THE ARNOLD CONJECTURE FOR SINGULAR SYMPLECTIC MANIFOLDS

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ABSTRACT. In this article, we study the Hamiltonian dynamics on singular symplectic manifolds and prove the Arnold conjecture for a large class of b^m-symplectic manifolds. More precisely, we prove a lower bound on the number 1-periodic Hamiltonian orbits for b^{2m}-symplectic manifolds depending only on the topology of the manifold. Moreover, for b^m-symplectic surfaces, we improve the lower bound depending on the topology of the pair (M, Z). We then venture into the study of Floer homology to this singular realm and we conclude with a list of open questions.

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1. INTRODUCTION

Symplectic structures on manifolds with boundary [NT] led naturally to the investigation of a class of Poisson manifolds which are symplectic manifolds away from a hypersurface but degenerate along this hypersurface (see [GMP] and [GL][1]). In the literature, these structures are called b^m- or $\log$-symplectic manifolds. They also show up in the space of geodesics of the Lorentz plane [KT] and furnish a natural phase space for regularized problems in celestial mechanics such as the restricted 3-body problem [KM, KMS, DKM, MO2, MO, BDMOP]. Another example where Hamiltonian dynamics on b-symplectic manifolds is outstanding is in the context of Painlevé transcendents ([M], [MM]).

The investigation of the dynamics on the odd-dimensional sibling of these singular manifolds started in [MO] and [MOP]. In this article we initiate the exploration of singular Hamiltonian dynamics on the even-dimensional counterpart and compare it to Hamiltonian dynamics for symplectic manifolds. In order to do so, we adopt the revolutionary approach due to Andreas Floer to prove the Arnold conjecture on compact symplectic manifolds.

Conjecture 1.1 (Arnold conjecture). *Let (M, ω) be a compact symplectic manifold and let $H : \mathbb{R} \times M \to \mathbb{R}$ be a time-dependent Hamiltonian function. Suppose that the solutions of period 1 of the associated Hamiltonian system are non-degenerate. Let $\mathcal{P}(H)$ denote the set of 1-periodic*

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[1]Other contributions are included in [MP, GMP2, GMPS, GMPS2, GMW1, GMW3, GMW2, GL, KM, KMS, Ca, MOT, MOT2]

orbits. Then

$$\#\mathcal{P}(H) \geq \sum_i \dim HM^i(M;\mathbb{Z}_2).$$

To tackle this long-standing conjecture, Floer defined a homology whose generators are the Hamiltonian periodic orbits and showed that this homology is isomorphic to the standard homology of the manifold M, thus depending only on the topology of M. Floer's exploration opened the door to a new set of techniques which was successful in proving the Arnold conjecture in many different settings (we direct the reader to [Sal, Section 1.1] for a comprehensive chronology of the work on the Arnold conjecture with the specific references).

In this paper, we pave the way to adapt Hamiltonian Floer theory to b-symplectic manifolds. This will allow us to apply the resulting theory to many natural examples endowed with a *singular* symplectic structure in celestial mechanics, Lorentzian geometry and the study of Painlevé transcendents. The geometry on these manifolds can be described as open symplectic manifolds equipped with a cosymplectic structure on the open ends. In a way, our theory provides a generalization of Floer theory to non-compact symplectic manifolds but it is different from the theory of tentacular Hamiltonians in [PW, PVW]. The cosymplectic structure on the ends is determined by a symplectic vector field that is transverse to the boundary. To avoid compactness issues in the definition of the Floer complex, the set of Hamiltonian functions needs to be restricted.

This is reminiscent of symplectic manifolds with convex boundaries: for them, the Hamiltonian dynamics was thoroughly studied in [FS]. In order to define a Floer theory for those open symplectic manifolds, special care needs to be taken at the boundary so as to have a well-defined complex.

In the class of b^m-symplectic manifolds, we are interested in periodic Hamiltonian orbits away from the critical set. The b^m-symplectic structure induces on the critical set a codimension one symplectic foliation. In the case where the leaves are compact and the Hamiltonian is smooth, the Hamiltonian vector field restricts to a smooth Hamiltonian vector field on each leaf and therefore there are infinitely many periodic orbits on the leaves. The existence of periodic orbits away from the critical set however is much more subtle. To study this question, we will argue that the appropriate Hamiltonian functions are the ones that are linear along the symplectic vector field and preserved by the modular vector field. Those Hamiltonian functions are called admissible Hamiltonian functions and they are not smooth functions (as they present log-terms or higher order singularities) but they belong to the class of functions naturally associated to b^m-symplectic manifolds, b^m-Hamiltonian functions.

In the case of b^{2m}-symplectic manifolds (without any restriction on the dimension), we can bound from below the number of time-1 periodic orbits of such Hamiltonian vector fields by the topology of the ambient manifold. This bound is achieved by proving that its associated dynamics is indeed symplectic for a *smooth* geometric structure, called the desingularized symplectic structure. The desingularization was previously introduced and studied in [GMW1]. We associate to the b-manifold a weighted graph (for the precise definition see Definition 4.1), whose vertices correspond to the connected components of $M \setminus Z$ and the edges to the connected components of Z. In the case where this graph is acyclic, the associated dynamics is in fact Hamiltonian. As a corollary, we get that the number of non-degenerate periodic orbits is bounded from below by the topology of the manifold, as given by the Arnold conjecture.

Theorem A. Let (M, Z, ω) be a compact b^{2m}-symplectic manifold whose associated graph is acyclic. Assume that $H_t \in {}^{b^{2m}}C^\infty(M)$ is a time-dependent admissible Hamiltonian function. Suppose that the solutions of period 1 of the associated Hamiltonian system are non-degenerate. Then

$$\#\mathcal{P}(H) \geq \sum_i \dim H_i(M;\mathbb{Z}_2).$$

We remark that for manifolds with trivial $H^1(M)$ the condition of having an acyclic graph is automatically satisfied. In the case where the manifold is cyclic, an alternative statement is presented.

In the case of orientable b^m-symplectic surfaces, this lower bound can be substantially improved. We do not only show that the above method can be adapted for m odd, we also prove an improved lower bound (in comparison to the previous one), that depends on the topology of the manifold together with the relative position of the critical curves.

Hence, we obtain the following.

Theorem B. Let (Σ, Z, ω) a b^m-symplectic orientable surface. Let H_t an admissible Hamiltonian in (Σ, Z, ω) whose periodic orbits are all non-degenerate. Then

$$\#\mathcal{P}(H) \geq \sum_{v\in V} \max\{2 + 2g_v - \deg(v), 0\}.$$

Under the admissibility condition of the Hamiltonian functions, we show that the associated Floer solutions satisfy a minimal principle:

Theorem C. Let (M, Z, ω) a b^m-symplectic manifold, and let $u : \Omega \subset \mathbb{C} \to \mathcal{N} \subset M$, where $\mathcal{N}$ is a tubular neighbourhood of Z in $M \setminus Z$. Let also $f : \mathcal{N} \to \mathbb{R}$ given by $\log|z|$ if $m = 1$ and $-\frac{1}{(m-1)z^{m-1}}$ if $m > 1$. Let u a solution of the Floer equation for an admissible Hamiltonian $H \in \mathcal{C}^\infty(S^1 \times \mathcal{N})$. If $f \circ u$ attains its maximum or minimum on Ω, then $f \circ u$ is constant.

From this result we deduce compactness of the space of solutions. We are thus able to introduce a well-defined a Hamiltonian Floer-type homology on b^m-symplectic manifolds.

Our definition of Hamiltonian Floer theory on b^m-symplectic manifold together with the lower bounds in Theorem A and B, gives rise to new open questions which we describe at the end of this article.

Organization of the paper. In Section 2 we include a brief introduction to b^m-symplectic geometry and Floer homology. We also include some known results concerning symplectic manifolds with convex boundaries, which are in some sense at the opposite end of the spectrum to b^m-symplectic manifolds. We then study in Section 3 some preliminary results concerning the Hamiltonian dynamics around the critical set of b^m-symplectic manifolds. This section will serve as a motivation to study this particular class of Hamiltonian functions on b^m-symplectic manifolds. In Section 4 we prove in Theorem A and Theorem B a lower bound for the number of periodic orbits for this class of Hamiltonian functions. The Hamiltonian functions are suitable to define a Hamiltonian Floer type homology on b^m-symplectic manifolds, as is seen in Section 5. We end the article with open questions in Section 6 that we will consider in future work.

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2. Preliminaries

In this section we provide a summary of the background material required for our setting. We begin by giving an outline of the proof of the Arnold conjecture in the case of closed and aspherical symplectic manifolds using Floer theory, following mostly the contents of [AD] and [Sal].

After that we will collect the basic results on b- and b^m-symplectic manifolds that we will need.

Finally, we will reproduce shortly the construction developed in [FS], which has been an important inspiration for our version of the Floer complex on b^m-symplectic manifolds.

2.1. **Basics on Floer homology.** Let (M,ω) be a compact symplectic manifold. Let us assume that the first Chern class of M vanishes on $\pi_2(M)$ and also that $\langle[\omega],\pi_2(M)\rangle = 0$.

Definition 2.1. Let $H_t : \mathbb{R}\times M \to \mathbb{R}$ be a time-1 periodic smooth Hamiltonian. The action functional is defined on the space of contractible loops $\mathcal{L}M := \{x \in \mathcal{C}^\infty(S^1, M) \mid x \text{ contractible}\}$ and is given by

$$\mathcal{A}_{H_t}(x) = \int_{S^1} H_t(x(t))dt - \int_{D^2} v^*\omega,$$

where v is a filling of x within M.

Critical points of the action functional correspond to 1-periodic orbits of the Hamiltonian vector field. For a generic[2] Hamiltonian H, the Floquet multiplier of a critical point x of $\mathcal{A}_H$ is different from zero and, as M is compact, there exist only a finite number of such 1-periodic orbits. Such a Hamiltonian is usually called *admissible* and we denote the set of admissible Hamiltonian functions by $\mathcal{H}_{\text{adm}}$.

