ISSN 1330–0008
CODEN FIZAE4

ON THE ENVELOPE SOLITONS AND MODULATIONAL INSTABILITY IN AN ELECTRON – POSITRON PLASMA

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Received 23 June 1998; revised manuscript received 15 March 1999
Accepted 4 October 1999

We have studied the formation of envelope soliton in a electron-positron plasma. The deduced nonlinear Schrödinger equation yields information about the modulational instability of the system for the different ranges of the plasma parameters. It is important to note that the modulational stability depends on the positron-electron density ratio. The theory is applicable to solitary waves in space plasmas.

PACS numbers: 52.35.-g, 52.35.Sb, 02.30.Jr UDC 531.742

Keywords: electron-positron plasma, envelope soliton, nonlinear Schrödinger equation, modulational stability, dependence on positron-electron density ratio, solitary waves in space plasmas

1. *Introduction*

It has been observed recently that nonlinear waves in electron-positron plasma behave differently from those in plasma with electrons and ions [1]. The electron-positron plasma occured in the early universe [2], and is present in active galactic nuclei [3] and in pulsar magnetosphere [4]. The different nature of an electron-positron plasma is due to the fact that the constituents have almost the same mass. People have already observed small amplitude solitons in plasma with significant percentage of positrons [5]. Here we study the formation of the envelope solitons in an electron-positron plasma. This is done with the help of a methodology advocated earlier by Fried and Ichikawa [6] and Roy Chowdhury et al. [7]. In this approach, one first derives a nonlinear dispersion relation which is then utilised to deduce the nonlinear Schrödinger (NLS) equation. The NLS equation so deduced yields the explicit form of the envelope soliton and also yields information about the modulational stability.

2. Formulation

We consider a plasma consisting of electrons, positrons and positive ions. It is also assumed that the usual hydrodynamic description is valid. Let n_α (α =i, e, p) stand for the ion, electron and positron density and v_α (α =i, e, p) be their respective velocities. Let ϕ denote the electrostatic potential, (x,t) the space and time coordinates, respectively, normalized by $k_{\rm B}T_{\rm e}/e$, $\lambda_D = \sqrt{\epsilon_0 k_{\rm B} T_{\rm e}/(n_0 e^2)}$ the Debye length, and $\omega_i^{-1} = \sqrt{\epsilon_0 m_i/(n_0 e^2)}$ the ion plasma period. The ion and electron velocities are normalized to the sound velocity $C_s = \sqrt{k_{\rm B}T_{\rm e}/m_i}$, where $k_{\rm B}$ is the Boltzmann constant.

The continuity equation and equation of motion for the electrons are

$$\frac{\partial n_{\rm e}}{\partial t} + \frac{\partial}{\partial x}(n_{\rm e} v_{\rm e}) = 0\,, \tag{1}$$

$$(\frac{\partial}{\partial t} + v_{\rm e}\frac{\partial}{\partial x})v_{\rm e} - \frac{\partial \phi}{\partial x} = 0\,. \tag{2}$$

We assume that the positrons form a background with density $n_{\rm p}$,

$$n_{\rm p} = \alpha \exp(-\phi/\beta)\,. \tag{3}$$

The ions are described by

$$\frac{\partial n_{\rm i}}{\partial t} + \frac{\partial}{\partial x}(n_{\rm i} v_{\rm i}) = 0\,, \tag{4}$$

$$(\frac{\partial}{\partial t} + v_{\rm i}\frac{\partial}{\partial x})v_{\rm i} + \frac{1}{Q}\frac{\partial \phi}{\partial x} = 0\,. \tag{5}$$

The Poisson's equation is given by

$$\frac{\partial^2 \phi}{\partial x^2} = n_{\rm e} - n_{\rm p} - n_{\rm i}\,. \tag{6}$$

Though the species electron and positron are very similar in nature, yet the main reason for the assumption (3) is to simplify the complexity of the problem. Otherwise, we would have two more equations of motion for the positrons, and the whole analysis would be very complicated.

To study the general problem of slow amplitude variation due to nonlinear effects, we proceed along the line of Fried and Ichikawa [6]. Before proceeding to the actual problem, we discuss in short the basics of the approach under consideration.

