \{s \in X(\mathrm{V}) \mid \forall e : \mathrm{V} \to \mathrm{V}.\ X(e)(s) = s\}.$$

For any $n \in \mathbb{N} \cup \{\infty\}$:

$$A(\star) \xrightarrow{A(!)} A(\mathrm{V}) \xrightarrow{A(n)} A(\star) = \mathrm{id},$$

so $A(!)$ is mono. Thus $A(\star) \cong \mathsf{im}\big(A(!)\big)$.

For any $e : \mathrm{V} \to \mathrm{V}$:

$$A(\star) \xrightarrow{A(!)} A(\mathrm{V}) \xrightarrow{A(e)} A(\mathrm{V}) = A(\star) \xrightarrow{A(!)} A(\mathrm{V}),$$

so $\mathsf{im}\big(A(!)\big) \subseteq \overline{X}(\star)$. For any $x \in \overline{X}(\star)$, we also have $A(n\circ! : \mathrm{V} \to \mathrm{V})(x) = x$, so x must be in the image of $A(!)$, so we are done. $\square$

Definition 5.3.4. Consider the following class of pre-admissible monos in $\mathbb{V}_0$:

$$\mathcal{M}_{\mathbb{V}_0}(\mathrm{V}) = \{(\lambda x.x + n) \in \mathbb{V}_0(\mathrm{V}, \mathrm{V}) \mid n \in \mathbb{N}\} \cup \{! : 0 \to \mathrm{V}\}$$
$$\mathcal{M}_{\mathbb{V}_0}(0) = \{! : 0 \to 0\} \qquad \mathcal{M}_{\mathbb{V}_0}(\star) = \{\mathrm{id}_\star : 1 \to 1, ! : 0 \to 1\}.$$

It is easy to see that $\mathcal{M}_{\mathbb{V}_0}$ is a stable system of monos. Let $\Delta_{\mathbb{V}_0}$ be the presheaf associated to $\mathcal{M}_{\mathbb{V}_0}$, as explained in Definition 5.1.1. Then $\Delta_{\mathbb{V}_0}$ is a $J_{\mathbb{V}_0}$-sheaf because the sheaf condition only forces $\Delta_{\mathbb{V}_0}(0) \cong 1$. Thus, $\mathcal{M}_{\mathbb{V}_0}$ is indeed a class of pre-admissible monos and $\Delta_{\mathbb{V}_0}$ is a dominance. Denote by $L_{\mathbb{V}_0}$ the lifting monad given by $\Delta_{\mathbb{V}_0}$.

Proposition 5.3.5. *The dominance in* $\mathsf{PSh}(\mathbb{V})$ *corresponds to the one in* $\mathsf{Sh}(\mathbb{V}_0, J_{\mathbb{V}_0})$*:*

$$\overline{\Delta_{\mathbb{V}}} \cong \Delta_{\mathbb{V}_0}.$$

Proof. We already know (see Remark 4.4.6) that $\Delta_{\mathbb{V}}(\mathrm{V})$ contains the sieves generated by $\lambda x.\, x + n$, for each $n \in \mathbb{N}$, and the empty sieve, so this gives a direct correspondence to $\mathcal{M}_{\mathbb{V}_0}(\mathrm{V})$. It remains to show that $\overline{\Delta_{\mathbb{V}}}(\star) \cong 2$. But $\overline{\Delta_{\mathbb{V}}}(\star)$ contains those sieves on V invariant under the action of any $\Delta_{\mathbb{V}}(e)$, which means the total and empty sieve. $\square$

Chapter 6

Building normal models: conditions for completeness

In this chapter, as in the previous one, we continue to work towards a recipe for building normal models (Definition 4.3.1). So far, we have shown how to build a dominance in $\mathsf{Sh}(\mathbb{C}, J)$ from a class $\mathcal{M}$ of pre-admissible monos in $(\mathbb{C}, J)$ (Theorem 5.1.6).

Recall that to satisfy the definition of normal model, we also need the lifted natural numbers object to be complete. This is to allow us to interpret recursion in the model using Corollary 3.2.5. However, completeness of the lifted naturals is not true in general for any site with a class of pre-admissible monos. Thus, we make further assumptions, which can be roughly summarized as:

- We impose some concreteness conditions, which include Assumption 5.2.1.
- We strengthen the notion of class of pre-admissible monos $\mathcal{M}$ to a *class of admissible monos* (Definition 6.2.1).
- We combine a triple $(\mathbb{C}, J, \mathcal{M})$ with the vertical natural numbers site $(\mathbb{V}_0, J_{\mathbb{V}_0}, \mathcal{M}_{\mathbb{V}_0})$ from Section 5.3.

The chapter can be outlined as follows: in Section 6.1, we prove a useful result that characterizes completeness in sheaves in terms of completeness in the ambient presheaf category. In Section 6.2, we introduce the notion of admissible monos (Definition 6.2.1) and define what it means to combine sites (Definition 6.2.3). Then we state the main theorem of the chapter, Theorem 6.2.5, about the lifted naturals being complete. Section 6.3 is dedicated to proving this theorem, and Section 6.3.6, to discussing more precisely where each of the assumptions are used. The results in this chapter appeared at LICS 2022 [MMS22, Section 7.1].

The main contributions of the chapter are:

- introducing the notion of class of admissible monos
- and proving the theorem about completeness of the lifted natural numbers (Theorem 6.2.5), which is spread over several sections.

Our running example of normal model, the category vSet, of presheaves on the vertical natural numbers, appears again in this chapter. In Example 6.2.2, we show that the dominance we defined in vSet arises from a class of admissible monos. In Proposition 6.2.6, we show that the object of lifted natural numbers in vSet is complete as a consequence of Theorem 6.2.5.

In Theorem 7.1.1, we will summarize the recipe for building normal models developed in this chapter and the previous one. We will then show that the models we build with our recipe are adequate (Theorem 7.1.3) for interpreting call-by-value PCF ($\mathsf{PCF_v}$).

6.1 Completeness in sheaves and presheaves

Recall that by $L_{\mathcal{M}}$ we denote the lifting monad obtained from a class of monos $\mathcal{M}$ (Definition 5.1.7). To simplify the proof that $L_{\mathcal{M}}(\coprod_{\mathbb{N}} 1)$ is an $L_{\mathcal{M}}$-complete object, we prove that completeness in a sheaf category is equivalent to completeness in the corresponding presheaf category (Proposition 6.1.1).

This makes sense because we could instantiate the definition of $L_{\mathcal{M}}$-completeness (Definition 3.2.2) for both the presheaf category $\mathsf{PSh}(\mathbb{C})$ and the sheaf category $\mathsf{Sh}(\mathbb{C}, J)$. The instantiation for presheaves is possible because the dominance $\Delta_{\mathcal{M}} \rightarrowtail \Omega_J \rightarrowtail \Omega$ is also a dominance in $\mathsf{PSh}(\mathbb{C})$. Here Ω_J is the subobject classifier in $\mathsf{Sh}(\mathbb{C}, J)$, and Ω is the subobject classifier in $\mathsf{PSh}(\mathbb{C})$.

As noted in Remark 2.4.10, the lifting monad $L_{\mathcal{M}}$ has the general description in a topos $\mathcal{E}$:

$$L_{\mathcal{M}} \coloneqq \Sigma_{\Delta_{\mathcal{M}}} \circ \Pi_{\top},$$

where $\Pi_{\top} : \mathcal{E} \to \mathcal{E}/\Delta_{\mathcal{M}}$ is the right adjoint to the pullback functor $\top^* : \mathcal{E}/\Delta_{\mathcal{M}} \to \mathcal{E}/1 \cong \mathcal{E}$ and $\Sigma_{\Delta_{\mathcal{M}}} : \mathcal{E}/\Delta_{\mathcal{M}} \to \mathcal{E}$ maps $f : A \to \Delta_{\mathcal{M}}$ to A. The topos $\mathcal{E}$ can be instantiated with both $\mathsf{Sh}(\mathbb{C}, J)$ and $\mathsf{PSh}(\mathbb{C})$, thus giving two lifting monads.

In the presheaves case, the functor $\Pi_{\top} : \mathsf{PSh}(\mathbb{C}) \to \mathsf{PSh}(\mathbb{C})/\Delta_{\mathcal{M}}$ preserves sheaves, so we can regard the monad $L_{\mathcal{M}}$ on $\mathsf{PSh}(\mathbb{C})$ as an extension of the one on $\mathsf{Sh}(\mathbb{C}, J)$. (We could prove that $\Pi_{\top}$ preserves sheaves by viewing the sheaf condition as a family of orthogonality conditions with respect to covering sieves, seen as subobjects of representables, and then using the adjunction $\top^* \dashv \Pi_{\top}$.) From now on we will use $L_{\mathcal{M}}$ to refer to both of the monad on sheaves and presheaves depending on the context.

The constructions of vertical natural numbers ω and $\overline{\omega}$ from Section 3.1 have versions both in presheaves and sheaves. Let ω_P and ω_S be the colimit calculated in presheaves and sheaves respectively. Notice that they are both colimits of the same diagram because each $L^n_{\mathcal{M}}1$ is already a sheaf. We therefore have:

$$\omega_S \cong a(\omega_P).$$

The limit $\overline{\omega}$ is the same in both presheaves and sheaves. We also obtain the following two inclusion maps in presheaves and sheaves respectively:

$$i_P : \omega_P \to \overline{\omega} \qquad\qquad i_S : \omega_S \to \overline{\omega}.$$

We can now state the lemma about completeness that we are interested in: completeness in presheaves is equivalent to completeness in sheaves. This lemma will make calculations easier because ω_P has a simpler explicit description than ω_S.

Proposition 6.1.1. *Let $\mathsf{Sh}(\mathbb{C}, J)$ be a sheaf category with a class of pre-admissible monos $\mathcal{M}$. Let X be an object in $\mathsf{Sh}(\mathbb{C}, J)$. Then the following are equivalent:*

1. *For all $A \in \mathsf{PSh}(\mathbb{C})$, and all maps $f : \omega_P \times A \to X$, f has a unique extension along $i_P \times \mathrm{id}_A$:*

$$\begin{array}{ccc} \omega_P \times A & \xrightarrow{f} & X \\ {\scriptstyle i_P \times \mathrm{id}_A}\downarrow & \nearrow_{\exists!} & \\ \overline{\omega} \times A & & \end{array}$$

2. *For all $B \in \mathsf{Sh}(\mathbb{C}, J)$, and all maps $g : \omega_S \times B \to X$, g has a unique extension along $i_S \times \mathrm{id}_B$:*

$$\begin{array}{ccc} \omega_S \times B & \xrightarrow{g} & X \\ {\scriptstyle i_S \times \mathrm{id}_B}\downarrow & \nearrow_{\exists!} & \\ \overline{\omega} \times B & & \end{array}$$

3. *For all c in $\mathbb{C}$, and all maps $f : \omega_P \times y(c) \to X$, f has a unique extension along $i_P \times \mathrm{id}_{y(c)}$.*

Proof. First we show we can identify $a(i_P) : i(\omega_P) \to \overline{\omega}$ with $i_S : \omega_S \cong a(\omega_P) \to \overline{\omega}$.

From Lemma 3.1.6 we see that both i_P and i_S are constructed from a family of maps (between sheaves):

$$(L^m_{\mathcal{M}} 1 \to L^n_{\mathcal{M}} 1)_{m,n \in \mathbb{N}}$$

such that for each m they form a cocone with apex $L^m_{\mathcal{M}} 1$ for the diagram:

$$1 \xrightarrow{\perp_1} L_{\mathcal{M}} 1 \xrightarrow{L_{\mathcal{M}}(\perp_1)} L^2_{\mathcal{M}} 1 \xrightarrow{L^2_{\mathcal{M}}(\perp_1)} \dots.$$

Since $\omega_S \cong a(\omega_P)$, we get comparison maps $(f_m : \omega_P \to L^m_{\mathcal{M}} 1)_{m \in \mathbb{N}}$ and $(g_m : \omega_S \to L^m_{\mathcal{M}} 1)_{m \in \mathbb{N}}$ such that we can identify g_m with $a(f_m)$.

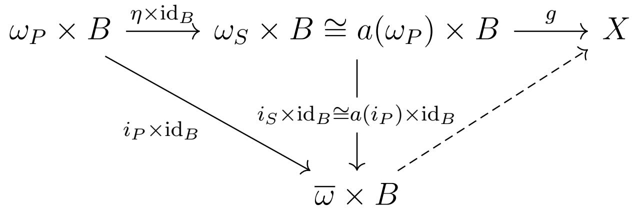
Both $(f_m : \omega_P \to L^m_{\mathcal{M}} 1)_{m \in \mathbb{N}}$ and $(g_m : \omega_S \to L^m_{\mathcal{M}} 1)_{m \in \mathbb{N}}$ give cones for the diagram:

$$1 \xleftarrow{!} L_{\mathcal{M}} 1 \xleftarrow{L_{\mathcal{M}}(!)} L^2_{\mathcal{M}} 1 \xleftarrow{L^2_{\mathcal{M}}(!)} \dots$$

for which $\overline{\omega}$ is the limit, and comparison maps $i_P : \omega_P \to \overline{\omega}$ and $i_S : \omega_S \to \overline{\omega}$.

Because we know we can identify the cone $(a(f_m) : a(\omega_P) \to L^m_{\mathcal{M}}1)_{m\in\mathbb{N}}$ with the cone $(g_m : \omega_S \to L^m_{\mathcal{M}}1)_{m\in\mathbb{N}}$, then we are able to identify the comparison maps $a(i_P) : a(\omega_P) \to \overline{\omega}$ and $i_S : \omega_S \to \overline{\omega}$ as well.

Item 1 implies Item 2. Consider a sheaf B and a map $g : \omega_S \times B \to X$. We can transform the extension problem into one for presheaves by precomposing with the unit η of the adjunction $a \dashv i$:



So we get a unique extension of $g \circ (\eta \times \mathrm{id}_B)$. By naturality of η with respect to a, we get that:

$$a(i_P) \circ \eta = i_P$$

so the left triangle commutes.

By using η to move across the isomorphism

$$\mathsf{PSh}(\omega_P \times B, X) \cong \mathsf{Sh}(a(\omega_P) \times B, X),$$

we deduce that the right triangle commutes as well, so we have an extension of g. This extension is unique because the extension of $g \circ (\eta \times \mathrm{id}_B)$ is too.

Item 2 implies Item 1. We start with an extension problem for presheaves $f : \omega_P \times A \to X$. We can transform it into an extension problem for sheaves by sheafifying f, as shown in Figure 6.1.

So we obtain a unique extension of $a(f)$. The left square commutes by naturality of η with respect to a. Therefore, we obtain an extension of f, which we can show is unique. If we have another extension of f, then by sheafifying the triangle we get the unique extension of $a(f)$, therefore this second extension of f must be equal to the first.

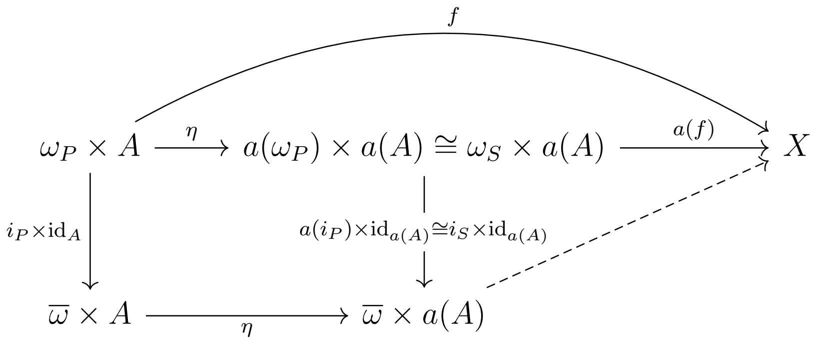


Figure 6.1: Diagram for proof of Proposition 6.1.1.

Item 1 equivalent to Item 3. This is true because in presheaves every object A is a colimit of representables.

□

6.2 Admissible monos

As we discussed at the beginning of the chapter, we define a strengthening of the notion of class of pre-admissible monos. Each of the conditions in the definition is explained in detail in Remark 6.3.12, but notice that they include Assumption 5.2.1.

Definition 6.2.1. Let $(\mathbb{C}, J)$ be a concrete site (Definition 2.1.6), satisfying axioms (M) and (L) (Remark 2.1.2), with an initial object covered by the empty set, and a class of pre-admissible monos $\mathcal{M}$ (Definition 5.1.2). Then $\mathcal{M}$ is a *class of admissible monos* if the following conditions are satisfied:

1. For every object c in $\mathbb{C}$, the map $0 \to c$ is in $\mathcal{M}$.

2. $\Delta_{\mathcal{M}}$ is *concrete*. We saw in Lemma 5.2.3 that this means $\mathcal{M}$-subobjects $(m : c' \rightarrowtail c)$ are determined by the set of points of c that factorize through them, $|m| \subseteq |c|$, and the order $m \leq m'$ is given by inclusion $|m| \subseteq |m'|$.

3. For every increasing chain of monos on c, $(m_n : c_n \rightarrowtail c)_{n\in\mathbb{N}} \in \mathcal{M}$, we assume the subobject $m_\infty : c_\infty \rightarrowtail c$ determined by the set of points

$\bigcup_{n\in\mathbb{N}}|m_n|$ exists and is in $\mathcal{M}$.

4. Given an increasing chain of monos $(m_n : c_n \rightarrowtail c)_{n\in\mathbb{N}} \in \mathcal{M}$, the closure under precomposition (with any morphism) of the set $\{m_n : c_n \rightarrowtail c_\infty\}_{n\in\mathbb{N}}$ contains a covering family of c_∞.

Example 6.2.2. The class of pre-admissible monos $\mathcal{M}_{\mathbb{V}_0}$ (Definition 5.3.4), in the site of $\mathsf{Sh}(\mathbb{V}_0, J_{\mathbb{V}_0})$, is admissible. We omit checking that $\Delta_{\mathbb{V}_0}$ is concrete. Every increasing chain of monos in $\mathcal{M}_{\mathbb{V}_0}$ is eventually constant, so we already know its least upper bound is in $\mathcal{M}_{\mathbb{V}_0}$. Moreover, the cover of the least upper bound needed for Item 4 is always $\{\mathrm{id}\}$.

Recall from Section 5.3 that the $\mathsf{Sh}(\mathbb{V}_0, J_{\mathbb{V}_0})$ and vSet model of $\mathsf{PCF_v}$ are actually equivalent. Thus we can see that in our running example of normal model, the presheaves on the vertical natural numbers vSet, the dominance $\Delta_{\mathbb{V}}$ is actually generated by a class of admissible monos.

Definition 6.2.3 (Combining concrete sites)**.** Let $(\mathbb{C}_1, J_1, \mathcal{M}_1)$ and $(\mathbb{C}_2, J_2, \mathcal{M}_2)$ be concrete sites, satisfying axioms (M) and (L), with initial objects covered by the empty set, and with classes of admissible monos $\mathcal{M}_1$, $\mathcal{M}_2$.

Let $\mathbb{C}_1 + \mathbb{C}_2$ be the category obtained from $\mathbb{C}_1$ and $\mathbb{C}_2$ as follows:

- We identify their respective terminal objects and respective initial objects; we keep all other objects form $\mathbb{C}_1$ and $\mathbb{C}_2$ as distinct objects in $\mathbb{C}_1 + \mathbb{C}_2$.

- The maps in $\mathbb{C}_1 + \mathbb{C}_2$ contain all the maps from $\mathbb{C}_1$ and $\mathbb{C}_2$, plus any *new* maps of the form $c \to \star \to c'$ that have appeared. These are maps going from an object of $\mathbb{C}_1$ to an object of $\mathbb{C}_2$ or vice-versa; we think of them as constant maps.

From this description we define the obvious inclusion functors $F_1 : \mathbb{C}_1 \to \mathbb{C}_1 + \mathbb{C}_2$ and $F_2 : \mathbb{C}_2 \to \mathbb{C}_1 + \mathbb{C}_2$.

Define the coverage $J_1 \cup J_2$ on $\mathbb{C}_1 + \mathbb{C}_2$ to contain the images of all the covers from $\mathbb{C}_1$ and $\mathbb{C}_2$ under the functors F_1 and F_2 respectively. Similarly, define the class of monos $\mathcal{M}_1 \cup \mathcal{M}_2$ to contain the images under F_1 and F_2

of all the monos in $\mathcal{M}_1$ and $\mathcal{M}_2$. (Hence our use of the union symbol is intuitive.)

Define the combination of the sites $(\mathbb{C}_1, J_1, \mathcal{M}_1)$ and $(\mathbb{C}_2, J_2, \mathcal{M}_2)$ to be $(\mathbb{C}_1 + \mathbb{C}_2, J_1 \cup J_2, \mathcal{M}_1 \cup \mathcal{M}_2)$.

Proposition 6.2.4. *The combined site* $(\mathbb{C}_1+\mathbb{C}_2, J_1 \cup J_2, \mathcal{M}_1 \cup \mathcal{M}_2)$ *is also a concrete site with an initial object covered by the empty set,* $J_1 \cup J_2$ *satisfies axioms (M) and (L), and* $\mathcal{M}_1 \cup \mathcal{M}_2$ *is a class of admissible monos.*

Proof. We need to check that $(\mathbb{C}_1 + \mathbb{C}_2, J_1 \cup J_2)$ is a concrete site. When the terminal objects of $\mathbb{C}_1$ and $\mathbb{C}_2$ are identified in $\mathbb{C}_1 + \mathbb{C}_2$, the resulting object is still terminal. From this we can see that $\mathbb{C}_1 + \mathbb{C}_2$ is still a well-pointed category.

The (M) and (L) axioms still hold because we have not introduced any new objects or covering families. The initial object in $\mathbb{C}_1 + \mathbb{C}_2$ is the one obtained by identifying the initial objects of $\mathbb{C}_1$ and $\mathbb{C}_2$, and it is still covered by the empty set, like in J_1 and J_2. Moreover, every covering family still contains all the points of the object it covers.

For axiom (C), we need to check it holds for each constant map $f : d \to c$ with $d \in \mathbb{C}_2$ and $c \in \mathbb{C}_1$. This works because by axiom (M), d is covered by the identity, and any cover of c must contain all points of c by concreteness, therefore f factors through such a cover.

We need to show that $\mathcal{M}_1 \cup \mathcal{M}_2$ is a stable system of monos. Consider a constant map $f : d \to c$ with $d \in \mathbb{C}_2$ and $c \in \mathbb{C}_1$, and an $\mathcal{M}$-subobject $c' \rightarrowtail c$. If $c = 0$, then from Lemma 5.2.2 we know $c' \rightarrowtail c$ must be $[\mathrm{id}_0]$ so the pullback along f is $0 \to d$ which is in $\mathcal{M}_2$ by assumption.

The presheaf $\Delta_{\mathcal{M}_1 \cup \mathcal{M}_2}$ is still a sheaf because $\Delta_{\mathcal{M}_1 \cup \mathcal{M}_2}(c) = \Delta_{\mathcal{M}_1}(c)$ if $c \in \mathbb{C}_1$, and similarly for $\mathbb{C}_2$, and the sheaf condition has not changed. Therefore, $\mathcal{M}_1 \cup \mathcal{M}_2$ is a class of pre-admissible monos. Concreteness of $\Delta_{\mathcal{M}_1 \cup \mathcal{M}_2}$ follows from concreteness of $\Delta_{\mathcal{M}_1}$ and $\Delta_{\mathcal{M}_2}$.

The increasing chains of monos in $\mathcal{M}_1 \cup \mathcal{M}_2$, on an object c, are the chains from $\mathcal{M}_1$ and $\mathcal{M}_2$ depending on whether $c \in \mathbb{C}_1$ or $c \in \mathbb{C}_2$, so the last two conditions in the definition of class of admissible monos are still satisfied. Thus $\mathcal{M}_1 \cup \mathcal{M}_2$ is a class of admissible monos. □

We can now state the main theorem of this chapter, which says that using a site like the one we identified in Definition 6.2.1, we can obtain a sheaf category with the right complete objects for modelling $\mathsf{PCF_v}$. The proof of this theorem appears in the next section.

An additional premise of the theorem is that the site contains the triple $(\mathbb{V}_0, J_{\mathbb{V}_0}, \mathcal{M}_{\mathbb{V}_0})$ (see Section 5.3), which is a site that describes the category of presheaves on the vertical natural numbers, vSet. This requirement is discussed in Remark 6.3.13.

Theorem 6.2.5. *Let $(\mathbb{C}, J, \mathcal{M})$ be a concrete site, satisfying axioms (M) and (L), with an initial object covered by the empty set, and a class of admissible monos $\mathcal{M}$. In the sheaf category $\mathsf{Sh}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$ the dominance $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ and the lifted natural numbers $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\coprod_{\mathbb{N}} 1)$ are $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$-complete objects.*

As a consequence of Theorem 6.2.5, we obtain a result about our running example of normal model, the category vSet, first introduced in Section 4.4. This result concludes the argument that vSet is indeed a normal model of $\mathsf{PCF_v}$ (Definition 4.3.1) without type constants:

Proposition 6.2.6. *In vSet (Definition 4.4.1), the dominance $\Delta_{\mathbb{V}}$ and the lifted natural numbers $L_{\mathbb{V}}(\mathsf{Nat}_{\mathbb{V}})$ are $L_{\mathbb{V}}$-complete objects.*

Proof. In Theorem 6.2.5, choose $\mathbb{C}$, J and $\mathcal{M}$ to be empty. Then we obtain that the dominance $\Delta_{\mathbb{V}_0}$ and the lifted natural numbers $L_{\mathbb{V}_0}(\mathsf{Nat}_{\mathbb{V}_0})$ in $\mathsf{Sh}(\mathbb{V}_0, J_{\mathbb{V}_0})$ are complete objects.

Using the equivalences of categories from Proposition 5.3.3, and $\overline{\Delta_{\mathbb{V}}} \cong \Delta_{\mathbb{V}_0}$ from Proposition 5.3.5, it is easy to show that the maps $\top : 1 \to \Delta_{\mathbb{V}}$ and $\top : 1 \to \Delta_{\mathbb{V}_0}$ correspond to each other. Thus $L_{\mathbb{V}}$ and $L_{\mathbb{V}_0}$ also correspond to each other because they are obtained from the respective dominances. Thus $L_{\mathbb{V}}(\mathsf{Nat}_{\mathbb{V}}) \cong L_{\mathbb{V}_0}(\mathsf{Nat}_{\mathbb{V}_0})$, and we can deduce that $\Delta_{\mathbb{V}}$ and $L_{\mathbb{V}}(\mathsf{Nat}_{\mathbb{V}})$ from vSet are complete. $\square$

6.3 Proving the dominance and lifted naturals are complete (Theorem 6.2.5)

According to Proposition 6.1.1, to prove Theorem 6.2.5 it is enough to show that $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ and $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\coprod_{\mathbb{N}} 1)$ are right-orthogonal to the map $(i_P \times \mathrm{id}_{y(c)}) : \omega_P \times y(c) \to \overline{\omega} \times y(c)$ for every $c \in \mathbb{C}$, where ω_P is a colimit calculated in $\mathsf{PSh}(\mathbb{C})$. Therefore we start this section by calculating ω_P explicitly.

We then prove Lemma 6.3.3 which says that the maps $\omega_P \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ are the infinite monotone binary sequences. This result is used to prove that $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ is orthogonal to $i_P : \omega_P \to \overline{\omega}$ (Lemma 6.3.4). Lemma 6.3.4 is used to show that $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ is orthogonal to $(i_P \times \mathrm{id}_{y(c)}) : \omega_P \times y(c) \to \overline{\omega} \times y(c)$ (Lemma 6.3.7).

For the lifted naturals $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\coprod_{\mathbb{N}} 1)$ we follow a similar pattern. We first show $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\coprod_{\mathbb{N}} 1)$ is orthogonal to $i_P : \omega_P \to \overline{\omega}$ (Lemma 6.3.9) using the analogous result for $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$, Lemma 6.3.4. Then we use Lemma 6.3.9, and crucially also Lemma 6.3.7 about $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$, to show that $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\coprod_{\mathbb{N}} 1)$ is orthogonal to $(i_P \times \mathrm{id}_{y(c)}) : \omega_P \times y(c) \to \overline{\omega} \times y(c)$ (Lemma 6.3.11).

6.3.1 Explicit description of ω and $\overline{\omega}$ in presheaves

We defined ω_P to be the colimit of:

$$1 \xrightarrow{\perp_1} L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} 1 \xrightarrow{L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\perp_1)} L^2_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} 1 \xrightarrow{L^2_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\perp_1)} \cdots . \tag{6.1}$$

in presheaves.

The map $\perp_1 : 1 \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} 1$ was defined in Lemma 2.4.12 to be the total map corresponding to the partial map $(0 \rightarrowtail 1, 0 \to 1)$. In Remark 2.4.7, we showed that $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} 1 \cong \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$, so $\perp_1$ must be the classifying map of $0 \rightarrowtail 1$, which we constructed in the proof of Proposition 5.2.6. Therefore $\perp_1 : 1 \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} 1$ is defined as:

$$(\perp_1)_c(*) = [0 \to c] \in \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(c) \cong (L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} 1)(c).$$

Applying the formula for the lifting monad from Proposition 5.1.8 we see that:

$$(L^n_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1)(c) = \sum_{(c_1 \rightarrowtail c)\in \mathsf{Sub}_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(c)} \cdots \sum_{(c_n \rightarrowtail c_{n-1})\in \mathsf{Sub}_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(c_{n-1})} 1(c_n).$$

Using Remark 5.1.9, we see that the map $\left(L^n_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\bot_1)\right)_c : L^n_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1 \to L^{n+1}_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1$ sends $(c_n \rightarrowtail \dots c_1 \rightarrowtail c)$ to

$$(0 \rightarrowtail c_n \rightarrowtail \dots c_1 \rightarrowtail c).$$

So the elements of $\omega_P(c)$ can be thought of as *eventually* 0 *infinite chains* of $(\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0})$-subobjects

$$(\dots \rightarrowtail c_n \rightarrowtail \dots c_1 \rightarrowtail c).$$

The action of $\omega_P(f : d \to c)$ is to pull back along f to obtain a chain of $(\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0})$-subobjects starting at d.

We can show that ω_P is concrete. A chain $(\dots \rightarrowtail c_n \rightarrowtail \dots c_1 \rightarrowtail c)$ becomes a function which maps a point $x : \star \to c$ to the pullback of the chain along x. If two chains become the same function, they must be equal because $(\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0})$-subobjects are determined by their points (Lemma 5.2.3).

We can also explicitly calculate $\overline{\omega}$, the limit of the chain:

$$1 \xleftarrow{!} L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1 \xleftarrow{L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(!)} L^2_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1 \xleftarrow{L^2_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(!)} \dots$$

Using Remark 5.1.9, we see that the map $\left(L^n_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(!)\right)_c : (L^{n+1}_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1)(c) \to (L^n_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1)(c)$ sends $(c_{n+1} \rightarrowtail c_n \rightarrowtail \dots c_1 \rightarrowtail c)$ to:

$$(c_n \rightarrowtail \dots c_1 \rightarrowtail c),$$

so we can think of the elements of $\overline{\omega}(c)$ as *infinite chains* of $(\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0})$-subobjects $(\dots \rightarrowtail c_n \rightarrowtail \dots c_1 \rightarrowtail c)$. The action of $\overline{\omega}(f : d \to c)$ is to pull back a chain along f. Because $\overline{\omega}$ is the limit of concrete objects it is also concrete, since concrete presheaves are a reflective subcategory of presheaves.

Remark 6.3.1. The concreteness assumptions from Assumption 5.2.1 are satisfied by $(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$ and $\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}$, so from Lemma 5.2.2 the only $\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}$-subobjects of $\star$ are $[\star \to \star]$ and $[0 \to \star]$. Therefore, we will often think of $\omega_P(\star)$ as the set of natural numbers $\mathbb{N}$, where the number n is represented by the chain of subobjects that becomes 0 after n steps:

$$(\dots 0 \rightarrowtail \underbrace{\star \rightarrowtail \dots}_{n \text{ times}} \rightarrowtail \star)$$

Similarly, we think about $\overline{\omega}(\star)$ as the natural numbers with infinity $\mathbb{N} \cup \{\infty\}$, where infinity is represented by the chain of subobjects that is constant $\mathrm{id}_\star$.

6.3.2 The dominance $\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$ is orthogonal to the map $i_P : \omega_P \to \overline{\omega}$

Definition 6.3.2. Let a function

$$\mathcal{F} : \mathsf{Hom}(\omega_P, \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}) \to [\mathbb{N} \to 2]$$

between morphisms $\omega_P \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$ and infinite binary sequences be defined by mapping $\alpha : \omega_P \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$ to its component at the terminal object $\star$:

$$\alpha_\star : \big(\omega_P(\star) \cong \mathbb{N}\big) \to \big(\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\star) \cong 2\big).$$

The definition above makes sense because the concreteness assumptions from Assumption 5.2.1 hold, so we know from Lemma 5.2.2 that:

$$\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\star) \cong \{[\star \to \star], [0 \to \star]\} \cong 2.$$

We explained in Remark 6.3.1 how an element in $\omega_P(\star)$, which is a chain of subobjects $(\dots 0 \rightarrowtail \underbrace{\star \rightarrowtail \dots}_{n \text{ times}} \rightarrowtail \star)$, corresponds to $n \in \mathbb{N}$.

Lemma 6.3.3. *The function*

$$\mathcal{F} : \mathsf{Hom}(\omega_P, \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}) \to [\mathbb{N} \to 2]$$

from Definition 6.3.2 is injective and sends each map $\omega_P \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ to an infinite monotone *binary sequence, where $2 = \{0 \leq 1\}$. Moreover, every infinite monotone binary sequence $\mathbb{N} \to 2$ is the image of some map $\omega_P \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$.*

Proof. We know that $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ is concrete so maps into it are uniquely determined by their component at $\star$, therefore $\mathcal{F}$ is injective.

Showing that $\mathcal{F}(\alpha)$ is monotone. We will show that for any $n \geq 1$:

$$\alpha_\star(\dots 0 \rightarrowtail 0 \rightarrowtail \underbrace{\star \rightarrowtail \dots}_{n-1 \text{ times}} \rightarrowtail \star) = [\star \to \star]$$

implies

$$\alpha_\star(\dots 0 \rightarrowtail \star \rightarrowtail \underbrace{\star \rightarrowtail \dots}_{n-1 \text{ times}} \rightarrowtail \star) = [\star \to \star].$$

The class of monos at V was defined in Section 5.3 to be:

$$\mathcal{M}_{\mathbb{V}_0}(\mathrm{V}) = \{(\lambda x.x + n) \in \mathbb{V}_0(\mathrm{V}, \mathrm{V}) \mid n \in \mathbb{N}\} \cup \{! : 0 \to \mathrm{V}\},$$

and there is a point $\star \xrightarrow{n} \mathrm{V}$ in the site for each $n \in \mathbb{N} \cup \{\infty\}$.

Consider the chain of monos on V:

$$f = (\dots 0 \to \mathrm{V} \xrightarrow{(+1)} \underbrace{\mathrm{V} \xrightarrow{(+1)} \dots}_{(n-1) \text{ times}} \mathrm{V})$$

and the chain on $\star$:

$$d_{n-1} = (\dots 0 \to 0 \to \underbrace{\star \to \dots}_{(n-1) \text{ times}} \star).$$

Then d_{n-1} is the pullback of f along the map $\star \xrightarrow{n-1} \mathrm{V}$.

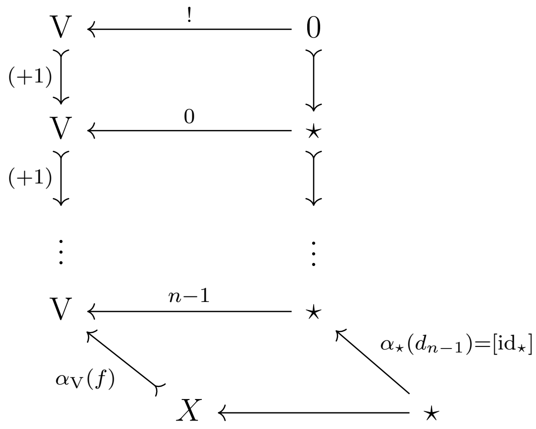


Figure 6.2: Diagram for proof of Lemma 6.3.3.

Because ω_P and $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ act by pullback, and α is natural, $\alpha_\star(d_{n-1})$ is the pullback of $\alpha_{\mathrm{V}}(f)$ along $(n-1)$. Thus, we obtain the diagram in Figure 6.2 where all the squares are pullbacks.

By assumption $\alpha_\star(d_{n-1}) = [\mathrm{id}_\star]$, so it must be the case that:

$$\alpha_{\mathrm{V}}(f) = \mathrm{V} \xrightarrow{(+k)} \mathrm{V}$$

for some $k < n$.

Now consider the diagram in Figure 6.3, where we let $d_n = (\dots 0 \to \underbrace{\star \to \dots}_{n \text{ times}} \star)$ be the pullback of f along $\star \xrightarrow{n} \mathrm{V}$. Again all the squares in the diagram are pullbacks. Because we know $k < n$, $\alpha_\star(d_n)$ must equal $[\mathrm{id}_\star]$, which we were trying to show.

Using the fact that we have just shown, namely that:

$$\alpha_\star(d_{n-1}) = [\star \to \star] \implies \alpha_\star(d_n) = [\star \to \star],$$

we can show by induction that once the infinite binary sequence $\mathcal{F}(\alpha) = \alpha_\star$ becomes 1, it has to remain 1, so the sequence is monotone.

Showing that any infinite binary sequence is the image of a map $\alpha : \omega_P \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ under $\mathcal{F}$. Let the action of $\alpha_\star$ be given by the binary

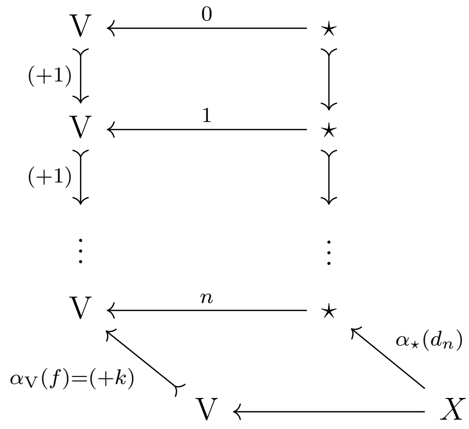


Figure 6.3: Diagram for proof of Lemma 6.3.3.

sequence. Since $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ is concrete, the other components of α are uniquely determined. We need to show that the induced α is natural.

Because ω_P is concrete, we can think of a chain of subobjects in $\omega_P(c)$ as a function from points $x : \star \to c$ to chains in $\omega_P(\star)$ obtained by pullback. Similarly an $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobject in $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(c)$ is a subset of the points of c. Checking naturality amounts to checking that when we postcompose a function from $\omega_P(c)$ by $\alpha_\star$, we obtain an element of $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(c)$.