The Floer complex consists of the graded $\mathbb{Z}_2$-vector space $CF_*(M,H)$ generated by the critical points of $\mathcal{A}_H$ and graded by an index called the Conley-Zehnder index (see for instance [Sal, Section 2.4]).

Definition 2.2. The Floer equation is obtained by computing the negative gradient flow along a loop x of the action functional. The metric here is the one induced on the loops from the compatible metric g on (M,ω,J), where J is a compatible almost complex structure. It is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0. \tag{2.1}$$

The energy is defined by

$$E(u) = \int_{S^1\times\mathbb{R}} u^* d\mathcal{A}_H,$$

and the set of finite energy solutions is defined by

$$\mathcal{M} = \left\{u : \mathbb{R}\times S^1 \to M | u \text{ is a solution to the Floer equation and } E(u) < \infty\right\}. \tag{2.2}$$

[2]Here generic means that it belongs to a countable intersection of sets which are open and dense in the C^∞-topology.

The properties of $\mathcal{M}$ were studied by Floer using the techniques pioneered by Gromov in his analysis of pseudo-holomorphic curves. In the particular context of the Floer equation, very similar methods are used to show the following theorem:

Theorem 2.3 ([AD, Theorems 6.5.4 and 6.5.6])**.** *$\mathcal{M}$ is compact. Moreover, for any $u \in \mathcal{M}$, there exist two critical points of $\mathcal{A}_H$, x and y, such that*

$$\lim_{s \to -\infty} u(s, \cdot) = x, \quad \lim_{s \to +\infty} u(s, \cdot) = y$$

in $\mathcal{C}^\infty(S^1, M)$, and

$$\lim_{s \to \pm\infty} \frac{\partial u}{\partial s}(s, \cdot) = 0.$$

Therefore, for any pair $x, y \in \mathrm{Crit}(\mathcal{A}_H)$ the set $\mathcal{M}(x, y) \subset \mathcal{M}$ of orbits connecting x with y is compact. Conversely, $\mathcal{M} = \bigcup_{x,y \in \mathrm{Crit}(\mathcal{A}_H)} \mathcal{M}(x, y)$.

In order to define a boundary map for the Floer complex, the dimension of each manifold $\mathcal{M}(x, y)$ needs to be determined. The linear approximation of the Floer operator acting on the set of perturbations of elements of $\mathcal{M}(x, y)$ is defined as

$$\mathcal{P}^{1,p}(x, y) = \left\{ P : (s, t) \mapsto \exp_{u(s,t)} Y(s, t) \mid u \in \mathcal{M}(x, y), Y \in W^{1,p}(u^*(TM)) \right\}.$$

Here $W^{1,p}(u^*(TM))$ denotes the set of maps $Y : \mathbb{R} \times S^1 \to TM$ such that $\pi \circ Y = u$ (where $\pi : TM \to M$ is the natural projection of the tangent bundle) and such that their local trivializations belong to the Sobolev space $W^{1,p}$.

Then, the Floer operator is defined as

$$\begin{array}{rccc} \mathcal{F} : & \mathcal{P}^{1,p}(x, y) & \longrightarrow & L^p(\mathbb{R}, S^1) \\ & w & \longmapsto & \frac{\partial w}{\partial s} + J_w \frac{\partial w}{\partial t} + \mathrm{grad}_w H_t, \end{array}$$

and its linearization has the expression

$$d\mathcal{F}_u(Y) = \frac{\partial Y}{\partial s} + J_u \frac{\partial Y}{\partial t} + (\mathcal{L}_Y J)_u \frac{\partial u}{\partial t} + \mathcal{L}_Y(\mathrm{grad}_u H).$$

One can check (see Theorem 8.1.5 in [AD] or Theorem 2.2 in [Sal]) that $d\mathcal{F}_u$ is a Fredholm map for any $u \in \mathcal{M}(x, y)$ and that it has Fredholm index $\mathrm{Ind}(d\mathcal{F}_u) = \mu_{CZ}(x) - \mu_{CZ}(y)$, the difference of Conley-Zehnder indices. Further, it can be seen that it is a surjective map for a generic choice of H and J. From this, the expected result regarding the dimension of $\mathcal{M}(x, y)$ follows:

Theorem 2.4. *For $p > 2$, $\mathcal{F}^{-1}(0)$ is a finite dimensional compact manifold of dimension $\mu_{CZ}(x) - \mu_{CZ}(y) - 1$.*

The boundary map of the Floer complex is then defined as $\partial_k : CF_{k+1}(M; H, J) \to CF_k(M; H, J)$ with

$$\partial_k(x) = \sum_{y \in CF_k(M;H,J)} n(x, y) y,$$

where $n(x, y)$ is the number of points in the zero-dimensional and compact manifold $\mathcal{M}(x, y)$, modulo 2.

From the established properties it follows that the boundary map is well-defined.

Theorem 2.5 ([F2, Theorem 4])**.**

$$\partial_k \circ \partial_{k+1} = 0.$$

Thus, the Floer complex $(CF_\bullet(M; H, J), \partial_\bullet)$ is well defined and it induces a homology. It is clear that we used both H and J to define this complex, so it is clear that the complex (and therefore the homology) can depend on these choices. However, as the following theorem shows, this turns out not to be the case for the homology.

Theorem 2.6 ([F2, Theorem 5], [AD, Chapter 11]). *The homology induced by the Floer complex does not depend on the choice of a pair (H, J).*

Indeed, this complex can be identified with the Morse homology:

Theorem 2.7 ([F2, Theorem 1], [AD, Theorem 10.1.1]). *The Floer homology is isomorphic to the Morse homology,*

$$HF_\bullet(M) \cong HM_{\bullet+n}(M).$$

This concludes the proof of the Arnold Conjecture in the aspherical standard case.

2.2. **Geometrical structures on b^m-manifolds.** We begin by providing a quick summary of the theory of b^m-symplectic manifolds. For a more detailed development on this topic we direct the reader to [GMP2, BDMOP, Sco].

Definition 2.8. Let M be a compact and connected manifold, and let $Z \subset M$ be an embedded submanifold of codimension 1. We call such pairs **b-manifolds**. In this paper we will assume that Z is defined by the regular zero-level set of a smooth, globally defined function $f : M \to \mathbb{R}$, that is $Z = f^{-1}(0)$.

We define the set of b-vector fields as

$${}^b\mathfrak{X}(M) := \{v \in \mathfrak{X}(M) \mid \ v|_Z \in \mathfrak{X}(Z)\},$$

this means, the set of vector fields on M that are tangent to Z. This has the structure of a projective module, so using the Serre-Swan theorem (see for instance [Sw]) we can define the vector bundle whose sections are ${}^b\mathfrak{X}(M)$, the b-tangent bundle ${}^bTM \to M$.

The b-cotangent bundle is its dual, ${}^bTM^* := \left({}^bTM\right)^*$. The forms on this vector bundle form a complex, denoted by ${}^b\Omega^\bullet(M)$, which contains $\Omega^\bullet(M)$ canonically, and moreover, we can extend the notion of differential to this complex, $d : {}^b\Omega^\bullet(M) \to {}^b\Omega^{\bullet+1}(M)$.

A **b-symplectic manifold** is a 3-tuple (M, Z, ω), where (M, Z) is a b-manifold and $\omega \in {}^b\Omega^2(M)$ is a closed and non-degenerate b-form. We will often omit the critical set Z in the notation.

More generally, we can consider singularities such as b^m-symplectic manifolds, as was done by [Sco]. Mimicking the b-case we start by introducing the analogue of sections of the b^m-tangent bundle.

Definition 2.9. A **b^m-vector field** is a vector field v on a b-manifold (M, Z) vanishing to order m at Z.

The b^m-vector fields are locally generated by $\{x_1^m \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\}$ where $x_1 = 0$ describes Z and $(x_1, \dots, x_n)$ are coordinates on M. Thanks to the Serre-Swan theorem [Sw], given a b-manifold (M, Z), there exists a unique vector bundle ${}^{b^m}TM$ all whose smooth sections are b^m-vector fields. Such a bundle is called the **b^m-tangent bundle**. Analogously, the **b^m-cotangent bundle** can be defined either as dual to the tangent one:

$${}^{b^m}T^*M = ({}^{b^m}TM)^*.$$

We can then introduce the set of b^m-forms ${}^{b^m}\Omega^k(M)$ as sections of the exterior product of this bundle $\bigwedge^k({}^{b^m}T^*M)$. The forms on this vector bundle form a complex, denoted by ${}^{b^m}\Omega^\bullet(M)$, which contains $\Omega^\bullet(M)$ canonically, and moreover, we can extend the notion of differential to this complex, $d : {}^{b^m}\Omega^\bullet(M) \to {}^{b^m}\Omega^{\bullet+1}(M)$. We denote by ${}^{b^m}H^*(M)$ the associated b^m-cohomology. The zero-degree terms of this complex are the motivation to introduce the definition of b^m-functions which we denote as ${}^{b^m}\mathcal{C}^\infty(M)$ as

$${}^{b^m}\mathcal{C}^\infty(M) = \left(\bigoplus_{i=1}^{m-1} t^{-i}\mathcal{C}^\infty(t)\right) \oplus {}^b\mathcal{C}^\infty(M)$$

where ${}^b\mathcal{C}^\infty(M)$ denote the space of b-functions (see [GMPS]), ${}^b\mathcal{C}^\infty(M) = \{a \log|t| + g, a \in \mathbb{R}\, g \in \mathcal{C}^\infty(M)\}$.

Among b^m-forms, we consider forms of degree two and define the analog of symplectic form in this set-up.

