Chapter 1

Complex Excitable Media: Activator designs, while inhibitors embellish

Sergio Alonso Muñoz
Department of Physics, Universitat Politecnica de Catalunya
s.alonso@upc.edu

A system composed by coupled reaction-diffusion equations is one of the most classical scenarios of pattern formation. There are also other mechanisms of pattern formation,[1] however, all have in common the interplay between a mechanism of transport and the non-linearities of the system.[2] Probably, the simplest example of pattern formation involving reaction-diffusion equations is a bistable front moving with constant velocity. A certain concentration U experiences an autocatalytic non-linear growth and the corresponding increase of U travels along the media by diffusion.[3] Such process is simply modelled by a single reaction-diffusion equation:

$$\frac{\partial U}{\partial t} = D\nabla^2 U + F(U), \tag{1}$$

where $F(U)$ is a nonlinear function, typically a cubic function on U, giving rise to two stable solutions, i.e. bistable system. Under an adequate initial condition a front changes the stability from the metastable to the more stable solution. In such case, the concentration U activates his own production and can be noted as *activator*. Such activation propagates by diffusion through the medium forming a travelling front with a fix velocity which depends of the parameters on Eq.(1). The resulting fronts are robust and very stable waves which propagate through the system

We can change the stability of the stable solutions by the addition of an extra variable V which inhibits the production of U. The new variable V is usually noted as *inhibitor*. Such extra variable is produced due to the presence of U, and it gives rise to a typical activator-inhibitor system[4] of two coupled equations:

$$\frac{\partial U}{\partial t} = D\nabla^2 U + F(U) - V, \qquad (2)$$
$$\frac{\partial V}{\partial t} = D_v\nabla^2 V + \epsilon(U - V),$$

The effect of the inhibitor is to return to the initial stable state after the increase due to the autocatalytic part of the activator, resulting in an excitation, a large excursion in the variable space, instead of a definitive change on the variable value. This type of dynamics corresponds to excitable media.[5] This effect produces the generation of a travelling wave[6,7] instead of the travelling front typical in bistable media. The perturbation of the neighbouring corresponds to an excitation which propagates forming the wave. The temporal scale of the return to the rest state is given by the parameter ϵ in Eqs.(2). While the velocity and curvature are designed by the dynamics of the activator U, the inhibitor V outlines the thickness and shape of the wave.

In two dimensions, the breakup of the travelling wave generates a free end. The resulting wave rotates around this free end, giving rise to a rotating spiral wave, see an example in Fig.1. The beauty of spirals is out of discussion, it has been a typical motive for human ornaments forever and ever. Spiral waves in excitable media have been observed in chemical systems like the Belousov-Zhabotinsky reaction,[8] and CO Oxidation in catalytic surfaces;[9] and in biological systems like calcium waves inside Xenopus laevis oocytes[10] and action potential propagation in cardiac tissue.[11]

Under random initial condition excitable media can produce multiple spiral waves, see Fig.2, a set of interacting rotating spiral waves which keep the relative distances among the centres of rotation.

The complexity of the spatio-temporal patterns produced in excitable media is increased with additional inhibitors. Following the structure of Eqs.(2) we add an additional inhibitor:

$$\frac{\partial U}{\partial t} = D\nabla^2 U + F(U,V,W) - V - W,$$
$$\frac{\partial V}{\partial t} = D_v\nabla^2 V + \epsilon_1 G_1(U,V), \qquad (3)$$
$$\frac{\partial W}{\partial t} = D_w\nabla^2 W + \epsilon_2 G_2(U,W),$$

where ϵ_1 and ϵ_2 are two temporal scales which may be very different.[12] Therefore, each inhibitor controls the dynamics at different temporal scales.