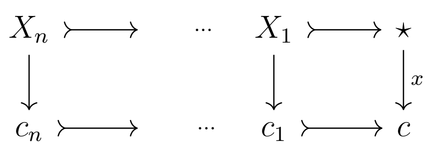
Consider a chain in $\omega_P(c)$:

$$\ldots 0 \rightarrowtail c_n \rightarrowtail \ldots c_1 \rightarrowtail c.$$

If there exists $0 \leq k \leq n$ such that at position k the infinite monotone binary sequence becomes 1, then $x : \star \to c$ gets mapped to

- $[\star \to \star]$, if $X_k = \star$ $(x \in |c_k|)$;
- $[0 \to \star]$, if $X_k = 0$ $(x \notin |c_k|)$

where the X_k is part of the pullback diagram below:



So the subobject of c being carved out is $c_k \rightarrowtail \dots c$, which is an $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobject because these subobjects are closed under composition.

Otherwise, if the sequence does not become 1 at an index less or equal to n, every $x : \star \to c$ is sent to $[0 \to \star]$, so the subobject of c being carved out is $0 \to c$.

□

Lemma 6.3.4. *The dominance $\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$ is right-orthogonal to the comparison map $i_P : \omega_P \to \overline{\omega}$.*

Proof. In Definition 6.3.2 and Lemma 6.3.3, we showed that a map $\alpha : \omega_P \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$ can be identified with a monotone binary sequence, which gives the action of $\alpha_\star$. The function $(i_P)_\star$ is an inclusion, and the only element of $\overline{\omega}(\star)$ that is not in its image is the constant $\star$ infinite sequence:

$$d_\infty = (\dots \star \rightarrowtail \dots \star \rightarrowtail \star).$$

Since $\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$ is concrete, in order to define an extension $\overline{\alpha} : \overline{\omega} \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$ of α, we only need to choose $\overline{\alpha}_\star(\dots \star \rightarrowtail \dots \star \rightarrowtail \star)$, and the rest of $\overline{\alpha}_\star$ is the same as $\alpha_\star$. If the monotone binary sequence corresponding to α becomes 1 at position n, then choose:

$$\overline{\alpha}_\star(\dots \star \rightarrowtail \dots \star \rightarrowtail \star) = [\star \to \star].$$

If the sequence is always 0, choose:

$$\overline{\alpha}_\star(\dots \star \rightarrowtail \dots \star \rightarrowtail \star) = [0 \to \star].$$

In both cases we need to prove **naturality** and **uniqueness** of the candidate extension.

Case $\alpha_\star$ becomes 1 at n. Uniqueness of $\overline{\alpha}$. Define a map $\beta : L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1 \to \overline{\omega}$. The object $\overline{\omega}$ is a limit of a diagram of concrete presheaves, so it is also concrete. Therefore it is enough to specify $\beta_\star$. Let:

$$\beta_\star([\star \to \star]) = (\ldots \star \rightarrowtail \ldots \star \rightarrowtail \star) = d_\infty$$
$$\beta_\star([0 \to \star]) = (\ldots 0 \rightarrowtail \underbrace{\star \rightarrowtail \ldots}_{n \text{ times}} \rightarrowtail \star) = d_n.$$

We defer showing naturality of β.

Consider the composite $\overline{\alpha} \circ \beta : L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1 \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$. This must be a monotone map, like we showed in Definition 6.3.2 and Lemma 6.3.3. Therefore:

$$[\star \to \star] = \overline{\alpha}_\star(\beta_\star([0 \to \star])) \leq \overline{\alpha}_\star(\beta_\star([\star \to \star])) = \overline{\alpha}_\star(\ldots \star \rightarrowtail \star).$$

So indeed, $[\star \to \star]$ is the only value we can choose for $\overline{\alpha}_\star(\ldots \star \rightarrowtail \star)$.

Now we need to show that $\beta : L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1 \to \overline{\omega}$ is indeed natural. We can obtain it as a comparison map into the limit $\overline{\omega}$, where the cocone:

$$(\beta_k : L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1 \to L^k_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1)_{k \in \mathbb{N}}$$

on the diagram:

$$1 \xleftarrow{!} L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1 \xleftarrow{L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(!)} L^2_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1 \xleftarrow{L^2_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(!)} \ldots$$

is defined below. Again we only need to give $(\beta_k)_\star$ thanks to concreteness.

Let:

$$(\beta_k)_\star([\star \to \star]) = d_k$$
$$(\beta_k)_\star([0 \to \star]) = \begin{cases} d_n & \text{if } n \leq k \\ d_k & \text{otherwise.} \end{cases}$$

We can easily check that $(\beta_k)_{k \in \mathbb{N}}$ does form a cone because each $\left(L^k_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(!)\right)_\star$: $(L^{k+1}_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1)(\star) \to (L^k_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1)(\star)$ sends d_{k+1} to d_k and is identity otherwise. To show β is indeed the comparison map for this cone, use the fact that the projections $\pi_k : \overline{\omega} \to L^k_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1$ truncate an infinite chain of monos to the first k monos.

Now, show that each $\beta_k : L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1 \to L^k_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1$ is natural. Notice that it is a map between two concrete objects. If $k \leq n$, then $(\beta_k)_\star$ is constant, so it "preserves the relations", since concrete objects contain all constant functions.

The map $(\beta_{n+1}) : L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1 \to L^{n+1}_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1$ is given by:

$$(\beta_{n+1})_\star(d_0) = d_n \qquad\qquad (\beta_{n+1})_\star(d_1) = d_{n+1}.$$

Observe that this is the same as:

$$(\eta_{L^n_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1})_\star \circ \ldots (\eta_{L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1})_\star$$

because, for example:

$$(\eta_{L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1})_\star(d_0) = d_1 \qquad\qquad (\eta_{L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1})_\star(d_1) = d_2.$$

So because η is natural, we obtain that β_{n+1} is as well.

The last case is $(\beta_{n+p}) : L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1 \to L^{n+p}_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1$, with $p > 1$. This has the form:

$$(\beta_{n+p})_\star(d_0) = d_n \qquad\qquad (\beta_{n+p})_\star(d_1) = d_{n+p}$$

so we can decompose it into:

$$L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1(\star) \xrightarrow{(\beta_{n+1})_\star} L^{n+1}_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1(\star) \xrightarrow{(\gamma_p)_\star} L^{n+p}_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1(\star).$$

Where we have chosen $(\gamma_p)_\star$ to be:

$$(\gamma_p)_\star(d_{n+1}) = d_{n+p}$$
$$(\gamma_p)_\star(d_{(k\leq n)}) = d_k.$$

But $(\gamma_p)_\star$ is actually equal to:

$$\left(L^{n+(p-1)}_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\eta_1 : 1 \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1)\right)_\star \circ \ldots \circ \left(L^{n+1}_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\eta_1 : 1 \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1)\right)_\star,$$

so γ_p must be natural.

Case $\alpha_\star$ is always 0. Uniqueness of $\overline{\alpha}$. We need to show that we cannot choose $\overline{\alpha}_\star(d_\infty) = [\star \to \star]$, so it must be $[0 \to \star]$.

Consider a map $\delta : y(\mathrm{V}) \to \overline{\omega}$, defined as:

$$\delta_\star(n) = d_n$$
$$\delta_\star(\infty) = d_\infty,$$

since we know the points of V are $\mathbb{N} \cup \{\infty\}$. We will show later that δ is natural.

Consider the composite $(\overline{\alpha} \circ \delta) : y(\mathrm{V}) \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$:

$$\overline{\alpha}_\star(\delta_\star(n)) = [0 \to \star]$$
$$\overline{\alpha}_\star(\delta_\star(\infty)) = [\star \to \star].$$

By the Yoneda lemma $(\overline{\alpha} \circ \delta)$ corresponds to a mono $m : X \rightarrowtail \mathrm{V}$ in $\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\mathrm{V})$. Also by Yoneda, precomposing $(\overline{\alpha} \circ \delta)$ by a point $y(k) : y(\star) \to y(\mathrm{V})$ is the same as taking pullback of m with $k : \star \to \mathrm{V}$.

So we have the following two pullback squares that $m : X \rightarrowtail \mathrm{V}$ must satisfy, for any $k \in \mathbb{N}$:

$$\begin{array}{ccc} X & \longleftarrow & 0 \\ {\scriptstyle m}\downarrow & & \downarrow \\ \mathrm{V} & \underset{k}{\longleftarrow} & \star \end{array} \qquad\qquad \begin{array}{ccc} X & \longleftarrow & \star \\ {\scriptstyle m}\downarrow & & \downarrow \\ V & \underset{\infty}{\longleftarrow} & \star \end{array}$$

However, m can be either $\mathrm{V} \xrightarrow{(+p)} \mathrm{V}$ for some $p \in \mathbb{N}$ or $0 \to \mathrm{V}$, and none if them satisfies both these conditions. Therefore, we must choose $\overline{\alpha}_\star(d_\infty) = [0 \to \star]$ instead.

Now we must show $\delta : y(\mathrm{V}) \to \overline{\omega}$ is natural. We will show it is the comparison map into the limit $\overline{\omega}$.

Consider a cone $(\delta_k : y(\mathrm{V}) \to L^k_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} 1)_{k\in\mathbb{N}}$ given by:

$$\begin{aligned}(\delta_k)_\star(n \leq k) &= d_n\\ (\delta_k)_\star(n > k) &= d_k\\ (\delta_k)_\star(\infty) &= d_k.\end{aligned}$$

We can easily check that this forms a cone, given the action of $L^k_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(!)$, and that δ is indeed the comparison map for this cone, given the projections $\pi_k : \overline{\omega} \to L^k_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(1)$.

To show each $\delta_k : y\mathrm{V} \to L^k_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} 1$ is natural, it is enough by Yoneda to find a chain of subobjects of V, $v \in L^k_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} 1(\mathrm{V})$. Because $L^k_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} 1$ is concrete, v is determined by its pullback, living in $L^k_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} 1(\star)$, with each $n : \star \to \mathrm{V}$, and $\infty : \star \to \mathrm{V}$.

Again by Yoneda, taking such a pullback, with say n, is the same as precomposing δ_k by $y(n) : y(\star) \to y(\mathrm{V})$. This further corresponds to the value of $(\delta_k)_\star(n) \in L^k_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} 1(\star)$. Given the definition of $(\delta_k)_\star(n)$, we can see that the chain that corresponds to it is:

$$v = (\underbrace{\mathrm{V} \xrightarrow{(+1)} \ldots}_{k \text{ times}} \mathrm{V}).$$

Case $\alpha_\star$ is always 0. Naturality of $\overline{\alpha}$. Because $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ is concrete, we only need to show that each $\overline{\alpha}_c$ is valued in $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(c)$. We defined $\overline{\alpha}_\star : \overline{\omega}(\star) \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\star)$ to be constant $[0 \to \star]$, so every element of $\overline{\omega}(c)$ gets mapped by $\overline{\alpha}_c$ to $0 \to c$, which is in $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(c)$.

Case $\alpha_\star$ becomes 1 at n. Naturality of $\overline{\alpha}$. We have defined $\overline{\alpha}_\star : \overline{\omega}(\star) \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\star)$ to be:

$$\begin{aligned}\overline{\alpha}_\star(d_k) &= \begin{cases}[0 \to \star] & \text{if } k < n\\ [\star \to \star] & \text{if } k \geq n\end{cases}\\ \overline{\alpha}_\star(d_\infty) &= [\star \to \star].\end{aligned}$$

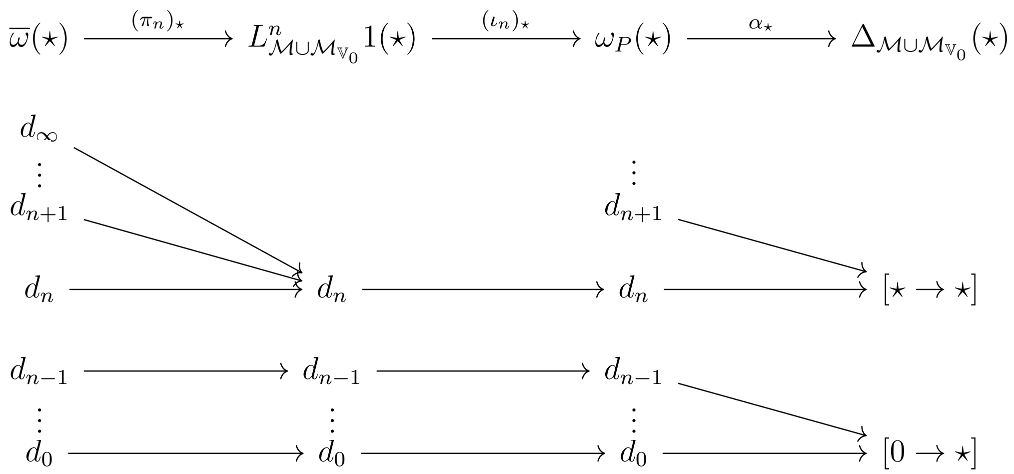


Figure 6.4: Diagram for the proof of Lemma 6.3.4.

We can therefore obtain $\overline{\alpha}_\star$ as the composition of the three maps from Figure 6.4. In the figure, the first map is the projection from the limit $\overline{\omega}$, the second map is the inclusion into the colimit ω_P, and the third is α. We already know all these maps are natural, so $\overline{\alpha}$ must be natural too.

□

Corollary 6.3.5. *Consider an infinite monotone binary sequence with a top element, such that the sequence is continuous, that is:*

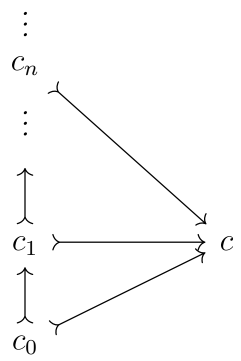
- *if the sequence is constant* 0*, then the top element is* 0*;*
- *if the sequence becomes* 1 *at position* n*, then the top element is* 1*.*

Such a sequence determines a natural transformation $\overline{\omega} \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$*.*

Proof. This follows immediately from the way the extension of $\alpha : \omega_P \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ along $i_P : \omega_P \to \overline{\omega}$ is constructed in the proof of Lemma 6.3.4. □

6.3.3 The dominance $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ is orthogonal to the maps $\big(i_P \times y(c)\big) : \omega_P \times y(c) \to \omega \times y(c)$

Lemma 6.3.6. *For any object* c *in* $\mathbb{C} + \mathbb{V}_0$*, a map* $\alpha : \omega_P \times y(c) \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ *can be described as an increasing infinite chain of* $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$*-subobjects of* *c:*



Moreover, the maps $c_k \rightarrowtail c_{k+1}$ *must be* $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$*-subobjects as well.*

Proof. Since $\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$ is concrete, maps into it are determined by their component at $\star$.

We can show that for any point $x : \star \to c$ the function:

$$\alpha_\star(-, x) : \omega_P(\star) \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\star)$$

induces a natural transformation:

$$\alpha(-, x) : \omega_P \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}.$$

Because ω_P is also concrete it is enough to show that postcomposition by $\alpha_\star(-, x)$ "preserves the relations". By naturality of α, postcomposition by $\alpha_\star(-, x)$ is actually the same as the action of $\alpha_{c'}\big(-, \lambda(\_ \in |c'|).\,(x \in |c|)\big)$:

$$\begin{array}{ccc}
\omega_P(\star) & \xrightarrow{\alpha_\star(-,x)} & \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\star) \\
{\scriptstyle (-)\circ(y:\star \to c')}\uparrow & & \uparrow \\
\omega_P(c') & \xrightarrow[\alpha_{c'}\left(-,\lambda(_\in|c'|).\,(x\in|c|)\right)]{} & \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(c')
\end{array}$$

and we know by its type that this gives a result in $\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$.

We have shown in Definition 6.3.2 and Lemma 6.3.3 that the map:

$$\alpha(-, x) : \omega_P \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}.$$

corresponds to an infinite monotone binary sequence.

Similarly, we can show that for any chain d_k in $\omega_P(\star)$ the function:

$$\alpha_\star(d_k, -) : \mathsf{Hom}(\star, c) \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\star)$$

induces a natural transformation:

$$\alpha(d_k, -) : y(c) \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}.$$

Notice that $y(c)$ is concrete because the site $\mathbb{C}+\mathbb{V}_0$ is concrete. To show that $\alpha_\star(d_k, -)$ preserves relations it is enough to notice that the following square commutes by naturality of α:

$$\begin{array}{ccc}
\mathsf{Hom}(\star, c) & \xrightarrow{\alpha_\star(d_k,-)} & \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\star) \\
{\scriptstyle (-)\circ(y:\star\to c')}\uparrow & & \uparrow \\
\mathsf{Hom}(c', c) & \xrightarrow[\alpha_{c'}(\ldots 0\rightarrowtail \underbrace{c'\rightarrowtail\ldots}_{k\text{ times}}\, c',-)]{} & \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(c')
\end{array}$$

By the Yoneda lemma, for each $k \in \mathbb{N}$, the map

$$\alpha(d_k, -) : y(c) \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$$

corresponds to an $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobject:

$$m_k : c_k \rightarrowtail c,$$

which is the value of $\big(\alpha(d_k, -)\big)_c$ at id_c. We will now show that

$$(m_k : c_k \rightarrowtail c)_{k\in\mathbb{N}}$$

is the increasing chain of subobject that the map α determines.

From the naturality square above, we can see that any point $y : \star \to c$ factors through m_k if and only if:

$$\alpha_\star(d_k, y : \star \to c) = [\star \to \star].$$

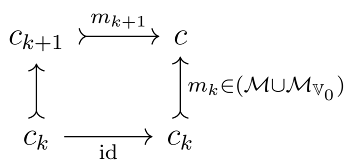
Because we have shown $\alpha_\star(d_k, y : \star \to c)$ is a *monotone* binary sequence, it follows that:

$$\alpha_\star(d_{k+1}, y : \star \to c) = [\star \to \star]$$

as well, which is equivalent to y factoring through m_{k+1}. Hence by Lemma 5.2.3 we obtain that:

$$m_k \leq m_{k+1}.$$

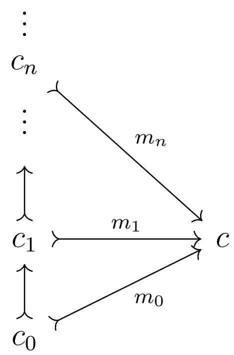
Finally, to show that each $c_k \rightarrowtail c_{k+1}$ is an $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobject, observe that the following square is a pullback:



Because $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$ is closed under pullback, it must be the case that $(c_k \rightarrowtail c_{k+1}) \in (\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$. $\square$

Lemma 6.3.7. *The dominance $\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$ is right-orthogonal to the comparison map $(i_P \times \mathrm{id}_{y(c)}) : \omega_P \times y(c) \to \overline{\omega} \times y(c)$, for any object c in the site $\mathbb{C} + \mathbb{V}_0$.*

Proof. Consider a map $\alpha : \omega_P \times y(c) \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$. We know from Lemma 6.3.6 that α can be described as an increasing chain of $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobjects of c:



From assumption (3) in the definition of class of admissible monos, we know that this chain has an upper bound:

$$m_\infty : c_\infty \rightarrowtail c.$$

This is an $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobject whose points are exactly the union of all the points in the chain.

Define a candidate extension for α:

$$\overline{\alpha}_\star : \overline{\omega}(\star) \times \mathsf{Hom}(\star, c) \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\star)$$

as:

$$\begin{aligned} \overline{\alpha}_\star(d_k, -) &= \alpha_\star(d_k, -) \\ \overline{\alpha}_\star(d_\infty, x) &= \begin{cases} [\star \to \star] & \text{if } x \in |m_\infty| \\ [0 \to \star] & \text{otherwise.} \end{cases} \end{aligned}$$

Naturality of $\overline{\alpha}$. Consider another object c' in the site $\mathbb{C} + \mathbb{V}_0$. We need to show that given an element of $\overline{\omega}(c') \times \mathsf{Hom}(c', c)$, which is a function $|c'| \to \overline{\omega}(\star) \times |c|$, postcomposing by $\overline{\alpha}_\star$ determines an $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobject of c'.

Consider an element in $\overline{\omega}(c')$, which is an infinite decreasing chain of subobjects of c':

$$b = \ldots c'_2 \rightarrowtail c'_1 \rightarrowtail c'.$$

This can be turned into a function by pulling back along each point $x : \star \to c'$:

$$\begin{array}{ccccccc} \cdots & X_2 & \rightarrowtail & X_1 & \rightarrowtail & \star \\ & \downarrow & & \downarrow & & \downarrow{\scriptstyle x} \\ \cdots & c'_2 & \rightarrowtail & c'_1 & \rightarrowtail & c' \end{array}$$

And consider also a map $h \in (\mathbb{C} + \mathbb{V}_0)(c', c)$.

The subset of $|c'|$ determined by the function:

$$|c'| \xrightarrow{\langle b,h \rangle} \overline{\omega}(\star) \times |c| \xrightarrow{\overline{\alpha}_\star} \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\star)$$

can be described as:

$$S = \bigcup_{n \in \mathbb{N}} \big\{ x \in |c| \bigm| x \in |c'_n| \text{ and } h(x) \in |m_n| \big\}$$

which includes the case when $b(x) = d_\infty$ and $h(x) \in |m_\infty|$.

We can obtain the same subset of $|c'|$ by taking the union for all $k \in \mathbb{N}$ of the subsets:

$$S_k = \bigcup_{n \leq k} \big\{ x \in |c| \bigm| x \in |c'_n| \text{ and } h(x) \in |m_n| \big\}.$$

It is clear that for any k:

$$S_k \subseteq S_{k+1}.$$

We will now show that each S_k determines an $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobject of c'.

Consider the function $(r_k)_\star : \overline{\omega}(\star) \to \omega_P(\star)$, which truncates a chain to its first k components:

$$(r_k)_\star(d_n) = \begin{cases} d_n & \text{if } n \leq k \\ d_k & \text{if } n > k \end{cases}$$
$$(r_k)_\star(d_\infty) = d_k.$$

The function $(r_k)_\star$ can be written in terms of the projection out of the limit $\overline{\omega}$ and the inclusion into the colimit ω_P:

$$(r_k)_\star = \overline{\omega}(\star) \xrightarrow{(\pi_k)_\star} L^k_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\star) \xrightarrow{(\iota_k)_\star} \omega_P(\star).$$

This means $(r_k)_\star$ must determine a natural transformation $r_k : \overline{\omega} \to \omega_P$, i.e. $(r_k)_\star$ preserves the relations.

Now notice that $S_k \subseteq |c'|$ is determined by the following function:

$$|c'| \xrightarrow{\langle b,h \rangle} \overline{\omega}(\star) \times |c| \xrightarrow{(r_k)_\star \times |c|} \omega_P(\star) \times |c| \xrightarrow{\alpha_\star} \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\star).$$

Because we know both r_k and $\alpha_\star$ are natural, this function must determine an $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobject of c':

$$s_k : S_k \rightarrowtail c'.$$

Thus, we have an increasing chain of $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobject of c':

$$(s_k : S_k \rightarrowtail c')_{k \in \mathbb{N}}.$$

Using assumption (3) for a class of admissible monos, the subobject determined by the union:

$$S = \bigcup_{k \in \mathbb{N}} S_k$$

is also in $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$, so we are done.

Uniqueness of $\overline{\alpha}$. For any point $x : \star \to c$, the function:

$$\overline{\alpha}_\star(-, x) : \overline{\omega}(\star) \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\star)$$

determines a natural transformation:

$$\overline{\alpha}(-, x) : \overline{\omega} \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$$

which is an extension of $\alpha(-, x) : \omega_P \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$. Thanks to Lemma 6.3.4 this must he the unique extension.

If we had another extension $\overline{\alpha}'$ of α, we would get that for all $x : \star \to c$:

$$\overline{\alpha}'(-, x) = \overline{\alpha}(-, x),$$

and so $\overline{\alpha}' = \overline{\alpha}$, as required.

□

6.3.4 The lifted naturals are orthogonal to the map $i_P : \omega_P \to \overline{\omega}$

Lemma 6.3.8. *Natural transformations of the form*

$$\omega_P \to L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} \left(\coprod_{\mathbb{N}} 1 \right) \cong L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$$

$$\overline{\omega} \to L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} \left(\coprod_{\mathbb{N}} 1 \right)$$

factor through $\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} \cong L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(1)$.

Proof. First describe the elements of $L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})(\mathrm{V})$. From Remark 5.2.7, because V is only covered by identity:

$$\mathsf{Nat}(\mathrm{V}) = \{f : |\mathrm{V}| \to \mathbb{N} \mid f \text{ constant}\}$$
$$\mathsf{Nat}(0) = \{\bot\}.$$

Using Proposition 5.1.8, we see that an element in $L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})(\mathrm{V})$ is either of the form

$$(\mathrm{V} \xrightarrow{(+p)} \mathrm{V}, n)$$

for some $p, n \in \mathbb{N}$, or $\bot$, which corresponds to the $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobject $0 \to \mathrm{V}$. The element $(\mathrm{V} \xrightarrow{(+p)} \mathrm{V}, n)$ can be described as a function $|\mathrm{V}| \to \mathbb{N} + \{\bot\}$, where:

$$k \mapsto \begin{cases} n & \text{if } k \geq p \\ \bot & \text{otherwise} \end{cases}$$
$$\infty \mapsto n,$$

and $\bot \in L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})(\mathrm{V})$ can be described as the constant $\bot$ function.

Every map $\alpha : \overline{\omega} \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$ factors through $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$. Consider the infinite chain in $\overline{\omega}(\mathrm{V})$:

$$b = (\ldots \mathrm{V} \xrightarrow{(+1)} \mathrm{V} \xrightarrow{(+1)} \ldots \mathrm{V}).$$

As a function $|\mathrm{V}| \to \overline{\omega}(\star)$, b can be described as:

$$b(k) = d_k$$
$$b(\infty) = d_\infty.$$

Because α is a natural transformation into a concrete object, the function $\alpha_\star \circ b : |\mathrm{V}| \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})(\star)$ must correspond to exactly one element of $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})(\mathrm{V})$.

If $\alpha_\star \circ b$ corresponds to $\bot$, then it must be the case that:

$$\alpha_\star(d_k) = \alpha_\star(d_\infty) = \bot.$$

Therefore, $\alpha_\star$ factors as the constant 0 function:

$$\overline{\omega}(\star) \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\star) \cong L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(1)(\star)$$

followed by any of the coproduct inclusions $\left(L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{inc}_k : 1 \to \coprod_{\mathbb{N}} 1)\right)_\star$:

$$L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(1)(\star) \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})(\star) \cong \mathbb{N} + \{\bot\}.$$

A constant function always determines a natural transformation, so we have found a factoring of α through $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$.

If $\alpha_\star \circ b$ corresponds to $(\mathrm{V} \xrightarrow{(+p)} \mathrm{V}, n)$, then:

$$\alpha_\star(d_\infty) = n$$
$$\alpha_\star(d_{q \geq p}) = n$$
$$\alpha_\star(d_{q<p}) = \bot.$$

We can then factor $\alpha_\star$ as the function:

$$f : \overline{\omega}(\star) \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\star) \cong L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(1)(\star)$$

where:

$$\begin{aligned} f_\star(d_\infty) &= 1 \\ f_\star(d_{q \geq p}) &= 1 \\ f_\star(d_{q < p}) &= 0, \end{aligned}$$

followed by the inclusion map $\left(L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\mathsf{inc}_n : 1 \to \coprod_{\mathbb{N}} 1)\right)_\star$. We can see that f has the form of an infinite continuous monotone binary sequence, so by Corollary 6.3.5 it determines a natural transformation $\overline{\omega} \to \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$.

Every map $\beta : \omega \to L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$ factors through $\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$. We will use the description of ω_P as the colimit of the objects $L^n_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1$. Each $L^n_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1$ is a retract of $\overline{\omega}$ because:

$$L^n_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1 \xrightarrow{\mathsf{inc}} \overline{\omega} \xrightarrow{\pi_n} L^n_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1 = \mathrm{id}.$$

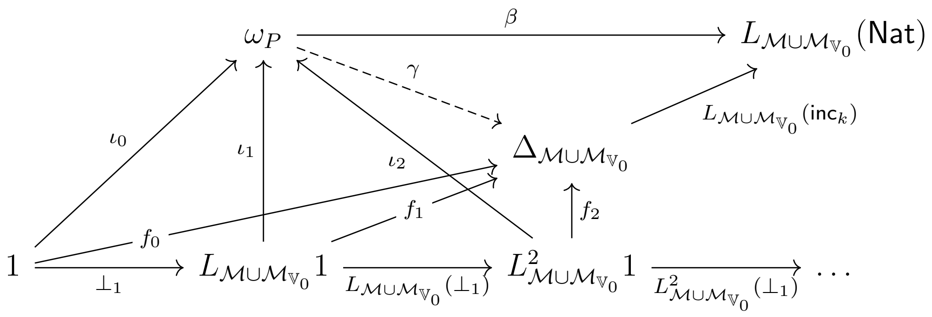
Then every map $L^n_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1 \to L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$ factors through $\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$, because we have already proved that maps from $\overline{\omega}$ factor:

$$\begin{array}{ccccccc} L^n_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1 & \xrightarrow{\mathsf{inc}} & \overline{\omega} & \xrightarrow{\pi_n} & L^n_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} 1 & \longrightarrow & L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat}) \\ & & & \searrow^{f} & & \nearrow_{L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\mathsf{inc}_k)} & \\ & & & & \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} & & \end{array}$$

We know f is given by a monotone binary sequence, and inc_k is a coproduct inclusion into $\coprod_{\mathbb{N}} 1$.

A map $\beta : \omega_P \to L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$ determines a cocone over the diagram for

ω_P, such that every map in the cocone factors through $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$:



We can easily see that all the maps $\beta \circ \iota_n$ in the cocone factor through the same coproduct inclusion $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{inc}_k)$ because of the shape of the diagram. Each f_n is given by a monotone binary sequence of length $n+1$, such that each sequence is an extension of the previous one.

Because $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{inc}_k)$ is mono, $(f_n)_n$ forms a cocone as well. So there is a comparison map $\gamma : \omega_P \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$.

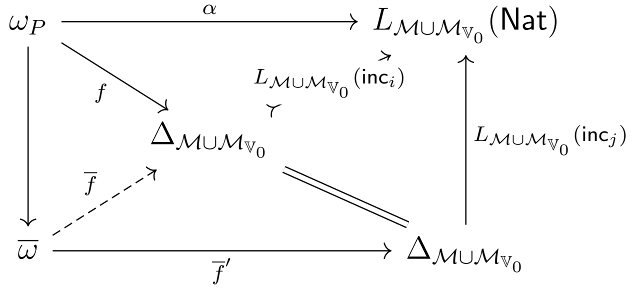
Both $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{inc}_k) \circ \gamma$ and β are comparison maps for the cocone:

$$\left(L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{inc}_k) \circ f_n\right)_{n\in\mathbb{N}} = (\beta \circ \iota_n)_{n\in\mathbb{N}},$$

and so they must be equal. Thus we have a factoring of β through $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$. □

Lemma 6.3.9. *The object of lifted natural numbers $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$ is right-orthogonal to the comparison map $i_P : \omega_P \to \overline{\omega}$.*

Proof. Consider a map $\alpha : \omega_P \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$. From Lemma 6.3.8 we know that it factors through $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$:



From Lemma 6.3.4, f has a unique extension $\overline{f}$, so

$$L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{inc}_i) \circ \overline{f}$$

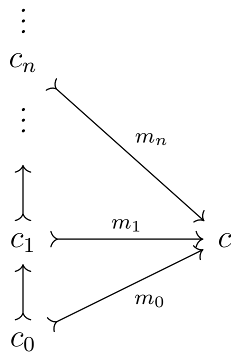
is an extension of α. We need to prove that it is the unique extension.

Consider another extension $\overline{\omega} \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$. From Lemma 6.3.8 it must factor through $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ as well, but possibly through a different coproduct inclusion inc_j.

In the proof of Lemma 6.3.8, we have shown $\overline{f}'$ must be either the constant 0 infinite binary sequence with top element 0, or a monotone binary sequence which becomes 1 at some position $n \in \mathbb{N}$ and has top element 1. In both cases, because both $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{inc}_i) \circ \overline{f}$ and $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{inc}_j) \circ \overline{f}'$ are extensions of α, we see they must be equal. □

6.3.5 The lifted naturals are orthogonal to the maps $\big(i_P \times y(c)\big) : \omega_P \times y(c) \to \overline{\omega} \times y(c)$

Lemma 6.3.10. *A map $\alpha : \omega_P \times y(c) \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$ can be described as an increasing chain of $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobjects of c:*



together with a family of functions $(g_n : |c_n| \to \mathbb{N})_{n\in\mathbb{N}}$ which extend each other, that is:

$$\forall x \in |c_n|.\ g_{n+1}(x) = g_n(x),$$

and such that each $g_n : |c_n| \to \mathbb{N}$ is locally constant on a cover of c_n.

Proof. For each element $d_n \in \omega_P(\star)$, $\alpha_\star(d_n, -)$ determines a natural transformation:

$$\alpha(d_n, -) : y(c) \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat}).$$

By the Yoneda lemma and using Remark 5.2.7, $\alpha(d_n, -)$ corresponds to a pair:

$$\big(m_n : (c_n \rightarrowtail c) \in \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(c),\ g_n : |c_n| \to \mathbb{N}\big)$$

of an $\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}$-subobject and a function g_n locally constant on a cover of c_n.

If we fix a point $x \in |c|$, then we obtain a natural transformation:

$$\alpha(-, x) : \omega_P \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat}),$$

such that

$$\alpha_\star(d_n, x) = \begin{cases} g_n(x) & \text{if } x \in |c_n| \\ \bot & \text{otherwise.} \end{cases}$$

From Lemma 6.3.8 we know $\alpha(-, x)$ factors through $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$, as an infinite monotone binary sequence $\omega_P \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ followed by a coproduct inclusion

$$L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{inc}_k) : \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} \cong L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}} 1 \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat}) \cong L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}\left(\coprod_{\mathbb{N}} 1\right).$$

Thus, we see that:

$$x \in |c_n| \implies x \in |c_{n+1}| \qquad\qquad g_n(x) = g_{n+1}(x),$$

so we get the increasing chain of subobjects and functions that we expect. □

Lemma 6.3.11. *The object of lifted natural numbers $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$ is right-orthogonal to the comparison map $(i_P \times \mathrm{id}_{y(c)}) : \omega_P \times y(c) \to \overline{\omega} \times y(c)$, for any object c in the site $\mathbb{C} + \mathbb{V}_0$.*

Proof. Consider a map $\alpha : \omega_P \times y(c) \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$. From Lemma 6.3.10, we can describe it as a chain of subobjects and functions:

$$\big(m_n : (c_n \rightarrowtail c) \in \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(c),\ g_n : |c_n| \to \mathbb{N}\big)_{n\in\mathbb{N}}.$$

Defining a candidate extension $\overline{\alpha} : \overline{\omega} \times y(c) \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$ of α. It is enough to specify $\overline{\alpha}_\star(d_\infty, -)$, while $\overline{\alpha}_\star(d_k, -)$ is determined by α. By Yoneda, this means giving an element of $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})(c)$.

Consider the tuple:

$$(m_\infty : c_\infty \rightarrowtail c,\ g_\infty : |c_\infty| \to \mathbb{N})$$

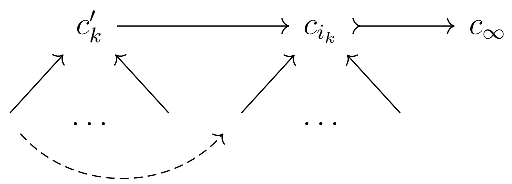
where m_∞ is the subobject determined by the set of points $\bigcup_{n\in\mathbb{N}}|m_n|$, and $g_\infty = \bigcup_{n\in\mathbb{N}} g_n$. By assumption (3) for a class of admissible monos, m_∞ is indeed an $\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}$-subobject. We can show that g_∞ is locally constant on a cover of c_∞, and thus the tuple is an element of $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})(c)$.

From assumption (4), we know that the closure under precomposition of the set $\{m_n : c_n \rightarrowtail c_\infty\}_{n\in\mathbb{N}}$ contains a covering family of c_∞:

$$\{c'_k \rightarrowtail c_\infty\}_{k\in I}.$$

Each $c'_k \rightarrowtail c_\infty$ must factor through some $m_{i_k} : c_{i_k} \rightarrowtail c_\infty$. Consider the cover of c_{i_k} on which g_{i_k} is locally constant. Then g_∞ is also locally constant on this cover.

By axiom (C) of coverage, we obtain a cover of c'_k on which g_∞ is locally constant:



Using this cover for each $k \in \mathbb{N}$ and using axiom (L), we can refine the coverage $\{c'_k \rightarrowtail c_\infty\}_{k\in\mathbb{N}}$ of c_∞ to one on which g_∞ is locally constant.

Proving the candidate extension for α is natural. To show that the function $\overline{\alpha}_\star : \overline{\omega}(\star) \times |c| \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})(\star)$ that we just defined determines a natural transformation, we will factorize it as:

$$\overline{\omega}(\star) \times |c| \xrightarrow{\langle h_\star, \mathrm{id}_{|c|} \circ \pi_2 \rangle} \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\star) \times |c| \xrightarrow{\beta_\star} \mathbb{N} + \{\bot\},$$

and show each of these two functions determines a natural transformation.

We choose $h_\star$ to be determined by the increasing chain of subobjects with a top element

$$(m_n : c_n \rightarrowtail c)_{n \in \mathbb{N} \cup \{\infty\}},$$

and define $\beta_\star$ as:

$$\beta_\star(0, x) = \bot$$
$$\beta_\star(1, x) = \begin{cases} g_\infty(x) & \text{if } x \in |c_\infty| \\ \bot & \text{if } x \notin |c_\infty|. \end{cases}$$

It is simple to see that $\overline{\alpha}_\star = \beta_\star \circ \langle h_\star, \mathrm{id}_{|c|} \circ \pi_2 \rangle$.

To show $h_\star$ is natural, notice that it is the unique extension constructed in Lemma 6.3.7 of the map:

$$\omega_P(\star) \times |c| \xrightarrow{\alpha_\star} L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})(\star) \xrightarrow{\left(L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\coprod_{\mathbb{N}} 1 \xrightarrow{!} 1)\right)_\star} \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\star)$$

The above map can be described by the increasing chain of subobjects $(m_n : c_n \rightarrowtail c)_{n \in \mathbb{N}}$, according to Lemma 6.3.6. Thus h must be natural.