Definition 2.10. Let (M^{2n}, Z) be a b-manifold. Let $\omega \in {}^{b^m}\Omega^2(M)$ be a closed b^m-form. We say that ω is **b^m-symplectic** if ω_p is of maximal rank as an element of $\Lambda^2\left({}^{b^m}T_p^*M\right)$ for all $p \in M$. We call a **b^m-symplectic manifold** a triple (M, Z, ω).

When $m = 1$, these forms are simply called b-symplectic forms.

These forms can be described more precisely in a neighbourhood U of the critical set Z ($U = Z \times (-\epsilon, \epsilon)$) as

$$\omega = \sum_{j=1}^{m} \frac{df}{f^j} \wedge \pi^*(\alpha_j) + \beta \tag{2.3}$$

where the α_j are closed one forms on Z, β is a closed 2-form on U, and $\pi : U \longrightarrow Z$ is the projection. Non-degeneracy of the form ω implies that $\beta|_Z$ is of maximal rank and α_m is nowhere vanishing. The form α_m defines the symplectic foliation of the Poisson structure associated with ω, and β gives the symplectic form on the leaves of this foliation. As shown in [GMP2] and [GMW1], when a leaf of the symplectic foliation is compact, then the critical set is given by the mapping torus

$$Z = \frac{\mathcal{L} \times [0, T]}{(x, 0) \sim (f(x), T)}. \tag{2.4}$$

Given a b^m-function H, the Hamiltonian vector field X_H is defined by the non-degeneracy of the b^m-symplectic form by $\iota_{X_H}\omega = -dH$. We highlight here that the vector field X_H is a b^m-vector field, and therefore when we are interested in the dynamics of this vector field, we view it as a smooth vector field as we did before.

The only local invariant for b^m-symplectic forms is the dimension as the following theorem shows:

Theorem 2.11 (b^m-Darboux Theorem, [GMW1])**.** *Let ω be a b^m-symplectic form on (M^{2n}, Z). Let $p \in Z$. Then we can find a local coordinate chart $(z, t, x_2, y_2, \dots, x_n, y_n)$ centered at p such that hypersurface Z is locally defined by $z = 0$ and*

$$\omega = \frac{dz}{z^m} \wedge dt + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Natural examples of b-symplectic manifolds are given by symplectic manifolds with boundary (see [NT]). Other examples are given by the space of geodesics on the Lorentz plane (as described in [KT] and discussed in [MO2]). Let us recall this construction: Let M be a smooth manifold endowed with a pseudo-Riemannian metric g. Recall that the geodesic flow in T^*M is the Hamiltonian vector field X_H of $H(q,p) = g(p,p)/2$. Then the set of all oriented geodesic lines $\mathcal{L}$ can be decomposed as $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_- \cup \mathcal{L}_0$ where $\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0$ stand for the space of oriented non-parameterized space-, time- and light-like geodesics respectively . The set $\mathcal{L}_0$ is the common boundary of $\mathcal{L}_\pm$. Khesin and Tabachnikov proved in [KT] that the manifolds $\mathcal{L}_\pm$ carry symplectic structures which are described from T^*M by Hamiltonian reduction on the level hypersurfaces $H = \pm 1$. The manifold $\mathcal{L}_0$ carries a contact structure whose symplectization is the Hamiltonian reduction of the symplectic structure in T^*M (without the zero section) on the level hypersurface $H = 0$. In dimension 2 (the Lorenz plane case) this structure is just a b-symplectic surface (as in dimension 1 a contact structure is the same as a cosymplectic one).

The following vector field will play an important role in this paper.

Definition 2.12. Let (M, Z, ω) be a b^m-symplectic manifold. By definition, there exists a natural epimorphism of vector bundles $\varphi : {}^{b^m}TM\big|_Z \to TZ$ induced by the restriction of b^m-vector fields into Z. The kernel of this map, ${}^{b^m}N(M,Z) := \ker(\varphi)$, is a line bundle over Z which is trivializable (see [GMP, Proposition 4]).

We call a b^m-vector field $X \in \Gamma({}^{b^m}N(M,Z)) \subset {}^{b^m}\mathfrak{X}(M,Z)$ trivializing the line bundle a **normal b^m-vector field**.

We will call it a **normal symplectic b^m-vector field** and denote it as X^σ if it is also symplectic with respect to ω, this means, if $\mathcal{L}_{X^\sigma}\omega = 0$.

b^m-Symplectic manifolds can be seen as *open* symplectic manifolds with a certain geometric structure prescribed at the open ends. A similar situation occurs with *convex symplectic manifolds*, which we will discuss briefly in Subsection 2.3, where near the boundary the existence of a transverse Liouville vector field is required. In [FS] the Floer homology for convex symplectic manifolds is defined. From this perspective, b-symplectic manifolds represent the other end of the spectrum: they can be seen as symplectic manifolds with boundary equipped with a symplectic vector field (instead of Liouville) that is transverse to the boundary. The role of the symplectic vector field is played by the normal symplectic b-vector field (as in Definition 2.12).

Example 2.13. Consider a b^m-symplectic manifold (M, Z, ω) and let z be a defining function such that in local coordinates around a point in Z we have the decomposition $\omega = \frac{dz}{z^m} \wedge \alpha + \beta$, $\alpha \in \Omega^1(M)$ and $\beta \in \Omega^2(M)$ as in Theorem 2.11. The normal symplectic b^m-vector field can then be given by $X^\sigma = z^m \frac{\partial}{\partial z}$.

Example 2.14. Consider the b^m-symplectic torus $(\mathbb{T}^2, \{\sin\theta_1 = 0\}, \omega = \frac{d\theta_1}{\sin^m\theta_1} \wedge d\theta_2)$. A normal symplectic b^m-vector field is given globally by $X^\sigma = \sin^m\theta_1 \frac{\partial}{\partial\theta_1}$.

The following lemma shows how the case that we are studying is actually quite general. In particular, we can construct a b-symplectic structure for any symplectic manifold with boundary provided we choose a normal symplectic vector field.

Lemma 2.15 ([FMM]). *Let $(M, \partial M)$ be a manifold with boundary ∂M and let $\omega \in \Omega^2(M)$ a symplectic form on $M \setminus \partial M$ such that there exists a symplectic vector field that points outwards or inwards of the boundary. Then, for each $m \in \mathbb{N}, m \geq 1$ there exists a b^m-symplectic structure on $(M, \partial M)$ with critical set given by ∂M that coincides with the symplectic structure outside of a tubular neighbourhood of the boundary ∂M.*

Proof. We start by showing that the boundary ∂M can be endowed with a cosymplectic structure.

We assume without loss of generality that X^σ points inwards at ∂M. Let $\varphi_t : U \subset M \times \mathbb{R} \to M$ denote the flow of X^σ. As X^σ is transverse to ∂M, there exists a tubular neighbourhood $V_\varepsilon := \{(x, z) \mid 0 \leq z < \varepsilon(x)\} \subset \partial M \times \mathbb{R}$ for some function $\varepsilon : \partial M \to \mathbb{R}$ such that

$$\begin{array}{rccc} c: & V_\varepsilon & \longrightarrow & M \\ & (x,z) & \longmapsto & \varphi_z(x) \end{array}$$

is an embedding.

Let $\theta := c^*(\iota_{X^\sigma}\omega)$ and $\eta := c^*\omega$. Both are well-defined forms in ∂M because they are invariant with respect to z. We can show this for η under the observation that the vector fields $\frac{\partial}{\partial z}$ and X^σ are c-related, so $\mathcal{L}_{\frac{\partial}{\partial z}}\eta = c^*\left(\mathcal{L}_{X^\sigma}\omega\right) = 0$. Conversely,

$$\mathcal{L}_{\frac{\partial}{\partial z}}\theta = c^*\left(\mathcal{L}_{X^\sigma}\iota_{X^\sigma}\omega\right) = c^*\left(d\iota_{X^\sigma}\iota_{X^\sigma}\omega + \iota_{X^\sigma}d\iota_{X^\sigma}\omega\right) = c^*\left(\iota_{X^\sigma}\mathcal{L}_{X^\sigma}\omega\right) = 0,$$

where the first term vanishes because ω is skewsymmetric, and the second does because X^σ is a symplectic vector field.

Moreover, $\theta \wedge \eta^{n-1}$ is a volume form for ∂M, because

$$\theta \wedge \eta^{n-1} = c^*\left(\iota_{X^\sigma}\omega \wedge \omega^{n-1}\right) = \frac{1}{n}c^*\left(\iota_{X^\sigma}\omega^n\right) = \frac{1}{n}\iota_{\frac{d}{dt}}c^*\omega^n,$$

which is a well-defined non-degenerate form due to its z-invariance.

Therefore, $(\partial M, \theta, \eta)$ is a cosymplectic manifold, and V_ε is an embedding into M. Moreover, by our definitions it is clear that

$$c^*\omega = dz \wedge \theta + \eta.$$

Let $\psi : V_\varepsilon \to \mathbb{R}$ a bump function such that

- For $0 \leq z < \frac{\varepsilon(x)}{3}$, $\psi(x, z) = 1$.
- For $z > \frac{2\varepsilon(x)}{3}$, $\psi(x, z) = 0$.

Then the b-form

$$\overline{\omega} = d(\psi \log z + (1 - \psi)z) \wedge \theta + \eta$$

is a b-symplectic form on V_ε. If we push it forward to M, it coincides with ω outside of a tubular neighbourhood of ∂M.

In the same way, for $m > 1$ we can use the b^m-form

$$\overline{\omega} = d\left(-\frac{\psi}{m-1}\frac{1}{z^{m-1}} + (1 - \psi)z\right) \wedge \theta + \eta.$$

□

The tubular neighbourhood and the defining function will always be chosen to satisfy the following lemma.