The two temporal scales of the inhibitors in Eqs.(3) can be tunned to produce very different patterns. The inclusion of such second inhibitor produces complex excitable media, which have been employed in the context of computational modeling of cardiac tissue[13] and on the modeling of chemical reactions in microemulsions.[12,14]

The interaction between consecutive waves of a rotating spiral wave can induce an alternation among the thickness of the waves. Finally, it gives rise to unstable spiral waves and produces a turbulence of waves which reminds the cardiac fibrillation,[15] see Fig.3. On the other hand, spiral waves can be unstable and produce multiple spiral waves due to interaction with other kind of instabilities typical of reaction-diffusion systems, see in Fig.4 the interplay between the rotating spiral waves, for example see Fig.2 and a Turing instability, see Fig.5. Turing patterns result from large values of the diffusion coefficient of one of the inhibitors in comparison with the diffusion coefficient of the activator.[4]

The dynamics is more complex in three-dimensional systems. Scroll waves are the three-dimensional equivalents to spiral waves.[16] In its cross section, a scroll wave has the shape of a spiral.[17] Such spirals are stacked one upon another and rotate around a filament that occupies the centre of the scroll.[18] The scroll may be straight, see Fig.6, or it may also be closed into a ring, form knots or even generate a complex turbulent state,[19,20] see an example in Fig.7.

Finally, there are several computational tools to solve the reaction-diffusion equations in a restricted region of the space. The use of a phase field facilitates the reconstruction of the borders of the irregular domains in the heart[18] or the cylindrical shape of a three-dimensional chemical reactor.[21] For example, we can fit the complex dynamics depicted by the excitable waves inside regular or irregular domains, see, for example, the turbulent waves inside two neighboring spheres in Fig.8.

In summary, several pretty patterns obtained in complex excitable media have been shown in this chapter. While the activator marks the direction of the propagation and even the velocity, the inhibitor can produce important interactions responsible of the formation of complex patterns. The increase of the number of inhibitors can directly increase the complexity of the patterns. We have also shown that the pass to three dimensions unveils hidden instabilities.

I hope the reader will enjoy the beauty of the patterns obtained in complex excitable media as much as I have enjoyed with their production.

Fig. 1. **Rotating spiral wave in a generic model of excitable media.** A rotating spiral wave is the most typical spatio-temporal structure appearing in excitable media. It rotates with constant velocity around the free end of the spiral giving rise to a continuous emision of traveling waves. The spiral wave is obtained using a FitzHugh-Nagumo[6,7] model consisting in an activator and an inhibitor, see Eqs.(2). We plot the value of the variable U but a similar pattern is obtained if we plot the variable V.

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Fig. 2. **Multiple spiral waves formation under random initial conditions in an excitable model.** Several spiral waves rotate simultaneously with the same frequency in an homogeneous extended system generated by a set of random initial conditions. Waves are obtained in a model of chemical waves in microemulsions, however, similar behavior can be optained in simpler models with adequate initial conditions.

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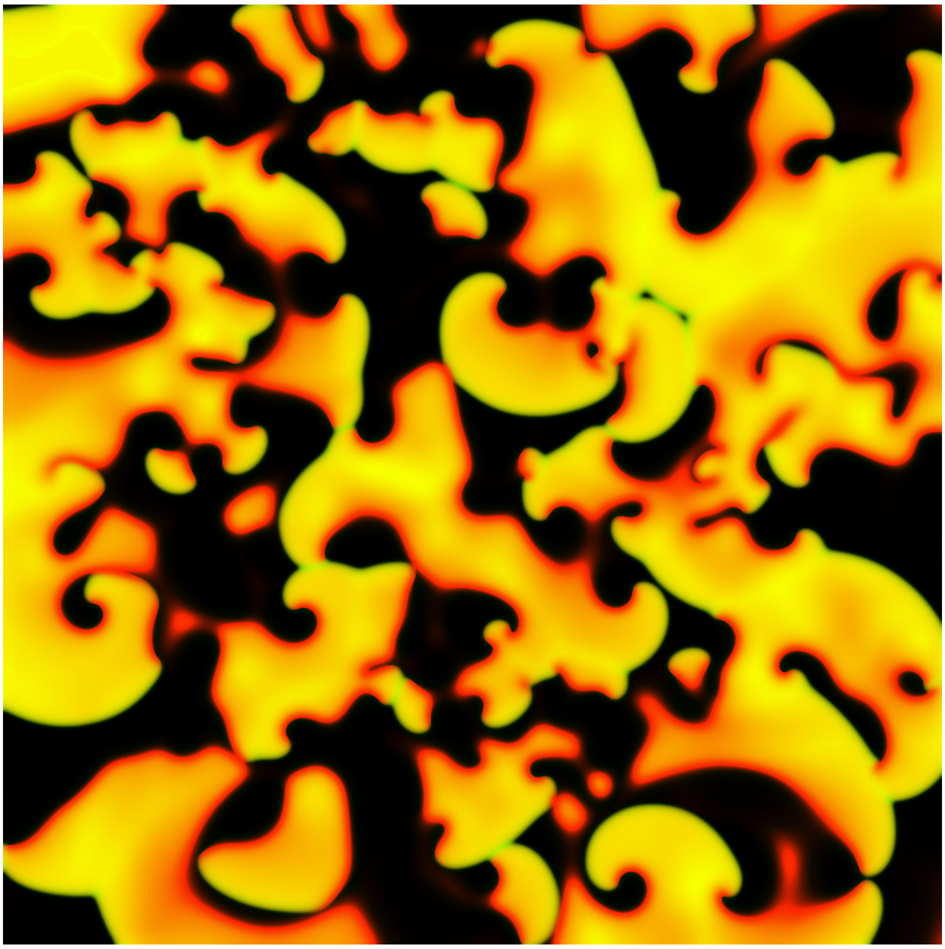