To show $\beta_\star$ preserves relations, consider an element of $\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(c') \times \mathsf{Hom}(c', c)$:

$$(m : c'' \rightarrowtail c',\ f : c' \to c).$$

Thinking of the tuple above as a function, we need to show that the function:

$$|c'| \xrightarrow{\langle m, f \rangle} \Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(\star) \times |c| \xrightarrow{\beta_\star} \mathbb{N} + \{\bot\}$$

1. determines an $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobject $X \rightarrowtail c'$,

2. and the restricted function $|X| \to \mathbb{N}$ is locally constant on a cover of X.

Notice that $x \in |X|$ if and only if $x \in |c''|$ and $f(x) \in |c_\infty|$. We can obtain X by taking the following two pullbacks:

$$\begin{array}{ccccc} c'' & \overset{m}{\rightarrowtail} & c' & \overset{f}{\longrightarrow} & c \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle m_\infty} \\ X & \rightarrowtail & \cdot & \longrightarrow & c_\infty \end{array}$$

Because $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobjects are closed under pullback and composition, $X \rightarrowtail c'$ is also an $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobject.

Consider the cover of c_∞ on which g_∞ is locally constant. Using axiom (C) of coverage we can obtain a cover of X that factors through this cover of c_∞, and hence g_∞ is locally constant on this cover of X. Since $\beta_\star \circ \langle m, f \rangle$ restricted to $|X|$ agrees with g_∞, we are done.

Proving the extension of $\alpha : \omega_P \times y(c) \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$ is unique. For any extension $\overline{\alpha}$ of α, and each $x \in |c|$, $\overline{\alpha}(-, x)$ is an extension of $\alpha(-, x) : \omega_P \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$. From Lemma 6.3.9 we know that such an extension of $\alpha(-, x)$ is unique, so $\overline{\alpha}$ must be unique. □

6.3.6 Putting it all together

Finally, we can use all the results in this section to prove Theorem 6.2.5. We also discuss where the conditions in the theorem are used precisely.

Theorem 6.2.5. *Let $(\mathbb{C}, J, \mathcal{M})$ be a concrete site, satisfying axioms (M) and (L), with an initial object covered by the empty set, and a class of admissible monos $\mathcal{M}$. In the sheaf category $\mathsf{Sh}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$ the dominance $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ and the lifted natural numbers $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\coprod_{\mathbb{N}} 1)$ are $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$-complete objects.*

Proof. Using Proposition 6.1.1, we see that it is enough to show orthogonality with respect to $(i_P \times y(c)) : \omega_P \times y(c) \to \overline{\omega} \times y(c)$, for every $c \in \mathbb{C}$. For $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ we showed this in Lemma 6.3.7, and for $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$, where $\mathsf{Nat} \cong \coprod_{\mathbb{N}} 1$, in Lemma 6.3.11.

□

Remark 6.3.12. In the definition of admissible monos (Definition 6.2.1), restricting $(\mathbb{C}, J)$ to be a concrete site and asking for the dominance to be concrete (condition (2)) is crucial for proving Lemma 6.3.3, which describes the maps $\omega_P \to \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ as monotone binary sequences. We do not know a proof of Theorem 6.2.5 without these assumptions.

Asking that J satisfies axioms (M) and (L) is a matter of convenience, because we could always replace J with another coverage that satisfies these axioms and determines the same sheaves.

Asking for all maps $0 \to c$ to be in $\mathcal{M}$ (condition (1)), and for 0 to be covered by the empty set, are both used to make the monad generated by the dominance pointed (via Proposition 5.2.6). This is needed to satisfy Assumption 3.0.1, which allows us to define ω and $\overline{\omega}$ and prove a fixed point theorem.

Condition (3) from the definition of admissible monos was used in the proofs of Lemma 6.3.7 and Lemma 6.3.11 to show $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ and $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$ are orthogonal to $i_P \times y(c)$. Intuitively, this condition can be explained by thinking of an $\mathcal{M}$-subobject $m : c' \rightarrowtail c$ as a *semidecidable* subset of c, in the computability sense discussed at the beginning of Section 2.4, in which there is a Turing machine that accepts the inputs from c' and diverges otherwise. Then condition (3) asks that the union of an increasing chain of semidecidable subsets is also semidecidable, which intuitively should be true.

Condition (4) for admissible monos is used in the proof of Lemma 6.3.11. Recall that the Yoneda embedding preserves monos but not general colimits. By asking for a covering family for the union of an increasing chain of $\mathcal{M}$-subobjects, condition (4) ensures that this union remains the colimit of the chain after applying the Yoneda embedding.

Remark 6.3.13. The proofs of Lemma 6.3.3 and Corollary 6.3.5, describing maps into $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ as monotone binary sequences, both rely on the fact that the site contains $(\mathbb{V}_0, J_{\mathbb{V}_0}, \mathcal{M}_{\mathbb{V}_0})$, defined in Section 5.3.

Recall that $(\mathbb{V}_0, J_{\mathbb{V}_0}, \mathcal{M}_{\mathbb{V}_0})$ is a site that generates the category vSet, and that there is an embedding from $\omega\mathsf{CPO}$ into vSet (Proposition 4.4.10). Intuitively, adding $(\mathbb{V}_0, J_{\mathbb{V}_0}, \mathcal{M}_{\mathbb{V}_0})$ to the site in Theorem 6.2.5 imposes a condition analogous to the continuity of maps between ωcpos, and is thus an alternative to considering $\omega\mathsf{CPO}$-valued sheaves.

Chapter 7

Adequate models for $\mathsf{PCF_v}$ in categories of sheaves

In this chapter, we combine the results from Chapters 5 and 6 to give a recipe for building normal models (Definition 4.3.1) of call-by-value PCF ($\mathsf{PCF_v}$) in categories of sheaves. This recipe is described in Theorem 7.1.1 and is the most important contribution of the thesis. In Section 7.2 we show that this recipe can be used to unify models of higher-order recursion that have so far been developed separately.

Normal models are closely related to Simpson's natural models [Sim98], as explained in Section 7.4. This previous work, however, does not give ways of building such models, hence our recipe is a contribution in that direction.

In the remainder of Section 7.1, we prove Proposition 7.1.2 which says that the models we construct with our recipe land in the subcategory of *concrete* sheaves (Definition 2.1.8). We use this fact to spell out explicitly the interpretation of $\mathsf{PCF_v}$ types, in Figure 7.1.

We then state an adequacy theorem (Theorem 7.1.3) for normal models obtained from our recipe, with respect to the operational semantics of $\mathsf{PCF_v}$ with type and term constants, defined in Section 4.2.1. The proof of adequacy is deferred to Section 7.3, where a logical relations argument is used. Recall that we already proved normal models are sound in Theorem 4.3.5. The material summarized so far was published at LICS 2022 [MMS22, Section 7.2].

In Example 7.1.5, we explain how our running example vSet, of presheaves on the vertical natural numbers, can be built via the recipe for normal models from Theorem 7.1.1. We then deduce via Theorem 7.1.3 that the vSet model for $\mathsf{PCF_v}$ is adequate.

In Section 7.2, we explain how three existing models are an instance of our recipe for normal models (Theorem 7.1.1), and are thus adequate for $\mathsf{PCF_v}$. The three models are the categories of ω-quasi-Borel spaces, ω-diffeological spaces, and $\omega\mathbf{PAP}$ spaces. These examples were sketched in [MMS22, Examples 3.6-3.8]. In Chapter 8, we will present yet another example of our recipe, with a different flavour, a fully abstract model for $\mathsf{PCF_v}$.

Finally, in Section 7.4, we discuss related work.

7.1 A recipe for building adequate normal models

In this section, we use the main theorems in the previous two chapters, Theorem 5.1.6 and Theorem 6.2.5, to give a way of constructing normal models (Definition 4.3.1) for $\mathsf{PCF_v}$ with type and term constants. Recall from Example 4.2.1 that a type constant of interest might be real with a set of term constants $Val_{\mathsf{real}} \cong \mathbb{R}$ and also with a term constant for each measurable function $\mathbb{R}^n \to \mathbb{R}$.

The most important data we need to be able to construct a normal model is a triple $(\mathbb{C}, J, \mathcal{M})$ of a concrete site (Definition 2.1.6) with a class of admissible monos (Definition 6.2.1). The construction of normal models is detailed in Theorem 7.1.1 and its proof. It crucially relies on combining, via Definition 6.2.3, the site $(\mathbb{C}, J, \mathcal{M})$ with the site for vertical natural numbers $(\mathbb{V}_0, J_{\mathbb{V}_0}, \mathcal{M}_{\mathbb{V}_0})$, defined in Section 5.3. As explained in Remark 6.3.13, adding the vertical natural numbers to the site is an, arguably more modular, alternative to considering ωcpo valued sheaves. We then prove in Theorem 7.1.3 that the normal models we obtain in this way are adequate.

Theorem 7.1.1 (Recipe for building normal models)**.** *Consider a concrete site* $(\mathbb{C}, J)$*, satisfying the (M) and (L) axioms (Remark 2.1.2), with an initial object covered by the empty set, and with a class of admissible monos* $\mathcal{M}$ *(Definition 6.2.1).*

Assume that for each type constant α*, there is a* concrete sheaf A_α *in the category* $\mathsf{Sh}(\mathbb{C}, J)$*, such that the points of* A_α *are in bijection with* Val_α *(the set of values of type* α *introduced in Section 4.2.1):*

$$|A_\alpha| \cong \mathit{Val}_\alpha.$$

Assume that for each term constant $f : (\alpha_1 \times \ldots \times \alpha_n) \to \beta$*, there is a morphism of concrete sheaves*

$$\phi_f : (A_{\alpha_1} \times \ldots \times A_{\alpha_n}) \to L_{\mathcal{M}}(A_\beta)$$

that agrees on points with the partial function $f : (\mathit{Val}_{\alpha_1} \times \ldots \times \mathit{Val}_{\alpha_n}) \rightharpoonup \mathit{Val}_\beta$*. Here* $L_{\mathcal{M}}$ *is the lifting monad on* $\mathsf{Sh}(\mathbb{C}, J)$ *generated by* $\mathcal{M}$*.*

Then the category $\mathsf{Sh}(\mathbb{C}+\mathbb{V}_0, J \cup J_{\mathbb{V}_0})$ *is a normal model (Definition 4.3.1) of* $\mathsf{PCF_v}$ *with the given type and term constants.*

Proof. In this case, the Grothendieck topos from the definition of normal model (Definition 4.3.1) is the category of sheaves $\mathsf{Sh}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$.

Both $\mathcal{M}$ and $\mathcal{M}_{\mathbb{V}_0}$ are classes of admissible monos, so, by Proposition 6.2.4, their combination $\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}$ is also a class of admissible monos in the site $(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$. Thus, $\mathsf{Sh}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$ has a dominance $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ generated by $\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}$ via Theorem 5.1.6. From Proposition 5.2.6, it follows that the dominance $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ classifies the subobject $0 \rightarrowtail 1$.

Again thanks to Proposition 6.2.4, we can apply Theorem 6.2.5 to deduce that the lifted natural numbers object $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\coprod_{\mathbb{N}} 1)$ in $\mathsf{Sh}(\mathbb{C}+\mathbb{V}_0, J \cup J_{\mathbb{V}_0})$ is $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$-complete, (where $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ is the lifting monad obtained from the dominance $\Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$).

Now we deal with the requirements for type constants. For each type constant α, we need to choose an object A'_α in $\mathsf{Sh}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$, and a mapping of Val_α into the points of A'_α. Define for any other object c in $\mathbb{C}+\mathbb{V}_0$

different from V:

$$A'_\alpha(c) = A_\alpha(c),$$

and

$$A'_\alpha(\mathrm{V}) = \{h : |\mathrm{V}| \to |A'_\alpha| \mid h \text{ constant}\}.$$

To show that A'_α is a sheaf, it is enough to show it satisfies separately the sheaf conditions given by J and $J_{\mathbb{V}_0}$ respectively. The conditions for J are satisfied because A'_α agrees with A_α. The condition for $J_{\mathbb{V}_0}$ (Definition 5.3.2) is trivial. Moreover, A'_α is a *concrete* sheaf because A_α is concrete. Therefore, from the assumptions there is a bijection:

$$|A'_\alpha| = |A_\alpha| \cong \mathit{Val}_\alpha.$$

It remains to show that $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$ is $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$-complete. Recall that from Proposition 6.1.1, it is enough to show that $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$ is orthogonal to the maps $\omega_P \times y(c) \to \overline{\omega} \times y(c)$ from the ambient presheaf category. As an intermediate step we show that $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$ is orthogonal to $\omega_P \to \overline{\omega}$. This is the same strategy we followed for $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$ in Section 6.3, and indeed the proof for $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$ that follows subsumes that proofs of Lemmas 6.3.9 to 6.3.11 for $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(\mathsf{Nat})$.

Showing $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$ is orthogonal to $\omega_P \to \overline{\omega}$. Recall the isomorphism $\omega_P(\star) \cong \mathbb{N}$ (Remark 6.3.1), so maps from ω_P into a concrete presheaf can be thought of as sequences of points of that presheaf.

Consider a map $f : \omega_P \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$. If we postcompose by

$$L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(!) : L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha) \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(1)$$

we know that $\left(L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(!) \circ f\right) : \omega_P \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(1) \cong \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ is an infinite monotone binary sequence, from Definition 6.3.2 and Lemma 6.3.3. If the sequence is always 0, then $f_\star$ is always $\bot$, so its extension to $\overline{\omega} \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$ is also always $\bot$. (Recall from Remark 6.3.1 that $\overline{\omega}(\star) \cong \mathbb{N} + \{\infty\}$.)

If the sequence becomes 1, then $f_\star$ is a sequence made of a finite number of $\bot$'s followed by values from $|A'_\alpha|$. We show that in fact the value from

$|A'_\alpha|$ must be constant, so f has an obvious extension to $\overline{\omega} \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$, with the same value from $|A'_\alpha|$.

Consider the map:

$$f' : \omega_P \to A'_\alpha$$

obtained from f by removing the $\perp$'s from the beginning of the sequence. We show that $f'_\star$ is constant.

Recall from Section 6.3.1 that the elements of $\omega_P(\mathrm{V})$ are eventually 0 infinite chains of $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobjects of V. Alternatively, because ω_P is concrete, they can be regarded as functions $|\mathrm{V}| \to \omega_P(\star)$. From the definition of $\mathcal{M}_{\mathbb{V}_0}$ (Definition 5.3.4), we can see that $\omega_P(\mathrm{V})$ contains exactly the eventually constant monotone sequences valued in $\mathbb{N}$.

By concreteness, the action of f'_V on a function $|\mathrm{V}| \to \omega_P(\star)$ is given by post-composition with $f'_\star$. Because $A'_\alpha(\mathrm{V})$ contains only constant functions $|\mathrm{V}| \to |A'_\alpha|$, it must be the case that $f'_\star$ is constant.

Showing that maps $\alpha : \omega_P \times y(c) \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$ are increasing chains. Recall that an element of $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)(c)$ is an $(\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0})$-subobject $c' \rightarrowtail c$ together with a function $h : |c'| \to |A'_\alpha|$ in $A'_\alpha(c')$. Therefore by Yoneda, for each $n \in \omega_P(\star)$, $\alpha(n, -)$ is given by a pair:

$$\big(m_n : c_n \rightarrowtail c,\ g_n : |c_n| \to |A'_\alpha|\big).$$

Consider the following composite map:

$$\omega_P \times y(c) \xrightarrow{\alpha} L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha) \xrightarrow{L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(!)} L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(1) \cong \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}.$$

For each $n \in \omega_P(\star)$, it is given by the subobject $m_n : c_n \rightarrowtail c$. We already proved in Lemma 6.3.6 that these subobjects must form an increasing chain $\{m_n : c_n \rightarrowtail c\}_{n\in\mathbb{N}}$.

If we fix a point $x \in |c|$, then we proved in the previous paragraph that the map $\alpha(-, x) : \omega_P \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$ is a chain that is either constant $\perp$ or a finite number of $\perp$'s followed by a constant value from $|A'_\alpha|$. This means that each function $g_{n+1} : |c_{n+1}| \to |A'_\alpha|$ must extend the previous one:

$$\forall x \in |c_n|.\ g_{n+1}(x) = g_n(x).$$

Thus the map $\alpha : \omega_P \times y(c) \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$ is actually an increasing chain of tuples, each consisting of an $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$-subobject and a function from $A'_\alpha(c_n)$:

$$\big(m_n : c_n \rightarrowtail c,\ g_n : |c_n| \to |A'_\alpha|\big)_{n\in\mathbb{N}}.$$

Showing $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$ is orthogonal to $\omega_P \times y(c) \to \overline{\omega} \times y(c)$. To define a candidate extension $\overline{\alpha}$ for $\alpha : \omega_P \times y(c) \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$, we only need to choose a value for $\overline{\alpha}(\infty, -)$. Let this be given by the tuple:

$$\big(m_\infty : c_\infty \rightarrowtail c,\ g_\infty : |c_\infty| \to |A'_\alpha|\big)$$

where m_∞ is the subobject determined by the points $|c_\infty| = \bigcup_{n\in\mathbb{N}}|c_n|$; m_∞ is in $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$ by assumption (3) in the definition of admissible monos. The function g_∞ is defined as $\bigcup_{n\in\mathbb{N}} g_n$.

To show that g_∞ is in $A'_\alpha(c_\infty)$ we use the fact that A'_α is a sheaf. From assumption (4) for admissible monos, we know that the sieve obtained by closing under precomposition with any map the set $\{m_n : c_n \rightarrowtail c_\infty\}_{n\in\mathbb{N}}$ contains a covering family of c_∞. Therefore, we know from Proposition 2.1.11 that the sheaf condition must hold for the whole sieve as well. The set $\{g_n : c_n \to |A'_\alpha|\}_{n\in\mathbb{N}}$ is a matching family for $\{m_n : c_n \rightarrowtail c_\infty\}_{n\in\mathbb{N}}$ and hence for the whole sieve. The amalgamation of this matching family is g_∞, so because A'_α is a sheaf, g_∞ must be in $A'_\alpha(c_\infty)$.

We need to show that the proposed extension $\overline{\alpha}$ is natural. Let $\overline{h}$ be the unique extension calculated in Lemma 6.3.7 of the composite:

$$\omega_P \times y(c) \xrightarrow{\alpha} L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\alpha) \xrightarrow{L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(!)} L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(1) \cong \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}.$$

So $\overline{h}$ is described by the increasing chain with sup $\{m_n : c_n \rightarrowtail c\}_{n\in\mathbb{N}\cup\{\infty\}}$.

Then $\overline{\alpha}_\star$ factorizes as:

$$|\overline{\omega}| \times |c| \xrightarrow{\langle \overline{h}_\star, \mathrm{id}_{|c|} \circ \pi_2 \rangle} |\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}| \times |c| \xrightarrow{\beta_\star} |L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)|$$

where $\beta_\star$ is defined to be:

$$\beta_\star(0, x) = \bot$$
$$\beta_\star(1, x) = \begin{cases} g_\infty(x) & \text{if } x \in |c_\infty| \\ \bot & \text{if } x \notin |c_\infty|. \end{cases}$$

Therefore, it is enough to show that $\beta_\star$ determines a natural transformation.

For this, consider a pair $(m : c'' \rightarrowtail c',\ f : c' \to c)$ from $\Delta_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(c') \times \mathsf{Hom}(c', c)$. It gets mapped by $\beta_\star$ to the pair

$$\big(X \rightarrowtail c',\ (g_\infty \circ f)\big|_{|X|} : |X| \to |A'_\alpha|\big),$$

where $X \rightarrowtail c'$ is the subobject obtained in the following pullback diagram:

$$\begin{array}{ccccc} c'' & \overset{m}{\rightarrowtail} & c' & \overset{f}{\longrightarrow} & c \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle m_\infty} \\ X & \overset{z_2}{\rightarrowtail} & \cdot & \overset{z_1}{\longrightarrow} & c_\infty \end{array}$$

Because admissible monos are closed under pullback and composition we know that $X \rightarrowtail c'$ is in $(\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0})$. The map $(g_\infty \circ f)\big|_{|X|}$ is obtained by applying the functorial action of $A'_\alpha(z_1 \circ z_2)$ to g_∞; the action is precomposition because A'_α is concrete. Thus $(g_\infty \circ f)\big|_{|X|}$ is in $A'_\alpha(X)$.

Next, we show that $\overline{\alpha}$ is the unique extension of α. For any $x \in |c|$, we proved already that the map $\alpha(-, x) : \omega_P \to L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$ has a unique extension. But $\overline{\alpha}(-, x)$ is an extension, therefore $\overline{\alpha}$ must be unique. This concludes the proof that $L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}(A'_\alpha)$ is an $L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$-complete object.

The last requirement from the definition of normal model is to find, for each term constant $f : (\alpha_1 \times \ldots \times \alpha_n) \to \beta$, a morphism

$$\phi'_f : (A'_{\alpha_1} \times \ldots \times A'_{\alpha_n}) \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\beta)$$

that agrees on points with the corresponding partial function

$$f : (\mathit{Val}_{\alpha_1} \times \ldots \times \mathit{Val}_{\alpha_n}) \rightharpoonup \mathit{Val}_\beta.$$

Let $(\phi'_f)_\star = (\phi_f)_\star$; we already know $(\phi_f)_\star$ agrees with f.

We already know ϕ_f is a natural transformation, and that for any $c \in \mathbb{C}$:

$$L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\beta)(c) = L_{\mathcal{M}}(A_\beta)(c),$$

so we only need to check that $(\phi_f)_\star$ sends, by post-composition, elements of $A'_{\alpha_1}(\mathrm{V}) \times \ldots \times A'_{\alpha_n}(\mathrm{V})$ to elements of $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\beta)(\mathrm{V})$. From Proposition 5.1.8 and the fact that A'_β is concrete, we can see that $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(A'_\beta)(\mathrm{V})$ contains functions $|\mathrm{V}| \to |A'_\beta| \uplus \{\bot\}$, which are either constant $\bot$ or a finite number of $\bot$'s followed by a value from $|A'_\beta|$. Because $A'_{\alpha_1}(\mathrm{V}) \times \ldots \times A'_{\alpha_n}(\mathrm{V})$ contains only constant functions $|\mathrm{V}| \to |A'_{\alpha_1}| \times \ldots \times |A'_{\alpha_n}|$ we are done.

□

The interpretation in the $\mathsf{Sh}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$ model of $\mathsf{PCF_v}$ with new type constants is the one described in Section 4.3, where the lifting monad is $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$. From Theorem 4.3.5, we know that this interpretation is sound. We know from Proposition 4.3.4 that for any $\mathsf{PCF_v}$ type τ, its (lifted) interpretation $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau]\!]$ is an $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$-complete object.

We can prove further that every type is interpreted as a *concrete* sheaf in $\mathsf{Sh}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$. Therefore, we will sometimes (e.g in Section 7.2) refer to the normal model from Theorem 7.1.1 as the model in the category of *concrete* sheaves $\mathsf{Conc}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$.

Proposition 7.1.2. *Under the assumption in Theorem 7.1.1, in the normal model* $\mathsf{Sh}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$*, the sheaf* $[\![\tau]\!]$ *interpreting a* $\mathsf{PCF_v}$ *type is concrete.*

Proof. The base type nat is interpreted as the infinite coproduct $\coprod_{\mathbb{N}} 1$. In Remark 5.2.7 we described this explicitly as:

$$\llbracket \mathsf{nat} \rrbracket(\star) = \mathbb{N}$$
$$\llbracket \mathsf{nat} \rrbracket(c) = \{f : |c| \to \mathbb{N} \mid f \text{ locally constant on a cover of } c\}.$$

So we can immediately see that $\llbracket \mathsf{nat} \rrbracket$ is concrete. The initial and terminal objects 0 and 1 are also concrete.

The type constants are interpreted as $\llbracket \alpha \rrbracket = A'_\alpha$, where A'_α was constructed in the proof of Theorem 7.1.1 to be concrete.

We know from Proposition 5.2.5 that the lifting monad $L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$ preserves concreteness . It is a standard fact that concreteness is preserved by product and coproduct, and that concrete sheaves are an exponential ideal (Proposition 2.1.9). Therefore, all $\mathsf{PCF_v}$ types must be concrete. □

Because types are concrete sheaves, they admit an explicit description in terms of sets and relations, which is shown in Figure 7.1. The lifting monad $L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$ also has an explicit description, as in Figure 7.2, obtained using Proposition 5.1.8. The strong monad structure at $\star$ is the same as that of the lifting monad on Set.

Theorem 7.1.3 (Adequacy)**.** *Under the assumptions in Theorem 7.1.1, the normal model* $\mathsf{Sh}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$ *(or* $\mathsf{Conc}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$*) is an adequate model for* $\mathsf{PCF_v}$ *with type and term constants.*

More precisely, if τ is a ground type ($\mathsf{0}$, $\mathsf{1}$, nat *or α), and t is a closed computation of type τ, $\vdash^{\mathbf{c}} t : \tau$, then*

$$\llbracket t \rrbracket = \eta_{\llbracket \tau \rrbracket} \circ \llbracket v \rrbracket \qquad : 1 \to L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} \llbracket \tau \rrbracket$$

implies that t reduces to v, $t \Downarrow v$.

Remark 7.1.4. For ground types, adequacy implies that if t diverges, then $\llbracket t \rrbracket$ corresponds to $\perp$ from $|L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} \llbracket \tau \rrbracket|$. This is because of concreteness and the fact that $L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$ only adds a $\perp$ element to the underlying set of each ground type.

$$|[\![\mathsf{1}]\!]| \cong 1 \qquad [\![\mathsf{1}]\!](c) \cong 1$$

$$|[\![\mathsf{0}]\!]| = \emptyset \qquad [\![\mathsf{0}]\!](c \neq 0) = \emptyset \qquad [\![\mathsf{0}]\!](0) \cong 1$$

$$|[\![\mathsf{nat}]\!]| = \mathbb{N}$$

$$[\![\mathsf{nat}]\!](c) = \big\{ f : |c| \to \mathbb{N} \;\big|\; \exists \{g_i : c_i \to c\}_{i \in I} \in (J \cup J_{\mathbb{V}_0})(c) \text{ s.t. each } f \circ g_i \text{ is constant} \big\}$$

$$|[\![\alpha]\!]| \cong \mathit{Val}_\alpha \qquad [\![\alpha]\!](c' \neq \mathrm{V}) = (A_\alpha)(c')$$

$$[\![\alpha]\!](\mathrm{V}) = \big\{ h : |\mathrm{V}| \to \mathit{Val}_\alpha \;\big|\; h \text{ constant} \big\}$$

$$|[\![\tau \to \tau']\!]| = \mathsf{Sh}([\![\tau]\!],\, L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!])$$

$$[\![\tau \to \tau']\!](c) = \big\{ f : |c| \to \mathsf{Sh}([\![\tau]\!],\, L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!]) \;\big|\; \forall h : d \to c \in (\mathbb{C} + \mathbb{V}_0),\ \forall g : |d| \to |[\![\tau]\!]| \in [\![\tau]\!](d).\ \lambda x \in |d|.\big(f(h(x))\ g(x)\big) \in L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!](d) \big\}$$

$$|[\![\tau \times \tau']\!]| = |[\![\tau]\!]| \times |[\![\tau']\!]|$$

$$[\![\tau \times \tau']\!](c) = \big\{ \langle f, g \rangle : |c| \to |[\![\tau]\!]| \times |[\![\tau']\!]| \;\big|\; f \in [\![\tau]\!](c),\ g \in [\![\tau']\!](c) \big\}$$

$$|[\![\tau + \tau']\!]| = |[\![\tau]\!]| + |[\![\tau']\!]|$$

$$[\![\tau + \tau']\!](c) = \big\{ f : |c| \to |[\![\tau]\!]| + |[\![\tau']\!]| \;\big|\; \exists \{g_i : c_i \to c\}_{i \in I} \in (J \cup J_{\mathbb{V}_0})(c) \text{ s.t. for each } i,\ (f \circ g_i) \in [\![\tau]\!](c_i) \text{ or } (f \circ g_i) \in [\![\tau']\!](c_i) \big\}$$

Figure 7.1: Explicit description of the interpretation of $\mathsf{PCF_v}$ in the category $\mathsf{Sh}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$, where c is any object in $\mathbb{C} + \mathbb{V}_0$.

$$|L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau]\!]| = |[\![\tau]\!]| \uplus \{\bot\}$$

$$(L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau]\!])(c) = \big\{g : |c| \to |[\![\tau]\!]| \uplus \{\bot\} \;\big|\; \exists c' \rightarrowtail c \in (\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}) \text{ s.t. } g^{-1}(|[\![\tau]\!]|) = \mathsf{Im}(|c'|) \text{ and } g|_{\mathsf{Im}(|c'|)} \in [\![\tau]\!](c')\big\}.$$

$$|\eta_{[\![\tau]\!]}| : |[\![\tau]\!]| \to |L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau]\!]|$$

$$|\eta_{[\![\tau]\!]}|(x) = x$$

$$|\mu_{[\![\tau]\!]}| : |L^2_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau]\!]| \to |L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau]\!]|$$

$$|\mu_{[\![\tau]\!]}|(x) = x \qquad |\mu_{[\![\tau]\!]}|(\bot_1) = |\mu_{[\![\tau]\!]}|(\bot_2) = \bot$$

$$|\mathrm{str}_{[\![\tau]\!],[\![\sigma]\!]}| : |[\![\tau]\!]| \times |L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\sigma]\!]| \to |L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}([\![\tau]\!] \times [\![\sigma]\!])|$$

$$|\mathrm{str}_{[\![\sigma]\!],[\![\tau]\!]}|(x,y) = (x,y), \qquad |\mathrm{str}_{[\![\sigma]\!],[\![\tau]\!]}|(x,\bot) = \bot$$

Figure 7.2: Explicit description of the lifting monad on the interpretation of $\mathsf{PCF_v}$ types in $\mathsf{Sh}(\mathbb{C}+\mathbb{V}_0, J\cup J_{\mathbb{V}_0})$.

Example 7.1.5. The category $\mathsf{Sh}(\mathbb{V}_0, J_{\mathbb{V}_0})$ (defined in Section 5.3) is a normal model for $\mathsf{PCF_v}$ without type constants obtained via the recipe in Theorem 7.1.1 in a trivial way. To satisfy the premises of the theorem, we can choose a trivial concrete site with an initial and terminal object only.

From Theorem 7.1.3 we can deduce that the $\mathsf{Sh}(\mathbb{V}_0, J_{\mathbb{V}_0})$ model is an adequate model of $\mathsf{PCF_v}$ without any type constants. Since $\mathsf{Sh}(\mathbb{V}_0, J_{\mathbb{V}_0})$ is equivalent to the vSet model, defined in Section 4.4, vSet is also an adequate model of $\mathsf{PCF_v}$. This is not surprising since we showed in Proposition 4.4.10 that the vSet model is essentially the traditional ωCPO model.

7.2 Examples of concrete sheaf models

In this section, we consider three examples of normal models which are already known and explain how they can each be seen as an instance of our recipe from Theorem 7.1.1 of constructing normal models. The three examples are:

- the category of ω-quasi-Borel spaces, ω**Qbs** [VKS19, HKSY17], a model of probabilistic programming;
- the category of ω-diffeological spaces, ω**Diff** [HSV20, Vák20], a model of differentiable programming;
- the category of ω**PAP** spaces [LHM21], a variation on ω-diffeological spaces allowing some non-smoothness.

In all cases we consider the interpretation of $\mathsf{PCF_v}$ with the additional type constant real and appropriate term constants, as discussed in Example 4.2.1.

In each case, we prove that there is an embedding of the original model into a category of concrete sheaves generated using our recipe (Propositions 7.2.9, 7.2.18 and 7.2.28). Therefore, we can deduce soundness and adequacy for $\mathsf{PCF_v}$ with real in the three original models via Theorem 4.3.5 and Theorem 7.1.3 respectively.

In all examples, the language being modelled is actually richer than $\mathsf{PCF_v}$ with real, so the models have additional structure that we do not discuss, such as a probability monad on ω-quasi-Borel spaces, or an AD macro in the case of ω-diffeological spaces. Instead, our aim is to show that these three examples all model higher-order recursion in essentially the same way.

7.2.1 ω-quasi-Borel spaces

We sketch how the category of ω-quasi-Borel spaces [VKS19, HKSY17], a model for probabilistic computation, is an instance of our recipe for building normal models from Theorem 7.1.1. The category of ω-quasi-Borel spaces was introduced to model the combination of higher-order recursive programs and programs that can express continuous distributions over types.

Definition 7.2.1. The concrete site $(\mathbf{Sbs}, J_{\mathbf{Sbs}})$ consists of the category of standard Borel spaces, **Sbs** (e.g. [Kal02, Appendix A1]), which has objects the Borel subsets of $\mathbb{R}$ and morphisms all measurable functions between them. For each Borel subset U, the coverage $J_{\mathbf{Sbs}}(U)$ is made up of the countable sets of measurable inclusion functions

$$\{U_i \hookrightarrow U\}_{i \in I}$$

such that $U = \bigcup_{i \in I} U_i$ and the U_i's are disjoint.

Proposition 7.2.2. *The pair* $(\mathbf{Sbs}, J_{\mathbf{Sbs}})$ *is indeed a concrete site with an initial object covered by the empty set, and satisfies axioms (M) and (L) (Remark 2.1.2) as well.*

Proof. To prove the main coverage axiom consider a covering family $\{U_i\}_{i \in I}$ of U and a measurable function $f : V \to U$. The pullback of each : $U_i \hookrightarrow U$ with f is $f^{-1}(U_i)$, as each $f^{-1}(U_i)$ is a measurable subset of V, so $\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$ is a covering family of V, where each $f^{-1}(U_i) \to V$ factors through $U_i \hookrightarrow U$.

Axiom (M) is satisfied because $\{U \hookrightarrow U\}$ is a covering family of U. Axiom (L) follows from the fact that composing measurable inclusions gives a measurable inclusion. The initial object in **Sbs** is the empty set $\emptyset$ and so it is indeed covered by the empty family.

To prove that $(\mathbf{Sbs}, J_{\mathbf{Sbs}})$ is concrete notice that the terminal object is the singleton set 1. It is easy to see that maps of the form $\mathbf{Sbs}(V, U) \to \mathsf{Set}(V, U)$ are injective because **Sbs** morphisms are functions. For every covering family $\{U_i\}_{i\in I}$ of U the map

$$\coprod_{i\in I} \mathbf{Sbs}(1, U_i) \to \mathbf{Sbs}(1, U)$$

is surjective because $\bigcup_{i\in I} U_i = U$. □

Definition 7.2.3. Let $\mathcal{M}_{\mathbf{Sbs}}$ be the class of all monomorphisms in $(\mathbf{Sbs}, J_{\mathbf{Sbs}})$.

In fact, all monos in **Sbs** map Borel sets to Borel sets, so they are all isomorphic to an inclusion. Therefore, we will treat any mono as an inclusion.

Proposition 7.2.4. *The class $\mathcal{M}_{\mathbf{Sbs}}$ is a class of admissible monos.*

Proof. One can check that $\mathcal{M}_{\mathbf{Sbs}}$ contains all isomorphisms and is closed under composition. Pullbacks of monos are again monos so $\mathcal{M}_{\mathbf{Sbs}}$ is a stable class.

Denote by $\Delta_{\mathbf{Sbs}}$ the presheaf obtained from $\mathcal{M}_{\mathbf{Sbs}}$. To show $\Delta_{\mathbf{Sbs}}$ is a sheaf for $J_{\mathbf{Sbs}}$ consider a covering family $\{U_i\}_{i\in I}$ of U, and a family of monos $\{U_i' \rightarrowtail U_i\}_{i\in I}$. Assuming U_i' is a Borel subset of U_i, then $\bigcup_{i\in I} U_i'$ is also a Borel subset, because they are closed under countable union. The mono $\bigcup_{i\in I} U_i' \rightarrowtail U$ is the unique amalgamation that we need.

It remains to check the conditions in Definition 6.2.1 for a class of admissible monos. Every map $\emptyset \to U$ is a mono so is part of $\mathcal{M}_{\mathbf{Sbs}}$.

The map $\Delta_{\mathbf{Sbs}}(U) \to [U \to \Delta_{\mathbf{Sbs}}(\star)]$ is an injection because $\Delta_{\mathbf{Sbs}}(\star)$ has exactly two elements and $\Delta_{\mathbf{Sbs}}(U)$ contains the Borel subsets of U. Hence $\Delta_{\mathbf{Sbs}}$ is concrete.

For the third condition, consider a countable chain of Borel subsets $(U_n \hookrightarrow U)_{n\in\mathbb{N}} \in \mathcal{M}_{\mathbf{Sbs}}$. Then the union $\bigcup_{n\in\mathbb{N}} U_n$ is also a Borel subset. For the fourth condition, notice that because Borel subsets are closed under complement the following set is a cover of $\bigcup_{n\in\mathbb{N}} U_n$:

$$\{U_0\} \cup \{U_{i+1} \setminus U_i \mid i \in \mathbb{N}\}.$$

These subsets are part of the closure under precomposition of the set $\{U_i \hookrightarrow \bigcup_{n\in\mathbb{N}} U_n \mid i \in \mathbb{N}\}$, as required. $\square$

Recall the type real from Example 4.2.1, with $\mathit{Val}_{\mathsf{real}} \cong \mathbb{R}$, which we add to $\mathsf{PCF_v}$ in the ω-quasi-Borel spaces case.

Definition 7.2.5. Define the interpretation of the type constant real in the category of sheaves $\mathsf{Sh}(\mathbf{Sbs}, J_{\mathbf{Sbs}})$ to be:

$$A_{\mathsf{real}} \cong y(\mathbb{R}).$$

For each term constant $f : \mathsf{real}^n \to \mathsf{real}$, corresponding to a measurable function $\mathbb{R}^n \to \mathbb{R}$, let

$$\phi_f : (A_{\mathsf{real}} \times \ldots \times A_{\mathsf{real}}) \to L_{\mathcal{M}_{\mathbf{Sbs}}}(A_{\mathsf{real}})$$

be given at $\star$ by this measurable function.