To start with, we assume the existence of a suitable nonlinear dispersion relation

$$\epsilon(k, \omega, A) = 0\,, \tag{7}$$

k being the wave vector, ω the frequency and A the amplitude. Therefore, ϵ denotes the functional relation between k, ω and A for the plasma under consideration. When there is no dependence on A, the relation becomes the usual linear dispersion relation. If initially the wave amplitude of the electrostatic potential ϕ is $\phi(x,0)$,

$$\phi(x,0) = \int \mathrm{d}k\ \phi_k \exp(\mathrm{i}kx) + c.c.\,, \tag{8}$$

with ϕ_k being peaked around $k = k_0$. In the usual linear theory, the long-time behaviour is given as

$$\phi(x,t) = \int \mathrm{d}k\ \phi_k \exp\{\mathrm{i}(kx - \omega t)\} + c.c\,. \tag{9}$$

In the basic formulation of Fried and Ichikawa [6], it was assumed that one can still use (9) in some approximation. We consider the situation when the amplitude and the spread of k values are small. Then, it is meaningful to talk about an expansion of the dispersion relation both in A^2 and $\bar{k} = k - k_0$. So

$$\epsilon(k,\omega,A) = \epsilon(k,\omega,0) + A^2 \frac{\partial \epsilon}{\partial A^2} + \ldots = 0\,. \tag{10}$$

For ω, we have

$$\omega = \Omega(k) + MA^2 = \omega_0 + \Gamma\,, \tag{11}$$

and

$$\Gamma = v_g \bar{k} + v_g' \frac{\bar{k}^2}{2} + MA^2\,, \tag{12}$$

Ω (k) being the solution of the linear dispersion relation

$$\epsilon[k, \Omega(k), 0] = 0\,,$$

$$\omega_0 = \Omega(k_0), \qquad v_g = \left.\frac{\partial \Omega}{\partial k}\right|_{k=k_0}\,, \tag{13}$$

$$v_g' = \left.\frac{\partial^2 \Omega}{\partial k^2}\right|_{k=k_0}, \qquad M = \left.\frac{\partial \omega}{\partial A^2}\right|_{k=k_0}, \qquad \omega = \omega_0, \qquad A = 0\,.$$

Equations (10) and (11) were used by Fried and Ichikawa [6] to deduce the nonlinear Schrödinger equation

$$\mathrm{i}\left(\frac{\partial \phi}{\partial t} + v_g \frac{\partial \phi}{\partial x}\right) + \frac{v_g'}{2}\frac{\partial^2 \phi}{\partial x^2} + q_1 \phi\,|\phi|^2 = 0\,, \tag{14}$$

$$q_1 = -M = -\frac{\partial \omega}{\partial A^2} \,.$$

To apply the above methodology in the case under consideration, we first of all deduce the (linear) dispersion relation of the system represented by Eqs. (1) to (6). For this we set

$$n_\mathrm{e} = 1 + n'_\mathrm{e}, \qquad v_\mathrm{e} = v'_\mathrm{e} \,,$$
$$n_\mathrm{p} = n^0_\mathrm{p} + n'_\mathrm{p}, \qquad n_\mathrm{i} = n^0_\mathrm{i} + n'_\mathrm{i} \,, \tag{15}$$
$$v_\mathrm{i} = v'_\mathrm{i} \qquad \phi = \phi' \,,$$

where each primed variable denotes perturbed quantity which is proportional to $\exp\{\mathrm{i}(kx - \omega t)\}$. Proceeding in the usual manner, we get

$$\omega^2 = \frac{1 + (1-\alpha)/Q}{\alpha/\beta} \cdot \frac{k^2}{1 + (\beta/\alpha)k^2} \,. \tag{16}$$

So, this is our $\epsilon(k, \omega, 0)$. To introduce corrections for small variation of the amplitude, we proceed to a frame of reference moving with velocity V and set $\xi = x - Vt$. Whence, upon integration, the equations yield

$$v_\mathrm{e} = V \frac{n_\mathrm{e} - 1}{n_\mathrm{e}} - 2V^2 \frac{n_\mathrm{e} - 1}{n_\mathrm{e}} + V^2 \frac{(n_\mathrm{e} - 1)^2}{n_\mathrm{e}^2} - 2\phi = 0 \,, \tag{17}$$
$$v_\mathrm{i} = V \frac{n_\mathrm{i} - 1 + \alpha}{n_\mathrm{i}} \,,$$

from which one can solve for n_e and n_i

$$n_\mathrm{i} = \frac{1-\alpha}{\sqrt{1 - 2\phi/(QV^2)}}, \qquad n_\mathrm{e} = \frac{1}{\sqrt{1 + 2\phi/(V^2)}} \,. \tag{18}$$

The Laplace equation leads to

$$\frac{\partial^2 \phi}{\partial x^2} = n_\mathrm{e} - n_\mathrm{p} - n_\mathrm{i} = \frac{1}{\sqrt{1 + 2\phi/V^2}} - \alpha \mathrm{e}^{-\phi/\beta} - \frac{1-\alpha}{\sqrt{1 - 2\phi/(QV^2)}} \,. \tag{19}$$