Proposition 2.16 ([GMW2, Theorem 2]). *There exists a choice of defining function z for the critical set and a projection $\pi : \mathcal{N}(Z) \to Z$ such that*

$$\omega = \sum_{i=1}^{m} \frac{dz}{z^i} \wedge \pi^*\alpha_i + \pi^*\beta$$

where $\alpha_i \in \Omega^1(Z)$ is closed and $\beta \in \Omega^2(Z)$ is symplectic on the foliation defined by α_m.

The notion of modular vector field of a b^m-symplectic manifold, which can be seen as a conjugate of X^σ with respect to ω will play an important role in our constructions. The notion of modular vector field is well-defined for any Poisson manifold and measures how far Hamiltonian vector fields are from preserving a given volume form.

Definition 2.17. Let (M, Z, ω) be b^m-symplectic manifold equipped with a smooth volume form $\Omega \in \Omega^{2n}(M)$. The **modular vector field** in this context is the vector field defined as a derivation by

$$f \mapsto \frac{\mathcal{L}_{X_f}\Omega}{\Omega}.$$

We denote it as v_{mod}. Recall that when one of the leaves of the symplectic foliation is compact, then the critical set is given as in Equation (2.4) by the mapping torus

$$Z = \frac{\mathcal{L} \times [0, T]}{(x, 0) \sim (f(x), T)}.$$

The modular vector field is then given by $v_{mod} = \frac{\partial}{\partial \theta}$, where θ is the translation in the second coordinate (see [GMPS]). We call the period of the modular vector field in the mapping torus the **modular period** or **modular weight** of that component of Z.

Remark 2.18. The modular vector field is a b^m-vector field, and its restriction to Z is transverse to the symplectic foliation. In particular, in a decomposition as in Proposition 2.16 it can[3] be defined by the system of equations

$$\iota_{v_{mod}}\widetilde{\alpha} = 1,\ \iota_{v_{mod}}\beta = 0,$$

where $\widetilde{\alpha} = \sum_{i=1}^{m} z^{m-i}\alpha_i$ in the expression of the Laurent series (Equation 2.3).

$$\omega = \frac{dz}{z^m} \wedge \widetilde{\alpha} + \beta. \tag{2.5}$$

2.3. **Symplectic manifolds with convex boundary.**

Floer theory in the non-compact set-up has already been studied previously in [FS], most notably when there exists a Liouville vector field (instead of a symplectic one as described in the previous section) transverse to the boundary.

Definition 2.19.

(1) A compact symplectic manifold (W, ω) with boundary ∂W is **convex** if there exists a Liouville vector field X near the boundary, this means that $\mathcal{L}_X\omega = \omega$.
(2) A non-compact symplectic manifold is convex if there exists an increasing sequence of compact convex submanifolds $W_i \subset W$ exhausting W.

By convention, we assume that X always points outwards towards the boundary.

This induces a contact 1-form on the boundary, given by $\alpha = \iota_X\omega$ and the Hamiltonian dynamics on the boundary are described by the dynamics of the Reeb vector field R_α on ∂M. Important examples are given by cotangent bundles or Stein manifolds.

Compact convex symplectic manifolds can be viewed as non-compact symplectic manifolds. Indeed, one can attach a so-called *proboscis* to the boundary: using the Liouville vector field, one can glue

$$(P_\varepsilon := \partial W \times (-\varepsilon, \infty), d(e^r\alpha))$$

to the convex symplectic manifold (W, ω). The resulting manifold, denoted by $(\widehat{W}, \widehat{\omega})$ is called the *completion* of (W, ω). The flow of the Liouville vector field generates an $\mathbb{R}^*$-action on the proboscis P_ε. The compatible almost complex structure in the proboscis is chosen to be invariant with respect to this action and on the boundary, the Liouville vector field is sent to the Reeb vector field.

The non-compactness is an issue for the Floer trajectories, as they may run along the proboscis to infinity. A wise choice of the notion of admissibility for the Hamiltonian functions assures that the Floer trajectories remain in the original manifold. Namely, the Hamiltonian function H is asked to satisfy on P_ε that $H = h \circ f : P_\varepsilon \to \mathbb{R}$, where $f(x, r) = e^r$ (here $(x, r) \in \partial M, (-\varepsilon, \infty)$) and $h \in C^\infty(\mathbb{R})$ satisfies that $0 \leq h'(\rho) < \kappa$, where

$$\kappa := \inf\{c > 0 \mid \dot{x}(t) = cR(x(t)) \text{ has a 1-periodic orbit}\}.$$

By the choice of the compatible almost complex structure, it follows that the Hamiltonian equation on ∂M is given by

$$\dot{x}(t) = h'(1)R(x(t)),$$

and thus that there are no 1-periodic solutions on $\partial M \times [0, \infty)$.

The authors in [FS] apply the maximum principle to the function h to assure that the Floer trajectories do not escape to infinity but remain inside M.

This gives rise to a Floer-type homology defined for convex symplectic manifolds. In order to compute this Floer homology, the authors proceed by proving that this homology is isomorphic to a Morse homology adapted to this non-compact situation. Morse theory

[3]Modular vector fields are defined up to addition by a Hamiltonian vector field, see for instance [GMP2].

for non-compact manifolds is generally not well-defined and additional assumptions on the Morse functions under consideration need to be taken. In this particular setting, an admissible Morse function in this setting is a smooth Morse function on $\widehat{M}$ whose restriction to P_ε is given by

$$F(x,r) = e^{-r}, \quad x \in \partial M, r \in [0,\infty).$$

The negative gradient of this Morse function fits well with the dynamics of a Hamiltonian vector field associated with an admissible time-independent Hamiltonian vector field. Indeed, by the choice of the almost complex structure, the negative gradient of an admissible Morse function in P_ε is just the Liouville vector field X and therefore transverse to the boundary.

This homology gives rise to a well-defined Morse-type homology and it is classically known that this does not depend on the choice of F (within the class of admissible Morse functions considered), see [Sch].

The authors prove that a PSS-isomorphism between the above-defined Floer homology and Morse homology holds. Thus, this turns the Floer homology on convex symplectic manifold into a homology that can be computed rather easily.

3. Dynamics around the critical hypersurface and admissible Hamiltonian functions

In this section we focus on the definition of admissible Hamiltonian that gives rise to well-behaved dynamics close to the critical set of the b-symplectic (and b^m-symplectic) manifold. We will present the conditions on our functions incrementally to lay out the motivation underlying them, particularly in relation to the dynamics of the induced Hamiltonian vector fields.

Note that if we consider $M\backslash Z$ as an open symplectic manifold with cosymplectic behaviour at infinity in the sense introduced in Lemma 2.15, these are a subcategory of Hamiltonians on the open manifold with a prescribed behaviour towards infinity. In this, our work here has parallels to the construction in [PW, PVW] and more recently in [K], where the dynamics of the Hamiltonian needs to be subscribed near the singular locus or infinity.

We also provide a detailed definition of the concept of admissible Hamiltonians.

We begin by defining the notion of an almost complex structure compatible with the b^m-symplectic structure.

Definition 3.1. Let (M,Z,ω) be a b^m-symplectic manifold. An almost complex structure on the b^m-tangent bundle is an endomorphism

$$J : {}^{b^m}TM \to {}^{b^m}TM$$

such that $J^2 = -\operatorname{Id}$. As $({}^{b^m}TM,\omega)$ is a symplectic vector bundle, there exist almost complex structures that are compatible with the b^m-symplectic structure, i.e.

$$g_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$$

defines a b^m-Riemannian metric (that is a bilinear, symmetric and non-degenerate 2-form on ${}^{b^m}TM$). We furthermore ask the almost complex structure to be compatible with the cosymplectic structure in the tubular neighbourhood around the critical set. More precisely, the almost complex structure must satisfy the following:

1. The restriction of J to the symplectic leaves is a smooth almost complex structure.
2. It leaves the distribution $\langle X^\sigma, v_{mod}\rangle$ invariant, and in particular

$$J(X^\sigma) = v_{mod}.$$

(3) The almost complex structure commutes with the flow of the normal symplectic b^m-vector field, i.e.

$$d\varphi_t(z,x)J(z,x) = J(\varphi_t(z),x))d\varphi_t(z,x).$$

This definition is very similar to the definition of an almost complex structure compatible with a symplectic structure around a contact-type hypersurface. The only difference here is that, instead of linking the Reeb vector field with the Liouville vector field, we have a relation between the symplectic vector field and the modular vector field. We give here an example that serves as a semi-local model around the critical set.

Example 3.2. In a tubular neighbourhood $\mathcal{N}(Z)$ around the critical set Z, the b^m-symplectic form given by

$$\omega = \sum_{i=1}^{m} \frac{dz}{z^i} \wedge \alpha_i + \beta$$

where $\alpha_i \in \Omega^1(Z)$ and $\beta \in \Omega^2(Z)$ are closed forms as in Proposition 2.16. The modular vector field is defined by the equations $\iota_{v_{mod}}\alpha_m = 1$ and $\iota_{v_{mod}}\beta = 0$ and we consider its extension to the tubular neighbourhood. At each point around the tubular neighbourhood, we have the splitting

$$^{b^m}TM = \langle X^\sigma \rangle \oplus \langle v_{mod} \rangle \oplus \mathcal{F}.$$

We define the almost complex structure by the following:

- $J(X^\sigma) = v_{mod}$
- $J|\mathcal{F} = J_{\mathcal{F}}$, where $J_{\mathcal{F}}$ is an almost complex structure on the foliation $\mathcal{F}$, compatible with the induced symplectic structure by β.

The b^m-metric $g_J(\cdot,\cdot)$ is given in this decomposition by

$$g_J = \left(\frac{dz}{z^m}\right)^2 + \alpha \otimes \alpha + g_{\mathcal{F}}, \tag{3.1}$$

where $g_{\mathcal{F}}$ is the leafwise Riemannian metric on the foliation F.

Riemannian metrics on ^bTM as in Equation (3.1) have appeared previously in [MOP, Definition 2.12] under the name of asymptotically exact b-metric.