Proposition 7.2.6. *The interpretation of* real *from Definition 7.2.5, given by* A_{real} *and* ϕ_f*, satisfies the assumptions for type and term constants from Theorem 7.1.1.*

Proof. First of all, the representable $y(\mathbb{R})$ is a *concrete* sheaf because $y(\mathbb{R})(U)$ contains all measurable functions $U \to \mathbb{R}$. Therefore ϕ_f is uniquely determined by its component at $\star$ and it is a morphism of concrete sheaves because measurable functions are closed under composition. Moreover, we have isomorphisms:

$$|A_{\mathsf{real}}| \cong |y(\mathbb{R})| \cong \mathbb{R} \cong \mathit{Val}_{\mathsf{real}}$$

so all the assumptions in Theorem 7.1.1 are satisfied. $\square$

Proposition 7.2.7. *The category* $\mathsf{Sh}(\mathbf{Sbs}+\mathbb{V}_0, J_{\mathbf{Sbs}} \cup J_{\mathbb{V}_0})$ *is a normal model of* $\mathsf{PCF_v}$ *with* real*, obtained using the recipe from Theorem 7.1.1. The interpretation of the language is contained in the subcategory of concrete sheaves* $\mathsf{Conc}(\mathbf{Sbs} + \mathbb{V}_0, J_{\mathbf{Sbs}} \cup J_{\mathbb{V}_0})$, *Proof.* Combining the results from Propositions 7.2.2, 7.2.4 and 7.2.6 we see that the site $(\mathbf{Sbs}, J_{\mathbf{Sbs}})$, with the class of monos $\mathcal{M}_{\mathbf{Sbs}}$ and the interpretation of real from Definition 7.2.5, satisfies all the assumptions of Theorem 7.1.1. We proved in Proposition 7.1.2 that in any model obtained from Theorem 7.1.1, the interpretation actually lives in the subcategory of concrete sheaves. □

Recall from Figure 7.1 that in the $\mathsf{Conc}(\mathbf{Sbs} + \mathbb{V}_0, J_{\mathbf{Sbs}} \cup J_{\mathbb{V}_0})$ model the type $[\![\mathsf{real}]\!]$ at V contains only the constant functions $|\mathrm{V}| \to \mathbb{R}$. This is analogous to giving $\mathbb{R}$ the discrete ordering, if we were to use cpo-valued sheaves instead of the vertical natural numbers site.

Proposition 7.2.9 makes precise the idea that the model of $\mathsf{PCF_v}$ in $\omega\mathbf{Qbs}$ is the same as the $\mathsf{Conc}(\mathbf{Sbs}+\mathbb{V}_0, J_{\mathbf{Sbs}} \cup J_{\mathbb{V}_0})$ model, and therefore an example of Theorem 7.1.1. We first recall the definition of $\omega\mathbf{Qbs}$ and of a lifting monad L on $\omega\mathbf{Qbs}$ from [VKS19, Section 3].

Definition 7.2.8 ([VKS19])**.** A quasi-Borel space (qbs) $X = (|X|, M_X)$ consists of a set $|X|$ and a set of functions $M_X \subseteq [\mathbb{R} \to |X|]$ called random elements, such that:

1. all constant functions are in M_X;
2. M_X is closed under precomposition with measurable functions on $\mathbb{R}$;
3. if $\mathbb{R} = \bigcup_{n \in \mathbb{N}} U_n$, where U_n are pairwise disjoint and Borel measurable, and $(\alpha_n : \mathbb{R} \to |X|) \in M_X$ for all n, then the co-pairing $[\alpha_n|_{U_n}]_{n\in\mathbb{N}}$ is in M_X.

A morphism $f : X \to Y$ between quasi-Borel spaces is a structure-preserving function $f : |X| \to |Y|$: if $\alpha \in M_X$ then $(f \circ \alpha) \in M_Y$. Denote by $\mathbf{Qbs}$ the category of qbses and their morphisms.

An ω-quasi-Borel space (ωqbs) X is a triple $X = (|X|, M_X, \leq_X)$ where $(|X|, M_X)$ is a qbs and $(|X|, \leq_X)$ is an ωcpo, and M_X, endowed with the pointwise order, is closed under pointwise sups of ω-chains. A morphism $f : X \to Y$ between ωqbses is a Scott-continuous function between their underlying ωcpos that is also a **Qbs** morphism between their underlying qbses. Denote by $\omega\mathbf{Qbs}$ the category of ωqbses and their morphisms.

A lifting monad L on $\omega\mathbf{Qbs}$ is defined on objects to be

$$LX = (|LX|, M_{LX}, \leq_{LX})$$

where:

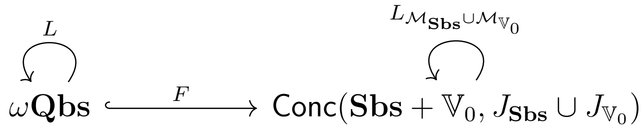
$$\begin{aligned} |LX| &= |X| + \{\bot\} \\ M_{LX} &= \big\{\beta : \mathbb{R} \to |X| + \{\bot\} \;\big|\; U = \beta^{-1}(|X|) \text{ is a Borel subset of } \mathbb{R} \\ &\qquad \text{and there exists } \alpha \in M_X \text{ such that } \alpha|_U = \beta|_U\big\} \end{aligned}$$

and the order $\leq_{LX}$ extends the order on $|X|$ by making $\bot$ the bottom element of $|LX|$.

For morphisms $f : X \to Y$, Lf maps $\bot$ to $\bot$. The strong monad structure of L is given by underlying functions between sets that act like those for the $(-) + 1$ monad on Set.

Proposition 7.2.9. *There is a functor* $F : \omega\mathbf{Qbs} \to \mathsf{Conc}(\mathbf{Sbs} + \mathbb{V}_0, J_{\mathbf{Sbs}} \cup J_{\mathbb{V}_0})$ *which is full, faithful, preserves products, coproducts and exponentials, and commutes with the lifting monad:*

$$FL = L_{\mathcal{M}_{\mathbf{Sbs}} \cup \mathcal{M}_{\mathbb{V}_0}} F$$



Moreover, for every $\omega\mathbf{Qbs}$ *object* X, FX *is a* $L_{\mathcal{M}_{\mathbf{Sbs}} \cup \mathcal{M}_{\mathbb{V}_0}}$*-complete sheaf, and* F *commutes with the interpretation of* $\mathsf{PCF_v}$ *(with the type constant* real*) in* $\omega\mathbf{Qbs}$ *and in* $\mathsf{Conc}(\mathbf{Sbs} + \mathbb{V}_0, J_{\mathbf{Sbs}} \cup J_{\mathbb{V}_0})$.

Proof. We sketch the definition of the functor F and omit the proof that it satisfies all the properties above. Consider an $\omega\mathbf{Qbs}$ object X with underlying ωcpo structure $|X|$ and random elements $M_X \subseteq [\mathbb{R} \to |X|]$.

If $|X|$ is non-empty, define the concrete sheaf FX at an object U in **Sbs** to be:

$$FX(U) = \{f : U \to |X| \mid \exists g \in M_X.\ g|_U = f\}.$$

At V, F is defined as:

$$FX(\mathrm{V}) = \{f : |\mathrm{V}| \to |X| \mid f \text{ is an } \omega \text{ chain from } |X| \text{ with its sup}\}.$$

If $|X|$ is empty the definition of F is almost the same except we choose $FX(\emptyset) \cong 1$. The action of a map $FX(g : V \to U)$ is precomposition.

When checking that F commutes with the interpretation of $\mathsf{PCF_v}$ types, the most interesting part is to check that it preserves fixed points. This can be done the same way as in the proof that the ωCPO model embeds into the vSet model (Proposition 4.4.10). □

It is worth mentioning that there are other sites which give rise to a category of sheaves equivalent to ω**Qbs**. For example, we could consider the one object category of the real numbers and all its measurable endomorphisms, and a suitable coverage. However, this would not be a concrete site, and so it would not immediately fit into our recipe of generating a normal model.

The role of the coverage $J_{\mathbf{Sbs}}$ from Definition 7.2.1 is to ensure that, for example, given a measurable subset U of $\mathbb{R}$, the coproduct $(U + (\mathbb{R} \setminus U)) \cong \mathbb{R}$ is preserved when passing from **Sbs** to sheaves via the sheafified Yoneda embedding ay (see Proposition 2.1.5). We would like such coproducts to be preserved so that $ay(\mathbb{R})$ has suitable structure for interpreting the type real.

The idea of using sheaf conditions to preserve colimits, that are then used to interpret datatypes, goes back to Fiore and Simpson [FS99], who studied full definability for a typed lambda-calculus with sums. The sheaf conditions in the ω-diffeological spaces and ω**PAP** examples will play a similar role.

7.2.2 ω-diffeological spaces

We show how the category of ω-diffeological spaces [HSV20, Vák20], a model of differentiable programming, is an instance of our recipe for building normal models (Theorem 7.1.1). This model was introduced in order to prove correctness of automatic differentiation methods on higher-order languages with recursion.

Definition 7.2.10. Consider the site $(\mathbf{Cart}, J_{\mathbf{Cart}})$ where the objects of $\mathbf{Cart}$ are the open subsets $U \subseteq \mathbb{R}^n$ for any $n \in \mathbb{N}$, and the morphisms are smooth maps between these open subsets. The coverage at an object U, $J_{\mathbf{Cart}}(U)$, contains all the countable sets of inclusion functions $\{U_i \hookrightarrow U\}_{i \in I}$ such that

$$U = \bigcup_{i \in I} U_i.$$

Proposition 7.2.11. *The pair* $(\mathbf{Cart}, J_{\mathbf{Cart}})$ *is a concrete site with an initial object covered by the empty set, satisfying the (M) and (L) axioms (Remark 2.1.2).*

Proof. For the main coverage axiom consider a covering family $\{U_i\}_{i \in I}$ of U and a smooth map $f : V \to U$. The pullback of $U_i \hookrightarrow U$ along f is $f^{-1}(U_i)$. Then $\{f^{-1}(U_i)\}_{i \in I}$ covers V and that each $f^{-1}(U_i)$ factors through U_i. Axioms (M) and (L) are satisfied because $\{U \hookrightarrow U\}$ is a cover and we can compose inclusion functions.

In $\mathbf{Cart}$, every object $\emptyset \subseteq \mathbb{R}^n$ is an initial object, all isomorphic to each other; they are all covered by the empty family according to Definition 7.2.10. The terminal is the singleton set $\mathbb{R}^0 \subseteq \mathbb{R}^0$. The category $\mathbf{Cart}$ is well-pointed and every covering family of U contains all the points of U, so the pair $(\mathbf{Cart}, J_{\mathbf{Cart}})$ is a concrete site. ☐

Definition 7.2.12. Let $\mathcal{M}_{\mathbf{Cart}}$ be the class of monomorphisms in the site $(\mathbf{Cart}, J_{\mathbf{Cart}})$ defined as:

$$\mathcal{M}_{\mathbf{Cart}}(U) = \{m : U' \rightarrowtail U \mid m \text{ isomorphic to an open inclusion}\}.$$

Proposition 7.2.13. *The class* $\mathcal{M}_{\mathbf{Cart}}$ *is a class of admissible monos.*

Proof. We can check that $\mathcal{M}_{\mathbf{Cart}}$ is a stable system of monos, according to Definition 2.4.1.

Let $\Delta_{\mathbf{Cart}}$ be the presheaf obtained from $\mathcal{M}_{\mathbf{Cart}}$. To show $\Delta_{\mathbf{Cart}}$ is a sheaf, consider a covering family $\{U_i\}_{i \in I}$ of U and a matching family $\{V_i \hookrightarrow U_i\}_{i \in I}$

for it. Notice that the U_i's do not have to be disjoint, but they cover all of U. The candidate amalgamation is

$$(V = \bigcup_{i \in I} V_i) \hookrightarrow U,$$

which is an open set because it is the union of open subsets. When we pull back $V \hookrightarrow U$ along $U_i \hookrightarrow U$ we get V_i thanks to the matching condition.

The proof that $\mathcal{M}_{\mathbf{Cart}}$ satisfies the conditions of a class of admissible monos is similar to the proof for $\mathcal{M}_{\mathbf{Sbs}}$ from Proposition 7.2.4. One difference is that when proving the fourth condition, given an increasing chain $(U_n \hookrightarrow U)_{n \in \mathbb{N}} \in \mathcal{M}_{\mathbf{Cart}}$, the covering family we choose for $\bigcup_{n \in \mathbb{N}} U_n$ is $\{U_n \mid n \in \mathbb{N}\}$. □

Definition 7.2.14. Recall the type constant real from Example 4.2.1, with $Val_{\mathsf{real}} \cong \mathbb{R}$. Define the interpretation of real in the category of sheaves $\mathsf{Sh}(\mathbf{Cart}, J_{\mathbf{Cart}})$ to be:

$$A_{\mathsf{real}} \cong y(\mathbb{R}).$$

For each term constant $f : \mathsf{real}^n \to \mathsf{real}$, corresponding to a smooth function $\mathbb{R}^n \to \mathbb{R}$, let

$$\phi_f : (A_{\mathsf{real}} \times \ldots \times A_{\mathsf{real}}) \to L_{\mathcal{M}_{\mathbf{Cart}}}(A_{\mathsf{real}})$$

be given at $\star$ by this smooth function.

Proposition 7.2.15. *The interpretation of* real *from Definition 7.2.14 satisfies the assumptions for type and term constants from Theorem 7.1.1.*

Proof. The representable $y(\mathbb{R})$ is a *concrete* sheaf because $y(\mathbb{R})(U)$ contains all smooth maps $U \to \mathbb{R}$. Therefore ϕ_f is uniquely determined by its component at $\star$ and is a morphism of concrete sheaves. Moreover, we have an isomorphism:

$$|A_{\mathsf{real}}| \cong Val_{\mathsf{real}}$$

so all the assumptions in Theorem 7.1.1 are satisfied. □

Proposition 7.2.16. *The category* $\mathsf{Sh}(\mathbf{Cart} + \mathbb{V}_0, J_{\mathbf{Cart}} \cup J_{\mathbb{V}_0})$ *is a normal model of* $\mathsf{PCF_v}$ *with* real*, obtained using the recipe from Theorem 7.1.1. The* *interpretation of the language is contained in the subcategory of concrete sheaves* $\mathsf{Conc}(\mathbf{Cart}+\mathbb{V}_0, J_{\mathbf{Cart}} \cup J_{\mathbb{V}_0})$.

Proof. By combining Propositions 7.2.11, 7.2.13 and 7.2.15, and finally using Proposition 7.1.2 for concreteness. $\square$

From Figure 7.1, we know that in the $\mathsf{Conc}(\mathbf{Cart}+\mathbb{V}_0, J_{\mathbf{Cart}} \cup J_{\mathbb{V}_0})$ model, the sheaf $[\![\mathsf{real}]\!]$ at V contains only the constant functions $|\mathrm{V}| \to \mathbb{R}$, which is analogous to equipping $\mathbb{R}$ with the discrete ordering.

Proposition 7.2.18 states that the model of $\mathsf{PCF_v}$ in $\omega\mathbf{Diff}$ is essentially the same as the model in $\mathsf{Conc}(\mathbf{Cart}+\mathbb{V}_0, J_{\mathbf{Cart}} \cup J_{\mathbb{V}_0})$, and thus is an example of our recipe for building normal models (Theorem 7.1.1). First, we recall the definition of $\omega\mathbf{Diff}$ and the lifting monad L on it from [Vák20].

Definition 7.2.17 ([Vák20])**.** A diffeological space $X = (|X|, \mathcal{P}_X)$ is a set $|X|$ together with, for each $n \in \mathbb{N}$ and each open subset U of $\mathbb{R}^n$, a set of functions $\mathcal{P}_X^U \subseteq [U \to |X|]$ called plots, such that:

1. all constant functions are in $\mathcal{P}_X^U$;
2. if $f : V \to U$ is a smooth function and $p \in \mathcal{P}_X^U$, then $p \circ f \in \mathcal{P}_X^V$;
3. if $\left(p_i \in \mathcal{P}_X^{U_i}\right)_{i \in I}$ is a compatible family, meaning that $\left(x \in U_i \cap U_j \implies p_i(x) = p_j(x)\right)$, and $U = \bigcup_{i \in I} U_i$, then the function $p : U \to |X|$, $p(x \in U_i) = p_i(x)$ is in $\mathcal{P}_X^U$.

A morphism $f : X \to Y$ between diffeological spaces is a function $f : |X| \to |Y|$ which preserves the structure: for all plots p in $\mathcal{P}_X^U$, the composite $f \circ p$ is in $\mathcal{P}_Y^U$. Denote by **Diff** the category of diffeological spaces and their morphisms.

An ω-diffeological space (ω-ds) is a triple $X = (|X|, \mathcal{P}_X, \leq_X)$ such that $(|X|, \mathcal{P}_X)$ is a diffeological space and $(|X|, \leq_X)$ is an ωcpo, satisfying the additional condition that, for any ω-chain $\left(\alpha_i \in \mathcal{P}_X^U\right)_{i \in \mathbb{N}}$ in the pointwise order, its pointwise sup is in $\mathcal{P}_X^U$. A morphism $f : X \to Y$ between ω-dses is a function $f : |X| \to |Y|$ which is Scott-continuous and is a morphism of diffeological spaces. Denote by $\omega\mathbf{Diff}$ the category of ω-dses and morphisms between them.

A lifting monad L on $\omega\mathbf{Diff}$ can be defined as follows. For an ω-ds X, let LX have underlying ωcpo structure $(|X| + \{\bot\}, \leq_{LX})$, where $\leq_{LX}$ extends the order $\leq_X$ by making $\bot$ the bottom element. The underlying diffeological space structure $(|X| + \{\bot\}, \mathcal{P}_{LX})$ of LX is defined as:

$$\mathcal{P}^U_{LX} = \big\{\alpha : U \to |X| + \{\bot\} \;\big|\; V = \alpha^{-1}(|X|) \text{ is an open subset of } U \\ \text{and } \alpha|_V \in \mathcal{P}^V_X\big\}.$$

The strong monad structure of L is the same as that of the $(-) + 1$ monad on Set.

Proposition 7.2.18. *There is a functor $F : \omega\mathbf{Diff} \to \mathsf{Conc}(\mathbf{Cart}+\mathbb{V}_0, J_{\mathbf{Cart}}\cup J_{\mathbb{V}_0})$ which is full, faithful, preserves products, coproducts and exponentials, and commutes with the lifting monad:*

$$FL = L_{\mathcal{M}_{\mathbf{Cart}} \cup \mathcal{M}_{\mathbb{V}_0}} F$$

$$\overset{L}{\circlearrowleft}\;\; \omega\mathbf{Diff} \xhookrightarrow{\;\;F\;\;} \mathsf{Conc}(\mathbf{Cart} + \mathbb{V}_0, J_{\mathbf{Cart}} \cup J_{\mathbb{V}_0}) \;\; \overset{L_{\mathcal{M}_{\mathbf{Cart}} \cup \mathcal{M}_{\mathbb{V}_0}}}{\circlearrowleft}$$

Moreover, for every $\omega\mathbf{Diff}$ object X, FX is a $L_{\mathcal{M}_{\mathbf{Cart}} \cup \mathcal{M}_{\mathbb{V}_0}}$-complete sheaf, and F commutes with the interpretation of $\mathsf{PCF_v}$ *(with the type constant* real*) in $\omega\mathbf{Diff}$ and in $\mathsf{Conc}(\mathbf{Cart} + \mathbb{V}_0, J_{\mathbf{Cart}} \cup J_{\mathbb{V}_0})$.*

Proof. Again, we only sketch the definition of F and omit the other details. An object X in $\omega\mathbf{Diff}$ comes with an underlying ωcpo $|X|$ and a set of plots $P^U_X \subseteq [U \to |X|]$ for each open subset U of each $\mathbb{R}^n$. Therefore we can define F to be:

$$FX(U) = P^U_X.$$

The definition of $FX(\mathrm{V})$ is similar to the $\omega\mathbf{Qbs}$ case (Proposition 7.2.9). $\square$

7.2.3 ωPAP spaces

The category of $\omega\mathbf{PAP}$ spaces is a variation of ω-diffeological spaces, proposed in [LHM21] to allow a certain degree of non-smoothness to be modelled. Some non-smoothness is desirable to accommodate programs that are not differentiable but widely used, like the ReLU function used in neural networks, $f(x) = \mathsf{max}(0, x)$, or an if-then-else construct. The category of $\omega\mathbf{PAP}$ spaces can also be seen as extending the work in [LYRY20] from first-order to higher-order programs; [LYRY20] were the first to introduce the notion of **PAP** function.

We now sketch how $\omega\mathbf{PAP}$ spaces are an instance of our recipe for building normal models (Theorem 7.1.1).

Definition 7.2.19. Define the site $(\mathbf{PAP}, J_{\mathbf{PAP}})$ to have as objects c-analytic subsets $U \subseteq \mathbb{R}^n$, for any $n \in \mathbb{N}$. A c-analytic subset is a countable union of analytic subsets. An analytic set is a subset of an open set carved out using a finite number of analytic inequalities [LHM21]. Stated differently, analytic sets are obtained by closing under finite intersections the following sets:

$$\{x \in U \mid f(x) \leq 0 \text{ for some analytic } f : U \to \mathbb{R}\}.$$

Notice that this includes open subsets.

Morphisms in **PAP** are functions $f : V \to U$ between c-analytic sets, meaning that the domain V has a disjoint partition of analytic subsets $\{A_i\}_{i\in I}$ and each $f|_{A_i}$ is analytic [LYRY20, LHM21].

The coverage $J_{\mathbf{PAP}}(U)$ is defined to contain the countable sets of (**PAP**) inclusions $\{A_i \hookrightarrow U\}_{i\in I}$ such that $U = \bigcup_{i\in I} A_i$ and the A_i's are disjoint c-analytic sets.

Remark 7.2.20. It can be shown that any c-analytic set is covered by a disjoint partition of analytic subsets [LH21]. Therefore, any inclusion $A_i \hookrightarrow U$ of c-analytic sets is **PAP**.

Proposition 7.2.21. *The pair* $(\mathbf{PAP}, J_{\mathbf{PAP}})$ *is a concrete site with an initial object covered by the empty set, satisfying the (M) and (L) axioms (Remark 2.1.2).*

Proof. For the main coverage axiom consider a cover $\{A_i \hookrightarrow U\}_{i\in I}$ of U, where the A_i's are c-analytic, and consider a **PAP** function $f : V \to U$. Because f is a co-pairing of analytic functions, and analytic functions pull back analytic subsets to analytic subsets, then $f^{-1}(A_i)$ is the union of analytic subsets, and hence is c-analytic. So the cover of V that we are looking for is $\{f^{-1}(A_i)\}_{i\in I}$. The function $f^{-1}(A_i) \to A_i$ is **PAP** because it is the restriction of f to a c-analytic subset.

Axioms (M) and (L) are easy to check.

Both the empty set $\emptyset \subseteq \mathbb{R}^n$, for any n, and the singleton $\mathbb{R}^0 \subseteq \mathbb{R}^0$ are c-analytic subsets, so they are the initial and terminal object respectively. Notice that the empty set is covered by the empty family. Showing that $(\mathbf{PAP}, J_{\mathbf{PAP}})$ is a concrete site is then done similarly to the $\omega\mathbf{Qbs}$ case (Proposition 7.2.2). □

Definition 7.2.22. Let $\mathcal{M}_{\mathbf{PAP}}$ be the class of monos in $(\mathbf{PAP}, J_{\mathbf{PAP}})$ where:

$$\mathcal{M}_{\mathbf{PAP}}(U) = \{m : U' \rightarrowtail U \mid U' \text{ c-analytic}, m \text{ isomorphic to an inclusion}\}.$$

Proposition 7.2.23. *The class $\mathcal{M}_{\mathbf{PAP}}$ is a class of admissible monos.*

Proof. Like in the $\omega\mathbf{Qbs}$ and $\omega\mathbf{Diff}$ cases, one can check that $\mathcal{M}_{\mathbf{PAP}}$ is a stable system of monos. Let $\Delta_{\mathbf{PAP}}$ be the presheaf obtained from $\mathcal{M}_{\mathbf{PAP}}$. Showing the sheaf condition holds for $\Delta_{\mathbf{PAP}}$ is similar to the $\omega\mathbf{Qbs}$ case, using the fact that c-analytic sets are closed under countable unions.

We omit the proof that $\mathcal{M}_{\mathbf{PAP}}$ satisfies the first three conditions needed for it to be a class of admissible monos. To show the fourth condition, given an increasing chain of monos $(U_n \hookrightarrow U)_{n\in\mathbb{N}}$ from $\mathcal{M}_{\mathbf{PAP}}$, the cover of $\bigcup_{n\in\mathbb{N}} U_n$ that we need to consider is the one from Remark 7.2.20 i.e. the disjoint partition by analytic subsets that every c-analytic set admits. □

Definition 7.2.24. Recall the type constant real from Example 4.2.1, with $Val_{\mathsf{real}} \cong \mathbb{R}$. Define the interpretation of real in the category of sheaves $\mathsf{Sh}(\mathbf{PAP}, J_{\mathbf{PAP}})$ to be:

$$A_{\mathsf{real}} \cong y(\mathbb{R}).$$

For each term constant $f : \mathsf{real}^n \to \mathsf{real}$, corresponding to a **PAP** function $\mathbb{R}^n \to \mathbb{R}$, let

$$\phi_f : (A_{\mathsf{real}} \times \ldots \times A_{\mathsf{real}}) \to L_{\mathcal{M}_{\mathbf{PAP}}}(A_{\mathsf{real}})$$

be given at $\star$ by this **PAP** function.

Proposition 7.2.25. *The interpretation of* real *from Definition 7.2.24 satisfies the assumptions for type and term constants from Theorem 7.1.1.*

Proof. The proof is similar to the ω**Qbs** case (Proposition 7.2.6). It is enough to notice that $y(\mathbb{R})$ is a sheaf and is concrete, and that $|A_{\mathsf{real}}| \cong \mathit{Val}_{\mathsf{real}}$. □

Proposition 7.2.26. *The category* $\mathsf{Sh}(\mathbf{PAP} + \mathbb{V}_0, J_{\mathbf{PAP}} \cup J_{\mathbb{V}_0})$ *is a normal model of* $\mathsf{PCF_v}$ *with* real*, obtained using the recipe from Theorem 7.1.1. The interpretation of the language is contained in the subcategory of concrete sheaves* $\mathsf{Conc}(\mathbf{PAP} + \mathbb{V}_0, J_{\mathbf{PAP}} \cup J_{\mathbb{V}_0})$.

Proof. By combining Propositions 7.2.21, 7.2.23 and 7.2.25, and finally using Proposition 7.1.2 for concreteness. □

From Figure 7.1, we know that in the $\mathsf{Conc}(\mathbf{PAP}+\mathbb{V}_0, J_{\mathbf{PAP}} \cup J_{\mathbb{V}_0})$ model, the sheaf $[\![\mathsf{real}]\!]$ at V contains only the constant functions $|\mathrm{V}| \to \mathbb{R}$, equipping $\mathbb{R}$ with the discrete ordering.

As we did in the ω**Qbs** and ω**Diff** cases, we can now state (Proposition 7.2.28) that the model of $\mathsf{PCF_v}$ in the category ω**PAP** [LHM21] is essentially the $\mathsf{Conc}(\mathbf{PAP} + \mathbb{V}_0, J_{\mathbf{PAP}} \cup J_{\mathbb{V}_0})$ model, and therefore an instance of our recipe for building normal models (Theorem 7.1.1). First, we recall the definition of ω**PAP** from [LHM21] and of the lifting monad L on ω**PAP** used there.

Definition 7.2.27 ([LHM21])**.** An ω-pap space $X = (|X|, \mathcal{P}_X, \leq_X)$ is a triple where $|X|$ is a set, $(|X|, \leq_X)$ is an ω-cpo and, for each $n \in \mathbb{N}$ and each c-analytic subset U of $\mathbb{R}^n$, $\mathcal{P}_X^U \subseteq [U \to |X|]$ is a set of functions such that:

1. all constant functions are in $\mathcal{P}_X^U$;
2. for any c-analytic set $V \subseteq \mathbb{R}^m$, and any **PAP** morphism $f : V \to U$, if ϕ is in $\mathcal{P}_X^U$, then the composite $\phi \circ f$ is in $\mathcal{P}_X^V$;

3. if $\{A_i\}_{i \in I}$ is a countable disjoint c-analytic partition of U, and given $\left(\phi_i \in \mathcal{P}_X^{A_i}\right)_{i \in I}$, the co-pairing $[\phi_i]_{i \in I} : U \to |X|$ is in $\mathcal{P}_X^U$;

4. given an ω-chain in $\mathcal{P}_X^U$ with respect to the pointwise order, its pointwise sup is in $\mathcal{P}_X^U$.

A morphism $f : X \to Y$ between ω-pap spaces is a function $f : |X| \to |Y|$ which is Scott-continuous and structure-preserving: if $\phi \in \mathcal{P}_X^U$, then $f \circ \phi \in \mathcal{P}_Y^U$. Denote by $\omega\mathbf{PAP}$ the category of ω-pap spaces and morphisms between them.

To define a lifting monad L on $\omega\mathbf{PAP}$, let LX have underlying ω-cpo $(|X| + \{\bot\}, \leq_{LX})$ where $\leq_{LX}$ extends the order on $|X|$ by adding $\bot$ as the bottom element. For each c-analytic subset U, $\mathcal{P}_{LX}^U$ is defined as:

$$\mathcal{P}_{LX}^U = \big\{\alpha : U \to |X| + \{\bot\} \;\big|\; V = \alpha^{-1}(|X|) \text{ is a c-analytic subset of } U \text{ and } \alpha|_V \in \mathcal{P}_X^V\big\}.$$

The strong monad structure of L is the same as that for the $(-) + 1$ monad on Set.

Proposition 7.2.28. *There is a functor $F : \omega\mathbf{PAP} \to \mathsf{Conc}(\mathbf{PAP}+\mathbb{V}_0, J_{\mathbf{PAP}} \cup J_{\mathbb{V}_0})$ which is full, faithful, preserves products, coproducts and exponentials, and commutes with the lifting monad:*

$$FL = L_{\mathcal{M}_{\mathbf{PAP}} \cup \mathcal{M}_{\mathbb{V}_0}} F$$

$$\overset{L}{\circlearrowleft}\ \omega\mathbf{PAP} \xhookrightarrow{\;F\;} \mathsf{Conc}(\mathbf{PAP} + \mathbb{V}_0, J_{\mathbf{PAP}} \cup J_{\mathbb{V}_0})\ \overset{L_{\mathcal{M}_{\mathbf{PAP}} \cup \mathcal{M}_{\mathbb{V}_0}}}{\circlearrowleft}$$

Moreover, for every $\omega\mathbf{PAP}$ object X, FX is a $L_{\mathcal{M}_{\mathbf{PAP}} \cup \mathcal{M}_{\mathbb{V}_0}}$-complete sheaf, and F commutes with the interpretation of $\mathsf{PCF_v}$ (with the type constant real) in $\omega\mathbf{PAP}$ and in $\mathsf{Conc}(\mathbf{PAP} + \mathbb{V}_0, J_{\mathbf{PAP}} \cup J_{\mathbb{V}_0})$.

Proof notes. The functor F is constructed exactly like in the $\omega\mathbf{Diff}$ case (Proposition 7.2.18). We omit the details. $\square$

7.3 Proving adequacy (Theorem 7.1.3)

From Section 7.1, it remains to prove that the model in $\mathsf{Sh}(\mathbb{C}+\mathbb{V}_0, J \cup J_{\mathbb{V}_0})$ is adequate with respect to the operational semantics of $\mathsf{PCF_v}$:

Theorem 7.1.3 (Adequacy)**.** *Under the assumptions in Theorem 7.1.1, the normal model* $\mathsf{Sh}(\mathbb{C}+\mathbb{V}_0, J \cup J_{\mathbb{V}_0})$ *(or* $\mathsf{Conc}(\mathbb{C}+\mathbb{V}_0, J \cup J_{\mathbb{V}_0})$*) is an adequate model for* $\mathsf{PCF_v}$ *with type and term constants.*

More precisely, if τ is a ground type ($\mathsf{0}$, $\mathsf{1}$, nat *or α), and t is a closed computation of type τ, $\vdash^{\mathsf{c}} t : \tau$, then*

$$[\![t]\!] = \eta_{[\![\tau]\!]} \circ [\![v]\!] \qquad : 1 \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau]\!]$$

implies that t reduces to v, $t \Downarrow v$.

We proved in Proposition 7.1.2 that every type is interpreted as a concrete sheaf. This means that morphisms between types are determined by their underlying function at the terminal $\star$. We have also seen in Figure 7.1 that at $\star$, the denotation of types has the same structure as in Set or $\omega\mathsf{CPO}$, e.g. $|[\![\tau+\tau']\!]| = |[\![\tau]\!]| + |[\![\tau']\!]|$, $|L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau]\!]| = |[\![\tau]\!]| + \{\bot\}$.

Using these observations, we prove adequacy using a logical relations approach, similar to the usual proof of adequacy for the traditional $\omega\mathsf{CPO}$ model (e.g. [Win93, Lemma 11.14]). One difference is that, for us, the logical relation deals with the underlying sets of concrete sheaves, rather than with cpos. Another difference appears in the proof of the fundamental property, where we have to deal with fixed points constructed by Corollary 3.2.5, rather than by Tarski's fixed point theorem.

Denote by Val_τ the set of all closed values of type τ and similarly by Comp_τ the set of closed computations. Recall from Section 4.2.1 that for type constants α, there is a new value constant for each element of Val_α. Therefore, we get an isomorphism:

$$\mathsf{Val}_\alpha \cong Val_\alpha \cong |[\![\alpha]\!]|.$$

$$
\begin{aligned}
\lhd^{\mathsf{v}}_{\mathsf{nat}} &= \{(n,n) \mid n \in \mathbb{N}\} \\
\lhd^{\mathsf{v}}_{1} &= \{(*,\star)\} \\
\lhd^{\mathsf{v}}_{0} &= \emptyset \\
\lhd^{\mathsf{v}}_{\alpha} &= \{(u,u) \mid u \in \mathit{Val}_{\alpha}\} \\
\lhd^{\mathsf{v}}_{\tau\times\tau'} &= \{(d,v) \mid \exists v_1, v_2.\, v = (v_1,v_2) \text{ and} \\
&\qquad \pi_1(d) \lhd^{\mathsf{c}}_{\tau} (\mathsf{return}\, v_1) \text{ and } \pi_2(d) \lhd^{\mathsf{c}}_{\tau'} (\mathsf{return}\, v_2)\} \\
\lhd^{\mathsf{v}}_{\tau+\tau'} &= \{(d,v) \mid \text{either } d = \mathsf{inl}\, a, v = \mathsf{inl}\, w \text{ and } a \lhd^{\mathsf{v}}_{\tau} w, \\
&\qquad \text{or } d = \mathsf{inr}\, a, v = \mathsf{inr}\, w \text{ and } a \lhd^{\mathsf{v}}_{\tau'} w\} \\
\lhd^{\mathsf{v}}_{\tau\to\tau'} &= \big\{(d,v) \mid \forall a \in |[\![\tau]\!]|,\, w \in \mathsf{Val}_{\tau}.\, a \lhd^{\mathsf{v}}_{\tau} w \implies (d\ a) \lhd^{\mathsf{c}}_{\tau'} (v\ w)\big\} \\
\lhd^{\mathsf{c}}_{\tau} &= \big\{(d,t) \mid \forall d' \in |[\![\tau]\!]|.\, \big(d = (\eta_{[\![\tau]\!]})_\star(d') \implies \exists w.\, t \Downarrow w \text{ and } d' \lhd^{\mathsf{v}}_{\tau} w\big)\big\}.
\end{aligned}
$$

Figure 7.3: Definition of the logical relation for adequacy.

Definition 7.3.1. Define a logical relation

$$
\begin{aligned}
\lhd^{\mathsf{v}}_{\tau} &\subseteq |[\![\tau]\!]| \times \mathsf{Val}_{\tau} \\
\lhd^{\mathsf{c}}_{\tau} &\subseteq |L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau]\!]| \times \mathsf{Comp}_{\tau}
\end{aligned}
$$

between the underlying set of each type and the sets of terms as shown in Figure 7.3.

In the definition of $\lhd^{\mathsf{c}}_{\tau}$ in Figure 7.3, recall that $(\eta_{[\![\tau]\!]})_\star$ is a function of type $|[\![\tau]\!]| \to |L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau]\!]|$.

Consider a typing context $\Gamma = x_1 : \tau_1, \ldots x_n : \tau_n$. Denote by Val_{Γ} the set of closed values that the variables in Γ can take:

$$
\mathsf{Val}_{\Gamma} = \mathsf{Val}_{\tau_1} \times \ldots \times \mathsf{Val}_{\tau_n}.
$$

Recall that $[\![\Gamma]\!] = [\![\tau_1]\!] \times \ldots \times [\![\tau_n]\!]$. Then we can extend the logical relation to Γ as:

$$\lhd^{\mathsf{v}}_{\Gamma} \subseteq |[\![\Gamma]\!]| \times \mathsf{Val}_{\Gamma}$$
$$\lhd^{\mathsf{v}}_{\Gamma} = \lhd^{\mathsf{v}}_{\tau_1} \times \ldots \times \lhd^{\mathsf{v}}_{\tau_n}.$$

If $\sigma \in \mathsf{Val}_{\Gamma}$ and $\Gamma \vdash^{\mathbf{v}} v : \tau$ is a value, we denote by $v[\sigma]$ the closed value obtained by substituting in v the values from σ.

We will prove the *fundamental property* of the logical relation which states that, given related contexts, every term is related to its denotation:

Lemma 7.3.2 (Fundamental property)**.** *For any $\rho \in |[\![\Gamma]\!]|$ and $\sigma \in \mathsf{Val}_{\Gamma}$:*

- *For values: $\rho \lhd^{\mathsf{v}}_{\Gamma} \sigma \implies [\![\Gamma \vdash^{\mathbf{v}} v : \tau]\!]_{\star}(\rho) \lhd^{\mathsf{v}}_{\tau} v[\sigma]$;*
- *For computations: $\rho \lhd^{\mathsf{v}}_{\Gamma} \sigma \implies [\![\Gamma \vdash^{\mathbf{c}} t : \tau]\!]_{\star}(\rho) \lhd^{\mathsf{c}}_{\tau} t[\sigma]$.*

Using the fundamental property we can prove adequacy. Consider each ground type in turn. For type nat, assume there is a value $- \vdash^{\mathbf{v}} v : \mathsf{nat}$ such that $[\![t]\!] = \eta_{[\![\mathsf{nat}]\!]} \circ [\![v]\!]$. From the description of $[\![\mathsf{nat}]\!]$ in Remark 5.2.7, we know that $|[\![\mathsf{nat}]\!]| \cong \mathbb{N}$, so we can see that $[\![v]\!]_{\star}(*)$ must be some $n \in \mathbb{N}$, so we know that:

$$[\![t]\!]_{\star}(*) = (\eta_{[\![\mathsf{nat}]\!]})_{\star}(n).$$

By the fundamental property, we obtain $[\![t]\!]_{\star}(*) \lhd^{\mathsf{c}}_{\mathsf{nat}} t$. So by the definition of the logical relation for computations, there must exist a value w such that $n \lhd^{\mathsf{v}}_{\mathsf{nat}} w$ and $t \Downarrow w$. By the definition of the logical relation for nat, w must be n, so we obtain $t \Downarrow v$ as required.