Fig. 3. **Spatio-temporal chaos from an alternan instability in a model of action potential propagation in cardiac tissue.** Single spiral waves may be unstable and may breakup because the interaction between consecutive waves. The rotation of the spiral wave naturaly produces a periodic pacing of the tissue which can produce a bifurcation to long-short waves. If the interaction is strong enough, it produces the local decay of the wave, conduction block and the production of two new spiral waves. Repeating the same mechanism and due to interaction among spirals, complex wave patterns are found.

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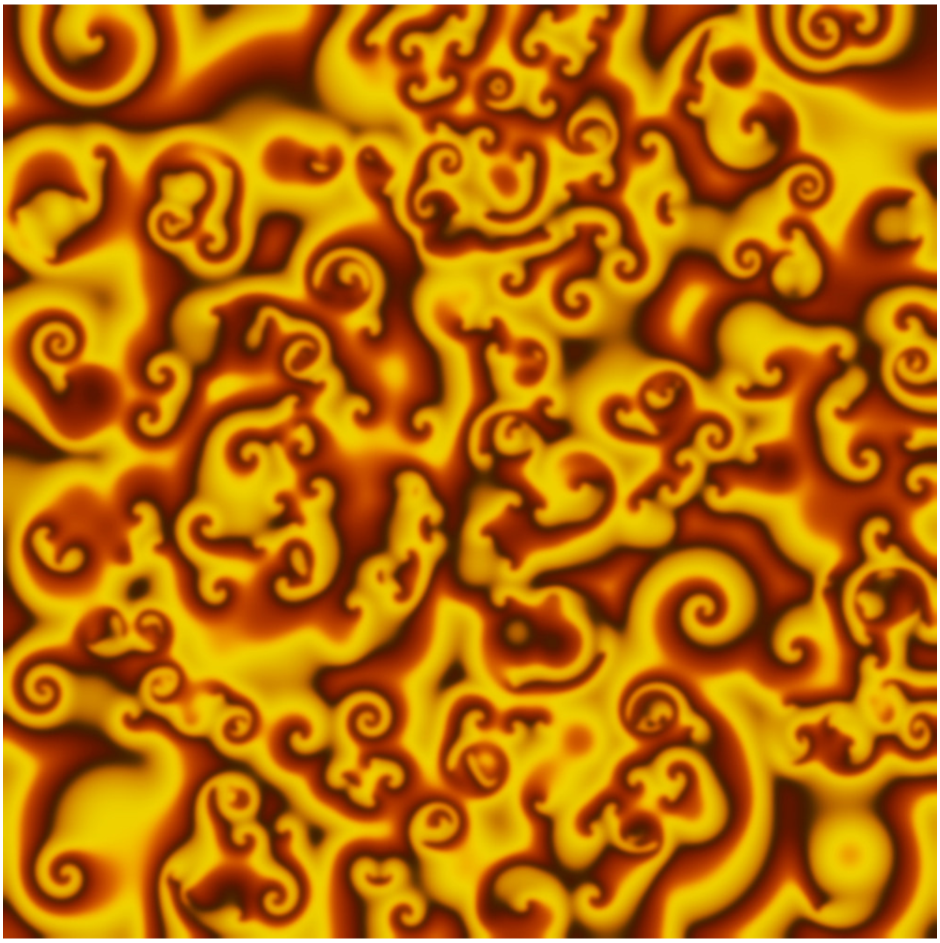

Fig. 4. **Spatio-temporal chaos from an interaction of Turing and wave instabilities a model of chemical waves in microemulsions.** The use of two inhibitors permits the simulateneous induction of two different instabilities which under certain window of parameters permits the cohexistence of spiral waves with Turing patterns. The resulting pattern is a complex mix of spatio-temporal structures.