Definition 3.3. A metric on $^{b^m}TM$ is called *asymptotically exact* if there exists a trivialization around the critical set Z

$$(z,P) : \mathcal{N}(Z) \to (-\epsilon,\epsilon) \times Z$$

such that $g = \left(\frac{dz}{z^m}\right)^2 + P^*h$, where h is a smooth metric on Z.

This example in fact serves as a local model around the critical set.

Lemma 3.4 ([BLS], Lemma 2.6). *Let (M,Z,ω) be a compact oriented b-symplectic manifold, with defining function and tubular neighbourhood $\mathcal{N}(Z)$ chosen as in Proposition 2.16.*

There exists a direct sum decomposition $^bTM|_{\mathcal{N}(Z)} = {}^bT\mathcal{N}(Z) = \pi^*E \oplus \pi^*\mathcal{F}$, where $\mathcal{F} = \ker(\alpha_Z)$ and $E = \langle X^\sigma, v_{mod}\rangle$ are subbundles of $^bTM|_Z$. v_{mod} is uniquely determined by $\alpha_Z(v_{mod}) = 1$, $\iota^*\beta(v_{mod}) = 0$, and X^σ is defined as in Definition 2.12.

There also exists a compatible complex structure J on the symplectic vector bundle $(^bTM,\omega)$ such that the metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ takes a product form near Z:

$$g|_{\mathcal{N}(Z)} = \left(\frac{dz}{z}\right)^2 + \alpha \otimes \alpha + \pi^* g_{\mathcal{F}}$$

where $g_{\mathcal{F}}$ is a compatible metric on the symplectic vector subbundle $(\mathcal{F},\beta_{\mathcal{F}}) \subset (^bTM|_Z, \omega|_Z)$.

The lemma can be generalized without any technical difficulties to b^m-symplectic manifolds and the proof will therefore be omitted. From this lemma, it readily follows that the associated b^m-metric to a compatible almost complex structure is an exact b^m-metric.

Lemma 3.5. *The space of compatible almost complex structures is non-empty and contractible.*

Proof. We start by defining a compatible almost complex structure in a tubular neighbourhood around the critical set as in Example 3.2 and we then apply Exercise 2.2.19 in [Wen]. □

Let (M, Z) a b-manifold, and take $z : M \to \mathbb{R}$ a defining function. We will denote the set of time-dependent b^m-Hamiltonians (respectiely b-Hamiltonians) as

$$
\begin{aligned}
{}^b\mathcal{C}^\infty(S^1 \times M) &:= \{k(t)\log|z| + h_t \mid k \in \mathcal{C}^\infty(S^1), h \in \mathcal{C}^\infty(S^1 \times M)\},\\
{}^{b^m}\mathcal{C}^\infty(S^1 \times M) &:= \left(\bigoplus_{i=1}^{m-1} z^{-i}\mathcal{C}^\infty(S^1 \times \mathbb{R})\right) \oplus {}^b\mathcal{C}^\infty(S^1 \times M),
\end{aligned}
$$

where $\mathcal{C}^\infty(S^1 \times \mathbb{R})$ refers to the set of functions taking values in time t (the S^1-component) and the defining function z (the $\mathbb{R}$-component).

Definition 3.6. Let (M, Z, ω) be a b^m-symplectic manifold and let X^σ the normal symplectic b^m-vector field. We say that a (time-dependent) Hamiltonian $H_t \in {}^{b^m}\mathcal{C}^\infty(S^1 \times M)$ is **linear along** X^σ if $\mathcal{L}_{X^\sigma} H = k(t)$, where $k \in \mathcal{C}^\infty(S^1)$ is a time-dependent function.

Remark 3.7. If $k \equiv 0$, this means that the Hamiltonian is a first integral of X^σ.

Example 3.8. Consider the b-symplectic manifold $(\mathbb{T}^2, \{\sin\theta_1 = 0\}, \omega = \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2)$ and the Hamiltonian given by

$$H = k(t)\log\left|\frac{\sin\theta_1}{1+\cos\theta_1}\right| + \sin\theta_2,$$

where $k \in \mathcal{C}^\infty(S^1)$ is a time-dependent function. In this case, we have that $\sin\theta_1 \frac{\partial}{\partial\theta_1}$ is the normal symplectic b-vector field, as we saw in Example 2.14. This Hamiltonian satisfies then that $\mathcal{L}_{X^\sigma} H = k(t)$. The Hamiltonian vector field is then given by $X_H = k(t)\frac{\partial}{\partial\theta_2} + \cos\theta_2 \sin\theta_1 \frac{\partial}{\partial\theta_1}$.

Considering the particular case in which k is a constant, if $k \notin \mathbb{Z}$ then the time 1-flow of X_H does not have any periodic orbits around the critical set.

We note that, in dimension 2, the Hamiltonian vector fields associated with b-Hamiltonian functions that are constant along X^σ will always have infinitely many periodic orbits around the critical set:

Proposition 3.9. *Let (Σ, Z, ω) be a compact b-symplectic surface with X^σ a normal symplectic vector field. Let H be a b-Hamiltonian function such that $\mathcal{L}_{X^\sigma} H = k \in \mathbb{R}$ is constant and different from zero. Then, for any $\varepsilon > 0$ small enough there exist periodic orbits of period $\frac{a}{k}$ in the tubular neighbourhood $\mathcal{N}_\varepsilon(Z)$ around the critical set with modular weight a. Furthermore, if $k \in a\mathbb{Z}$, then there exists 1-periodic Hamiltonian orbit in $\mathcal{N}_\epsilon(Z)$.*

Proof. By compactness, Z must be a finite union of circles. For the purposes of this proof we can assume that $Z = S^1 \cong [0,1]\big/(0 \sim 1)$ and a tubular neighbourhood around Z is diffeomorphic to $S^1 \times (-\epsilon, \epsilon)$.

In this tubular neighbourhood around Z, the b-symplectic form is given by

$$\omega = a\frac{dz}{z} \wedge d\theta.$$

In these coordinates the normal symplectic b-vector field is $X^\sigma = z\frac{\partial}{\partial z}$. If H_t satisfies the conditions of the proposition, then $H_t(z,\theta) = k\log|z| + h_t(\theta)$. Hence the Hamiltonian vector field is given by

$$aX_{H_t} = k\frac{\partial}{\partial\theta} - z\frac{\partial h_t}{\partial\theta}\frac{\partial}{\partial z}.$$

The flow of this vector field X_{H_t} is given by

$$\theta(t) = \theta_0 + \tfrac{1}{a}kt$$
$$z(t) = z_0\exp\left(-\int_0^t \frac{1}{a}\frac{\partial h_s}{\partial\theta}(\theta_0 + ks)ds\right)$$

where (z_0,θ_0) is the initial position. Notice that, as the image of θ lies in S^1, $\theta(t)$ is merely a periodic flow, and in particular $\theta(t+\frac{a}{k}) = \theta(t)$.

The z-component of the flow has a 1-periodic orbit if and only if there is some $\theta \in S^1$ such that

$$F(\theta) := \int_0^{\frac{a}{k}} \frac{\partial h_t}{\partial\theta}(\theta + kt)dt = 0.$$

Indeed, integrating the function F over S^1 we obtain by applying Fubini's theorem that

$$\int_{S^1} F(\theta)d\theta = \int_{S^1}\int_0^{\frac{a}{k}} \frac{\partial h_t}{\partial\theta}(\theta+kt)dtd\theta = \int_0^{\frac{a}{k}}\int_{S^1} \frac{\partial h_t}{\partial\theta}(\theta+kt)d\theta dt = \int_0^{\frac{a}{k}} [h_t(\theta+kt)]_{\theta=0}^{\theta=1}dt = 0,$$

and therefore $F(\theta) = 0$ for some θ. Then, for any z_0 small enough there exists some θ_0 such that (z_0,θ_0) belongs to a periodic orbit of period $\frac{1}{k}$. In particular, if $k \in \mathbb{Z}$, then there exists orbits of period $\frac{1}{k}$ in $\mathcal{N}_\epsilon(Z)$, and therefore also 1-periodic orbits. □

A similar statement holds for b^m-symplectic surfaces.

Proposition 3.10. *Let (Σ, Z, ω) be a compact b^m-symplectic surface with X^σ a normal symplectic vector field. Let H be a b^m-Hamiltonian function such that $\mathcal{L}_{X^\sigma}H = k \in \mathbb{R}$ is constant (different from zero). Then, for any $\varepsilon > 0$ there exist periodic orbits around the critical set.*

In contrast to the last proposition, in the b^m-symplectic case, we don't have a family of orbits with the same period. The proof will be as in Proposition 3.9, and we only repeat here the most important steps.

Proof. In the tubular neighbourhood, the b^m-symplectic form is given as the following Laurent series:

$$\omega = \left(\sum_{i=1}^m z^{m-i}a_i\right)\frac{dz}{z^m}\wedge d\theta.$$

By the condition on the Hamiltonian, it is given by $H_t(z,\theta) = -\frac{1}{m-1}\frac{k}{z^{m-1}} + h_t(\theta)$ if $m > 1$, where k is constant. The Hamiltonian vector field is therefore given by

$$X_{H_t} = \frac{1}{\sum_{i=1}^m z^{m-i}a_i}\left(k\frac{\partial}{\partial\theta} - z^m\frac{\partial h_t}{\partial\theta}\frac{\partial}{\partial z}\right).$$

As in a tubular neighbourhood, the function $\frac{1}{\sum_{i=1}^m z^{m-i}a_i}$ is non-vanishing, we consider the parametrization of the vector field X_H given by

$$\widetilde{X}_{H_t} = k\frac{\partial}{\partial\theta} - z^m\frac{\partial h_t}{\partial\theta}\frac{\partial}{\partial z}.$$

As this is a reparametrization, both vector field are orbitally equivalent and thus periodic orbits of $\widetilde{X}_{H_t}$ correspond to periodic orbits of X_{H_t}.