For type constants α, the proof is analogous to the nat case because of the bijection $|[\![\alpha]\!]| \cong \mathsf{Val}_{\alpha}$ and because $\lhd^{\mathsf{v}}_{\alpha}$ was defined to be the identity relation. For type $\mathsf{1}$, the proof is similar, but v can only be $\star$. For type $\mathsf{0}$, $[\![\mathsf{0}]\!](\star) = \emptyset$ so there is no value v to consider.

Proof of the fundamental property (Lemma 7.3.2). The proof is by induction on the typing rules of $\mathsf{PCF}_{\mathsf{v}}$. Most cases are proved easily using the induction hypothesis and the definition of the logical relation. We will focus on a few interesting cases.

Case $\Gamma \vdash^{\mathsf{v}} u : \alpha$**.** Assume that $\rho \lhd^{\mathsf{v}}_{\Gamma} \sigma$. We need to prove that:

$$[\![\Gamma \vdash^{\mathsf{v}} u : \alpha]\!]_{\star}(\rho) \lhd^{\mathsf{v}}_{\alpha} u[\sigma].$$

Because α is a type constant, the only values of type α are $u \in \mathit{Val}_{\alpha}$. So the denotation of u is $u \in |[\![\alpha]\!]|$ and we are done because $\lhd^{\mathsf{v}}_{\alpha}$ was defined to be the identity relation.

Case $\Gamma \vdash^{\mathsf{v}} f : (\alpha_1 \times \ldots \times \alpha_n) \to \beta$**.** Assume that $\rho \lhd^{\mathsf{v}}_{\Gamma} \sigma$. We need to prove that:

$$[\![\Gamma \vdash^{\mathsf{v}} f]\!]_{\star}(\rho) \lhd^{\mathsf{v}}_{(\alpha_1 \times \ldots \times \alpha_n) \to \beta} f[\sigma].$$

Recall that the interpretation of f is:

$$[\![\Gamma]\!] \xrightarrow{!} 1 \xrightarrow{\mathsf{curry}(\phi_f)} \big(([\![\alpha_1]\!] \times \ldots \times [\![\alpha_n]\!]) \Rightarrow L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}([\![\beta]\!])\big).$$

From the definition of the logical relation for function types, it is enough to prove that for any $\{a_i \lhd^{\mathsf{v}}_{\alpha_i} w_i\}_{i=1,n}$:

$$|\phi_f|\ (a_1, \ldots, a_n) \lhd^{\mathsf{c}}_{\beta} f\ (w_1, \ldots, w_n).$$

If $|\phi_f|\ (a_1, \ldots, a_n) = \bot$, then we are done.

If $|\phi_f|\ (a_1, \ldots, a_n) = \eta_{[\![\beta]\!]}(d)$, we need to prove that there exists a value w such that $f\ (w_1, \ldots, w_n) \Downarrow w$ and $d \lhd^{\mathsf{v}}_{\beta} w$. Because $\lhd^{\mathsf{v}}_{\alpha_i}$ was defined to be the identity relation, we can identify a_i and w_i. We know that ϕ_f agrees with $f : (\mathit{Val}_{\alpha_1} \times \ldots \times \mathit{Val}_{\alpha_n}) \to \beta$ on points. So it must be the case that $f\ (w_1, \ldots, w_n) = w \in \mathit{Val}_{\beta}$ where $d \lhd^{\mathsf{v}}_{\beta} w$. By the reduction rules, this means that $f\ (w_1, \ldots, w_n) \Downarrow w$ as well.

Case $\Gamma \vdash^{\mathsf{c}} \mathsf{case}\, v\, \mathsf{of}\ \{\} : \tau$**.** Assume that $\rho \lhd^{\mathsf{v}}_{\Gamma} \sigma$. By induction hypothesis

$$[\![\Gamma \vdash^{\mathsf{v}} v]\!]_{\star}(\rho) \lhd^{\mathsf{v}}_{0} v[\sigma].$$

But we defined $\lhd_0^{\mathsf{v}} = \emptyset$, so we get a contradiction from which we can deduce

$$[\![\Gamma \vdash^{\mathbf{c}} \mathsf{case}\, v\, \mathsf{of}\ \{\}]\!]_\star(\rho) \lhd_\tau^{\mathsf{c}} (\mathsf{case}\, v\, \mathsf{of}\ \{\})[\sigma].$$

Case $\Gamma \vdash^{\mathbf{c}} \mathsf{let}\, x = t\, \mathsf{in}\, t' : \tau'$. Assume that $\rho \lhd_\Gamma^{\mathsf{v}} \sigma$. Assume further that

$$[\![\mathsf{let}\, x = t\, \mathsf{in}\, t']\!]_\star(\rho) = (\eta_{[\![\tau']\!]})_\star(d_1)$$

for some $d_1 \in [\![\tau']\!](\star)$.

As mentioned in Section 4.3, the interpretation of let using the categorical structure is:

$$[\![\Gamma]\!] \xrightarrow{\langle \mathrm{id}, [\![t]\!]\rangle} [\![\Gamma]\!] \times L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau]\!] \xrightarrow{\mathrm{str}} L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}([\![\Gamma]\!] \times [\![\tau]\!])$$
$$\xrightarrow{L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![t']\!]} L^2_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!] \xrightarrow{\mu} L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!].$$

Because at $\star$ the monad $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ acts like the lifting monad on Set (see Figure 7.2), it must be the case that:

$$[\![t]\!]_\star(\rho) = (\eta_{[\![\tau]\!]})_\star(d_2)$$

for some $d_2 \in [\![\tau]\!](\star)$. Using the induction hypothesis for τ, there must exist a value w such that $t[\sigma] \Downarrow w$ and $d_2 \lhd_\tau^{\mathsf{v}} w$.

Now we know that $(\rho, d_2) \lhd_{\Gamma,\tau}^{\mathsf{v}} (\sigma, w)$, and we can see from the interpretation of let that

$$[\![t']\!]_\star(\rho, d_2) = (\eta_{[\![\tau']\!]})_\star(d_1).$$

From the induction hypothesis for τ' we deduce that there must be a value u of type τ' such that $t'[\sigma, w] \Downarrow u$ and $d_1 \lhd_{\tau'}^{\mathsf{v}} u$. This implies that $(\mathsf{let}\, x = t\, \mathsf{in}\, t')[\sigma] \Downarrow u$, so we are done.

Case $\Gamma \vdash^{\mathbf{v}} (\mathsf{rec}\, f\, x.\, t) : \tau \to \tau'$. Assume that $\rho \lhd_\Gamma^{\mathsf{v}} \sigma$ and $a \lhd_\tau^{\mathsf{v}} w$. We need to prove that

$$[\![\mathsf{rec}\, f\, x.\, t]\!]_\star(\rho)(a) \lhd_{\tau'}^{\mathsf{c}} (\mathsf{rec}\, f\, x.\, t[\sigma])\, w.$$

All $\mathsf{PCF_v}$ types are interpreted as well-complete sheaves, and the denotation $[\![\mathsf{rec}\, f\, x.\, t]\!] : [\![\Gamma]\!] \to ([\![\tau]\!] \Rightarrow L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!])$ is the fixed point of the map

$$[\![t]\!] : [\![\Gamma]\!] \times ([\![\tau]\!] \Rightarrow L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!]) \times [\![\tau]\!] \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!]$$

constructed in Corollary 3.2.5.

Like in the proof of Corollary 3.2.5 and Theorem 3.2.3, we use the fact that $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ is a strong monad to get an algebra structure

$$\alpha : L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}([\![\tau]\!] \Rightarrow L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!]) \to ([\![\tau]\!] \Rightarrow L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!])$$

which we then use to get a map

$$[\![t]\!]^* : [\![\Gamma]\!] \times L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}([\![\tau]\!] \Rightarrow L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!]) \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}([\![\tau]\!] \Rightarrow L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!]).$$

Then

$$[\![\mathsf{rec}\, f\, x.\, t]\!] = \alpha \circ \mathsf{ap}_{\overline{\omega}}(-, \infty)$$

where $\mathsf{ap}_{\overline{\omega}} : \Gamma \times \overline{\omega} \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}([\![\tau]\!] \Rightarrow L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!])$ is a chain, with a sup, of approximations of the fixed point of $[\![t]\!]^*$, constructed in Lemma 3.2.4. Here $\overline{\omega}$ is the extended vertical natural numbers calculated as a limit, as in Assumption 3.1.1, where the lifting monad used is $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$.

We are going to show that the fixed point $(\mathsf{rec}\, f\, x.\, t)$ is related to each of its finite approximations. More precisely, show by induction on n that for all a and w such that $a \lhd^{\mathsf{v}}_{\tau} w$:

$$\mathsf{ev}\big((\alpha \circ \mathsf{ap}_{\omega})_\star(\rho, n), a\big) \lhd^{\mathsf{c}}_{\tau'} (\mathsf{rec}\, f\, x.\, t[\sigma])\, w, \tag{7.1}$$

where $\mathsf{ap}_{\omega} : \Gamma \times \omega \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}([\![\tau]\!] \Rightarrow L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!])$ was defined in the proof of Lemma 3.2.4, and is the same chain as $\mathsf{ap}_{\overline{\omega}}$ but without the sup (i.e. $\mathsf{ap}_{\overline{\omega}}$ is the unique extension of ap_{ω}). Here ω is the vertical natural numbers constructed as in Assumption 3.1.4 using the lifting monad $L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$.

In the base case we know by definition that

$$(\mathsf{ap}_{\omega})_\star(\rho, 0) = \bot_{|L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}([\![\tau]\!] \Rightarrow L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!])|}$$

so $\mathsf{ev}\big((\alpha \circ \mathsf{ap}_\omega)_\star(\rho, n), a\big) = \bot_{|L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!]|}$. By the definition of the logical relation for computations, we have that

$$\bot_{|L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\tau']\!]|} \lhd^{\mathsf{c}}_{\tau'} (\mathsf{rec}\, f\, x.\, t[\sigma])\, w.$$

For the induction step, we know that

$$(\mathsf{ap}_\omega)(\rho, n+1) = ([\![t]\!]^*)_\star(\rho, \mathsf{ap}_\omega(\rho, n))$$

from Equation (3.5) in the proof of Lemma 3.2.4. By calculating, we obtain that:

$$\mathsf{ev}\big((\alpha \circ \mathsf{ap}_\omega)_\star(\rho, n+1), a\big) = [\![t]\!]_\star\big(\rho, (\alpha \circ \mathsf{ap}_\omega)_\star(\rho, n), a\big). \tag{7.2}$$

By induction hypothesis for n, we get that:

$$\mathsf{ev}\big((\alpha \circ \mathsf{ap}_\omega)_\star(\rho, n), a\big) \lhd^{\mathsf{c}}_{\tau'} (\mathsf{rec}\, f\, x.\, t[\sigma])\, w$$

for any related a and w, so

$$(\alpha \circ \mathsf{ap}_\omega)_\star(\rho, n) \lhd^{\mathsf{v}}_{\tau\to\tau'} (\mathsf{rec}\, f\, x.\, t[\sigma]).$$

Using the above, we can apply the induction hypothesis for t to get

$$[\![t]\!]_\star\big(\rho, (\alpha \circ \mathsf{ap}_\omega)_\star(\rho, n), a\big) \lhd^{\mathsf{c}}_{\tau'} t[\sigma, (\mathsf{rec}\, f\, x.\, t[\sigma])/f, w/x].$$

By Equation (7.2), this is enough to conclude the induction step because $t[\sigma, (\mathsf{rec}\, f\, x.\, t[\sigma])/f, w/x]$ and $(\mathsf{rec}\, f\, x.\, t[\sigma])\, w$ reduce to the same value according to the operational semantics.

Now that we have shown Equation (7.1), we need to show that the sup of the chain of approximations of $(\mathsf{rec}\, f\, x.\, t[\sigma])\, w$ is also related to it, namely that:

$$(\alpha \circ \mathsf{ap}_{\overline{\omega}})_\star(\rho, \infty)(a) \lhd^{\mathsf{c}}_{\tau'} (\mathsf{rec}\, f\, x.\, t[\sigma])\, w.$$

To deduce this we prove a more general fact first, which in the adequacy proof for $\omega\mathsf{CPO}$ corresponds to showing that the logical relation is an admissible subset.

Denote by $\gamma_v : \{(-) \triangleleft^{\mathsf{v}}_{\tau} v\} \rightarrowtail [\![\tau]\!]$ the greatest subobject of $[\![\tau]\!]$ whose points are those elements in $|[\![\tau]\!]|$ related to v. Greatest here means that the subobject has all the relations of $[\![\tau]\!]$ with values in $\{(-) \triangleleft^{\mathsf{v}}_{\tau} v\}$. Because $[\![\tau]\!]$ is concrete, the domain of the subobject $\gamma_v : \{(-) \triangleleft^{\mathsf{v}}_{\tau} v\} \rightarrowtail [\![\tau]\!]$ is also a concrete presheaf. For a computation t of type τ we can analogously define a subobject of $L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} [\![\tau]\!]$.

The following two facts are enough to conclude the proof of the fundamental property in the rec case:

Lemma 7.3.3. *For any type τ:*

1. *For values: given a chain $f : \omega \to [\![\tau]\!]$ that factors through the subobject $\{(-) \triangleleft^{\mathsf{v}}_{\tau} v\} \rightarrowtail [\![\tau]\!]$, its extension $\overline{f} : \overline{\omega} \to [\![\tau]\!]$ also factors.*

2. *For computations: the statement is analogous but instead of $[\![\tau]\!]$ we use $L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} [\![\tau]\!]$.*

The statement for values makes sense because, not only every type is well-complete (Proposition 4.3.4), but it is also $L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}$-complete using Proposition 4.1.5 and its preceding discussion in Section 4.1.

So far in this proof of the fundamental property, we have used ω (Assumption 3.1.4) to denote a colimit calculated in the category of sheaves $\mathsf{Sh}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$. From Proposition 6.1.1 we know that for any type τ, $[\![\tau]\!]$ and $L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}} [\![\tau]\!]$ are right-orthogonal to $\omega_P \to \overline{\omega}$ as well, where ω_P is the colimit calculated in $\mathsf{PSh}(\mathbb{C} + \mathbb{V}_0)$.

It is in fact sufficient to prove Lemma 7.3.3 above for ω_P rather than ω, as the following lemma shows.

Lemma 7.3.4. *Assume that given any map from presheaves $g' : \omega_P \to X$, where X is a complete sheaf, such that g' factors through a subobject $X' \rightarrowtail X$, then the unique extension $\overline{g'} : \overline{\omega} \to X$ of g' also factors through $X' \rightarrowtail X$.*

Then, for any map from sheaves $g : \omega \to X$, where X is complete, and that factors through a subobject $X' \rightarrowtail X$, the extension $\overline{g} : \overline{\omega} \to X$ factors through $X' \rightarrowtail X$ as well.

Notice that the assumption in the lemma above makes sense because according to Proposition 6.1.1, completeness in sheaves is equivalent to completeness in presheaves.

Proof of Lemma 7.3.4. Recall that ω is the sheafification of $a(\omega_P)$. Start with a map $g : \omega \to X$ that factors through $X' \rightarrowtail X$. Using the unit of the adjunction $a \dashv i$, construct a map $g' : \omega_P \to \omega \to X$ that also factors through $X' \rightarrowtail X$. Then the extension $\overline{g'} : \overline{\omega} \to X$ factors as well. From the proof of Proposition 6.1.1 (Item 1 implies Item 2) we know that $\overline{g'}$ is also the unique extension of g, so we are done. $\square$

Now we prove Lemma 7.3.3 above with ω replaced by ω_P, simultaneously by induction on types:

Case α, nat (Values). Consider a map $f : \omega_P \to [\![\alpha]\!]$ that factors through the subobject $\{(-) \lhd^{\mathsf{v}}_{\alpha} u\} \rightarrowtail [\![\alpha]\!]$. Because $\lhd^{\mathsf{v}}_{\alpha} \subseteq [\![\alpha]\!] \times \mathsf{Val}_{\alpha}$ was defined to be the identity relation, the subobject $\{(-) \lhd^{\mathsf{v}}_{\alpha} u\}$ has exactly one point. Therefore, because it is concrete, it must be the terminal object. Thus f factors through $\omega_P \to 1$, so it is a constant map. Its unique extension to $\overline{\omega}$ is also constant with the same value, so it factors as required.

Case α, nat (Computations). Consider a map $f : \omega_P \to L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}[\![\alpha]\!]$ that factors through the subobject

$$\{(-) \lhd^{\mathsf{c}}_{\alpha} t\} \rightarrowtail L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}[\![\alpha]\!].$$

If t diverges, then $\{(-) \lhd^{\mathsf{c}}_{\alpha} t\}$ has exactly one point, $\bot$. So the proof that the unique extension f factors is the same as in the previous case.

If t converges, it must converge to a unique value u by soundness. Because $\lhd^{\mathsf{v}}_{\alpha}$ was defined to be the identity relation, this value u is related only to itself. Therefore, $\{(-) \lhd^{\mathsf{c}}_{\alpha} t\}$ has exactly two points, one of which is $\bot$ and the other is $u \in |[\![\alpha]\!]|$.

Recall the explicit description of ω_P from Section 6.3.1. We will use it to show that f is an eventually constant sequence with values in $|L_{\mathcal{M} \cup \mathcal{M}_{\mathbb{V}_0}}[\![\alpha]\!]|$.

Thus, its extension to $\overline{\omega}$ is also eventually constant with the same value and factors as required.

Consider the composite map:

$$\omega_P \xrightarrow{f} L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\alpha]\!] \xrightarrow{L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(!)} L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}(1) \cong \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}.$$

We know from Definition 6.3.2 and Lemma 6.3.3 that this map is an infinite monotone binary sequence. If the sequence is always 0, then f must be the always $\perp$ sequence. If the sequence becomes 1, f must be a finite number of $\perp$'s followed only by values from $|L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\alpha]\!]|$. In the latter case, recall that f factors through the subobject $\{(-) \triangleleft^{\mathsf{c}}_{\alpha} t\} \rightarrowtail L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\alpha]\!]$ which has two points, $\perp$ and $u \in |[\![\alpha]\!]|$. Therefore, f must be eventually constant with value u.

Case 1 (Values). The subobject we need to consider is $\{(-) \triangleleft^{\mathsf{v}}_{1} \star\} \rightarrowtail 1$ which has only one point. All maps $f : \omega_P \to 1$ and $\overline{f} : \overline{\omega} \to 1$ are constant and therefore factor as expected.

Case 1 (Computations). Consider the subobject $\{(-) \triangleleft^{\mathsf{c}}_{1} t\} \rightarrowtail L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1 \cong \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$. From Definition 6.3.2 and Lemma 6.3.3, a map $f : \omega_P \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1$ is an infinite monotone binary sequence.

If the sequence is always 0, then $\overline{f}_\star(\infty)$ is also 0 (i.e. $\perp$), so $\overline{f}$ factors because $\perp$ is related to every computation. If the sequence is eventually 1 then $\overline{f}_\star(\infty)$ is also 1. This point is part of the subobject $\{(-) \triangleleft^{\mathsf{c}}_{1} t\} \rightarrowtail L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}1 \cong \Delta_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}$ because f factors, so $\overline{f}$ factors as well.

Case 0 (Values). In this case $|[\![0]\!]|$ is empty so there are no maps $f : \omega_P \to [\![0]\!]$ to consider.

Case 0 (Computations). From the definition of the logical relation we can see that the subobject $\{(-) \triangleleft^{\mathsf{c}}_{0} t\} \rightarrowtail L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![0]\!]$ has exactly one point, $\perp$, so any extension $\overline{f}$ must factor.

Case $\sigma \to \tau$ (Values). Consider a map $f : \omega_P \to (\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}\llbracket\tau\rrbracket)$ which factors through the subobject $\{(-) \lhd^{\mathsf{v}}_{\sigma\to\tau} v\} \to (\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}\llbracket\tau\rrbracket)$. We need to show that the extension $\overline{f} : \overline{\omega} \to (\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}\llbracket\tau\rrbracket)$ of f also factors, which means showing:

$$\overline{f}_\star(\infty) \lhd^{\mathsf{v}}_{\sigma\to\tau} v.$$

Consider the uncurrying of f:

$$f' : \omega_P \times \llbracket\sigma\rrbracket \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}\llbracket\tau\rrbracket.$$

This factors through the map:

$$\beta : \{(-) \lhd^{\mathsf{v}}_{\sigma\to\tau} v\} \times \llbracket\sigma\rrbracket \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}\llbracket\tau\rrbracket.$$

For any $a \in |\llbracket\sigma\rrbracket|$ the map $f'(-, a)$ factors through $\beta(-, a)$.

For any closed value w of type σ such that $a \lhd^{\mathsf{v}}_{\sigma} w$, $\beta(-, a)$ factors through

$$\{(-) \lhd^{\mathsf{c}}_{\tau} (v\, w)\} \rightarrowtail L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}\llbracket\tau\rrbracket$$

by mapping each $d \in \{(-) \lhd^{\mathsf{v}}_{\sigma\to\tau} v\}$ to $(d\, a)$. So $f'(-, a)$ also factors through the same subobject. Applying the induction hypothesis for computations of type τ, we get that

$$\overline{f'(-, a)} : \overline{\omega} \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}\llbracket\tau\rrbracket$$

factors as well, and thus

$$\overline{f'(-, a)}_\star(\infty) \lhd^{\mathsf{c}}_{\tau} (v\, w).$$

By uniqueness of extensions, we get the following equalities:

$$\overline{f'(-, a)}_\star(\infty) = \overline{f'}_\star(\infty, a) = \big(\overline{f}_\star(\infty)\big)(a).$$

Since a and w were arbitrary we can deduce the required result:

$$\overline{f}_\star(\infty) \lhd^{\mathsf{v}}_{\sigma\to\tau} v.$$

Case $\sigma \times \tau$ (Values). Consider $f : \omega_P \to [\![\sigma]\!] \times [\![\tau]\!]$ which factors through $\{(-) \lhd^{\mathsf{v}}_{\sigma\times\tau} (v_1, v_2)\} \rightarrowtail [\![\sigma]\!] \times [\![\tau]\!]$. We can use the definition of the logical relation and the induction hypothesis to prove that:

$$\pi_1(\overline{f}_\star(\infty)) \lhd^{\mathsf{v}}_\sigma v_1$$

and similarly for π_2. Then deduce that

$$\overline{f}_\star(\infty) \lhd^{\mathsf{v}}_\sigma (v_1, v_2).$$

Case $\sigma + \tau$ (Values). Consider without loss of generality the subobject $\{(-) \lhd^{\mathsf{v}}_{\sigma+\tau} \mathsf{inl}\, w\} \rightarrowtail [\![\sigma]\!] + [\![\tau]\!]$ and a map $f : \omega_P \to [\![\sigma]\!] + [\![\tau]\!]$ that factors through it. (The proof for inr is similar.)

Because of the factoring we can see that for any $n \in \omega_P(\star)$:

$$f_\star(n) = \mathsf{inl}\, a_n$$

where $a_n \in |[\![\sigma]\!]|$ and $a_n \lhd^{\mathsf{v}}_\sigma w$. Therefore, we can form a map $f_1 : \omega_P \to [\![\sigma]\!]$ where:

$$(f_1)_\star(n) = a_n.$$

The map f_1 factors through $\{(-) \lhd^{\mathsf{v}}_\sigma w\} \rightarrowtail [\![\sigma]\!]$, so using the induction hypothesis for σ, we see that:

$$(\overline{f_1})_\star(\infty) \lhd^{\mathsf{v}}_\sigma w$$

and so

$$\big(\overline{f}_\star(\infty) = \mathsf{inl}\,(\overline{f_1})_\star(\infty)\big) \lhd^{\mathsf{v}}_\sigma \mathsf{inl}\, w.$$

Case $\tau_1 \to \tau_2$, $\tau_1 \times \tau_2$, $\tau_1 + \tau_2$ (Computations). Consider a type σ which can be any of the above. Assume that a map $f : \omega_P \to L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\sigma]\!]$ factors through the subobject $\{(-) \lhd^{\mathsf{c}}_\sigma t\} \rightarrowtail L_{\mathcal{M}\cup\mathcal{M}_{\mathbb{V}_0}}[\![\sigma]\!]$. We need to show that $\overline{f}_\star(\infty) \lhd^{\mathsf{c}}_\sigma t$.

Consider $d \in |[\![\sigma]\!]|$ such that $\overline{f}_\star(\infty) = \eta_\star(d)$. Then it cannot be the case that $f_\star$ is constant $\bot$. Because f factors, by the definition of the logical relation for computations, there must be a value w such that $t \Downarrow w$.

Recall from Section 6.3.1 that the elements of $\omega_P(\star)$ can be thought of as the natural numbers. Like in **Case** α, we can prove that once $f_\star$ takes a non-$\bot$ value it cannot become $\bot$ again.

Let $k \in \omega_P(\star)$ be the point where $f_\star$ stops being $\bot$. We can form a map $f' : \omega_P \to [\![\sigma]\!]$ such that:

$$f'_\star(k' < k) = f_\star(k)$$
$$f'_\star(k' \geq k) = f_\star(k').$$

We omit the proof that f' is natural. In the usual $\omega\mathsf{CPO}$ proof, this step corresponds to removing a finite number of elements from the beginning of a chain.

Since for all n:

$$\eta_\star(f'_\star(n)) \lhd^{\mathsf{c}}_\sigma t,$$

we can deduce that f' factors through $\{(-) \lhd^{\mathsf{v}}_\sigma w\} \rightarrowtail [\![\sigma]\!]$. We have proved the cases for values already, so we can deduce that $\overline{f'} : \overline{\omega} \to [\![\sigma]\!]$ also factors, and therefore

$$\big((\overline{f'})_\star(\infty) = d\big) \lhd^{\mathsf{v}}_\sigma w.$$

This concludes the proof of both Lemma 7.3.3 and of the fundamental property (Lemma 7.3.2).

□

7.4 Related work

Our general adequacy theorem (Theorem 7.1.3) is closely related to other adequacy results in the axiomatic and synthetic domain theory literature, for example by Fiore and Plotkin [Fio94, Chapter 9], [FP94] and Simpson [Sim98, Sim04]. More broadly, another related strand of research is that of topological domain theory (e.g. [Bat06]). It provides a cartesian closed category which can model recursive types and computational effects, including probabilistic computation, and which makes a connection to computability; see [BSS07] for an overview. Our work is most closely related to that of Simpson [Sim98] as we explain in detail.

Simpson [Sim98] considers the notion of *natural model of synthetic domain theory*, which is an elementary topos with a dominance and a natural numbers object which is well-complete [Sim98, Definition 2]. He then considers the interpretation of call-by-value PCF in natural models. He proves that this interpretation is adequate if and only if the topos is 1-consistent [Sim98, Theorem 2]. Being 1-consistent is a statement about the internal logic of the topos explained in [Sim98, Section 6].

Natural models of synthetic domain theory are very similar to our notion of normal model (Definition 4.3.1). Two differences are that we consider only Grothendieck toposes, rather than elementary toposes, and we allow the extension of $\mathsf{PCF_v}$ with type and term constants, to accommodate the examples in Section 7.2.

Similarities include the fact that Simpson uses the dominance to construct a lifting monad L in the same way as we do in Theorem 2.4.9. In his case, the initial algebra $\mathbf{I}$ and final coalgebra $\mathbf{F}$ of L play the role of ω and $\overline{\omega}$ from Section 3.1. Well-completeness is defined similarly to the way we define it in Definition 3.2.2 and is used to prove a fixed point theorem ([Sim98, Proposition 2]) corresponding to our Theorem 3.2.3.

We do not know whether our adequacy theorem (Theorem 7.1.3), for the normal model $\mathsf{Sh}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathrm{V}})$ obtained via the recipe in Theorem 7.1.1, is in fact an instance of adequacy for natural models ([Sim98, Theorem 2]). When we proved our adequacy theorem, we were not aware of this result for natural models. A fact which suggests that our adequacy theorem might follow from adequacy for natural models is that any non-trivial Grothendieck topos, and hence $\mathsf{Sh}(\mathbb{C} + \mathbb{V}_0, J \cup J_{\mathbb{V}_0})$, is 1-consistent. However, we do not know if using our assumptions we can deduce that ω is an initial algebra; we discussed this issue in more detail in Section 3.3 in connection with existing work in axiomatic domain theory. For the same reason, we do not know whether the fact that normal models have the necessary structure to interpret $\mathsf{PCF_v}$ (Proposition 4.3.4) follows from the equivalent statement for natural models.

Despite the strong similarity between normal and natural models, a main novelty of our work is the method of constructing normal models summarized in Theorem 7.1.1, by specifying a concrete site (Definition 2.1.6) and a class of admissible monos (Definition 6.2.1). This recipe is important because it unifies the treatment of higher-order recursion in several examples which have not been discussed together before: the vSet model (Section 4.4), the three models from Section 7.2, and the fully abstract model of PCF_v from Chapter 8. The previous work of Simpson [Sim98, Sim04] does not show how to obtain models that satisfy the conditions of a natural model, so our recipe is a contribution in this direction.

Moreover, I think that our recipe is useful because the input data needed to specify a model (the concrete site with admissible monos) is easy to present in many cases, like in the examples from Section 7.2. In any case, this data is at least easier to present than describing the objects and morphisms of the model directly, like in the case of the fully abstract model in Chapter 8.

Proving adequacy (Theorem 7.1.3) for a restricted class of normal models only, namely the models $\mathsf{Sh}(\mathbb{C}+\mathbb{V}_0, J\cup J_{\mathbb{V}_0})$ obtained from our recipe, is justified because, as we have shown through our examples, it covers many cases of interest. Our adequacy proof is significantly different from Simpson's [Sim98] proof of adequacy for natural models. As explained in Section 7.3, we are able to use the same proof outline as for adequacy in $\omega\mathsf{CPO}$, replacing cpos with concrete sheaves. Simpson gives a proof of adequacy in the internal logic of the topos used to model the language, which requires an encoding of the operational semantics in the internal logic using Gödel numbering.

Chapter 8

A fully abstract model of $\mathsf{PCF_v}$

In this chapter, we give another example of a normal model (Definition 4.3.1) built using the recipe from Theorem 7.1.1, starting from a concrete site (Definition 2.1.6) with admissible monos (Definition 6.2.1). In Section 8.2, we show that this model, which we call $\mathcal{G}$, is fully abstract (see e.g. [Win93, Section 11.10]) for call-by-value PCF ($\mathsf{PCF_v}$) without any type or term constants. Soundness follows from soundness for normal models (Theorem 4.3.5) and adequacy follows from Theorem 7.1.3.

A consequence of soundness and adequacy is that denotational equality of programs implies contextual equivalence. A model is *fully abstract* if denotational equality coincides with contextual equivalence. Thus for the model $\mathcal{G}$ we define, it remains to show the converse: that contextual equivalence implies denotational equality (Theorem 8.2.11). This is usually much harder to achieve, for example the traditional ωCPO model of PCF is not fully abstract [Plo77].

Unlike the previous examples of our recipe, those from Section 7.2 and the category vSet from Section 4.4, this fully abstract model $\mathcal{G}$ of $\mathsf{PCF_v}$ is new. This shows the wide applicability of our method for building normal models: the method helps explain the connection between existing models and it also facilitates the discovery of new models.

The model in this chapter is inspired by the fully abstract models for PCF of O'Hearn, Riecke and Sandholm [OR95, RS02] and Streicher and

Marz [Mar00a, Mar00b, Str06], who used logical relations. Initially, our aim was to organize their models using concrete sheaves, starting from the observation that logical relations can be seen as concrete presheaves. While trying to do this however, we defined a different category which is not straightforward to relate formally to the previous models because of the technical differences discussed in Section 8.3.

We argue that our fully abstract model is easier to present than the previous logical relations models because it uses the recipe for building normal models from Theorem 7.1.1. The input data for this recipe is easier to specify than describing the objects and morphisms in the model directly as the previous work does [OR95, RS02, Mar00a, Mar00b, Str06]. Moreover, by being an instance of our recipe, we can easily deduce that the model is sound and adequate.

The outline of the chapter is as follows. In Section 8.1, we define a collection of concrete sites with admissible monos, then use them to instantiate Theorem 7.1.1 to build a normal model $\mathcal{G}$.

Section 8.2 is devoted to the proof that $\mathcal{G}$ is a fully abstract model. A crucial step in the proof is to show that enough morphisms in $\mathcal{G}$ are definable by $\mathsf{PCF_v}$ programs, and that hence, to prove equality of denotations of terms, we only need to compare them on definable inputs. In Section 8.2.1, we make precise what these inputs are, and in Section 8.2.2 we show they are definable, Corollary 8.2.10. In Section 8.2.3, we state the full abstraction theorem (Theorem 8.2.11) and prove it using Corollary 8.2.10.

Finally, in Section 8.3 we discuss the connection with related work. The material in this chapter appeared at FSCD 2021 [MMS21, Sections 6.3, 7]. We note that $\mathcal{G}$ is a slight simplification over the fully abstract model published in [MMS21], as explained at the end of Section 8.3.

8.1 Defining the model

We start this section by recalling the concept of structural system of partitions of a finite set (Definition 8.1.1 and Definition 8.1.3), which was used in previous work on the full abstraction problem for PCF [Mar00a, Mar00b, Str06].

To define the sheaf category that will be our fully abstract model, we define a *collection* of concrete sites. For this, we use structural systems of partitions. The sites are introduced in Definition 8.1.4, with the coverage being described in Definition 8.1.6 and the class of admissible monos in Definition 8.1.10. We combine this collection of sites using Definition 6.2.3, and use it to instantiate our recipe for normal models from Theorem 7.1.1 to obtain a sheaf category $\mathcal{G}$. Definition 8.1.12 states what $\mathcal{G}$ is precisely and in Proposition 8.1.13 we prove that it is indeed a normal model.

8.1.1 Structural systems of partitions

To define the site of our fully abstract model $\mathcal{G}$ we use the notion of a *structural system of partitions* (SSP) which is due to Marz and Streicher [Mar00a, Mar00b, Str06] In Definition 8.1.3, we recall the definition of the category SSP.

Definition 8.1.1. Given a finite set w, a structural system of partitions S^w is a set containing *sets of disjoint subsets of* w, that is, (partial) partitions of w, and satisfying the following axioms:

1. $\{w\}, \emptyset \in S^w$.
2. (Refinement) P, $Q \in S^w$ and $U \in P$ imply that:
$$(P \setminus \{U\}) \cup (\{U \cap V \mid V \in Q\} \setminus \{\emptyset\}) \in S^w.$$
3. (Union) $U, V \in P \in S^w$ implies that $(P \setminus \{U, V\}) \cup \{U \cup V\} \in S^w$.

Remark 8.1.2. Using the refinement axiom and the $\emptyset \in S^w$ axiom we can deduce a further SSP axiom which appeared in [Str06]:

4. (Dropping) $P \in S^w$ and $U \in P$ imply that $(P \setminus \{U\}) \in S^w$.

Definition 8.1.3. The category SSP has objects pairs (w, S^w), of a finite set w and a system of partitions S^w on it. A morphism $f : (v, S^v) \to$

(w, S^w) is a function between sets $f : v \to w$ such that partitions in S^w pull back to partitions in S^v. More precisely, if $P = \{w_1, \ldots, w_n\} \in S^w$, then $\{f^{-1}(w_1), \ldots, f^{-1}(w_n)\} \setminus \{\emptyset\} \in S^v$. Composition is given by function composition.

We will denote $\{f^{-1}(w_1), \ldots, f^{-1}(w_n)\} \setminus \{\emptyset\}$ by $f^{-1}(P)$.

As explained in [Str06, Chapter 11], the intuition is that an object (w, S^w) is a finite datatype with underlying set w together with partitions of w induced by partial functions

$$f : w \rightharpoonup \mathbb{N}.$$

Intuitively, the only partial functions considered are those that are definable by sequential functional programs. So the SSP axioms are meant to reflect closure properties of such programs.

For example, consider the type unit $\to$ unit. A program of this type either terminates with value $\star$ or diverges, therefore w has two elements, $\top$ and $\bot$. The partitions in S^w are $\emptyset$, $\{\top, \bot\}$ and $\{\top\}$; $\{\bot\}$ is not a partition because we cannot observe divergence.

8.1.2 Defining sites and admissible monos

We now define a collection of concrete sites $\mathcal{I}_{\mathcal{C},F}$ that we will use to define the fully abstract model $\mathcal{G}$ (Definition 8.1.12).

Definition 8.1.4. For any small category $\mathcal{C}$ and any faithful functor $F : \mathcal{C} \to \mathsf{SSP}$, define the category $\mathcal{I}_{\mathcal{C},F}$ to have:

- as objects pairs (c, U) of an object c from $\mathcal{C}$ and a set U such that $U = \bigcup P$ for some $P \in S^{F(c)}$; and also a distinguished terminal object denoted by $\star$;
- as morphisms $X \to Y$ certain functions $|X| \to |Y|$, where $|(c, U)| = U$ and $|\star| \cong 1$. If $X = (c, U)$ and $Y = (d, V)$, take those functions $f : U \to V$ such that either f is constant, or there exists a morphism $\phi : c \to d$ in $\mathcal{C}$ such that the image of $F(\phi) : F(c) \to F(d)$ restricted to U is included in V. If either X or Y is $\star$, take all functions.

Remark 8.1.5. Notice that for (c, U) to be an object in $\mathcal{I}_{\mathcal{C},F}$ it is enough to ask that either $\{U\} \in S^{F(c)}$, by the union axiom, or that $U = \emptyset$.

Informally,the category $\mathcal{I}_{\mathcal{C},F}$ is a "totalization" of $F : \mathcal{C} \to \mathsf{SSP}$ in the sense that it adds enough objects such that the partial maps between objects in the image of F can now be represented as total maps.

The intuition is that an object in the image of F is a prediction of what the points of *finite* $\mathsf{PCF_v}$ types are, together with a prediction of the admissible domains of partial maps going out of this finite type (given by the SSP structure). Morphisms are a prediction of certain definable functions between types. When proving full abstraction, we will choose to use certain functors $F : \mathcal{C} \to \mathsf{SSP}$ (Definition 8.2.5) that are in fact a correct prediction.

Now we define a coverage on $\mathcal{I}_{\mathcal{C},F}$ that will ensure that the natural numbers object $\coprod_{\mathbb{N}} 1$, which interprets the type nat in the sheaf category $\mathcal{G}$ (Definition 8.1.12), has the correct structure to allow us to prove that $\mathcal{G}$ is a fully abstract model. This approach of using a coverage to obtain suitable colimits appears in previous work of Fiore and Simpson [FS99] about full definability for a simply-typed lamba-calculus with sums.