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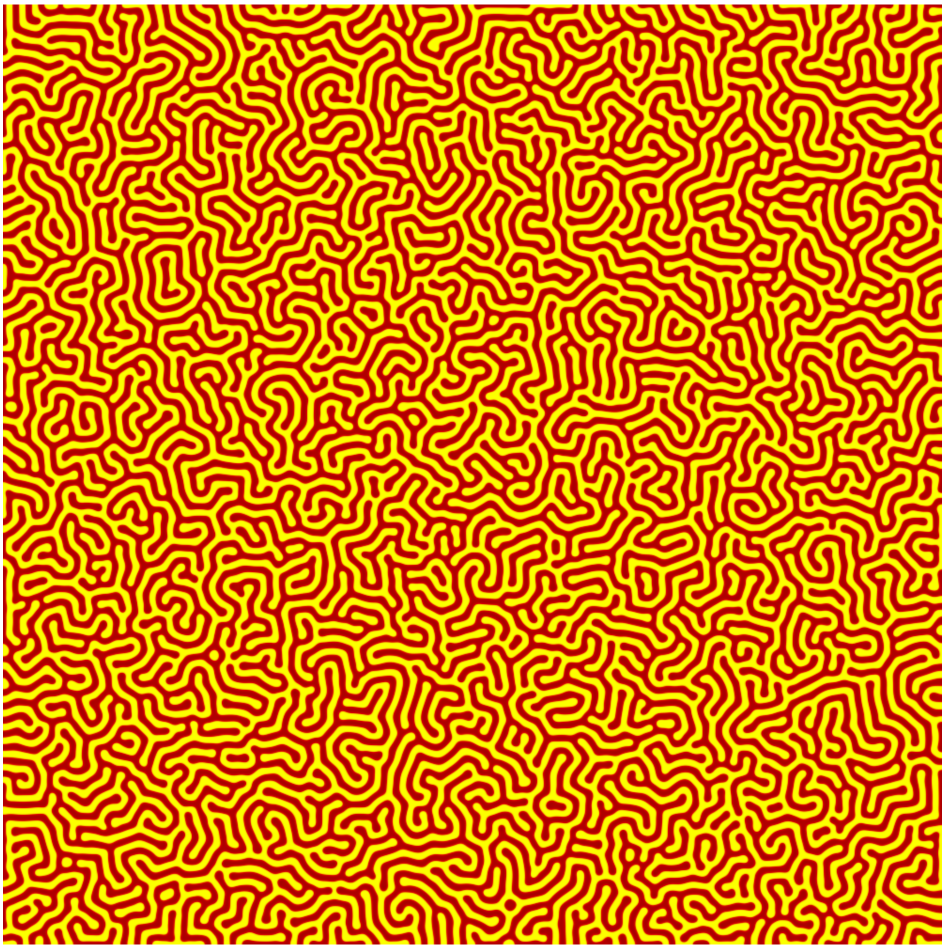

Fig. 5. **Turing patterns in reaction-diffusion systems.** The increase of the diffusion of the inhibitor in two coupled reaction-diffusion equations like Eqs.(2) can produce the formation of stable and static patterns. It is actually not a typical pattern for excitable media, however, its formation together with the presence of spiral waves in more complex systems, for example Eqs.(3), can give rise to the patterns obtained in Fig.4.