We now analyse the flow of the vector field $\widetilde{X}_{H_t}$ and show that this vector field has always infinitely many periodic orbits around the critical set.

$$\begin{cases} \theta(t) = \theta_0 + kt \\ z(t) = \left(\frac{1}{z_0^{m-1}} - (m-1) \int_0^t \frac{\partial h_t}{\partial \theta}(\theta_0 + ks) ds \right)^{-\frac{1}{m-1}} \end{cases}$$

The flow in the θ coordinate is periodic of period $\frac{1}{k}$ as in the proof of Proposition 3.9 and the same arguments apply as therein apply. This implies that for any z_0, there exists θ_0 such that the integral curve of the reparametrization $\widetilde{X}_{H_t}$ with initial condition (z_0, θ_0) is periodic. The same thus holds for the Hamiltonian vector field X_{H_t}. However, notice that due to the reparametrization, the periods of the periodic orbits are not constant. □

Remark 3.11. On Z, the flow described in Proposition 3.10 is as the flow in Proposition 3.9. Hence on Z, there is a periodic orbit of period $\frac{a_m}{k}$ on Z, where a_m is the modular weight.

This result is also true in higher dimensions if H does not depend on time.

Lemma 3.12. *Assume H is a time-independent Hamiltonian that is a first integral of X^σ. Then there is a 1-parametric family of critical points approaching the critical set.*

Proof. As H is a first integral of X^σ, H can be viewed as a function on $z^{-1}(\varepsilon)$, where $Z = z^{-1}(0)$. As Z is compact, there exist critical points of H on each hypersurface. The critical points translate to trivial periodic orbits which appear as a 1-parametric family. □

The same result holds also in higher dimensions if $\mathcal{L}_{X^\sigma} H = 0$ and the geometry of Z is that of a symplectic mapping torus.

Proposition 3.13. *Let (M, Z, ω) be a b-symplectic manifold such that Z is compact and Z is a trivial mapping torus. Then all X^σ-invariant Hamiltonian functions have periodic orbits arbitrarily close to Z.*

Remark 3.14. A sufficient condition so that (W, Z, ω) has a trivial mapping torus at Z is that the cohomology class $[\omega] \in {}^bH^2(M)$ is integral. See [GMW3, Section 2] for more details.

Proof. Around the critical set we have $\omega = \frac{dz}{z} \wedge \alpha + \beta$ as in Proposition 2.16. The tubular neighbourhood admits a codimension 2 symplectic foliation as in Proposition 3.4. Note the codimension 2 foliation around the critical set by $\mathcal{F}$. We want to solve Hamilton's equation $\iota_{X_H}\omega = -dH$. The Hamiltonian vector field can be computed using the b^m-Poisson structure, which is given by

$$\Pi = z\frac{\partial}{\partial z} \wedge v_{mod} + \pi_{\mathcal{L}}$$

and thus the Hamiltonian vector field is given by $\Pi(dH)$.

As H does not depend on z, the Hamiltonian vector field has the expression $X_H = (X_H)_L + \frac{\partial H}{\partial \theta} z \frac{\partial}{\partial z}$ where $\frac{\partial H}{\partial \theta}$ is a function which does not depend on z and $(X_H)_L$ is a Hamiltonian vector field along the leaf. On one hand, by the Arnold conjecture applied to a symplectic leaf $\mathcal{L}$, there always exists a 1-periodic orbit of $(X_H)_L$ in $\mathcal{L}$. On the other hand, we can apply Proposition 3.9 to the term $\frac{\partial H}{\partial \theta} z \frac{\partial}{\partial z}$ and thus we always find a 1-periodic orbits on the normal direction N. Hence there are periodic orbits for X_H. Furthermore, they come in 1-parametric families.

□

In contrast to the last propositions, we define *admissible Hamiltonian functions* to be linear along X^σ and without periodic orbits close to the critical hypersurface. Remember that when the foliation induced on the critical set has compact orbits, the symplectic foliation is a mapping torus. Thus, by means of the classical Arnold conjecture, a smooth Hamiltonian has an infinite number of periodic orbits. The condition of admissible Hamiltonian is on the other side of the spectrum and thus underlying the importance that the associated function is not smooth. In later sections, we will prove that this condition avoids that finite energy solutions to the Floer equation approach the critical hypersurface.

Definition 3.15. A Hamiltonian $H_t \in {}^{b^m}\mathcal{C}^\infty(M)$ is **admissible** if

(1) It is linear along X^σ: $\mathcal{L}_{X^\sigma} H_t = k(t)$.
(2) It is invariant with respect to the modular vector field: $\mathcal{L}_{v_{mod}} H_t = 0$.
(3) There are no 1-periodic orbits of X_H in a collar neighbourhood around Z.

We denote the set of admissible Hamiltonian functions by ${}^{b^m}\mathrm{Adm}(M, v_{mod}, X^\sigma)$.

Remark 3.16. Definition 3.15 depends on the choice of modular vector field and normal symplectic vector field. However, the lower bounds concerning the number of periodic Hamiltonian orbits in this paper are independent on these choices.

Remark 3.17. In the surface case, the local expression of an admissible Hamiltonian function is $H_t = k(t)\log|z|$ for $m = 1$ and $H_t = -k(t)\frac{1}{m-1}\frac{1}{z^{m-1}}$ for $m > 1$.

In higher dimensions there is an additional term, represented by a local function h_t such that $\mathcal{L}_{X^\sigma} h_t = 0$ and $\mathcal{L}_{v_{mod}} h_t = 0$, which means that h_t depends only on the coordinates on the symplectic leaves. The local expression of an admissible Hamiltonian functions is then $H_t = k(t)\log|z| + h_t$ for $m = 1$ and $H_t = -k(t)\frac{1}{m-1}\frac{1}{z^{m-1}} + h_t$ for $m > 1$.

Example 3.18. Consider $(\mathbb{T}^2, \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2)$ and let $H \in {}^bC^\infty(\mathbb{T}^2)$ be a Hamiltonian such that $\mathcal{L}_{X^\sigma} H = k(t)$ and $\mathcal{L}_{v_{\mathrm{mod}}} H = 0$. Under those conditions, the Hamiltonian is given by $H_t = k(t)\log|\tan(\frac{\theta_1}{2})|$ and thus the Hamiltonian vector field is $X_{H_t} = -k(t)\frac{\partial}{\partial\theta_2} = -k(t)v_{mod}$. This has 1-periodic orbits if and only if

$$\int_0^1 k(\tau)d\tau \in T\mathbb{Z}.$$

where T is the modular weight.

Thus the condition that there do not exist any periodic orbits in a collar neighbourhood around Z is necessary.

More generally, b-Hamiltonians that are linear along X^σ and are invariant with respect to the modular vector field do not admit any periodic orbits in a collar neighbourhood around Z if the following is satisfied:

Proposition 3.19. *Let $H_t \in {}^bC^\infty(M)$ such that $\mathcal{L}_{X^\sigma} H_t = k(t)$ and $\mathcal{L}_{v_{mod}} H_t = 0$. If*

$$\int_0^1 k(t)dt \in (0, T),$$

where T is the modular weight of Z, then there are no periodic orbits of X_H in a collar neighbourhood around Z.

Proof. In a neighbourhood of a point in Z, the b-Poisson structure can be described as $\Pi = X^\sigma \wedge v_{mod} + \pi_{\mathcal{L}}$. Under the assumptions of the proposition, the Hamiltonian vector field is given by $X_H = \Pi^\sharp(dH) = -k(t)v_{\mathrm{mod}} + (X_H)^{\mathcal{F}}$. The critical set is given by the mapping torus

$$Z = \frac{\mathcal{L} \times [0, T]}{(x, 0) \sim (f(x), T)}$$

and the modular vector field is $v_{mod} = \frac{\partial}{\partial\theta}$, where θ is the translation in the second coordinate (see [GMPS]). Observe that the description of the mapping torus holds in a neighbourhood as the mapping torus structure is transported by the normal symplectic b-vector field that we take. This means that the condition $0 < \int_0^1 k(t)dt < T$ suffices to guarantee that there are no periodic orbits in a tubular neighbourhood around the critical set, as the Hamiltonian vector field is not returning the same leaf of the symplectic foliation. □

Remark 3.20. In view of Proposition 3.10, in an ϵ-tubular neighbourhood around the critical set Z of a b^m-symplectic manifold, there exists an upper bound T_ϵ close to T such that if $\int_0^1 k(t)dt \in (0, T_\epsilon)$, then there are no periodic orbits of X_H in a ϵ-neighbourhood.

Example 3.21. Let us consider the $2n$-torus $\mathbb{T}^{2n}$ with coordinates $(\theta_1, ..., \theta_{2n})$ and with a b-symplectic form such that $Z = \{\sin\theta_1 = 0\}$ and such that globally we have

$$\omega = \frac{d\theta_1}{\sin\theta_1} \wedge \alpha + \beta,$$

for some closed $\alpha \in \Omega^1(\mathbb{T}^{2n})$ and $\beta \in \Omega^2(\mathbb{T}^{2n})$.

Let us then take $H_t = k\log\left|\frac{\sin\theta_1}{1+\cos\theta_1}\right|$ for k constant, so $X_H = kv_{mod}$. As we saw in Proposition 3.19 we can choose k small enough so that X_H does not have 1-periodic orbits. Thus, there exists a family of admissible Hamiltonians in $\mathbb{T}^{2n}$ with no 1-periodic orbits.