Definition 8.1.6. For a faithful functor $F : \mathcal{C} \to \mathsf{SSP}$, define a coverage $\mathcal{J}_{\mathcal{C},F}$ on $\mathcal{I}_{\mathcal{C},F}$ as follows:

- An object (c, U) is covered by the families of partial identity maps $\{(c, U_i) \to (c, U)\}_{1 \leq i \leq n}$ such that $\{U_1, \dots, U_n\} \in S^{F(c)}$ and $\bigcup_{1 \leq i \leq n} U_i = U$. If $U = \emptyset$, then $(c, \emptyset)$ is also covered by the identity $\{(c, \emptyset) \to (c, \emptyset)\}$.
- The terminal object $\star$ is covered only by the set containing the identity, $\{\star \to \star\}$.

Roughly speaking, this coverage ensures that in the category of sheaves on $\mathcal{I}_{\mathcal{C},F}$, the object $ay(c, U)$ is a coproduct of $\{ay(c, U_i)\}_{1 \leq i \leq n}$, as the set-theoretic intuition suggests, despite the fact that $\mathcal{I}_{\mathcal{C},F}$ might not have coproducts.

Remark 8.1.7. For any partition $\{U_1, \dots, U_n\} \in S^{F(c)}$ whose union is U, each (c, U_i) is a an object in $\mathcal{I}_{\mathcal{C},F}$ by the dropping SSP axiom (4). Therefore, we will often denote a cover $\{(c, U_i) \to (c, U)\}_{1 \le i \le n}$ by its corresponding partition $\{U_1, \ldots, U_n\} \in S^{F(c)}$.

Remark 8.1.8. Notice that in $\mathcal{I}_{\mathcal{C},F}$, the objects of the form $(c, \emptyset)$, corresponding to any c in $\mathcal{C}$ and $\emptyset \in S^{F(c)}$, are isomorphic and initial. According to the definition of $\mathcal{J}_{\mathcal{C},F}$, $(c, \emptyset)$ is covered by the empty family.

Because we want to use Theorem 7.1.1 to obtain a normal model, we prove that the sites $(\mathcal{I}_{\mathcal{C},F}, \mathcal{J}_{\mathcal{C},F})$ we have just defined are concrete, and choose a class of admissible monos (Definition 6.2.1) for each of them in Definition 8.1.10.

Proposition 8.1.9. *Each $(\mathcal{I}_{\mathcal{C},F}, \mathcal{J}_{\mathcal{C},F})$ is a concrete site satisfying the (M) and (L) axioms. Moreover, it has an initial object covered by the empty set.*

Proof. Recall the definition of coverage and concrete site from Section 2.1. To prove the main coverage axiom, consider a cover $P = \{U_1, \ldots, U_n\} \in S^{F(c)}$ of (c, U) and a map $f : (d, V) \to (c, U)$. Then P pulls back to a partition $f^{-1}(P) \in S^{F(d)}$, so we get a cover of (d, V). For each $V_i \in f^{-1}(P)$ there is a $U_j \in P$ which is its image under f, so there is a map $f|_{V_i} : (d, V_i) \to (c, U_j)$ in $\mathcal{I}_{\mathcal{C},F}$, as required by the coverage axiom.

Axiom (L) follows by applying the refinement axiom (2) repeatedly. To deduce axiom (M), use the union axiom (1) to see that for any object (c, U) with $U \neq \emptyset$, $\{U\} \in S^{F(c)}$. If $U = \emptyset$, we assumed explicitly that (c, U) is covered by identity.

To prove concreteness, notice that $\mathcal{I}_{\mathcal{C},F}\big(\star, (d, V)\big) \cong V$. Then maps of the form

$$\mathcal{I}_{\mathcal{C},F}\big((d, V), (c, U)\big) \to \mathsf{Set}(V, U)$$

are injective because morphisms in $\mathcal{I}_{\mathcal{C},F}$ are defined to be certain functions. For any cover $P = \{U_1, \ldots, U_n\}$ of (c, U), the map

$$\coprod_{1 \le i \le n} \mathcal{I}_{\mathcal{C},F}\big(\star, (c, U_i)\big) \to \mathcal{I}_{\mathcal{C},F}\big(\star, (c, U)\big)$$

is surjective because $\bigcup_{1 \le i \le n} U_i = U$. □

Definition 8.1.10. Let $\mathcal{M}_{\mathcal{C},F}$ be the class of monos in $\mathcal{I}_{\mathcal{C},F}$ given by:

$$\begin{aligned}\mathcal{M}_{\mathcal{C},F}(c,U) &= \{m : X \rightarrowtail (c,U) \mid m \text{ isomorphic to } (c,U') \to (c,U) \\ &\qquad \text{for some } U' \subseteq U\} \\ \mathcal{M}_{\mathcal{C},F}(\star) &= \{m : X \rightarrowtail \star \mid m \text{ isomorphic to } 0 \to \star \text{ or } \star \to \star\},\end{aligned}$$

where $(c,U') \to (c,U)$ is a partial identity map.

Proposition 8.1.11. *The class $\mathcal{M}_{\mathcal{C},F}$ is a class of admissible monos in the concrete site $(\mathcal{I}_{\mathcal{C},F}, \mathcal{J}_{\mathcal{C},F})$.*

Proof. First show that $\mathcal{M}_{\mathcal{C},F}$ is a stable system of monos. One can check that it contains all identities and is closed under composition. To show it is closed under pullback with arbitrary maps consider, $(c,U') \to (c,U) \in \mathcal{M}$ and a map $f : (d,V) \to (c,U)$. The pullback of f is a map $f|_{V'} : (d,V') \to (c,U')$ where V' is $f^{-1}(U')$. The pair (d,V') is a valid object in $\mathcal{I}_{\mathcal{C},F}$ because $\{U'\} \in S^{F(c)}$ and f pulls back partitions to partitions, and $(d,V') \to (d,V)$ is in $\mathcal{M}_{\mathcal{C},F}$ by definition.

To show that $\mathcal{M}_{\mathcal{C},F}$ is a class of pre-admissible monos we need to show that $\Delta_{\mathcal{M}_{\mathcal{C},F}}$ is a $\mathcal{J}_{\mathcal{C},F}$-sheaf. Let $P = \{U_1, \ldots, U_n\}$ be a covering family of (c,U). Then consider a family of $\mathcal{M}_{\mathcal{C},F}$-subobjects of the form

$$\{(c,U_i') \to (c,U_i) \mid 1 \leq i \leq n\}.$$

Because the U_i's are disjoint, this will always be a matching family.

We know that $P \in S^{F(c)}$ and $\{U_i'\} \in S^{F(c)}$ for $1 \leq i \leq n$. We can use the refinement axiom (2) repeatedly for each $U_i \in P$ and $\{U_i'\} \in S^{F(c)}$ to deduce that $\{U_1', \ldots, U_n'\} \in S^{F(c)}$. Therefore, $(c, \bigcup_{1\leq i\leq n} U_i')$ is a valid object in $\mathcal{I}_{\mathcal{C},F}$ and $(c, \bigcup_{1\leq i\leq n} U_i') \to (c,U)$ is the unique amalgamation we are looking for.

Now, we check the conditions for a class of admissible monos. Every map $0 \to (c,U)$ is in $\mathcal{M}_{\mathcal{C},F}$ because $0 \cong (c,\emptyset)$. To show $\Delta_{\mathcal{M}_{\mathcal{C},F}}$ is concrete consider the maps:

$$\Delta_{\mathcal{M}_{\mathcal{C},F}}(c,U) \to \mathsf{Set}(U, \Delta_{\mathcal{M}_{\mathcal{C},F}}(\star) \cong 2).$$

They are injective because each $\mathcal{M}_{\mathcal{C},F}$-subobject $(c,U') \to (c,U)$ can be identified with a subset $U' \subseteq U$.

Consider an increasing chain of $\mathcal{M}_{\mathcal{C},F}$-subobjects $\{(c, U_i) \to (c, U)\}_{i \in \mathbb{N}}$. The set U is finite so the chain must be eventually constant. Therefore we already know $(c, \bigcup_{i \in \mathbb{N}} U_i) \to (c, U)$ is in $\mathcal{M}_{\mathcal{C},F}$.

Let (c, U_k) be the object where the chain becomes constant. We know $\{U_k\} \in S^{F(c)}$, so $\{(c, U_k) \to (c, U_k)\}$ is a covering family of (c, U_k) which is included in the closure under precomposition of the set $\{(c, U_i) \to (c, U_k) \mid i \in \mathbb{N}\}$. □

We can now define a combined site using Definition 6.2.3, and define a sheaf category $\mathcal{G}$. In the next section, we will prove that $\mathcal{G}$ is a fully abstract model of $\mathsf{PCF_v}$ without any type or term constants.

Definition 8.1.12. Let $(\mathcal{I}_\mathcal{G}, \mathcal{J}_\mathcal{G}, \mathcal{M}_\mathcal{G})$ be the concrete site with admissible monos obtained by combining the sites $(\mathbb{V}_0, \mathcal{J}_{\mathbb{V}_0}, \mathcal{M}_{\mathbb{V}_0})$ and $(\mathcal{I}_{\mathcal{C},F}, \mathcal{J}_{\mathcal{C},F}, \mathcal{M}_{\mathcal{C},F})$ for each faithful functor $F : \mathcal{C} \to \mathsf{SSP}$, according to Definition 6.2.3. Let $\mathcal{G}$ be the category of sheaves on $(\mathcal{I}_\mathcal{G}, \mathcal{J}_\mathcal{G})$.

Proposition 8.1.13. *The sheaf category $\mathcal{G}$ is a normal model of $\mathsf{PCF_v}$ without any type or term constants, obtained by the recipe from Theorem 7.1.1.*

Proof. This is a straightforward application of our recipe for building normal models from Theorem 7.1.1. We instantiate the recipe for the site obtained by combining $(\mathcal{I}_{\mathcal{C},F}, \mathcal{J}_{\mathcal{C},F}, \mathcal{M}_{\mathcal{C},F})$ for each faithful functor $F : \mathcal{C} \to \mathsf{SSP}$. By combining Proposition 8.1.9, Proposition 8.1.11 and Proposition 6.2.4, we deduce that this combined site is a concrete site, satisfying axioms (M) and (L), with an initial object covered by the empty set and with a class of admissible monos. Thus, it satisfies the assumptions of Theorem 7.1.1. Since we are not considering any type constants the rest of the assumption in Theorem 7.1.1 are not relevant. □

Because $\mathcal{G} = \mathsf{Sh}(\mathcal{I}_\mathcal{G}, \mathcal{J}_\mathcal{G})$ is obtained using our recipe for normal models, we can deduce that the interpretation of $\mathsf{PCF_v}$ actually lives in the subcategory of concrete sheaves of $\mathcal{G}$, as explained in Section 7.1. Thus, the interpretation of types has the explicit description from Figure 7.1.

From Theorem 4.3.5 we know $\mathcal{G}$ is a sound model of $\mathsf{PCF_v}$, and from Theorem 7.1.3 we know it is adequate. In Theorem 8.2.11 we will prove it is also fully abstract. From now on, denote by $L_{\mathcal{G}}$ the lifting monad on $\mathcal{G}$ generated by $\mathcal{M}_{\mathcal{G}}$.

8.2 $\mathcal{G}$ is a fully abstract model of $\mathsf{PCF_v}$

The strategy we will use for proving full abstraction is to show that enough maps in the normal model $\mathcal{G}$ are $\mathsf{PCF_v}$-definable (Corollary 8.2.10). Roughly, this would mean we only need to compare denotations on definable elements of their domain, thus making it easier to prove that contextual equivalence implies denotational equality (Theorem 8.2.11). The converse can be deduced from soundness and adequacy, thus completing the proof of full abstraction.

We cannot expect to show that all maps in $\mathcal{G}$ are definable because, for example, there are uncountably many maps of type $\mathsf{Nat} \to \mathsf{Nat}$. To get around this problem we use the same method that Milner [Mil77] used when presenting a syntactic fully abstract model of PCF. We consider *finite approximations* of the sets of points of each type and show that all maps between these are definable.

In Section 8.2.1, we define finite approximations of the interpretation of types in Definition 8.2.1 and Proposition 8.2.2. Then in Proposition 8.2.3 we show that for each type these approximations form a chain, with the least upper bound being the type itself.

In Section 8.2.2 we first choose a collection $\{\mathcal{I}_{\mathcal{C}_n,F_n}\}_{n\in\mathbb{N}}$ of the concrete sites used to define the normal model $\mathcal{G}$ (Definition 8.1.12). The categories $\mathcal{C}_n$ are defined in Definition 8.2.4, and the functors F_n in Definition 8.2.5. Using these sites we state and prove our main definability result, Proposition 8.2.9, followed by Corollary 8.2.10. Section 8.2.3 is dedicated to proving the full abstraction result, Theorem 8.2.11.

$$
\begin{aligned}
\psi_n^{\mathsf{nat}} &= \text{“if } x \le n \text{ then } x \text{ else diverge”},\\
\psi_n^{\mathsf{1}} &= \mathsf{return}\, x,\\
\psi_n^{\mathsf{0}} &= \mathsf{return}\, x,\\
\psi_n^{\sigma\to\tau} &= \mathsf{return}\, \lambda u.\, \mathsf{let}\, v = \psi_n^{\sigma}[u/x]\, \mathsf{in}\, \mathsf{let}\, w = x\ v\, \mathsf{in}\, \psi_n^{\tau}[w/x],\\
\psi_n^{\sigma+\tau} &= \mathsf{case}\, x\, \mathsf{of}\, \{\mathsf{inl}\, y \to \mathsf{let}\, x' = \psi_n^{\sigma}[y/x]\, \mathsf{in}\, \mathsf{return}\, (\mathsf{inl}\, x'),\\
&\qquad\qquad\quad \mathsf{inr}\, z \to \mathsf{let}\, x' = \psi_n^{\tau}[z/x]\, \mathsf{in}\, \mathsf{return}\, (\mathsf{inr}\, x')\},\\
\psi_n^{\sigma\times\tau} &= \mathsf{let}\, y = \pi_1 x\, \mathsf{in}\, \mathsf{let}\, z = \pi_2 x\, \mathsf{in}\, \mathsf{let}\\
&\quad y' = \psi_n^{\sigma}[y/x]\, \mathsf{in}\, \mathsf{let}\, z' = \psi_n^{\tau}[z/x]\, \mathsf{in}\, \mathsf{return}\, (y', z').
\end{aligned}
$$

Figure 8.1: Definition of terms that truncate types.

8.2.1 Finite approximations of types

Definition 8.2.1. For each type σ and $n \in \mathbb{N}$, define a computation

$$x : \sigma \vdash^{\mathbf{c}} \psi_n^{\sigma} : \sigma$$

by recursion on σ as shown in Figure 8.1 [1].

Let h_n^{σ} be the denotation of ψ_n^{σ} in the normal model $\mathcal{G}$ (Definition 8.1.12):

$$h_n^{\sigma} = [\![\psi_n^{\sigma}]\!] : [\![\sigma]\!] \to L_{\mathcal{G}}[\![\sigma]\!].$$

The intuition is that the map h_n^{σ} truncates the interpretation of a type σ to level n, as we prove in the next proposition. In the proof of our main definability result (Proposition 8.2.9), it will be crucial that the h_n^{σ} maps are

[1] The term ψ_n^{nat} can be written explicitly as:

$$
\begin{aligned}
\psi_n^{\mathsf{nat}} = \mathsf{case}\, x\, \mathsf{of}\, \{&\underline{0} \to \mathsf{return}\, x,\\
&\mathsf{S}(x_1) \to\\
&\ldots\ \mathsf{case}\, x_n\, \mathsf{of}\, \{\underline{0} \to \mathsf{return}\, x,\\
&\qquad\qquad \mathsf{S}(x_{n+1}) \to (\mathsf{rec}\, f\, y.\, f\, y)\ \underline{0}\}\\
&\}
\end{aligned}
$$

definable.

Proposition 8.2.2. *Each map h_n^σ is idempotent in the Kleisli category of $L_\mathcal{G}$, more precisely:*

$$(h_n^\sigma)^\dagger \circ h_n^\sigma = h_n^\sigma.$$

Moreover, h_n^σ fixes finitely many points from $|[\![\sigma]\!]|$, where a point $x \in |[\![\sigma]\!]|$ is fixed if:

$$(h_n^\sigma)_\star(x) = (\eta_{[\![\sigma]\!]})_\star(x).$$

Let $[\![\sigma]\!]_n$ be the greatest subobject of $[\![\sigma]\!]$, such that $|[\![\sigma]\!]_n|$ contains the points fixed by h_n^σ. This means that the relations of $[\![\sigma]\!]_n$ are all the relations of $[\![\sigma]\!]$ which are valued in $|[\![\sigma]\!]_n|$. Then we get the following isomorphisms:

$$[\![\mathsf{nat}]\!]_n \cong [\![\underbrace{1 + \ldots + 1}_{n+1 \text{ times}}]\!] \qquad [\![0]\!]_n \cong [\![0]\!] \qquad [\![1]\!]_n \cong [\![1]\!]$$

$$[\![\sigma \to \tau]\!]_n \cong \big([\![\sigma]\!]_n \Rightarrow L_\mathcal{G}[\![\tau]\!]_n\big)$$

$$[\![\sigma \times \tau]\!]_n \cong [\![\sigma]\!]_n \times [\![\tau]\!]_n \qquad [\![\sigma + \tau]\!]_n \cong [\![\sigma]\!]_n + [\![\tau]\!]_n.$$

Proof. The proof is by induction on σ. To show that h_n^σ is idempotent it is enough to show that $(h_n^\sigma)_\star$ is, because h_n^σ is a natural transformation into a concrete presheaf.

Case nat. The underlying set of $[\![\mathsf{nat}]\!]$ is $\mathbb{N}$, so we can see that:

$$(h_n^{\mathsf{nat}})_\star(k) = \begin{cases} k & \text{if } k \leq n \\ \bot & \text{otherwise.} \end{cases}$$

The Kleisli extension $\big((h_n^{\mathsf{nat}})^\dagger\big)_\star$ sends $\bot$ to $\bot$, so we can immediately see that $(h_n^{\mathsf{nat}})_\star$ is idempotent.

The set of points $|[\![\mathsf{nat}]\!]_n|$ fixed by h_n^{nat} is

$$\{0, 1, \ldots, n\}.$$

Since $[\![\mathsf{nat}]\!]_n \rightarrowtail [\![\mathsf{nat}]\!]$ is the greatest subobject with these points, we can see, from the description of $[\![\mathsf{nat}]\!]$ and of coproduct in $\mathcal{G}$ (Figure 7.1), that:

$$[\![\mathsf{nat}]\!]_n \cong [\![\underbrace{1 + \ldots + 1}_{n+1 \text{ times}}]\!] \cong \underbrace{1 + \ldots + 1}_{n+1 \text{ times}}.$$

Case $\mathsf{0}$. In this case

$$h_n^{\mathsf{0}} = \eta_{[\![\mathsf{0}]\!]} : [\![\mathsf{0}]\!] \to L_{\mathcal{G}}[\![\mathsf{0}]\!].$$

Because of the sheaf condition, $[\![\mathsf{0}]\!](0) \cong 1$ and for any other object Y in the site, $[\![\mathsf{0}]\!](Y) = \emptyset$. So h_n^{0} is clearly idempotent and does not fix any points.

Case $\mathsf{1}$. Again $h_n^{\mathsf{1}} = \eta_{[\![\mathsf{1}]\!]}$. For any object Y in the site $[\![\mathsf{1}]\!](Y) \cong 1$, so h_n^{1} is idempotent and fixes one point.

Case $\sigma \to \tau$. We can see that $h_n^{\sigma\to\tau}$ has type:

$$h_n^{\sigma\to\tau} = [\![\psi_n^{\sigma\to\tau}]\!] : ([\![\sigma]\!] \Rightarrow L_{\mathcal{G}}[\![\tau]\!]) \to L_{\mathcal{G}}([\![\sigma]\!] \Rightarrow L_{\mathcal{G}}[\![\tau]\!]).$$

For any point of $[\![\sigma]\!] \Rightarrow L_{\mathcal{G}}[\![\tau]\!]$, that is, a morphism $f : [\![\sigma]\!] \to L_{\mathcal{G}}[\![\tau]\!]$ (see Figure 7.1), $h_n^{\sigma\to\tau}$ is defined as:

$$h_n^{\sigma\to\tau}(f) = \eta_{[\![\sigma]\!]\Rightarrow L_{\mathcal{G}}[\![\tau]\!]}\big((h_n^{\tau})^{\dagger} \circ f^{\dagger} \circ h_n^{\sigma}\big).$$

To show $h_n^{\sigma\to\tau}$ is **idempotent** consider the chain of equalities from Figure 8.2.

To show that $h_n^{\sigma\to\tau}$ **fixes finitely many points** it is enough to show that there are finitely many maps $f : [\![\sigma]\!] \to L_{\mathcal{G}}[\![\tau]\!]$, which will be the elements of $|[\![\sigma \to \tau]\!]_n|$, that satisfy:

$$f = (h_n^{\tau})^{\dagger} \circ f^{\dagger} \circ h_n^{\sigma}$$

because the unit of $L_{\mathcal{G}}$ is monic.

Because h_n^{τ} is idempotent, its image must be included in $|[\![\tau]\!]_n| \cup \{\bot\}$, where $|[\![\tau]\!]_n|$ is the set of points fixed by h_n^{τ}; and so does the image of f. We

$$
\begin{aligned}
(h_n^{\sigma\to\tau})^\dagger\big(h_n^{\sigma\to\tau}(f)\big) &= (h_n^{\sigma\to\tau})^\dagger\Big(\eta_{[\![\sigma]\!]\Rightarrow L_{\mathcal{G}}[\![\tau]\!]}\big((h_n^\tau)^\dagger \circ f^\dagger \circ h_n^\sigma\big)\Big)\\
&= (h_n^{\sigma\to\tau})^\dagger\big((h_n^\tau)^\dagger \circ f^\dagger \circ h_n^\sigma\big)\\
&= (h_n^\tau)^\dagger \circ \big((h_n^\tau)^\dagger \circ f^\dagger \circ h_n^\sigma\big)^\dagger \circ h_n^\sigma\\
&= (h_n^\tau)^\dagger \circ (h_n^\tau)^\dagger \circ f^\dagger \circ (h_n^\sigma)^\dagger \circ h_n^\sigma\\
&\qquad\qquad \text{using } (g_1^\dagger \circ g_2)^\dagger = g_1^\dagger \circ g_2^\dagger \text{ twice}\\
&= (h_n^\tau)^\dagger \circ f^\dagger \circ h_n^\sigma\\
&\qquad\qquad h_n^\tau,\ h_n^\sigma \text{ idempotent by induction hypothesis}\\
&= h_n^{\sigma\to\tau}(f).
\end{aligned}
$$

Figure 8.2: Proving that $h_n^{\sigma\to\tau}$ is idempotent.

can rewrite the equation above as:

$$f = f^\dagger \circ h_n^\sigma.$$

Similarly, the image of h_n^σ is included in $|[\![\sigma]\!]_n| \cup \{\bot\}$. This means that f is determined by its action on $|[\![\sigma]\!]_n|$. Since both $|[\![\tau]\!]_n|$ and $|[\![\sigma]\!]_n|$ are finite by induction hypothesis, there are a finite number of possibilities for f.

To show the isomorphism:

$$[\![\sigma \to \tau]\!]_n \cong \big([\![\sigma]\!]_n \Rightarrow L_{\mathcal{G}}[\![\tau]\!]_n\big),$$

it is enough to show that the two sides have the same points and the same relations. To show they have the same points, start with a map

$$g : [\![\sigma]\!]_n \to L_{\mathcal{G}}[\![\tau]\!]_n.$$

Because

$$L_{\mathcal{G}}[\![\tau]\!]_n \subseteq L_{\mathcal{G}}[\![\tau]\!],$$

we can extend g to the map

$$g^\dagger \circ h_n^\sigma : [\![\sigma]\!] \to L_{\mathcal{G}}[\![\tau]\!].$$

We can then show $g^\dagger \circ h_n^\sigma$ is a point of $[\![\sigma \to \tau]\!]_n$ because it is fixed by $h_n^{\sigma\to\tau}$:

$$\begin{aligned} h_n^{\sigma\to\tau}(g^\dagger \circ h_n^\sigma) &= \eta_{[\![\sigma]\!]\Rightarrow L_{\mathcal{G}}[\![\tau]\!]}\big((h_n^\tau)^\dagger \circ (g^\dagger \circ h_n^\sigma)^\dagger \circ h_n^\sigma\big) \\ &= \eta_{[\![\sigma]\!]\Rightarrow L_{\mathcal{G}}[\![\tau]\!]}\big((h_n^\tau)^\dagger \circ g^\dagger \circ (h_n^\sigma)^\dagger \circ h_n^\sigma\big) \\ &= \eta_{[\![\sigma]\!]\Rightarrow L_{\mathcal{G}}[\![\tau]\!]}\big((h_n^\tau)^\dagger \circ g^\dagger \circ h_n^\sigma\big) \\ &\qquad\qquad \text{because } h_n^\sigma \text{ idempotent} \\ &= \eta_{[\![\sigma]\!]\Rightarrow L_{\mathcal{G}}[\![\tau]\!]}\big(g^\dagger \circ h_n^\sigma\big) \\ &\qquad\qquad \text{the image of } g^\dagger \text{ is already in } [\![\tau]\!]_n \cup \{\bot\}. \end{aligned}$$

For the reverse inclusion, start with a map $f : [\![\sigma]\!] \to L_{\mathcal{G}}[\![\tau]\!]$ that is fixed by $h_n^{\sigma\to\tau}$, so we know:

$$f = (h_n^\tau)^\dagger \circ f^\dagger \circ h_n^\sigma.$$

This means that the image of f is included in $L_{\mathcal{G}}[\![\tau]\!]_n$, so by restricting we can get a map on points:

$$\big(f|_{[\![\sigma]\!]_n}\big)_\star : |[\![\sigma]\!]_n| \to |L_{\mathcal{G}}[\![\tau]\!]_n|.$$

This map preserves the relations of $[\![\sigma]\!]_n$ because they are a subset of the relations of $[\![\sigma]\!]$, and f preserves those.

Next, we show that $[\![\sigma \to \tau]\!]_n$ and $\big([\![\sigma]\!]_n \Rightarrow L_{\mathcal{G}}[\![\tau]\!]_n\big)$ agree at any object Y in the site, other than $\star$, (i.e. V, (c, U) or 0). According to Figure 7.1, their respective values at Y are:

$$\begin{aligned} &\big([\![\sigma]\!]_n \Rightarrow L_{\mathcal{G}}[\![\tau]\!]_n\big)(Y) = \big\{f : |Y| \to \mathcal{G}([\![\sigma]\!]_n, L_{\mathcal{G}}[\![\tau]\!]_n) \;\big|\; \\ &\qquad \forall h : X \to Y. \forall g \in [\![\sigma]\!]_n(X).\ \big(\lambda x \in |X|.\, f(h(x))\, g(x)\big) \in L_{\mathcal{G}}[\![\tau]\!]_n(X)\big\} \end{aligned}$$

and

$$\begin{aligned} &\big([\![\sigma \to \tau]\!]_n\big)(Y) = \big\{f : |Y| \to |[\![\sigma \to \tau]\!]_n| \subseteq \mathcal{G}([\![\sigma]\!], L_{\mathcal{G}}[\![\tau]\!]) \;\big|\; \\ &\qquad \forall h : X \to Y. \forall g \in [\![\sigma]\!](X).\ \big(\lambda x \in |X|.\, f(h(x))\, g(x)\big) \in L_{\mathcal{G}}[\![\tau]\!](X)\big\}. \end{aligned}$$

Start with $f \in \big([\![\sigma \to \tau]\!]_n\big)(Y)$. To get a function in $\big([\![\sigma]\!]_n \Rightarrow L_{\mathcal{G}}[\![\tau]\!]_n\big)(Y)$, it is enough to restrict each $f(y)$ to $f(y)|_{[\![\sigma]\!]_n}$, as we did at $\star$. To see that for any $g \in [\![\sigma]\!]_n(X)$

$$\big(\lambda x \in |X|.\, f(h(x))|_{[\![\sigma]\!]_n}\, g(x)\big) \in L_{\mathcal{G}}[\![\tau]\!]_n(X)$$

we use the description of $\big([\![\sigma \to \tau]\!]_n\big)(Y)$, the fact that the image of $f(h(x))$ is already included in $[\![\tau]\!]_n \cup \{\bot\}$, and the fact that $[\![\tau]\!]_n$ is the greatest subobject of $[\![\tau]\!]$.

Conversely, start with $f \in \big([\![\sigma]\!]_n \Rightarrow L_{\mathcal{G}}[\![\tau]\!]_n\big)(Y)$. To get an element of $\big([\![\sigma \to \tau]\!]_n\big)(Y)$ it is enough to precompose each $f(h(x))$ with h_n^σ. Then we need to show that for any $g \in [\![\sigma]\!](X)$:

$$\big(\lambda x \in |X|.\, f(h(x))^\dagger\ h_n^\sigma(g(x))\big) \in L_{\mathcal{G}}[\![\tau]\!](X).$$

Because $h_n^\sigma : [\![\sigma]\!] \to L_{\mathcal{G}}[\![\sigma]\!]$ is a morphism in $\mathcal{G}$, it preserves relations, so

$$(h_n^\sigma \circ g) \in L_{\mathcal{G}}[\![\sigma]\!](X).$$

But the image of h_n^σ is $[\![\sigma]\!]_n \cup \{\bot\}$, so

$$(h_n^\sigma \circ g) \in L_{\mathcal{G}}[\![\sigma]\!]_n(X).$$

From the description of the action of the lifting monad on relations (Figure 7.2), we see that there must be a mono $X' \rightarrowtail X \in \mathcal{M}_{\mathcal{G}}$ such that:

$$(h_n^\sigma \circ g)|_{X'} \in [\![\sigma]\!]_n(X').$$

From the explicit description of $\big([\![\sigma]\!]_n \Rightarrow L_{\mathcal{G}}[\![\tau]\!]_n\big)(Y)$ applied to $X' \rightarrowtail X \xrightarrow{h} Y$ and $(h_n^\sigma \circ g)|_{X'}$ we get that:

$$\big(\lambda x \in |X'|.\, f(h(x))\ h_n^\sigma(g(x))\big) \in L_{\mathcal{G}}[\![\tau]\!]_n(X').$$

So there must be a mono $X'' \rightarrowtail X' \in \mathcal{M}_{\mathcal{G}}$ such that

$$\big(\lambda x \in |X'|.\, f(h(x))\ h_n^\sigma(g(x))\big)|_{X''} \in [\![\tau]\!]_n(X'').$$

Using the mono $X'' \rightarrowtail X' \rightarrowtail X \in \mathcal{M}_{\mathcal{G}}$ we can see that

$$\big(\lambda x \in |X|.\, f(h(x))^{\dagger}\ h_n^{\sigma}(g(x))\big) \in L_{\mathcal{G}}[\![\tau]\!]_n(X) \subseteq L_{\mathcal{G}}[\![\tau]\!](X).$$

Case $\sigma + \tau$. The map

$$h_n^{\sigma+\tau} = [\![\psi_n^{\sigma+\tau}]\!] : [\![\sigma]\!] + [\![\tau]\!] \to L_{\mathcal{G}}([\![\sigma]\!] + [\![\tau]\!])$$

is equal to

$$h_n^{\sigma+\tau} = [L_{\mathcal{G}}(\mathsf{inj}_{[\![\sigma]\!]}) \circ h_n^{\sigma},\ L_{\mathcal{G}}(\mathsf{inj}_{[\![\tau]\!]}) \circ h_n^{\tau}].$$

It follows that $h_n^{\sigma+\tau}$ is idempotent because h_n^{σ} and h_n^{τ} are by induction hypothesis. Similarly, $h_n^{\sigma+\tau}$ fixes the points

$$|[\![\sigma + \tau]\!]_n| = |[\![\sigma]\!]_n| + |[\![\tau]\!]_n|.$$

To show the isomorphism

$$[\![\sigma + \tau]\!]_n \cong [\![\sigma]\!]_n + [\![\tau]\!]_n$$

it remains to show that the two sides have the same relations. This follows from the fact that $[\![\sigma + \tau]\!]_n$ is a greatest subobject of $[\![\sigma + \tau]\!]$, and from the description of coproducts in $\mathcal{G}$ (Figure 7.1).

Case $\sigma \times \tau$. In this case

$$h_n^{\sigma\times\tau} = [\![\psi_n^{\sigma\times\tau}]\!] : [\![\sigma]\!] \times [\![\tau]\!] \to L_{\mathcal{G}}([\![\sigma]\!] \times [\![\tau]\!])$$

is equal to

$$h_n^{\sigma\times\tau} = \mu_{[\![\sigma]\!]\times[\![\tau]\!]} \circ \mathrm{str}^2 \circ \langle h_n^{\sigma} \circ \pi_1, h_n^{\tau} \circ \pi_2 \rangle.$$

Notice that at $\star$, $\mu_{[\![\sigma]\!]\times[\![\tau]\!]} \circ \mathrm{str}^2$ identifies the two $\bot$ elements (see Figure 7.2). Then $h_n^{\sigma\times\tau}$ is idempotent because both h_n^{σ} and h_n^{τ} are. The points fixed by $h_n^{\sigma\times\tau}$ are $|[\![\sigma]\!]_n| \times |[\![\tau]\!]_n|$. The isomorphism

$$[\![\sigma \times \tau]\!]_n \cong [\![\sigma]\!]_n \times [\![\tau]\!]_n$$

is showed similarly to the coproduct case.

□

Recall the vertical natural numbers ω (Assumption 3.1.4), and the extended vertical natural numbers $\overline{\omega}$ (Assumption 3.1.1), which are calculated as a colimit and limit respectively, either in a category of sheaves or presheaves. From the discussion in Section 6.1, recall that when talking about completeness, it is sufficient to consider the colimit ω calculated in $\mathsf{PSh}(\mathcal{I}_{\mathcal{G}})$, rather than in $\mathcal{G}$. We will adopt this simplification in the rest of this chapter. The limit $\overline{\omega}$ is the same in both categories.

Recall from Remark 6.3.1 that the elements of $\omega(\star)$ are eventually 0, infinite chains of subobjects starting at $\star$:

$$\ldots 0 \rightarrowtail 0 \rightarrowtail \star \rightarrowtail \ldots \star,$$

so they can be identified with the natural numbers. The set $\overline{\omega}(\star)$ contains the same points plus the always $\star$ chain, which we denote by ∞.

The intuition behind the maps $h_n^\sigma : [\![\sigma]\!] \to L_{\mathcal{G}}[\![\sigma]\!]$, made precise in Proposition 8.2.2, is that for a point $x \in |[\![\sigma]\!]|$, $h_n^\sigma(x)$ is in some sense an approximation of x to level n. The next proposition shows that indeed the $h_n^\sigma(x)$'s form a chain, and that the least upper bound of this chain is x.

Recall from Proposition 4.3.4 that in a normal model, so also in $\mathcal{G}$, the interpretation of every type is a well-complete object (Definition 3.2.2). This justifies the existence of the unique extension H^σ in the statement of the next proposition.

Proposition 8.2.3. *The function on points:*

$$(h^\sigma)_\star(n, x) = (h_n^\sigma)_\star(x)$$

defines a morphism

$$h^\sigma : \omega \times [\![\sigma]\!] \to L_{\mathcal{G}}[\![\sigma]\!]$$

in $\mathcal{G}$, whose unique extension $H^\sigma : \overline{\omega} \times [\![\sigma]\!] \to L_{\mathcal{G}}[\![\sigma]\!]$ satisfies

$$(H^\sigma)_\star(\infty, x) = (\eta_{[\![\sigma]\!]})_\star(x).$$

Proof. The proof is by induction on σ.

Case nat. In a normal model $[\![\mathsf{nat}]\!] \cong \coprod_{\mathbb{N}} 1$, so

$$\omega \times [\![\mathsf{nat}]\!] \cong \coprod_{\mathbb{N}} \omega$$

and we can rewrite $(h^{\mathsf{nat}})_\star$ as:

$$(h^{\mathsf{nat}})_\star\big((n)_i\big) = (h_n^{\mathsf{nat}})_\star(i) = \begin{cases} i & \text{if } i \leq n \\ \bot & \text{otherwise.} \end{cases}$$

Notice that $(h^{\mathsf{nat}})_\star$ can be written as the co-pairing of the following functions $(f_i)_\star : \omega(\star) \to (L_{\mathcal{G}}[\![\mathsf{nat}]\!])(\star)$, for each $i \in \mathbb{N}$:

$$(f_i)_\star(n) = \begin{cases} i & \text{if } i \leq n \\ \bot & \text{otherwise.} \end{cases}$$

Because $L_{\mathcal{G}}[\![\mathsf{nat}]\!]$ is concrete, if we show $f_i : \omega \to L_{\mathcal{G}}[\![\mathsf{nat}]\!]$ is a well-defined morphism in $\mathcal{G}$, then this is enough to show that h^{nat} is the co-pairing $[f_i]_{i\in\mathbb{N}}$, and is thus a well-defined morphism.

The function $(f_i)_\star$ factors as:

$$\omega(\star) \xrightarrow{(g_i)_\star} L_{\mathcal{G}}1(\star) \xrightarrow{(L_{\mathcal{G}}\mathsf{inc})_\star} (L_{\mathcal{G}}[\![\mathsf{nat}]\!])(\star)$$

where

$$(g_i)_\star(n) = \begin{cases} * & \text{if } i \leq n \\ \bot & \text{otherwise.} \end{cases}$$

To show $g_i : \omega \to L_{\mathcal{G}}1$ is a well-defined morphism, consider a chain $b = (\ldots 0 \rightarrowtail X_k \rightarrowtail \ldots X_1 \rightarrowtail X)$ in $\omega(X)$. For each point $p : \star \to X$, denote by $p^*(b)$ the chain in $\omega(\star)$ which is the pullback of b along p. Then:

$$(g_i)_X(b) = \lambda p \in |X|. \begin{cases} * & \text{if } p^*(b) \geq i \\ \bot & \text{otherwise.} \end{cases}$$

To show $(g_i)_X(b) \in L_{\mathcal{G}}1(\star)$, we need to show it corresponds to an admissible mono $X' \rightarrowtail X$. If $k < i$, then the mono is $0 \to X$. Otherwise, it is $X_i \rightarrowtail X$.

Next, calculate the extension of h^{nat}. Because in a normal model $L_{\mathcal{G}}[\![\mathsf{nat}]\!]$ is complete, this extension must be unique.

From Lemma 6.3.9 we know that the unique extension of each $f_i : \omega \to L_{\mathcal{G}}[\![\mathsf{nat}]\!]$ is $\overline{f_i} : \overline{\omega} \to L_{\mathcal{G}}[\![\mathsf{nat}]\!]$ such that:

$$(\overline{f_i})_\star(\infty) = (\eta_{[\![\mathsf{nat}]\!]})_\star(i).$$

So we can let

$$H^{\mathsf{nat}} = [\overline{f_i}]_{i \in \mathbb{N}},$$

which has the desired property at ∞. Because $h^{\mathsf{nat}} = [f_i]_{i \in \mathbb{N}}$, it is clear that H^{nat} is its extension.