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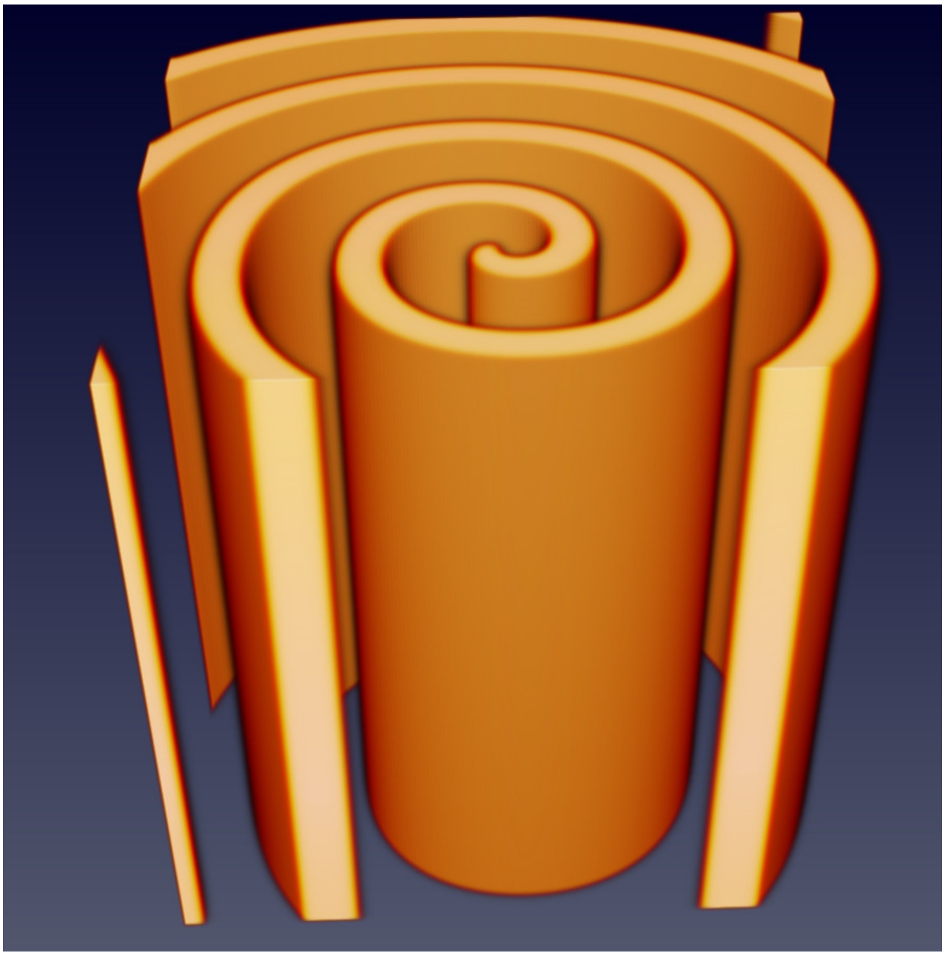

Fig. 6. **Rotating scroll wave in a generic model of excitable media.** Scroll waves are the three dimensional analogous to the two dimensional spiral waves. They rotate around a filament analogously as the spiral waves typically rotate around a circular core. Somehow, a scroll wave can be considered as a stack of rotating spiral waves that are organized by the filament, a line connecting the cores. A straight scroll wave is stable if the filament has positive tension and there are no strong interaction among consecutive waves.