Expanding the right-hand side of Eq. (19) in powers of ϕ, we get

$$\frac{\partial^2 \phi}{\partial x^2} = F_1 \phi + F_2 \phi^2 + F_3 \phi^3 \,, \tag{20}$$

where

$$F_1 = \frac{\alpha}{\beta} - \frac{1 + (1-\alpha)/Q}{V^2} \,,$$

$$F_2 = \frac{3}{2}\frac{1-(1-\alpha)/Q^2}{V^4} - \frac{\alpha}{2\beta^2}\,, \tag{21}$$

$$F_3 = \frac{\alpha}{6\beta^3} - \frac{5}{2}\frac{1+(1-\alpha)/Q^3}{V^6}\,.$$

We now try to analyse Eq. (20) by a Fourier decomposition of ϕ,

$$\phi = \sum_{-\infty}^{\infty} \phi_n \exp(inkx), \qquad \phi_n = \phi^\star_{-n}\,. \tag{22}$$

Whence, we get a set of equations of the following form

$$-n^2k^2\phi_n = F_1\phi_n + F_2 \sum_{p+m=n} \phi_p\phi_m + F_3 \sum_{p+m+l=n} \phi_p\phi_m\phi_l\,. \tag{23}$$

Furthermore, it is assumed that each of the Fourier modes ϕ_n can be expanded in powers of ϵ as

$$\phi_n = \sum_{1}^{\infty} \epsilon^p \phi_n^p, \qquad \lambda = -k^2 = \sum_{0}^{\infty} \epsilon^p \lambda_p\,, \tag{24}$$

where ϵ represents a scale length. We are actually computing the perturbative corrections to ϕ_n and k^2 in different orders of ϵ. Here, for convenience, we have set $\lambda = -k^2$. Whence, we get in various powers of ϵ

$$\begin{aligned}
&\phi^1_{\pm 1} = 1, \qquad \phi_n^{(1)} = 0, \qquad n \neq \pm 1\,,\\
&\lambda_0 = F_1 = \alpha/\beta - \{1+(1-\alpha)/Q\}/V^2\,,\\
&\phi_0^{(2)} = -2F_2/F_1\,,\\
&\phi_{\pm 2}^{(2)} = +F_2/(3F_1)\,,\\
&\lambda_2 = 2F_2\left(\phi_2^2 + \phi_0^2\right) + 3F_3\,.
\end{aligned} \tag{25}$$

From this information one can at once calculate the nonlinear (q_1) and dispersive ($v'_g/2$) coefficients appearing in the nonlinear Schrödinger equation (14). One obtains.

$$q_1 = -\frac{1}{2}\frac{\alpha/\beta \cdot G\{1+(1-\alpha)/Q\}}{\left(\alpha/\beta + F_1\right)^2}\,, \tag{26}$$

$$p_1 = +\frac{v'_g}{2} = \frac{1}{2}\frac{\partial^2\omega_0}{\partial k^2} = -\frac{3}{2}k\sqrt{1+\frac{1-\alpha}{Q}}\sqrt{\frac{\alpha}{\beta}+k^2}\,. \tag{27}$$

where $G = 2F_2(\phi_2^2 + \phi_0^2) + 3F_3$.

3. Discussion

It is now a well-known fact that the modulational instability can be studied very simply by looking at the product $p_1 q_1$ and analysing its variation with respect to the plasma parameters. In our case, the most important parameter is α which is equal to n_{p0}/n_0, the ratio between the equillibrium densities of positrons to electrons. In Figs. 1 and 2, we show the variation of p_1 and q_1 as functions of α. Two cases may arise, either $\alpha > 1$ or $\alpha < 1$. The two situations are exhibited in Figs. 1a and b, and 2a and b, respectively. The case when $\alpha > 1$ shows that