Case 0. We know that $[\![\mathsf{0}]\!](0) \cong 1$ and for any other object X in the site $[\![\mathsf{0}]\!](X) = \emptyset$. And $\omega(0) \cong 1$. Therefore

$$h^{\mathsf{0}} : \omega \times [\![\mathsf{0}]\!] \to L_{\mathcal{G}}[\![\mathsf{0}]\!]$$

is the empty function for any object X other than 0, and at 0 maps the only input element to the top element of $L_{\mathcal{G}}[\![\mathsf{0}]\!](0)$. Therefore, h^{0} is a well-defined morphism. The extension H^{0} is defined in the same way.

Case 1. We know that

$$h^{\mathsf{1}} : \omega \times [\![\mathsf{1}]\!] \to L_{\mathcal{G}}[\![\mathsf{1}]\!]$$

is defined as

$$(h^{\mathsf{1}})_\star(n, *) = (\eta_{[\![\mathsf{1}]\!]})_\star(*),$$

so it is a well-defined morphism. Its extension H^{1} is defined similarly, and satisfies:

$$(H^{\mathsf{1}})_\star(\infty, *) = *.$$

Case $\sigma \to \tau$. In this case we want to show there is a morphism

$$h^{\sigma\to\tau} : \omega \times (\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket) \to L_{\mathcal{G}}(\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket)$$

which is defined on points as:

$$(h^{\sigma\to\tau})_\star(n, f) = (\eta_{\llbracket\sigma\rrbracket\Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket})_\star\big((h^\tau_n)^\dagger \circ f^\dagger \circ h^\sigma_n\big).$$

To do this, start from an arbitrary morphism

$$x : X \to \omega \times (\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket)$$

and construct a morphism

$$X \to L_{\mathcal{G}}(\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket),$$

such that the construction is natural in X.

Let $x_1 : X \to \omega$ and $x_2 : X \times \llbracket\sigma\rrbracket \to L_{\mathcal{G}}\llbracket\tau\rrbracket$ be the two maps obtained from x. By the induction hypothesis, we have two maps:

$$\begin{aligned} h^\sigma &: \omega \times \llbracket\sigma\rrbracket \to L_{\mathcal{G}}\llbracket\sigma\rrbracket \\ h^\tau &: \omega \times \llbracket\tau\rrbracket \to L_{\mathcal{G}}\llbracket\tau\rrbracket. \end{aligned}$$

Then we can form the following map:

$$\begin{aligned} X \times \llbracket\sigma\rrbracket &\xrightarrow{\Delta_X \times \mathrm{id}_{\llbracket\sigma\rrbracket}} X \times X \times \llbracket\sigma\rrbracket \xrightarrow{\mathrm{id}_X \times x_1 \times \mathrm{id}_{\llbracket\sigma\rrbracket}} X \times \omega \times \llbracket\sigma\rrbracket \xrightarrow{\mathrm{id}_X \times h^\sigma} X \times L_{\mathcal{G}}\llbracket\sigma\rrbracket \\ &\xrightarrow{\Delta_X \times \mathrm{id}_{L_{\mathcal{G}}\llbracket\sigma\rrbracket}} X \times X \times L_{\mathcal{G}}\llbracket\sigma\rrbracket \xrightarrow{\mathrm{id}_X \times \mathrm{str}} X \times L_{\mathcal{G}}(X \times \llbracket\sigma\rrbracket) \xrightarrow{\mathrm{id}_X \times x_2^\dagger} X \times L_{\mathcal{G}}\llbracket\tau\rrbracket \\ &\xrightarrow{x_1 \times \mathrm{id}_{L_{\mathcal{G}}\llbracket\tau\rrbracket}} \omega \times L_{\mathcal{G}}\llbracket\tau\rrbracket \xrightarrow{\mathrm{str}} L_{\mathcal{G}}(\omega \times \llbracket\tau\rrbracket) \xrightarrow{(h^\tau)^\dagger} L_{\mathcal{G}}\llbracket\tau\rrbracket, \end{aligned}$$

where Δ_X is the copying map. By currying the map above and postcomposing with η we obtain

$$\alpha = X \to (\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket) \xrightarrow{\eta} L_{\mathcal{G}}(\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket).$$

If we let $X = \omega \times (\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket)$ and $x = \mathrm{id}_{\omega\times(\llbracket\sigma\rrbracket\Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket)}$ we obtain a map α_1 with the correct type for $h^{\sigma\to\tau}$. We need to show that at $\star$, α_1 agrees with $(h^{\sigma\to\tau})_\star$.

For this, let $X = 1$ and let $x = \langle n, f\rangle : 1 \to \omega \times (\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket)$, where $n \in \omega(\star)$ and $f : \llbracket\sigma\rrbracket \to L_{\mathcal{G}}\llbracket\tau\rrbracket$, and consider the induced map $\alpha_2 : 1 \to L_{\mathcal{G}}(\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket)$. Then by construction of α, $(\alpha_2)_\star$ is $(h^{\sigma\to\tau})_\star(n, f)$.

Now define the extension of $h^{\sigma\to\tau}$:

$$H^{\sigma\to\tau} : \overline{\omega} \times (\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket) \to L_{\mathcal{G}}(\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket).$$

This will be the unique extension because in a normal model the interpretation of any type is well-complete.

From the induction hypothesis we have maps H^σ and H^τ which are the extensions of h^σ and h^τ respectively. We can construct $H^{\sigma\to\tau}$ starting from an arbitrary map

$$x : X \to \overline{\omega} \times (\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket)$$

in the same way we constructed $h^{\sigma\to\tau}$, because no properties of ω were needed. Then for any $n \in \overline{\omega}(\star)$ and any $f : \llbracket\sigma\rrbracket \to L_{\mathcal{G}}\llbracket\tau\rrbracket$:

$$(H^{\sigma\to\tau})_\star(n, f) = (\eta_{\llbracket\sigma\rrbracket\Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket})_\star\big((H^\tau(n, -))^\dagger \circ f^\dagger \circ H^\sigma(n, -)\big).$$

So it is clear that $H^{\sigma\to\tau}$ is the extension of $h^{\sigma\to\tau}$, thanks to concreteness.

To show

$$(H^{\sigma\to\tau})_\star(\infty, f) = (\eta_{\llbracket\sigma\rrbracket\Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket})_\star(f)$$

it is enough to notice that by induction hypothesis

$$\big((H^\tau(n, -))^\dagger\big)_\star \circ (f^\dagger)_\star \circ \big(H^\sigma(n, -)\big)_\star = f_\star.$$

Case $\sigma + \tau$. We know that

$$(h^{\sigma+\tau})_\star(n, x) = (h_n^{\sigma+\tau})_\star(x) = [L_{\mathcal{G}}\mathrm{inc}_{\llbracket\sigma\rrbracket} \circ h_n^\sigma, L_{\mathcal{G}}\mathrm{inc}_{\llbracket\tau\rrbracket} \circ h_n^\tau]_\star(x).$$

To obtain a morphism that agrees on points with the above, notice that:

$$\omega \times (\llbracket\sigma\rrbracket + \llbracket\tau\rrbracket) \cong (\omega \times \llbracket\sigma\rrbracket) + (\omega \times \llbracket\tau\rrbracket),$$

so we can define

$$h^{\sigma+\tau} = [L_{\mathcal{G}}\mathrm{inc}_{\llbracket\sigma\rrbracket} \circ h^{\sigma}, L_{\mathcal{G}}\mathrm{inc}_{\llbracket\tau\rrbracket} \circ h^{\tau}],$$

and by induction hypothesis we are done.

Similarly, define

$$H^{\sigma+\tau} = [L_{\mathcal{G}}\mathrm{inc}_{\llbracket\sigma\rrbracket} \circ H^{\sigma}, L_{\mathcal{G}}\mathrm{inc}_{\llbracket\tau\rrbracket} \circ H^{\tau}],$$

which by induction hypothesis is the extension of $h^{\sigma+\tau}$, and has the required value at ∞.

Case $\sigma \times \tau$. We know

$$(h^{\sigma\times\tau})_\star(n,x) = (h_n^{\sigma\times\tau})_\star(x) = \big(\mu \circ \mathrm{str}^2 \circ \langle h_n^{\sigma} \circ \pi_1, h_n^{\tau} \circ \pi_2\rangle\big)_\star(x).$$

To get the same behaviour on points, define

$$h^{\sigma\times\tau} = \omega \times (\llbracket\sigma\rrbracket \times \llbracket\tau\rrbracket) \xrightarrow{\langle h^{\sigma}\circ(\mathrm{id}_\omega\times\pi_1),\, h^{\tau}\circ(\mathrm{id}_\omega\times\pi_2)\rangle} L_{\mathcal{G}}\llbracket\sigma\rrbracket \times L_{\mathcal{G}}\llbracket\tau\rrbracket \xrightarrow{\mu\circ\mathrm{str}^2} L_{\mathcal{G}}(\llbracket\sigma\rrbracket \times \llbracket\tau\rrbracket).$$

using the induction hypothesis. The extension $H^{\sigma\times\tau}$ can be defined similarly using H^{σ} and H^{τ}, and has the required properties by induction hypothesis. □

8.2.2 Finite approximations are definable

We now choose a collection of sites $\mathcal{I}_{\mathcal{C}_n,F_n}$, one for each $n \in \mathbb{N}$, from among the sites of $\mathcal{G}$, and use them to show that all points of truncated types $\llbracket\sigma\rrbracket_k$ (introduced in Proposition 8.2.2) are definable (Corollary 8.2.10).

To extend the truncated interpretation to typing contexts $\Gamma = x_1 : \tau_1, \ldots, x_l : \tau_l$, we define:

$$[\![\Gamma]\!]_k = [\![\tau_1]\!]_k \times \ldots \times [\![\tau_l]\!]_k.$$

Define $\mathcal{C}_n$ and F_n as follows:

Definition 8.2.4. Let $\mathcal{C}_n$ be the category where objects are typing contexts Γ and maps $\big((\Gamma, \Delta) \to \Gamma\big)$ correspond to projections:

$$[\![\Gamma, \Delta]\!]_n \cong [\![\Gamma]\!]_n \times [\![\Delta]\!]_n \to [\![\Gamma]\!]_n.$$

By projection we mean the morphism in $\mathcal{G}$ with underlying function:

$$\pi_1 : |[\![\Gamma]\!]_n| \times |[\![\Delta]\!]_n| \to |[\![\Gamma]\!]_n|.$$

Therefore, there can be at most one map between two objects of $\mathcal{C}_n$. Composition is given by composition in $\mathcal{G}$ and identities occur when the second component of the domain, Δ, is the empty context.

Definition 8.2.5. Let $F_n : \mathcal{C}_n \to \mathsf{SSP}$ be the functor defined by

$$F_n(\Gamma) = (|[\![\Gamma]\!]_n|, S^{\Gamma,n}),$$

where $P \in S^{\Gamma,n}$ if and only if there exists a computation

$$\Gamma \vdash^{\mathbf{c}} t : \mathsf{nat}$$

such that P is the set of non-empty fibres of the map

$$[\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t]\!]} L_{\mathcal{G}}[\![\mathsf{nat}]\!]$$

at $\star$, (excluding the fibre of $\bot$).

On maps, F_n sends $(\Gamma, \Delta) \to \Gamma$ from $\mathcal{C}_n$ to the Set projection:

$$F_n((\Gamma, \Delta) \to \Gamma) = (|[\![\Gamma, \Delta]\!]_n|, S^{(\Gamma,\Delta),n}) \to (|[\![\Gamma]\!]_n|, S^{\Gamma,n}),$$

which makes sense because $|[\![\Gamma, \Delta]\!]_n| \cong |[\![\Gamma]\!]_n| \times |[\![\Delta]\!]_n|$ from Proposition 8.2.2.

Remark 8.2.6. The definition above implies that $\{U\} \in S^{\Gamma,n}$ if and only if U is the domain of some computation $\Gamma \vdash^{\mathbf{c}} t : \mathsf{1}$ restricted to $[\![\Gamma]\!]_n$. By the union axiom of SSP this is also equivalent to $U \in P$ for some $P \in S^{\Gamma,n}$.

Lemma 8.2.7. *The mapping F_n from Definition 8.2.5 is a well-defined faithful functor.*

Proof. We need to show that $S^{\Gamma,n}$ satisfies the SSP axioms. The partition $\{|[\![\Gamma]\!]_n|\}$ is in $S^{\Gamma,n}$ because it corresponds to the computation:

$$\Gamma \vdash^{\mathbf{c}} \mathsf{return}\, \underline{0} : \mathsf{nat}.$$

The empty partition corresponds to, for example, the diverging computation:

$$\Gamma \vdash^{\mathbf{c}} (\mathsf{rec}\, f\, x.\, f\, x)\, \underline{0} : \mathsf{nat}.$$

To show $S^{\Gamma,n}$ satisfies the refinement axiom, suppose that $P, Q \in S^{\Gamma,n}$ and that $U \in P$. Suppose further that $\Gamma \vdash^{\mathbf{c}} t_P : \mathsf{nat}$ and $\Gamma \vdash^{\mathbf{c}} t_Q : \mathsf{nat}$ are the terms representing P and Q respectively. Let $k_1 \leq k_2 \leq \ldots \leq k_n \in \mathbb{N}$ be the values whose fibres give P, and suppose k_U among then is the image of U. Then to represent the partition $(P \setminus \{U\}) \cup (\{U \cap V \mid V \in Q\} \setminus \{\emptyset\})$ we can form the computation:

$$\Gamma \vdash^{\mathbf{c}} \mathsf{let}\, x = t_P\, \mathsf{in}\, \mathsf{if}\, x = k_U\, \mathsf{then}\, \mathsf{let}\, y = t_Q\, \mathsf{in}\, \mathsf{return}\, (y + k_n)\, \mathsf{else}\, \mathsf{return}\, x : \mathsf{nat}.$$

For the axiom about unioning partition classes, suppose $U, V \in P \in S^{\Gamma,n}$. Let $\Gamma \vdash^{\mathbf{c}} t : \mathsf{nat}$ be the computation that represents P and let k_U and k_V be the images of U and V respectively. To represent $(P \setminus \{U, V\}) \cup \{U \cup V\}$ consider the computation:

$$\Gamma \vdash^{\mathbf{c}} \mathsf{let}\, x = t\, \mathsf{in}\, \mathsf{if}\, x = k_V\, \mathsf{then}\, \mathsf{return}\, k_U\, \mathsf{else}\, \mathsf{return}\, x : \mathsf{nat}.$$

To show the projection $(|[\![\Gamma, \Delta]\!]_n|, S^{(\Gamma,\Delta),n}) \to (|[\![\Gamma]\!]_n|, S^{\Gamma,n})$ is a valid SSP map, we need to show that it pulls back partitions to partitions. Let $P \in S^{\Gamma,n}$

be represented by the computation $\Gamma \vdash^{\mathbf{c}} t : \mathsf{nat}$. Then P pulls back to:

$$\big\{U \times |[\![\Delta]\!]_n| \;\big|\; U \in P\big\}$$

which is induced by the term $\Gamma, \Delta \vdash^{\mathbf{c}} t : \mathsf{nat}$.

We omit the proof that F_n is functorial. It is faithful because there is at most one map between any two objects of $\mathcal{C}_n$. □

Remark 8.2.8. We can describe the site $\mathcal{I}_{\mathcal{C}_n,F_n}$ explicitly. The objects are either the terminal $\star$, or of the form (Γ, U) where Γ is a typing context and $U \subseteq |[\![\Gamma]\!]_n|$ where $\{U\} \in S^{\Gamma,n}$ or $U = \emptyset$. Morphisms $\big((\Gamma,\Delta),U\big) \to (\Gamma, V)$ are Set projections $U \to V$.

Now we prove the most important result about definability. It will allow us to deduce Corollary 8.2.10, which in turn plays a crucial role in the proof of full abstraction Theorem 8.2.11.

Proposition 8.2.9. *Consider the interpretation of* $\mathsf{PCF_v}$ *in the normal model* $\mathcal{G}$ *(Definition 8.1.12). For any type* σ*, for any* $n \in \mathbb{N}$*, and for any object* $(\Gamma, U)_{\mathcal{C}_n,F_n}$ *in* $\mathcal{I}_{\mathcal{C}_n,F_n}$*, a function*

$$g : U \to |[\![\sigma]\!]_n|$$

is in $[\![\sigma]\!]_n(\Gamma, U)_{\mathcal{C}_n,F_n}$ *if and only if there exists a computation*

$$\Gamma \vdash^{\mathbf{c}} t : \sigma$$

such that the domain at $\star$ *of*

$$\alpha = [\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t]\!]} L_{\mathcal{G}}[\![\sigma]\!] \xrightarrow{(h_n^\sigma)^\dagger} L_{\mathcal{G}}[\![\sigma]\!]_n$$

is U *and* $\alpha_\star|_U = g$*.*

When writing $(h_n^\sigma)^\dagger : L_{\mathcal{G}}[\![\sigma]\!] \to L_{\mathcal{G}}[\![\sigma]\!]_n$ we mean the map $(h_n^\sigma)^\dagger : L_{\mathcal{G}}[\![\sigma]\!] \to L_{\mathcal{G}}[\![\sigma]\!]$ with codomain restricted to $L_{\mathcal{G}}[\![\sigma]\!]_n$, which is possible because we know its image lies inside $L_{\mathcal{G}}[\![\sigma]\!]_n$. Notice that the right-to-left implication makes sense because $\{\mathsf{dom}(\alpha_\star)\} \in S^{\Gamma,n}$ since it is represented by the computation:

$$\Gamma \vdash^{\mathbf{c}} \mathsf{let}\, z = t \,\mathsf{in}\, \mathsf{let}\, y = \psi^\sigma_n[z/x] \,\mathsf{in}\, \mathsf{return} \star : \mathsf{1}.$$

Proof of Proposition 8.2.9. The proof is by induction on σ.

Case nat. Consider $g \in [\![\mathsf{nat}]\!]_n(\Gamma, U)_{\mathcal{C}_n,F_n}$. We proved in Proposition 8.2.2 that

$$[\![\mathsf{nat}]\!]_n = \underbrace{1 + 1 + \ldots + 1}_{n+1 \text{ times}}$$

so by the description of coproduct in $\mathcal{G}$ (Figure 7.1) it must be the case that the non-empty fibres of $g : U \to \sum_{n+1} 1$ form a partition $P \in S^{\Gamma,n}$, where $\bigcup P = U$.

Notice that P has at most $n+1$ elements.

By the definition of $S^{\Gamma,n}$, there must be a computation $\Gamma \vdash^{\mathbf{c}} t : \mathsf{nat}$ such that the non-empty fibres of

$$[\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t]\!]} L_{\mathcal{G}}[\![\mathsf{nat}]\!]$$

form P. Because P has finitely many elements, we can rearrange the output values of t to match those of g, and obtain another computation $\Gamma \vdash^{\mathbf{c}} t' : \mathsf{nat}$. Then

$$[\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t']\!]} L_{\mathcal{G}}[\![\mathsf{nat}]\!] \xrightarrow{(h^{\mathsf{nat}}_n)^\dagger} L_{\mathcal{G}}[\![\mathsf{nat}]\!]_n$$

has domain U and its restriction to U equals g. The postcomposition by $(h^\sigma_n)^\dagger$ does nothing because the image of $[\![\Gamma]\!]_n$ under $[\![t']\!]$ is already in $\{0, \ldots, n\}$.

For the converse, start with a term $\Gamma \vdash^{\mathbf{c}} t : \mathsf{nat}$. We need to show that the function

$$g = \left([\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t]\!]} L_{\mathcal{G}}[\![\Gamma]\!] \xrightarrow{(h^{\mathsf{nat}}_n)^\dagger} L_{\mathcal{G}}[\![\mathsf{nat}]\!]_n\right)_\star\Big|_U$$

is in $[\![\mathsf{nat}]\!]_n(\Gamma, U)_{\mathcal{C}_n,F_n}$, where U is the domain of definition of the function in brackets.

Recall the description of coproduct in concrete sheaves from Figure 7.2, where c is any object in the site:

$$\llbracket \tau + \tau' \rrbracket(c) = \big\{ f : |c| \to |\llbracket \tau \rrbracket| + |\llbracket \tau' \rrbracket| \;\big|\; \exists \{g_i : c_i \to c\}_{i \in I} \in J_{\mathcal{G}}(c) \\ \text{s.t. for each } i,\ (f \circ g_i) \in \llbracket \tau \rrbracket(c_i) \text{ or } (f \circ g_i) \in \llbracket \tau' \rrbracket(c_i) \big\}.$$

It is then enough to show that the non-empty fibres of g (excluding that of $\bot$) form a partition in $S^{\Gamma,n}$. For this, notice that the fibres of g are realised by the following computation restricted to $\llbracket \Gamma \rrbracket_n$:

$$\Gamma \vdash^{\mathbf{c}} \mathsf{let}\, z = t \,\mathsf{in}\, \psi_n^{\mathsf{nat}}[z/x] : \mathsf{nat}.$$

Case $\mathsf{0}$**.** We know that $\llbracket \mathsf{0} \rrbracket_n \cong \llbracket \mathsf{0} \rrbracket$. If $U \neq \emptyset$, then $\llbracket \mathsf{0} \rrbracket_n(\Gamma, U)_{\mathcal{C}_n, F_n} = \emptyset$ so there is no function g in $\llbracket \mathsf{0} \rrbracket_n(\Gamma, U)_{\mathcal{C}_n, F_n}$ to consider.

If $U = \emptyset$ then by the sheaf condition $\llbracket \mathsf{0} \rrbracket_n(\Gamma, \emptyset)_{\mathcal{C}_n, F_n}$ has exactly one element, the empty function $g : \emptyset \to |\llbracket \sigma \rrbracket_n| \cong \emptyset$. The following diverging computation corresponds to g:

$$\Gamma \vdash^{\mathbf{c}} (\mathsf{rec}\, f\, x.\, f\, x)\, \underline{0} : \mathsf{0}$$

because at $\star$ the morphism

$$\llbracket \Gamma \rrbracket_n \rightarrowtail \llbracket \Gamma \rrbracket \xrightarrow{\llbracket (\mathsf{rec}\, f\, x.\, f\, x)\, 1 \rrbracket} L_{\mathcal{G}} \llbracket \mathsf{0} \rrbracket \xrightarrow{h_n^{\mathsf{0}}} L_{\mathcal{G}} \llbracket \mathsf{0} \rrbracket_n$$

must be undefined everywhere since $|\llbracket \mathsf{0} \rrbracket_n| = \emptyset$.

For the converse, start with a computation $\Gamma \vdash^{\mathbf{c}} t : \mathsf{0}$. Because $|\llbracket \mathsf{0} \rrbracket_n|$ is empty, the underlying function of the following morphism must be everywhere undefined:

$$\llbracket \Gamma \rrbracket_n \rightarrowtail \llbracket \Gamma \rrbracket \xrightarrow{\llbracket t \rrbracket} L_{\mathcal{G}} \llbracket \mathsf{0} \rrbracket \xrightarrow{(h_n^{\mathsf{0}})^\dagger} L_{\mathcal{G}} \llbracket \mathsf{0} \rrbracket_n$$

so t corresponds to the unique element of $\llbracket \mathsf{0} \rrbracket_n(\Gamma, \emptyset)_{\mathcal{C}_n, F_n}$.

Case $\mathsf{1}$**.** We know $\llbracket \mathsf{1} \rrbracket_n \cong \llbracket \mathsf{1} \rrbracket$. This means that $\llbracket \mathsf{1} \rrbracket_n(\Gamma, U)_{\mathcal{C}_n, F_n}$ has exactly one element, the constant function $g : U \to |\llbracket \mathsf{1} \rrbracket_n| \cong 1$.

For the left to right implication, notice that there must be a term $\Gamma \vdash^{\mathbf{c}} t_U : \mathbf{1}$ whose domain when restricted to $[\![\Gamma]\!]_n$ is U. This term also corresponds to g, because postcomposition by $h_n^{\mathbf{1}}$ does nothing.

For the converse, consider a computation $\Gamma \vdash^{\mathbf{c}} t : \mathbf{1}$. It is clear that it corresponds to the unique element of $[\![\mathbf{1}]\!]_n(\Gamma, U)_{\mathcal{C}_n,F_n}$, where U is the domain of

$$[\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t]\!]} L_{\mathcal{G}}[\![\mathbf{1}]\!] \xrightarrow{(h_n^{\mathbf{1}})^{\dagger}} L_{\mathcal{G}}[\![\mathbf{1}]\!]_n.$$

Case $\sigma \to \tau$, left to right implication. Let $g \in [\![\sigma \to \tau]\!]_n(\Gamma, U)_{\mathcal{C}_n,F_n}$, so g has type

$$g : U \to |[\![\sigma]\!]_n \Rightarrow L_{\mathcal{G}}[\![\tau]\!]_n|$$

where $U \subseteq |[\![\Gamma]\!]_n|$.

Recall the description of function space in concrete sheaves from Figure 7.2, where c is any object in the site:

$$\begin{aligned}[\![\tau \to \tau']\!](c) = \big\{ f : |c| \to \mathsf{Sh}([\![\tau]\!],\, L_{\mathcal{G}}[\![\tau']\!]) \;\big|\; & \\ \forall h : d \to c \in \mathcal{I}_{\mathcal{G}},\ \forall g : |d| \to |[\![\tau]\!]| \in [\![\tau]\!](d). & \\ \lambda x \in |d|.\big(f(h(x))\ g(x)\big) \in L_{\mathcal{G}}[\![\tau']\!](d) \big\} &\end{aligned}$$

We will instantiate the description of $[\![\sigma \to \tau]\!]_n(\Gamma, U)_{\mathcal{C}_n,F_n}$ for

$$\pi_1 : \big((\Gamma, x : \sigma), U \times |[\![\sigma]\!]_n|\big)_{\mathcal{C}_n,F_n} \to (\Gamma, U)_{\mathcal{C}_n,F_n}$$

from $\mathcal{I}_{\mathcal{C}_n,F_n}$, and for

$$\pi_2 \in [\![\sigma]\!]_n\big((\Gamma, x : \sigma), U \times |[\![\sigma]\!]_n|\big)_{\mathcal{C}_n,F_n}, \quad \pi_2 : U \times |[\![\sigma]\!]_n| \to |[\![\sigma]\!]_n|.$$

To show that $\{U \times |[\![\sigma]\!]_n|\} \in S^{(\Gamma,\sigma),n}$, let $\Gamma \vdash^{\mathbf{c}} t_U : \mathbf{1}$ be any computation which when restricted to $[\![\Gamma]\!]_n$ has domain U. Then

$$\Gamma, x : \sigma \vdash^{\mathbf{c}} t_U : \mathbf{1}$$

restricted to $[\![\Gamma]\!]_n \times [\![\sigma]\!]_n$ has domain $U \times |[\![\sigma]\!]_n|$.

To show $\pi_2 \in [\![\sigma]\!]_n\big((\Gamma, x : \sigma), U \times |[\![\sigma]\!]_n|\big)_{\mathcal{C}_n, F_n}$ use the induction hypothesis for σ. Consider the computation:

$$\Gamma, x : \sigma \vdash^{\mathbf{c}} \mathsf{let}\, y = t_U \,\mathsf{in}\,\mathsf{return}\, x : \sigma.$$

When restricted to $[\![\Gamma]\!]_n \times [\![\sigma]\!]_n$ it has domain $U \times |[\![\sigma]\!]_n|$ and its postcomposition with h_n^σ (which does nothing) gives π_2.

So from the fact that $g \in [\![\sigma \to \tau]\!]_n(\Gamma, U)_{\mathcal{C}_n, F_n}$ we can deduce that:

$$\begin{aligned} A = \lambda(\rho_1 \times \rho_2) \in \big(U \times |[\![\sigma]\!]_n|\big).\, g(\pi_1(\rho_1, \rho_2))\ \pi_2(\rho_1, \rho_2) \\ \in L_{\mathcal{G}}[\![\tau]\!]_n\big((\Gamma, x : \sigma), U \times |[\![\sigma]\!]_n|\big)_{\mathcal{C}_n, F_n}. \end{aligned}$$

By the definition of $L_{\mathcal{G}}$, there exists $V \subseteq U \times |[\![\sigma]\!]_n|$ such that $\big((\Gamma, x : \sigma), V\big)_{\mathcal{C}_n, F_n}$ is an object in $\mathcal{I}_{\mathcal{C}_n, F_n}$ and

$$A|_V \in [\![\tau]\!]_n\big((\Gamma, x : \sigma), V\big)_{\mathcal{C}_n, F_n}.$$

We can use the induction hypothesis for τ to get a computation

$$\Gamma, x : \sigma \vdash^{\mathbf{c}} t_A : \tau$$

such that the map

$$[\![\Gamma]\!]_n \times [\![\sigma]\!]_n \rightarrowtail [\![\Gamma]\!] \times [\![\sigma]\!] \xrightarrow{[\![t_A]\!]} L_{\mathcal{G}}[\![\tau]\!] \xrightarrow{(h_n^\tau)^\dagger} L_{\mathcal{G}}[\![\tau]\!]_n$$

has domain V and its restriction to V equals $A|_V$. We can curry this map to get:

$$B : [\![\Gamma]\!]_n \to ([\![\sigma]\!]_n \Rightarrow L_{\mathcal{G}}[\![\tau]\!]_n)$$

whose domain is U and $B|_U = g$.

Consider the computation

$$\Gamma \vdash^{\mathbf{c}} t_g \stackrel{\text{def}}{=} (\mathsf{let}\, y = t_U \,\mathsf{in}\,\mathsf{return}\, \lambda x.\, t_A) : \sigma \to \tau.$$

The map

$$C = \llbracket\Gamma\rrbracket_n \rightarrowtail \llbracket\Gamma\rrbracket \xrightarrow{\llbracket t_g\rrbracket} L_{\mathcal{G}}(\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket) \xrightarrow{(h_n^{\sigma\to\tau})^\dagger} L_{\mathcal{G}}(\llbracket\sigma\rrbracket_n \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket_n)$$

has domain U and we can show its restriction to U equals $B|_U$ and hence g.

Unwinding the definition of $h_n^{\sigma\to\tau}$ we get that:

$$C = \lambda u \in |\llbracket\Gamma\rrbracket_n|. \begin{cases} \eta\big((h_n^\tau)^\dagger \circ \llbracket t_A\rrbracket(u,-)^\dagger \circ h_n^\sigma\big) & \text{if } u \in U \\ \bot & \text{if } u \notin U. \end{cases}$$

Because the input u comes from $|\llbracket\Gamma\rrbracket_n|$, h_n^σ has no effect so we can see C equals B.

Case $\sigma \to \tau$, right to left implication. Consider a computation $\Gamma \vdash^{\mathbf{c}} t : \sigma \to \tau$. Let $U \subseteq |\llbracket\Gamma\rrbracket_n|$ be its domain when restricted to $\llbracket\Gamma\rrbracket_n$ and postcomposed by $(h_n^{\sigma\to\tau})^\dagger$. Let

$$g = \big(\llbracket\Gamma\rrbracket_n \rightarrowtail \llbracket\Gamma\rrbracket \xrightarrow{\llbracket t\rrbracket} L_{\mathcal{G}}(\llbracket\sigma\rrbracket \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket) \xrightarrow{(h_n^{\sigma\to\tau})^\dagger} L_{\mathcal{G}}(\llbracket\sigma\rrbracket_n \Rightarrow L_{\mathcal{G}}\llbracket\tau\rrbracket_n)\big)\big|_U.$$

We need to prove that

$$g \in \llbracket\sigma \to \tau\rrbracket_n(\Gamma, U)_{\mathcal{C}_n,F_n}.$$

Consider $\pi_1 : \big((\Gamma,\Delta),V\big)_{\mathcal{C}_n,F_n} \to (\Gamma,U)_{\mathcal{C}_n,F_n}$ from $\mathcal{I}_{\mathcal{C}_n,F_n}$, so $\pi_1(V) \subseteq U$, and $a \in \llbracket\sigma\rrbracket_n\big((\Gamma,\Delta),V\big)_{\mathcal{C}_n,F_n}$. By induction hypothesis for σ, there must be a computation

$$\Gamma, \Delta \vdash^{\mathbf{c}} t_a : \sigma$$

such that when restricted to $\llbracket\Gamma\rrbracket_n \times \llbracket\Delta\rrbracket_n$ and $L_{\mathcal{G}}\llbracket\sigma\rrbracket_n$ its domain is V and

$$a = \big(\llbracket\Gamma\rrbracket_n \times \llbracket\Delta\rrbracket_n \rightarrowtail \llbracket\Gamma\rrbracket \times \llbracket\Delta\rrbracket \xrightarrow{\llbracket t_a\rrbracket} L_{\mathcal{G}}\llbracket\sigma\rrbracket \xrightarrow{(h_n^\sigma)^\dagger} L_{\mathcal{G}}\llbracket\sigma\rrbracket_n\big)\big|_V.$$

It is enough to prove that

$$\lambda v \in V.\, g(\pi_1(v))\ a(v) \quad \in (L_{\mathcal{G}}\llbracket\tau\rrbracket_n)\big((\Gamma,\Delta),V\big)_{\mathcal{C}_n,F_n}.$$

Using the description of the lifting monad $L_{\mathcal{G}}$ (Figure 7.2), we see that it suffices to find a subset $V' \subseteq V$, defined by some computation $\Gamma, \Delta \vdash^{\mathbf{c}} t_{V'} : 1$ when restricted to $[\![\Gamma]\!]_n \times [\![\sigma]\!]_n$, such that V' is the domain of $\lambda v \in V.\, g(\pi_1(v))\ a(v)$ and

$$\lambda v \in V'.\, g(\pi_1(v))\ a(v) \quad \in [\![\tau]\!]_n\big((\Gamma, \Delta), V'\big)_{\mathcal{C}_n, F_n}.$$

To define $t_{V'}$ use the fact that there is a term $\Gamma, \Delta \vdash^{\mathbf{c}} t_V : 1$ that defines V.

Then consider the following term which is roughly $\psi_n^{\sigma\to\tau}(t)\ \psi_n^{\sigma}(t_a)$:

$$\begin{aligned}\Gamma, \Delta \vdash^{\mathbf{c}} t_{V'} &\stackrel{\text{def}}{=} \mathsf{let}\, y = t_V \,\mathsf{in} \\ &\mathsf{let}\, z_1 = t \,\mathsf{in}\, \mathsf{let}\, z_2 = \psi_n^{\sigma\to\tau}[z_1/x] \,\mathsf{in}\, \mathsf{let}\, z_3 = t_a \,\mathsf{in}\, \mathsf{let}\, z_4 = \psi_n^{\sigma}[z_3/x] \,\mathsf{in} \\ &\qquad\qquad \mathsf{let}\, z_5 = (z_2\, z_4) \,\mathsf{in}\, \mathsf{return} \star : 1.\end{aligned}$$

Using the induction hypothesis for τ, the last step is to find a computation $\Gamma, \Delta \vdash^{\mathbf{c}} t' : \tau$ such that the following map has domain V':

$$[\![\Gamma]\!]_n \times [\![\sigma]\!]_n \rightarrowtail [\![\Gamma]\!] \times [\![\sigma]\!] \xrightarrow{[\![t']\!]} L_{\mathcal{G}}[\![\tau]\!] \xrightarrow{(h_n^{\tau})^{\dagger}} L_{\mathcal{G}}[\![\tau]\!]_n.$$

and equals $\lambda v \in V'.\, g(\pi_1(v))\ a(v)$.

Define t' similarly to $t_{V'}$:

$$\begin{aligned}\Gamma, \Delta \vdash^{\mathbf{c}} t' &\stackrel{\text{def}}{=} \mathsf{let}\, y = t_V \,\mathsf{in} \\ &\mathsf{let}\, z_1 = t \,\mathsf{in}\, \mathsf{let}\, z_2 = \psi_n^{\sigma\to\tau}[z_1/x] \,\mathsf{in}\, \mathsf{let}\, z_3 = t_a \,\mathsf{in}\, \mathsf{let}\, z_4 = \psi_n^{\sigma}[z_3/x] \,\mathsf{in} \\ &\qquad\qquad (z_2\, z_4) : \tau.\end{aligned}$$

Notice that the image of $(z_2\, z_4)$ is already in $L_{\mathcal{G}}[\![\tau]\!]_n$.

Case $\sigma + \tau$. For the left to right implication consider a function $g \in ([\![\sigma]\!]_n + [\![\tau]\!]_n)(\Gamma, U)_{\mathcal{C}_n, F_n}$. This means that there exists a cover P of U such that $P \in S^{\Gamma, n}$ and $\bigcup P = U$, and for each $V \in P$ either $g|_V \in [\![\sigma]\!]_n(\Gamma, V)_{\mathcal{C}_n, F_n}$ or $g|_V \in [\![\tau]\!]_n(\Gamma, V)_{\mathcal{C}_n, F_n}$.

By definition, $P \in S^{\Gamma, n}$ means that there is a computation $\Gamma \vdash^{\mathbf{c}} t_P : \mathsf{nat}$

such that the non-empty fibres of

$$[\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t_P]\!]} L_{\mathcal{G}}[\![\mathsf{nat}]\!]$$

(excluding that of $\perp$) give P.

For each $V \in P$ we can use the induction hypothesis for either σ or τ as appropriate. Assume without loss of generality that $g|_V \in [\![\sigma]\!]_n(\Gamma, V)_{\mathcal{C}_n, F_n}$. Then there is a computation $\Gamma \vdash^{\mathbf{c}} t_{g_V} : \sigma$ such that the domain of the following map is V

$$[\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t_{g_V}]\!]} L_{\mathcal{G}}[\![\sigma]\!] \xrightarrow{h_n^\sigma} L_{\mathcal{G}}[\![\sigma]\!]_n$$

and it equals $g|_V$.

Now we can construct the term that corresponds to g. There is a finite number of $V \in P$; let $n_V \in \mathbb{N}$ be the image of V under t_P. We can form the following computation written informally as:

$$\begin{aligned}\Gamma \vdash^{\mathbf{c}} \mathsf{let}\, y = t_P\, \mathsf{in}\, \{&\mathsf{if}\ y = n_V\ \mathsf{then}\ \mathsf{let}\, z = t_{g_V}\, \mathsf{in} \\ &\mathsf{return}\,(\mathsf{inl}\, z,\ \text{if } t_{g_V} : \sigma,\ \text{or } \mathsf{inr}\, z,\ \text{if } t_{g_V} : \tau)\}_{V \in P} : \sigma + \tau.\end{aligned}$$

It has domain U because t_P does, and behaves as expected when postcomposed by $h_n^{\sigma+\tau}$ because $h_n^{\sigma+\tau}$ is defined componentwise using h_n^σ and h_n^τ.