Remark 3.22. Consider the product of b^m-symplectic manifolds. This is no longer a b^m-symplectic manifold but admits a symplectic structure over a Lie algebroid. Those manifolds are called c-symplectic manifolds in [MS] and are a particular case of E-symplectic manifolds (see also, [NT2], [MMN]). More precisely, let (M_i, Z_i, ω_i) be a collection of b-symplectic manifolds, and consider $(M := \Pi_{i=1}^n M_i, \overline{Z}, \omega = \sum_i \pi_i^*\omega_i)$, where $\pi_i : M \to M_i$ denotes the projection and

$$\overline{Z} = \bigcup_{i=1}^n M_1 \times \cdots \times Z_i \times \cdots \times M_n.$$

The vector field $X^\sigma = \sum_{i=1}^n X_i^\sigma$ is a symplectic vector field on $M \setminus \overline{Z}$, which is transverse to the boundary. The same constructions as before thus work in this more general set-up.

As for Floer homology, we need to consider non-degenerate periodic orbits.

Definition 3.23. Consider a Hamiltonian $H_t \in {}^{b^m}\mathrm{Adm}(M,\omega)$. We denote by $\mathcal{P}(H)$ the set of 1-periodic orbits of the flow of X_H that are completely contained in $M \setminus Z$. We say that H is **regular** if for all periodic orbits $x \in \mathcal{P}(H)$ we have that

$$\det(\mathrm{Id} - d\phi^1_{X_H}(x(0))) \neq 0.$$

We denote the set of regular admissible Hamiltonians by ${}^{b^m}\mathrm{Reg}(M,\omega)$.

The set of regular Hamiltonians is open and dense in ${}^{b^m}\mathrm{Adm}(M,\omega)$ in the strong Whitney $\mathcal{C}^\infty$-topology.

4. The Arnold conjecture through desingularization

This section contains a proof of the Arnold conjecture for some b^m-symplectic manifolds. More precisely, we give a lower bound for the number of periodic orbits of an admissible Hamiltonian in terms of the topology of M and the relative position of Z in M. In the following sections, we will distinguish between the b^{2m}- and b^{2m+1}-symplectic case. First, we give a proof for b^{2m}-symplectic manifolds of any dimension. We then prove a lower bound on the number of periodic orbits for b^m-symplectic surfaces (Σ, Z, ω) for general m.

The arguments are based on the desingularization as proved in [GMW3]. The desingularization process introduced in [GMW3] associates a family of symplectic structures to a b^m-symplectic structure for even m. In this article we use it to relate the initial dynamics to the *desingularized dynamics*.

It will be useful to associate a graph to a b-manifold. We will be able to read some dynamical features concerning the dynamics from this graph.

Definition 4.1. The *associated graph* to a (M, Z) a b-manifold is the weighted graph whose vertices are the connected components of $M \setminus Z$ (denoted by M_i). Two vertices M_i and M_j are connected if there is a connected component of Z adjacent to both M_i and M_j. The weight associated to the vertex M_i is denoted by g_{M_i} and it is the genus of the component M_i seen as a surface (assuming that each open part is caped off with a disk).

We say that (M, Z) is *cyclic* or *acyclic* if its associated graph contains (or, respectively, does not contain) a cycle.

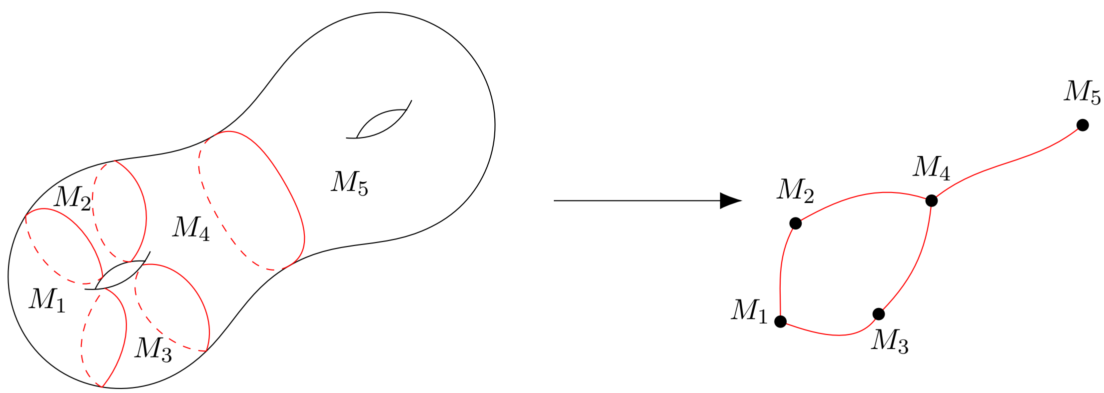

FIGURE 1. An example of the graph of a b^m-manifold (cyclic)

4.1. **The Arnold conjecture via desingularization.** In this subsection we prove an Arnold conjecture for b^{2m}-symplectic manifolds in any dimension. We do this using the desingularization procedure in [GMW1].

We recall it here for the sake of completeness:

Theorem 4.2 ([GMW1, Theorem 3.1])**.** *Let (M, Z, ω) be a b^{2m}-symplectic manifold. Then there exists a family of* ***symplectic*** *forms ω_ε which coincide with the b^{2m}-symplectic form ω outside an ε-neighborhood of Z and for which the family of bivector fields $(\omega_\varepsilon)^{-1}$ converges in the C^{2m-1}-topology to the Poisson structure ω^{-1} as $\varepsilon \to 0$.*

An immediate consequence of this theorem is that a b^{2m}-symplectic manifold is also symplectic. So, in particular, all symplectic obstructions apply for b^{2m}-symplectic manifolds. We will sketch the proof of Theorem 4.2 for the even case following [GMW1].

A Laurent series can be associated to any closed b^m-form in a tubular neighbourhood $\mathcal{N}(Z)$ of Z (see [Sco]):

$$\omega = \frac{dz}{z^m} \wedge \left(\sum_{i=0}^{m-1} \pi^*(\alpha_i) z^i \right) + \beta, \tag{4.1}$$

where $\pi : \mathcal{N}(Z) \to Z$ is the projection of the tubular neighborhood onto Z, α_i is a closed smooth de Rham 1-form on Z, and β is a de Rham 2-form on M and z is a defining function for Z.

Because of the formula 4.1, the b^{2m}-form can be written as

$$\omega = \frac{dz}{z^{2m}} \wedge \sum_{i=0}^{2m-1} \left(z^i \alpha_i\right) + \beta \tag{4.2}$$

on a tubular ε-neighbourhood of a given connected component of Z.

Definition 4.3. Let (M, ω), be a b^{2m}-symplectic manifold with critical set Z. Consider the decomposition given by the expression 4.2 on an ε-tubular neighborhood $\mathcal{N}_\varepsilon(Z)$ of a connected component of Z.

Let $f \in \mathcal{C}^\infty(\mathbb{R})$ be an odd smooth function satisfying $f'(x) > 0$ for all $x \in [-1, 1]$ and satisfying outside this interval that

$$f(x) = \begin{cases} \frac{-1}{(2m-1)x^{2m-1}} - 2 & \text{for } x < -1 \\ \frac{-1}{(2m-1)x^{2m-1}} + 2 & \text{for } x > 1. \end{cases}$$

We define f_ε as $\varepsilon^{-(2m-1)} f(x/\varepsilon)$.

The f_ε-**desingularization** ω_ε is a form that is defined on a neighbourhood $\mathcal{N}_\varepsilon(Z)$ as

$$\omega_\varepsilon = df_\varepsilon \wedge \left(\sum_{i=1}^{2m-1} z^i \alpha_i \right) + \beta. \tag{4.3}$$

Observe that ω_ε can be trivially extended to the whole manifold so that it coincides with ω outside $\mathcal{N}_\varepsilon(Z)$. Hence ω_ϵ is a smooth differential form on M.

We can now prove

Proposition 4.4. *Let (M, Z, ω) be a compact b^{2m}-symplectic manifold and let $H_t \in {}^{b^{2m}}\mathcal{C}^\infty(M)$ be an admissible Hamiltonian.*

(1) If the b^{2m}-symplectic manifold is acyclic, there exist a symplectic structure $\widetilde{\omega}$ and a smooth Hamiltonian $\widetilde{H}_t$ on M such that $X^{\widetilde{\omega}}_{\widetilde{H}}$ coincides with X^ω_H.

(2) If the b^{2m}-symplectic manifold is cyclic, there exists a symplectic structure $\widetilde{\omega}$ on M with respect to which X^ω_H is symplectic.

Before proving Proposition 4.4 we present two associated corollaries.

Corollary 4.5 (Arnold conjecture for acyclic b^{2m}-symplectic manifolds)**.** *Let (M, Z, ω) be a compact acyclic b^{2m}-symplectic manifold. Assume that $H_t \in {}^{b^{2m}}C^\infty(M)$ is a time-dependent admissible Hamiltonian function. Suppose that the solutions of period 1 of the associated Hamiltonian system are non-degenerate. Then*

$$\#\mathcal{P}(H) \geq \sum_i \dim H_i(M; \mathbb{Z}_2).$$

Corollary 4.6 (Arnold conjecture for cyclic b^{2m}-symplectic manifolds)**.** *Let (M, Z, ω) be a closed b^{2m}-symplectic manifold of dimension $2n$, which satisfies that*

$$c_1|_{\pi_2(M)} = \lambda\omega|_{\pi_2(M)}, \lambda \neq 0$$

and if $\lambda < 0$, the minimal Chern number N satisfies $N > n-3$. Suppose ϕ is a b^{2m}-symplectomorphism on (M, Z, ω) which is isotopic to the identity through b^{2m}-symplectomorphisms. If all the fixed points of ϕ are non-degenerate, then the number of fixed points of ϕ is at least the sum of the Betti numbers of the Novikov homology over $\mathbb{Z}_2$ associated to the Calabi invariant of ϕ.

This follows from the main theorem of [VO]. Note that this lower bound is zero in the case of $\mathbb{T}^2$, and therefore does not contradict Example 3.21.

We will now proceed to the proof of Proposition 4.4.