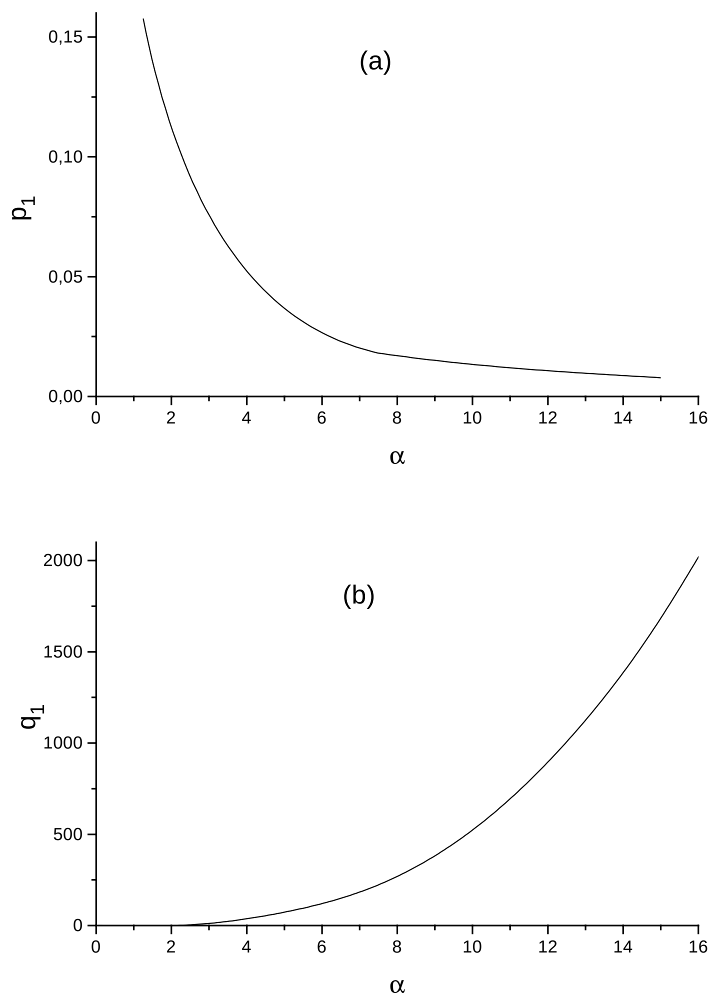
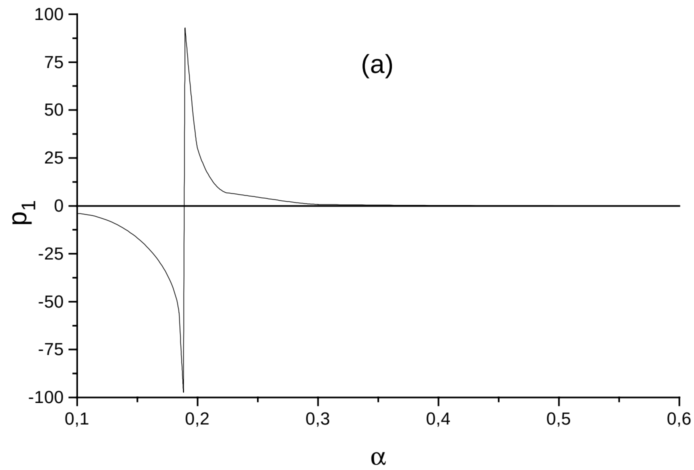


Fig. 1. Variation of a) p_1 and b) q_1 (see text) for $\alpha > 1$.



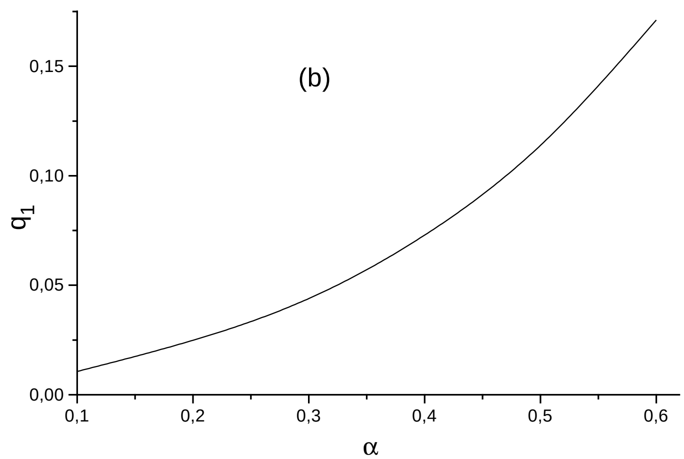


Fig. 2. Variation of a) p_1 and b) q_1 (see text) for $\alpha < 1$.

both p_1 and q_1 are of same sign and hence the product $p_1 q_1$ is always positive. On the other hand, for the case $\alpha < 1$, the situation as depicted in Figs. 2a and b, turns out to be different. Here we observe that for $0.1 < \alpha < 0.2$, the function p_1 is negative and sharply changes to positive as α becomes greater than 0.2. But q_1 remains positive althrough. So, here we observe a change of stability. Depending upon the relative concentration of the electrons and positrons, a stable situation may turn to be an unstable one. Due to the existence of a large amount of electrons and positrons in the space plasma, it is always possible to have a practical situation where such conditions can occur.

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ANVELOPNI SOLITONI I MODULACIJSKA NESTABILNOST U ELEKTRONSKO-POZITRONSKOJ PLAZMI

Proučavamo stvaranje anvelopnih solitona u elektronsko-pozitronskoj plazmi. Izvedena nelinearna Schrödingerova jednadžba daje podatke o modulacijskoj nestabilnosti sustava za razna područja parametara plazme. Važno je primijetiti kako modulacijska nestabilnost ovisi o omjeru gustoće elektrona i pozitrona. Teorija se može primijeniti na solitonske valove u svemirskoj plazmi.