For the right to left implication, start with a term $\Gamma \vdash^{\mathbf{c}} t : \sigma + \tau$. We can form two terms:

$$\Gamma \vdash^{\mathbf{c}} t_1 \stackrel{\text{def}}{=} \mathsf{let}\, x = t\, \mathsf{in}\, \mathsf{case}\, x\, \mathsf{of}\, \{\mathsf{inl}\, y \to \mathsf{return}\, y,\ \mathsf{inr}\, y \to \Omega_\sigma\} : \sigma$$
$$\Gamma \vdash^{\mathbf{c}} t_2 \stackrel{\text{def}}{=} \mathsf{let}\, x = t\, \mathsf{in}\, \mathsf{case}\, x\, \mathsf{of}\, \{\mathsf{inl}\, y \to \Omega_\tau,\ \mathsf{inr}\, y \to \mathsf{return}\, y\} : \tau,$$

where Ω_σ and Ω_τ are non-terminating computations, for example $\Omega_\sigma \stackrel{\text{def}}{=} (\mathsf{rec}\, f\, x.\, f\, x)\, \underline{0}$.

Consider the following morphisms:

$$g_1 = [\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t_1]\!]} L_{\mathcal{G}}[\![\sigma]\!] \xrightarrow{(h_n^\sigma)^\dagger} L_{\mathcal{G}}[\![\sigma]\!]_n$$
$$g_2 = [\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t_2]\!]} L_{\mathcal{G}}[\![\tau]\!] \xrightarrow{(h_n^\tau)^\dagger} L_{\mathcal{G}}[\![\tau]\!]_n$$
$$g = [\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t]\!]} L_{\mathcal{G}}[\![\sigma+\tau]\!] \xrightarrow{(h_n^{\sigma+\tau})^\dagger} L_{\mathcal{G}}[\![\sigma+\tau]\!]_n.$$

The domain of g is the union of the domains of g_1 and g_2 and g can be seen as the union of g_1 and g_2.

By induction hypothesis for σ and τ, we get that

$$g_1|_{\mathsf{dom}(g_1)} \in [\![\sigma]\!]_n(\Gamma, \mathsf{dom}(g_1))_{\mathcal{C}_n,F_n}$$

and similarly for g_2. We need to show that

$$g|_{\mathsf{dom}(g)} \in [\![\sigma+\tau]\!]_n(\Gamma, \mathsf{dom}(g))_{\mathcal{C}_n,F_n}$$

For this, it's enough to show that $\{\mathsf{dom}(g_1), \mathsf{dom}(g_2)\}$ is a cover for $(\Gamma, \mathsf{dom}(g))$, or equivalently that $\{\mathsf{dom}(g_1), \mathsf{dom}(g_2)\} \in S^{\Gamma,n}$. This follows by considering the computation:

$$\Gamma \vdash^{\mathbf{c}} \mathsf{let}\, z = t \,\mathsf{in}\, \mathsf{let}\, y = \psi_n^{\sigma+\tau}[z/x] \,\mathsf{in}$$
$$\mathsf{case}\, y \,\mathsf{of}\, \{\mathsf{inl}\, w \to \mathsf{return}\, 0,\ \mathsf{inr}\, w \to \mathsf{return}\, 1\} : \mathsf{nat}.$$

Case $\sigma \times \tau$. Let $g \in ([\![\sigma]\!]_n \times [\![\tau]\!]_n)(\Gamma, U)_{\mathcal{C}_n,F_n}$. This means $g = \langle g_1, g_2\rangle$ with $g_1 \in [\![\sigma]\!]_n(\Gamma, U)_{\mathcal{C}_n,F_n}$ and $g_2 \in [\![\tau]\!]_n(\Gamma, U)_{\mathcal{C}_n,F_n}$.

By induction hypothesis, there must be two computations $\Gamma \vdash^{\mathbf{c}} t_1 : \sigma$ and $\Gamma \vdash^{\mathbf{c}} t_2 : \tau$ such that the domain of

$$[\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t_1]\!]} L_{\mathcal{G}}[\![\sigma]\!] \xrightarrow{(h_n^\sigma)^\dagger} L_{\mathcal{G}}[\![\tau]\!]_n$$

is U, and its restriction to U equals g_1, and similarly for t_2 and g_2.

The following computation then corresponds to g:

$$\Gamma \vdash^{\mathbf{c}} \mathsf{let}\, x = t_1 \,\mathsf{in}\, \mathsf{let}\, y = t_2 \,\mathsf{in}\, \mathsf{return}\, (x, y) : \sigma \times \tau$$

because $h_n^{\sigma\times\tau}$ is defined using both h_n^{σ} and h_n^{τ}.

For the converse, start with a term $\Gamma \vdash^{\mathbf{c}} t : \sigma \times \tau$. Let

$$g = [\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t]\!]} L_{\mathcal{G}}([\![\sigma]\!] \times [\![\tau]\!]) \xrightarrow{(h_n^{\sigma\times\tau})^{\dagger}} L_{\mathcal{G}}([\![\tau]\!]_n \times [\![\sigma]\!]_n)$$

and let its domain be U. We need to show that $g|_U \in ([\![\sigma]\!]_n \times [\![\tau]\!]_n)(\Gamma, U)_{\mathcal{C}_n, F_n}$.

Form the term $\Gamma \vdash^{\mathbf{c}} \mathsf{let}\, x = t \,\mathsf{in}\, \pi_1(x) : \sigma$ which by induction hypothesis gives a function $g_1 \in [\![\sigma]\!]_n(\Gamma, U)_{\mathcal{C}_n, F_n}$. We can do the same for π_2 to get $g_2 \in [\![\tau]\!]_n(\Gamma, U)_{\mathcal{C}_n, F_n}$. Then it is enough to notice that $g|_U = \langle g_1, g_2 \rangle$. □

Corollary 8.2.10. *Consider the interpretation of* $\mathsf{PCF_v}$ *in the normal model* $\mathcal{G}$ *(Definition 8.1.12). For any type* σ *and for any* $n \in \mathbb{N}$*, every point of* $[\![\sigma]\!]_n$ *is definable. More precisely, for each point* $p : 1 \to [\![\sigma]\!]_n$ *there is a closed computation*

$$- \vdash^{\mathbf{c}} t : \sigma$$

with denotation $[\![t]\!] : 1 \to L_{\mathcal{G}}[\![\sigma]\!]$*, such that* $[\![t]\!]_{\star}(*) = p_{\star}(*)$.

Proof. Use Proposition 8.2.9 in the case where $(\Gamma, U)_{\mathcal{C}_n, F_n}$ is the empty context $(-)$, modelled by 1, and $U \cong 1$ is a one element set. Then $[\![\sigma]\!]_n\big((-), 1\big)_{\mathcal{C}_n, F_n}$ contains all the constant functions $p : 1 \to |[\![\sigma]\!]_n|$, and for each p there is a computation $\vdash^{\mathbf{c}} t_p : \sigma$, such that the following map picks out p:

$$1 \rightarrowtail 1 \xrightarrow{[\![t_p]\!]} L_{\mathcal{G}}[\![\sigma]\!] \xrightarrow{(h_n^{\sigma})^{\dagger}} L_{\mathcal{G}}[\![\sigma]\!]_n.$$

The computation t that we need can be constructed as:

$$- \vdash^{\mathbf{c}} \mathsf{let}\, z = t_p \,\mathsf{in}\, \psi_n^{\sigma}[z/x] : \sigma.$$

□

8.2.3 Full abstraction

Using the definability results from the previous section (Corollary 8.2.10) we can show that contextual equivalence implies denotational equality (Theorem 8.2.11). The converse follows from soundness and adequacy, thus completing the proof that the normal model $\mathcal{G}$ (Definition 8.1.12) is a fully abstract model of $\mathsf{PCF_v}$ without any type or term constants.

Theorem 8.2.11 (Full abstraction)**.** *Consider the interpretation of* $\mathsf{PCF_v}$ *without type or term constants in the normal model* $\mathcal{G}$*. This interpretation is fully abstract with respect to the operational semantics from Section 4.2 in the following sense:*

- *If two values* v_1 *and* v_2 *are contextually equivalent,* $\Gamma \vdash^{\mathbf{v}} v_1 \simeq v_2 : \sigma$*, then their denotations are equal:*

$$[\![v_1]\!] = [\![v_2]\!] : [\![\Gamma]\!] \to [\![\sigma]\!].$$

- *If two computations* t_1 *and* t_2 *are contextually equivalent,* $\Gamma \vdash^{\mathbf{c}} t_1 \simeq t_2 : \sigma$*, then their denotations are equal:*

$$[\![t_1]\!] = [\![t_2]\!] : [\![\Gamma]\!] \to L_{\mathcal{G}}[\![\sigma]\!].$$

Proof. The proof is by induction on σ, for values and computations simultaneously. The proof crucially relies on the fact that types are interpreted as *concrete* sheaves, so morphisms between them are determined by their underlying function at $\star$, i.e. by their action on points.

Case nat for values. Let $\Gamma \vdash^{\mathbf{v}} v_1 \simeq v_2 : \mathsf{nat}$ be two contextually equivalent values. Then either v_1 and v_2 are both the same variable from Γ, or they are the same natural number.

Case nat for computations. Consider two contextually equivalent computations $\Gamma \vdash^{\mathbf{c}} t_1 \simeq t_2 : \mathsf{nat}$. We will show first that for any $n \in \mathbb{N}$:

$$[\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t_1]\!]} L_{\mathcal{G}}[\![\mathsf{nat}]\!] = [\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t_2]\!]} L_{\mathcal{G}}[\![\mathsf{nat}]\!].$$

From Corollary 8.2.10, we know that for each point $p : 1 \to [\![\Gamma]\!]_n$ there is a computation $- \vdash^{\mathbf{c}} t_p : \Gamma$, so $[\![t_p]\!] : 1 \to L_{\mathcal{G}}[\![\Gamma]\!]$, such that

$$p_\star(*) = [\![t_p]\!]_\star(*).$$

Assume that $\Gamma = x_1 : \tau_1, \ldots x_k : \tau_k$. For every point p we can consider the following context of ground type:

$$C_p = \mathsf{let}\,(x_1, \ldots, x_k) = t_p\,\mathsf{in}\,\square.$$

Then by definition of contextual equivalence we know that

$$C_p[t_1] \Downarrow l \text{ if and only if } C_p[t_2] \Downarrow l.$$

If both computations diverge, then both $1 \xrightarrow{p} [\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t_1]\!]} L_{\mathcal{G}}[\![\mathsf{nat}]\!]$ and $1 \xrightarrow{p} [\![\Gamma]\!]_n \rightarrowtail [\![\Gamma]\!] \xrightarrow{[\![t_2]\!]} L_{\mathcal{G}}[\![\mathsf{nat}]\!]$ pick out $\perp$. This follows from adequacy as explained in Remark 7.1.4, because if $C_p[t_1]$ diverges, there can be no closed value v of type nat such that $[\![C_p[t_1]]\!] = \eta_{[\![\mathsf{nat}]\!]} \circ [\![v]\!]$. Since $|L_{\mathcal{G}}[\![\mathsf{nat}]\!]| = \mathbb{N} + \{\perp\}$, $[\![C_p[t_1]]\!]$ must be $\perp$.

The other case is when both computations return $l \in \mathbb{N}$. By soundness both denotations pick out $l \in |L_{\mathcal{G}}[\![\mathsf{nat}]\!]|$. So this suffices to show that $[\![t_1]\!]$ and $[\![t_2]\!]$ agree on $[\![\Gamma]\!]_n$.

Because the image of h_n^Γ is $L_{\mathcal{G}}[\![\Gamma]\!]_n$, we can deduce that for any $n \in \mathbb{N}$:

$$[\![t_1]\!]^\dagger \circ h_n^\Gamma = [\![t_2]\!]^\dagger \circ h_n^\Gamma : [\![\Gamma]\!] \to L_{\mathcal{G}}[\![\mathsf{nat}]\!]$$

and further, by concreteness, that

$$[\![t_1]\!]^\dagger \circ h^\Gamma = [\![t_2]\!]^\dagger \circ h^\Gamma : \omega \times [\![\Gamma]\!] \to L_{\mathcal{G}}[\![\mathsf{nat}]\!].$$

Because $L_{\mathcal{G}}[\![\mathsf{nat}]\!]$ is complete, the two maps above must have a unique extension, which must be

$$[\![t_1]\!]^\dagger \circ H^\Gamma = [\![t_2]\!]^\dagger \circ H^\Gamma : \overline{\omega} \times [\![\Gamma]\!] \to L_{\mathcal{G}}[\![\mathsf{nat}]\!],$$

since H^Γ is the extension of h^Γ.

Finally, we can evaluate both maps above at $\infty \in \overline{\omega}(\star)$, and by Proposition 8.2.3 obtain that, for any $x \in |[\![\Gamma]\!]|$:

$$[\![t_1]\!]_\star(x) = [\![t_2]\!]_\star(x).$$

So by concreteness $[\![t_1]\!] = [\![t_2]\!]$.

Case 1. In the value case, both values are either the same variable or $\star$, so they must have equal denotations. The computation case is exactly like the nat case for computations, because the only property of nat that we used is that it is a ground type.

Case 0. For values, they must both be the same variable from the context. For computations, suppose t_1 and t_2 are contextually equivalent, where $[\![t_1]\!], [\![t_2]\!] : [\![\Gamma]\!] \to L_{\mathcal{G}}[\![\mathsf{0}]\!]$. By concreteness we only need to compare the two denotations at $\star$, but $|L_{\mathcal{G}}[\![\mathsf{0}]\!]| \cong 1$, so $[\![t_1]\!]$ and $[\![t_2]\!]$ must have the same underlying constant function.

Case $\sigma \to \tau$ for values. Let $\Gamma \vdash^{\mathbf{v}} v_1 \simeq v_2 : \sigma \to \tau$. By congruence of contextual equivalence:

$$\Gamma, x : \sigma \vdash^{\mathbf{c}} (v_1\ x) \simeq (v_2\ x) : \tau.$$

Now apply the induction hypothesis for τ to deduce that the following two maps are equal:

$$\mathsf{uncurry}[\![v_1]\!] = \mathsf{uncurry}[\![v_2]\!] : [\![\Gamma]\!] \times [\![\sigma]\!] \to L_{\mathcal{G}}[\![\tau]\!],$$

so by currying we get that $[\![v_1]\!] = [\![v_2]\!]$.

Case $\sigma \to \tau$ for computations. Let $\Gamma \vdash^{\mathbf{c}} t_1 \simeq t_2 : \sigma \to \tau$. By congruence of contextual equivalence we can form the following two pairs of contextually equivalent terms:

$$\Gamma \vdash^{\mathbf{c}} t_1' \stackrel{\text{def}}{=} \mathsf{let}\, x = t_1 \,\mathsf{in\, return} \star : \mathsf{1} \qquad \Gamma \vdash^{\mathbf{c}} t_2' \stackrel{\text{def}}{=} \mathsf{let}\, x = t_2 \,\mathsf{in\, return} \star : \mathsf{1}$$

and

$$\Gamma \vdash^{\mathbf{v}} v_1 \stackrel{\text{def}}{=} \lambda x.\, \mathsf{let}\, f = t_1 \,\mathsf{in}\, f\, x : \sigma \to \tau$$
$$\Gamma \vdash^{\mathbf{v}} v_2 \stackrel{\text{def}}{=} \lambda x.\, \mathsf{let}\, f = t_2 \,\mathsf{in}\, f\, x : \sigma \to \tau.$$

Because we have already proved the $\mathbf{1}$ case and the $\sigma \to \tau$ case for values, we can deduce that the following denotations are equal:

$$[\![t_1']\!] = [\![t_2']\!] : [\![\Gamma]\!] \to L_{\mathcal{G}}[\![\mathsf{1}]\!]$$

and

$$[\![v_1]\!] = [\![v_2]\!] : [\![\Gamma]\!] \to ([\![\sigma]\!] \Rightarrow L_{\mathcal{G}}[\![\tau]\!]).$$

Consider the terms

$$\Gamma \vdash^{\mathbf{c}} t_1'' \stackrel{\text{def}}{=} \mathsf{let}\, y = t_1' \,\mathsf{in\, return}\, v_1 : \sigma \to \tau$$
$$\Gamma \vdash^{\mathbf{c}} t_2'' \stackrel{\text{def}}{=} \mathsf{let}\, y = t_2' \,\mathsf{in\, return}\, v_2 : \sigma \to \tau.$$

Because the denotational interpretation is compositional we can see that:

$$[\![t_1'']\!] = [\![t_2'']\!]$$

but also that $[\![t_1'']\!] = [\![t_1]\!]$ and $[\![t_2'']\!] = [\![t_2]\!]$. So $[\![t_1]\!] = [\![t_2]\!]$ and we are done.

Case $\sigma \times \tau$ for values. Let $\Gamma \vdash^{\mathbf{v}} v_1 \simeq v_2 : \sigma \times \tau$. Then the following two pairs of terms are contextually equivalent:

$$\Gamma \vdash^{\mathbf{c}} \pi_1(v_1) \simeq \pi_1(v_2) : \sigma \qquad \Gamma \vdash^{\mathbf{c}} \pi_2(v_1) \simeq \pi_2(v_2) : \tau.$$

By induction hypothesis, we get that

$$[\![\Gamma]\!] \xrightarrow{[\![v_1]\!]} [\![\sigma]\!] \times [\![\tau]\!] \xrightarrow{\pi_1} [\![\sigma]\!] \xrightarrow{\eta} L_{\mathcal{G}}[\![\sigma]\!] = [\![\Gamma]\!] \xrightarrow{[\![v_2]\!]} [\![\sigma]\!] \times [\![\tau]\!] \xrightarrow{\pi_1} [\![\sigma]\!] \xrightarrow{\eta} L_{\mathcal{G}}[\![\sigma]\!]$$

and similarly for π_2. Because π_1 and π_2 are jointly monic, and η is monic, we get that $[\![v_1]\!] = [\![v_2]\!]$.

Case $\sigma \times \tau$ for computations. Let $\Gamma \vdash^{\mathbf{c}} t_1 \simeq t_2 : \sigma \times \tau$. The proof is similar to the value case, but it uses terms of the form:

$$\Gamma \vdash^{\mathbf{c}} \mathsf{let}\, x = t_1 \,\mathsf{in}\, \pi_1(x) : \sigma$$

and the induction hypothesis for σ and τ.

Case $\sigma + \tau$ for values. Let $\Gamma \vdash^{\mathbf{v}} v_1 \simeq v_2 : \sigma + \tau$, and consider the following two pairs of contextual equivalent terms:

$$\Gamma \vdash^{\mathbf{c}} t_1 \stackrel{\text{def}}{=} \mathsf{case}\, v_1 \,\mathsf{of}\, \{\mathsf{inl}\, x \to \mathsf{return}\, x,\ \mathsf{inr}\, x \to \Omega_\sigma\} : \sigma$$
$$\Gamma \vdash^{\mathbf{c}} t_2 \stackrel{\text{def}}{=} \mathsf{case}\, v_2 \,\mathsf{of}\, \{\mathsf{inl}\, x \to \mathsf{return}\, x,\ \mathsf{inr}\, x \to \Omega_\sigma\} : \sigma$$

and

$$\Gamma \vdash^{\mathbf{c}} t_1' \stackrel{\text{def}}{=} \mathsf{case}\, v_1 \,\mathsf{of}\, \{\mathsf{inl}\, x \to \Omega_\tau,\ \mathsf{inr}\, x \to \mathsf{return}\, x\} : \tau$$
$$\Gamma \vdash^{\mathbf{c}} t_2' \stackrel{\text{def}}{=} \mathsf{case}\, v_2 \,\mathsf{of}\, \{\mathsf{inl}\, x \to \Omega_\tau,\ \mathsf{inr}\, x \to \mathsf{return}\, x\} : \tau.$$

By induction hypothesis we get that $[\![t_1]\!] = [\![t_2]\!]$ and $[\![t_1']\!] = [\![t_2']\!]$. So $[\![v_1]\!] = [\![v_2]\!]$ because $[\![v_1]\!]$ and $[\![v_2]\!]$ induce the same coproduct decomposition of $[\![\Gamma]\!]$ and agree on each component of that decomposition.

Case $\sigma + \tau$ for computations. Let $\Gamma \vdash^{\mathbf{c}} t_1 \simeq t_2 : \sigma + \tau$. The proof is similar to the value case, using the induction hypothesis for σ and τ. But instead it uses computations of the form:

$$\Gamma \vdash^{\mathbf{c}} \mathsf{let}\, z = t_1 \,\mathsf{in}\, \mathsf{case}\, z \,\mathsf{of}\, \{\mathsf{inl}\, x \to \mathsf{return}\, x,\ \mathsf{inr}\, x \to \Omega_\sigma\} : \sigma. \qquad \square$$

8.3 Related work

In this chapter, we defined a normal model $\mathcal{G}$ for $\mathsf{PCF_v}$ (without type and term constants), via the recipe from Theorem 7.1.1, using concrete sites and admissible monos. We then showed that $\mathcal{G}$ is a fully abstract model.

The first fully abstract model of PCF was a syntactic model given by Milner [Mil77]. Milner also showed that under mild assumptions extensional fully abstract models of PCF are unique up to isomorphism (see e.g. [Str06, Theorem 10.12]). Therefore, subsequent research has focused on finding better presentations of this fully abstract model that do not rely on the syntax of the language.

Game semantics models [AJM00, HO00] are one approach to solving the full abstraction problem. A different approach, by which our work is inspired, is to try to cut out non-definable maps from the cpo model (which is not fully abstract [Plo77]) by asking that they preserve certain logical relations. This approach has led to a lot of work attempting to capture definable or "sequential" functions using logical relations e.g. [Plo73, Sie92, JT93, KKS22]. Fully abstract models of PCF and FPC constructed using this method have been presented by O'Hearn, Riecke and Sandholm [OR95, RS02] and Streicher and Marz [Mar00a, Mar00b, Str06]. At the beginning of the chapter we discussed the relation between these models and our work in general terms, we now focus on some technical differences.

O'Hearn and Riecke [OR95] and Streicher [Str06] study call-by-name PCF, so their model is different from ours, although we did adopt the use of SSP from [Str06].

The model closest to ours is probably that by Riecke and Sandholm [RS02] who deal with FPC. Here are some of the similarities: our sites $\mathcal{I}_{\mathcal{C},F}$ correspond to the varying arities in [RS02]; the index category $\mathcal{C}$ corresponds to their $\mathcal{C}$ from [RS02, Section 3.4]; the SSP structure $S^{F(c)}$ corresponds to their path theory S^w; and our sheaf condition corresponds to the structure of a computational relation. Overall the Kleisli category of $L_{\mathcal{G}}$ plays the same role as the partial cartesian-closed category $\mathcal{RCPO}$ from [RS02].

There are however some technical differences. The objects in our model $\mathcal{G}$

have the structure of presheaves on $\mathbb{V}_0$, and since ωCPO embeds in vSet (Proposition 4.4.10), some of them, but not all, have the strucure of an ωcpo. The objects in $\mathcal{RCPO}$ however are dcpo's, with relations. We argue that our treatment of recursion in $\mathcal{G}$ is more modular: it requires combining the sites for full abstraction, $(\mathcal{I}_{\mathcal{C},F}, \mathcal{J}_{\mathcal{C},F})$, with an additional site, $(\mathbb{V}_0, J_{\mathbb{V}_0})$, rather than considering cpo-valued relations.

Another difference is that in the constructions of $\mathcal{RCPO}$, morphisms $f : v \to w$ from $\mathcal{C}$ are not required to pull back a partition from S^w to a partition from S^v. Roughly, this means that objects in $\mathcal{RCPO}$ are equipped with more relations than in $\mathcal{G}$. Thus, it is not straightforward to relate $\mathcal{RCPO}$ to the Kleisli category of $L_{\mathcal{G}}$, for example via an embedding like in Proposition 4.4.10. We conjecture that, at most, we could find a common subcategory of the two.

More recent work on full abstraction using logical relations is that by Kammar, Katsumata and Saville [KKS22]. We thank them for suggesting this topic of research to us. They present a method of constructing fully abstract models for languages with computational effects but without recursion. We do not know yet what the formal connection to our model is.

On a technical note, the model in this chapter is slightly different from the one in our published work [MMS21]. For defining the sites $\mathcal{I}_{\mathcal{C},F}$ of $\mathcal{G}$, [MMS21] considers functors $F : \mathcal{C} \to \mathsf{SSP}_\perp$, where $\mathsf{SSP}_\perp$ is the Kleisli category for a suitable lifting monad on SSP, rather than functors $F : \mathcal{C} \to \mathsf{SSP}$ as we did in Section 8.1.2. This means that more sites are used in the definition of $\mathcal{G}$ in [MMS21] compared to Section 8.1.2. The proof of definabilty in [MMS21] proceeds differently, taking advantage of these extra sites. In [MMS21], each $\mathcal{C}_n$ is defined to contain as objects all types, and morphisms are all the definable partial maps $[\![\sigma]\!]_n \to L_{\mathcal{G}}[\![\tau]\!]_n$. In Section 8.2.2 however, $\mathcal{C}_n$ was defined to only contain contexts and projections (Definition 8.2.4). Overall, the model $\mathcal{G}$ in this chapter is a slight simplification over [MMS21] because it requires fewer sites.

Chapter 9

Conclusion

In this thesis we built models of higher-order recursive programs that have the form of a category of concrete sheaves with a monad. The main contribution is a general method (Theorem 7.1.1) for obtaining such adequate models starting from a *concrete site* and a class of *admissible monomorphisms* (Definition 6.2.1) in the site. We showed how this method can be applied as a basis for modelling languages that combine higher-order recursion with other features, such as probability and differentiability.

More precisely, the examples of our method discussed in the thesis are:

- the usual $\omega\mathsf{CPO}$ model of call-by-value PCF (e.g. [Win93, Section 11.3]);
- the ω-quasi-Borel spaces model of probabilistic programming [VKS19];
- the ω-diffeological spaces model of differentiable programming [Vák20];
- the $\omega\mathbf{PAP}$ model [LHM21], a variation of ω-diffeological spaces allowing some non-smoothness;
- a fully abstract model of call-by-value PCF [MMS21].

Our method makes precise the idea that in all these models, which so far have been studied separately, the structure needed to model higher-order recursion can be obtained in the same way. This is non-trivial because in each case higher-order recursion interacts differently with the additional features of the language, or, in the last case, because we wish to capture a notion of "sequential" program.

The fully abstract sheaf model we presented in Chapter 8, although inspired by the work on full abstraction of O'Hearn and Riecke [OR95, RS02], is new. This model is an important contribution in itself because it explains, in terms of sheaves and admissible monos, some of the ideas about logical relations used by O'Hearn and Riecke. A similar goal was stated for example by Fiore and Simpson [FS99] in their work about using sheaf conditions to model sum types.

In designing our general method for building models, we used ideas from axiomatic and synthetic domain theory, such as the notion of dominance (e.g. [Ros86]). The synthetic domain theory literature studies toposes with a dominance, which are more general than the models obtained via our method. Nevertheless, our method is a contribution towards *obtaining* such toposes starting from more elementary data, a concrete site with admissible monos.

Future work

A next step is to investigate whether the sheaf models obtained using our method can interpret recursive types. There is evidence to suggest this might be possible: several of the examples of our method, such as the category of ω-quasi-Borel spaces and the category of ω-diffeological spaces, can already deal with recursive types; the fully abstract model of FPC of Riecke and Sandholm [RS02] includes recursive types. We conjecture that we can obtain an interpretation of recursive types in our sheaf categories by adapting the standard embedding-projection pairs argument from domain theory (see e.g. [AJ94]).

Recall that our recipe for constructing models interprets all types as concrete sheaves. This means that the models are well-pointed: all maps between types are determined by their action on the points of their domain. Possible future work is to relax the concreteness assumptions from the definition of admissible monos in order to allow models that are not well-pointed to be generated with our recipe. Examples of such models could include nominal sets [Pit13] and models of local state [PP02, KLMS17].

Another direction for future work would be to extend the fully abstract model from Chapter 8 with other computational effects beyond divergence, such as printing, global state or probabilistic choice. We expect this would require changing the sites $\mathcal{I}_{\mathcal{C},F}$ to account for the new notion of observation that a particular effect brings. In this context, it would be interesting to understand the connection with the work of Kammar, Katsumata and Saville [KKS22] on full abstraction for effects.

We think, however, that full abstraction for effects might be difficult to prove. Firstly, because the proof of full abstraction (Theorem 8.2.11) relies on the types being interpreted as *concrete* sheaves. Recall also that in the proof of Theorem 8.2.11, in the case of computations of type $\sigma \to \tau$, we proved $[\![t_1'']\!] = [\![t_1]\!]$ using the fact that it does not matter whether an application of t_1'' runs t_1 twice. This is a special property of non-termination, that might not hold for other effects. Even though we might not be able to achieve full abstraction for other effects by adapting our fully abstract model for PCF, exploring the space of models is still interesting because we might find models that validate more contextual equivalences than existing models.

Bibliography

[AJ94] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D.M. Gabbay, and T.S.E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, page 1–168. Clarendon Press, 1994.

[AJM00] S. Abramsky, R. Jagadeesan, and P. Malacaria. Full abstraction for PCF. *Inf. Comput.*, 163(2):409–470, 2000.

[AR94] J. Adamek and J. Rosicky. *Locally Presentable and Accessible Categories*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1994.

[Bat06] I. Battenfeld. Computational effects in topological domain theory. *Electronic Notes in Theoretical Computer Science*, 158:59–80, 2006.

[BH11] J. Baez and A. Hoffnung. Convenient categories of smooth spaces. *Trans. AMS*, 363(11), 2011.

[Bor94] F. Borceux. *Handbook of Categorical Algebra*, volume 1 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1994.

[BSS07] I. Battenfeld, M. Schröder, and A. Simpson. A convenient category of domains. *Electronic Notes in Theoretical Computer Science*, 172:69–99, 2007.

[CL02] J.R.B. Cockett and S. Lack. Restriction categories I: categories of partial maps. *Theoret. Comput. Sci.*, 270(1):223–259, 2002.

[CL03] J.R.B. Cockett and S. Lack. Restriction categories II: partial map classification. *Theoret. Comput. Sci.*, 294(1):61–102, 2003.

[Cur93] P.L. Curien. *Categorical Combinators, Sequential Algorithms, and Functional Programming*. Progress in Theoretical Computer Science. Birkhäuser, 1993.

[Dub79] E.J. Dubuc. Concrete quasitopoi. In M. Fourman, C. Mulvey, and D. Scott, editors, *Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis, Durham, July 9–21, 1977*. 1979.

[Ehr07] T. Ehrhard. On finiteness spaces and extensional presheaves over the Lawvere theory of polynomials. *J. Pure Appl. Algebra*, 2007.

[EX16] M. Escardó and C. Xu. A constructive manifestation of the Kleene-Kreisel continuous functionals. *Ann. Pure Appl. Log.*, 167(9):770–793, 2016.

[Fio94] M.P. Fiore. *Axiomatic domain theory in categories of partial maps*. PhD thesis, University of Edinburgh, UK, 1994.

[Fio96] M.P. Fiore. Enrichment and representation theorems for categories of domains and continuous functions. Available at the author's website, 1996.

[Fio97] M.P. Fiore. An enrichment theorem for an axiomatisation of categories of domains and continuous functions. *Math. Struct. Comput. Sci.*, 7(5):591–618, 1997.

[FP94] M.P. Fiore and G.D. Plotkin. An axiomatization of computationally adequate domain theoretic models of FPC. In *LICS*, 1994.

[FP96] M.P. Fiore and G.D. Plotkin. An extension of models of axiomatic domain theory to models of synthetic domain theory. In *CSL*, 1996.

[FPP97] M.P. Fiore, G.D. Plotkin, and A.J. Power. Complete cuboidal sets in axiomatic domain theory. In *LICS*, 1997.

[FR97] M.P. Fiore and G. Rosolini. Two models of synthetic domain theory. *J. Pure Appl. Algebra*, 116:151–162, 1997.

[FR01] M.P. Fiore and G. Rosolini. Domains in H. *Theoret. Comput. Sci.*, 264:171–193, 2001.

[FS99] M.P. Fiore and A.K. Simpson. Lambda definability with sums via Grothendieck logical relations. In *TLCA*, 1999.

[HKSY17] C. Heunen, O. Kammar, S. Staton, and H. Yang. A convenient category for higher-order probability theory. In *LICS*, 2017.

[HO00] J.M.E. Hyland and C.-H.L. Ong. On full abstraction for PCF: I, II, and III. *Inf. Comput.*, 163(2):285–408, 2000.

[HSV20] M. Huot, S. Staton, and M. Vákár. Correctness of automatic differentiation via diffeologies and categorical gluing. In *FOSSACS*, 2020.

[Joh02] P.T. Johnstone. *Sketches of an elephant: a Topos theory compendium.* Oxford logic guides. Oxford Univ. Press, 2002.

[JT93] A. Jung and J. Tiuryn. A new characterization of lambda definability. In *TLCA*, 1993.

[Kal02] O. Kallenberg. *Foundations of Modern Probability.* Probability and Its Applications. Springer New York, 2002.

[KKS22] O. Kammar, S. Katsumata, and P. Saville. Fully abstract models for effectful λ-calculi via category-theoretic logical relations. *Proc. ACM Program. Lang.*, 6(POPL):1–28, 2022.

[KLMS17] O. Kammar, P.B. Levy, S.K. Moss, and S. Staton. A monad for full ground reference cells. In *LICS*, 2017.

[LH21] A.K. Lew and M. Huot. Private communication. December 2021.

[LHM21] A.K. Lew, M. Huot, and V.K. Mansinghka. Towards denotational semantics of AD for higher-order, recursive, probabilistic languages. 2021. Presented at NeurIPS 2021 workshop on differentiable programming and POPL 2022 workshop on languages for inference.

[LPT03] P.B. Levy, J. Power, and H. Thielecke. Modelling environments in call-by-value programming languages. *Inform. Comput.*, 185(2):182–210, 2003.

[LS97] J.R. Longley and A.K. Simpson. A uniform approach to domain theory in realizability models. *Math. Struct. Comput. Sci.*, 7(5), 1997.

[LYRY20] W. Lee, H. Yu, X. Rival, and H. Yang. On correctness of automatic differentiation for non-differentiable functions. In *NeurIPS*, 2020.

[Mar00a] M. Marz. *A Fully Abstract Model for Sequential Computation*. PhD thesis, Technische Universität Darmstadt, 2000.

[Mar00b] M. Marz. A fully abstract model for sequential computation. *Electronic Notes in Theoretical Computer Science*, 35:133–152, 2000. Workshop on Domains IV.

[Mil77] R. Milner. Fully abstract models of typed λ-calculi. *Theoret. Comput. Sci.*, 4(1):1–22, 1977.

[MM92] S. MacLane and I. Moerdijk. *Sheaves in geometry and logic: a first introduction to topos theory*. Universitext. Springer, Berlin, 1992.

[MMS21] C. Matache, S. Moss, and S. Staton. Recursion and Sequentiality in Categories of Sheaves. In *FSCD*, 2021.

[MMS22] C. Matache, S.K. Moss, and S. Staton. Concrete categories and higher-order recursion: with applications including probability, differentiability, and full abstraction. In *LICS*, 2022.

[Mog91] E. Moggi. Notions of computation and monads. *Inf. Comput.*, 93:55–92, 1991.

[Mul92] P.S. Mulry. Monads and algebras in the semantics of partial data types. *Theoretical Computer Science*, 99(1):141–155, 1992.

[Mul94] P.S. Mulry. Partial map classifiers and partial cartesian closed categories. *Theoretical Computer Science*, 136(1):109–123, 1994.

[OR95] P.W. O'Hearn and J.G. Riecke. Kripke logical relations and PCF. *Inf. Comput.*, 120(1):107–116, 1995.

[Pit11] A. Pitts. Howe's method for higher-order languages. In D. Sangiorgi and J. Rutten, editors, *Advanced Topics in Bisimulation and Coinduction*, Cambridge Tracts in Theoretical Computer Science, page 197–232. Cambridge University Press, 2011.

[Pit13] A.M. Pitts. *Nominal Sets: Names and Symmetry in Computer Science*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2013.

[Plo73] G. Plotkin. Lambda definability and logical relations. Technical Report Memo SAI-RM-4, School of Artificial Intelligence, Edinburgh, 1973.

[Plo77] G.D. Plotkin. LCF considered as a programming language. *Theor. Comput. Sci.*, 5(3):223–255, 1977.

[PP02] G.D. Plotkin and J. Power. Notions of computation determine monads. In *FOSSACS*, 2002.

[Ros86] G. Rosolini. *Continuity and effectiveness in topoi*. PhD thesis, University of Oxford, 1986.

[RR88] E. Robinson and G. Rosolini. Categories of partial maps. *Inf. Comput.*, 79(2):95–130, 1988.

[RS99] B. Reus and T. Streicher. General synthetic domain theory – a logical approach. *Math. Struct. Comput. Sci.*, 9(2):177–223, 1999.

[RS02] J.G. Riecke and A. Sandholm. A relational account of call-by-value sequentiality. *Inf. Comput.*, 179(2):296–331, 2002.

[Sie92] K. Sieber. Reasoning about sequential functions via logical relations. In M.P. Fourman, P.T. Johnstone, and A.M. Pitts, editors, *Applications of Categories in Computer Science: Proceedings of the London Mathematical Society Symposium, Durham 1991*, London Mathematical Society Lecture Note Series, page 258–269. Cambridge University Press, 1992.

[Sim98] A.K. Simpson. Computational adequacy in an elementary topos. In *CSL*, 1998.

[Sim04] A.K. Simpson. Computational adequacy for recursive types in models of intuitionistic set theory. *Ann. Pure Appl. Log.*, 130(1-3):207–275, 2004.

[Str06] T. Streicher. *Domain-theoretic foundations of functional programming*. World Scientific, 2006.

[Vák20] M. Vákár. Denotational correctness of foward-mode automatic differentiation for iteration and recursion. *CoRR*, abs/2007.05282, 2020.

[VKS19] M. Vákár, O. Kammar, and S. Staton. A domain theory for statistical probabilistic programming. *Proc. ACM Program. Lang.*, 3(POPL):36:1–36:29, 2019.

[vOS00] J. van Oosten and A.K. Simpson. Axioms and (counter) examples in synthetic domain theory. *Ann. Pure Appl. Log.*, 104(1-3):233–278, 2000.

[Win93] G. Winskel. *The Formal Semantics of Programming Languages: An Introduction*. MIT Press, 1993.