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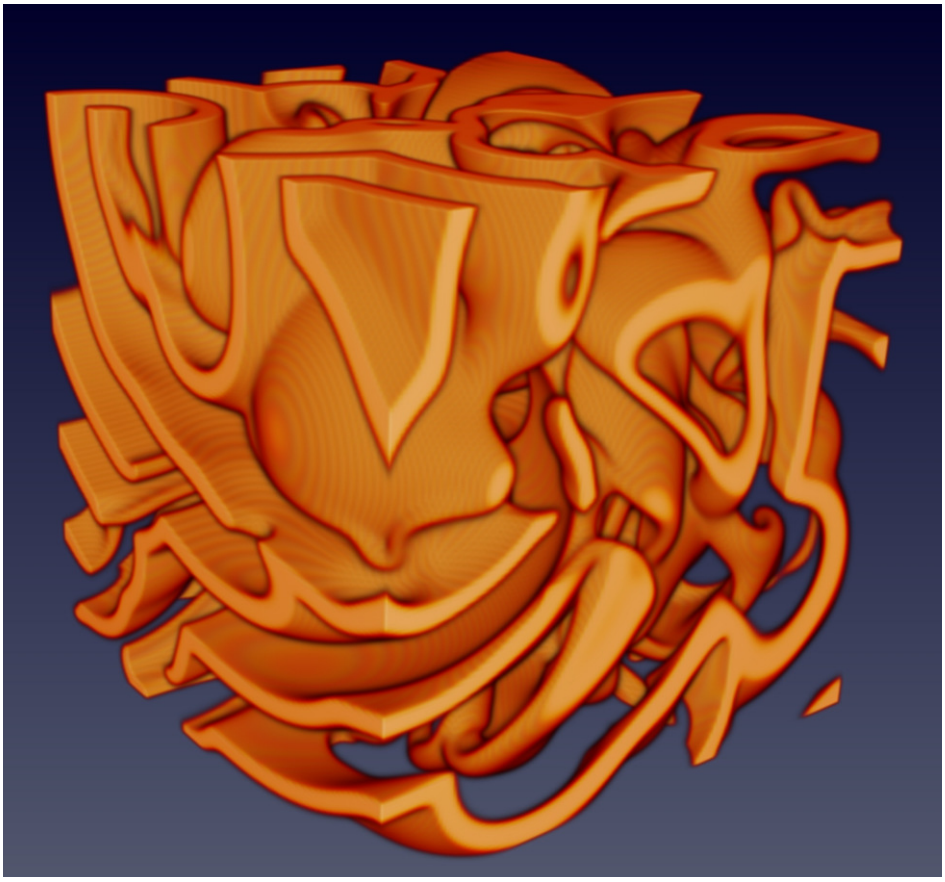

Fig. 7. **Negative filament tension of the filament of scroll waves in a generic model of excitable media.** If the filament tension of the scroll wave is negative, then the filament will tend to increase its length and the straight scroll wave shown in Fig.6 becomes unstable. The filament tends to stretch, bend, loop, and produce an expanding tangle that fills up the volume. Through the process of local stretching and bending of a filament, very complex and potentially chaotic wave patterns were expected to develop.

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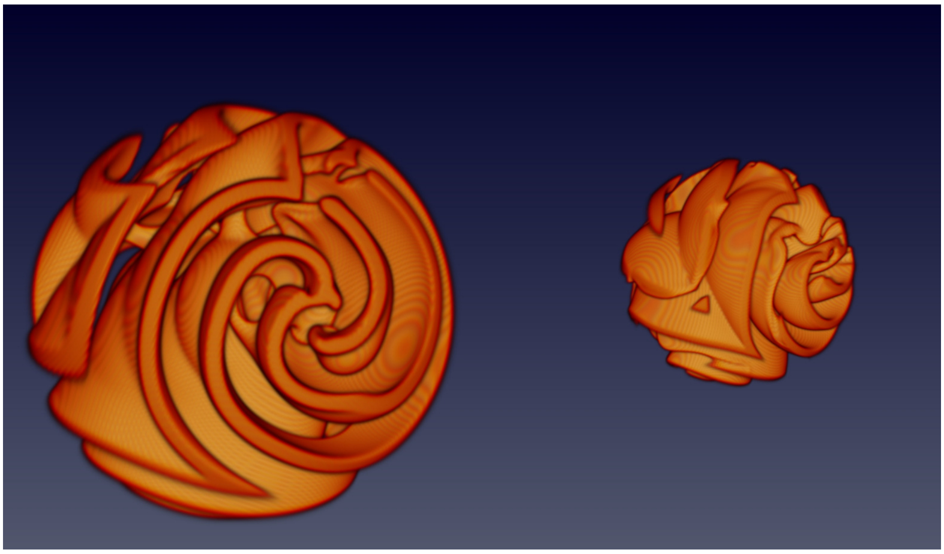

Fig. 8. **Negative filament tension of the filament of scroll waves in a generic model of excitable media under spatial confinement in two spheres.** We employ a phase field model which is an additional variable which value goes from 1 in the interior of a certain domain to 0 outside. The reaction-diffusion variables U and V are multiplied by the additional phase field. The transition between the two values is smooth and keep the non-flux boundary conditions in the irregular border. Two disconnected spheres are generated by the phase field and the dynamics of the turbulent behaviour computed in their interior.

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