Proof of Proposition 4.4. As already explained before, the proof desingularizes the b^{2m}-symplectic form. In the case where the b-manifold is acyclic, the admissible Hamiltonian can be desingularized to a smooth Hamiltonian function, and its associated Hamiltonian agrees with the initial b^{2m}-Hamiltonian. In the cyclic case, the admissible Hamiltonian can only be *locally* desingularized and therefore gives rise to symplectic dynamics. The desingularized dynamics still agrees with the initial one, and we will therefore conclude that in the cyclic case the associated dynamics is symplectic.

The desingularization of the b^{2m}-symplectic form as in Theorem 4.2 is given in a tubular neighbourhood $\mathcal{N}(Z)$ by

$$\omega_\varepsilon = df_\varepsilon \wedge \left(\sum_{i=0}^{2m-1} z^i \alpha_i\right) + \beta.$$

Away from this neighbourhood, this form agrees with the initial one.

By the assumptions of admissibility, the Hamiltonian is given in a tubular neighbourhood around Z (as in Remark 3.17) by

$$H = -k(t)\frac{1}{2m-1}\frac{1}{z^{2m-1}} + h_t.$$

We define in the tubular neighbourhood $\mathcal{N}(Z)$ the smooth Hamiltonian $\widehat{H}_\varepsilon = -k(t)f_\varepsilon + h_t$. A direction computation yields that the b^{2m}-Hamiltonian vector field agrees in this neighbourhood with the Hamiltonian vector field of $\widehat{H}_\epsilon$ with respect to ω_ϵ, i.e. $X_H^\omega = X_{\widehat{H}_\epsilon}^{\omega_\epsilon}$. As this is only a local construction, we conclude that the initial dynamics are given by a symplectic vector field.

We will now prove that the above construction can be done globally when the manifold (M, Z) is acyclic. First, we introduce some notation.

We denote by M^+, respectively M^-, the set of connected components where the globally defined defining function z is positive, respectively negative, i.e.

$$M^\pm = \{x \in M | \pm z > 0\}.$$

We start to define a smooth function $\widetilde{H}_\varepsilon$ on a connected component of $M^+ \setminus N_\varepsilon(Z)$ and we denote the corresponding vertex in the associated graph by $v_{initial}$. On this connected component, we set $\widetilde{H}_\varepsilon = H$. We extend the definition of the function $\widetilde{H}_\varepsilon$ by defining it on the connected components of $M \setminus Z$ corresponding to the adjacent vertices of $v_{initial}$. We then reiterate this step to continue to extend $\widetilde{H}_\varepsilon$ to the connected components of $M \setminus Z$ corresponding to the adjacent vertices. The condition on the associated graph being acyclic implies that the graph does not close up and therefore this gives rise to a globally defined function $\widetilde{H}_\varepsilon$. In order to simplify the wording, we use interchangeably the connected components of $M \setminus Z$ (respectively connected components of Z) and the vertices (respectively edges) of the associated graph V. For instance, when we write $v_{initial}$, we mean the connected component of $M \setminus Z$ corresponding to $v_{initial}$ and vice versa.

The extension of $\widetilde{H}_\varepsilon$ goes as follows. Having already defined $\widetilde{H}_\varepsilon$ on $v_{initial}$, we then define $\widetilde{H}_\varepsilon$ on the edges connected to $v_{initial}$.

In each edge e_i connected to v, we define the Hamiltonian by $\widetilde{H}_\varepsilon = k(t)f_\varepsilon + h_t$. On adjacent vertices v_i, we define it by $\widetilde{H}_\varepsilon = H + k(t)C(\varepsilon, m)$, where the constant $C(\varepsilon, m) = f_\varepsilon(\varepsilon)$ is such that the function $\widetilde{H}_\varepsilon$ glues to a smooth a function. To simplify notation, we simply denote it by C.

As the function f_ε satisfies $f_\varepsilon' > 0$, the function $\widetilde{H}_\varepsilon$ defines a smooth function on the domain of M corresponding to $v_{initial}$, e_i and v_i. Furthermore, $\widetilde{H}_\varepsilon$ does not admit any critical points on any tubular neighbourhood $N_\varepsilon(Z)$ corresponding to the edges e_i.

We iterate this construction step by step over all the edges connected and vertices neighbouring the vertices of the domain of definition until the domain of definition of the function $\widetilde{H}_\varepsilon$ is M. As the graph is acyclic, the function $\widetilde{H}_\varepsilon$ is a well-defined smooth function on M that does not have any critical points on $\mathcal{N}_\varepsilon(Z)$.

The reader can check that the Hamiltonian vector field of $\widetilde{H}_\varepsilon$ associated to $(\Sigma, \widetilde{\omega}_\varepsilon)$ equals to X_H, i.e. $X_H^\omega = X_{\widetilde{H}_\varepsilon}^{\widetilde{\omega}_\varepsilon}$. This finishes the proof of Proposition 4.4.

□

Note that the above result only holds for b^{2m}-symplectic manifolds. In the case of b^{2m+1}-symplectic manifolds, the singular form cannot be desingularized for obvious topological reasons. The desingularization theorem (Theorem 4.2) in that case yields the desingularization to a so-called *folded* symplectic form. In the next section, we will see that in the case of surfaces, the desingularization can still be adapted.

4.2. **The Arnold conjecture for b^m-symplectic surfaces.** In this section we prove an Arnold conjecture for b^m-symplectic surfaces. The classification and study of b^m-symplectic surfaces were accomplished in [MP]. As noticed higher up, the desingularization theorem (Theorem 4.2) does not hold for b^{2m+1}-symplectic manifolds.

In order to address the Arnold conjecture for general b^m-symplectic surfaces we start analyzing the dynamics of admissible Hamiltonians on the 2-disk where the critical curve is given by the boundary.

Remark 4.7. Given the 2-disk D^2 with critical curve $Z = \partial D^2$, the b-symplectic structure writes down in a neighbourhood around Z as Laurent series given by

$$\omega = \sum_{i=1}^{m} a_i \frac{dr}{(1-r)^i} \wedge d\theta,$$

where $a_i \in \mathcal{C}^\infty(S^1 \times D^2)$ are smooth functions and (r, θ) are polar coordinates. Given an admissible Hamiltonian, it is given in the tubular neighbourhood by

$$H(t, r, \theta) = \begin{cases} k(t) \log|1-r| & \text{if } m = 1 \\ -\frac{k(t)}{(m-1)(1-r)^{m-1}} & \text{if } m > 1 \end{cases}$$

for some $k : S^1 \to \mathbb{R}$ such that $k(t) > 0$ and $\int_0^1 k(t)dt < \frac{2\pi}{a_m}$. Here $\frac{2\pi}{a_m}$ is the modular weight.

Lemma 4.8. *Let $(D^2, Z = \partial D^2, \omega)$ be a b^m-symplectic disk and H_t an admissible Hamiltonian. Then the time-1 map of the flow of its Hamiltonian vector field $\varphi^1_{X_H} : D^2 \to D^2$ has at least one fixed point.*

Moreover, let $\mathcal{N}_\varepsilon(Z) = \{(r, \theta) \in D^2 \mid r > 1 - \varepsilon\}$ be an annulus around the boundary of the disk, and let ω a b^m-symplectic form on $\mathcal{N}_\varepsilon(Z)$ and H_t an admissible Hamiltonian on $\mathcal{N}_\varepsilon(Z)$. Then there exist extensions of ω and H_t to the whole disk such that the flow of X_H has exactly one fixed point.

Proof. The Hamiltonian vector field X_H is a vector field tangent to ∂D^2 so its flow is well defined. Applying Brouwer's theorem we conclude that the flow $\varphi^1_{X_H} : \overline{D^2} \to \overline{D^2}$ must have at least one fixed point. Moreover, semilocally around Z the vector field has the form $X_H = -a_m k(t) \frac{\partial}{\partial \theta}$, which does not vanish. Therefore, all fixed points of $\varphi^1_{X_H}$ will lie in the interior of the disk.

To prove the second claim it is enough to observe that both ω and H_t must be given on $\mathcal{N}_\varepsilon(Z)$ by the expressions described in Remark 4.7, which can be extended trivially to the whole disk. This means that we can choose such extensions in such a way that

$X_H = -a_m k(t)\frac{\partial}{\partial\theta}$ in the whole disk, where $\frac{\partial}{\partial\theta}$ has exactly one fixed point in $(0,0) \in D^2$ and the time 1 flow of X_H cannot have more fixed points because $0 < \int_0^1 k(t)dt < \frac{2\pi}{a_m}$. $\square$

This observation is useful to improve the lower bound of 1-periodic Hamiltonian orbits and thus is referred to as a Lemma in this article.

As will be proven, b^m-symplectic structures on orientable surfaces can be regularized into symplectic structures. While the case for even m is already covered in Proposition 4.4, we will show that the desingularization theorem can be adapted to the case of m odd in the case of surfaces. Similar to Proposition 4.4, we show that for m odd, the Hamiltonian dynamics of an admissible b^m-Hamiltonian on an acyclic b-symplectic surface is in fact given by the Hamiltonian dynamics of a smooth Hamiltonian on a symplectic manifold. In the case where the b^m-symplectic manifold is cyclic, the dynamics of an admissible b^m-Hamiltonian turns out to be symplectic.

Proposition 4.9. *Let (Σ, Z, ω) a b^m-symplectic surface and let H_t be an admissible Hamiltonian. Moreover, assume that Σ is orientable.*

1. *If the b^m-symplectic manifold is acyclic, there exist a symplectic structure $\widetilde{\omega}$ and a smooth Hamiltonian $\widetilde{H}_t$ on M such that $X^{\widetilde{\omega}}_{\widetilde{H}}$ coincides with X^{ω}_H.*
2. *If the b^m-symplectic manifold is cyclic, there exists a symplectic structure $\widetilde{\omega}$ on M with respect to which X^{ω}_H is symplectic.*

We repeat that the case when m is even is covered already by Proposition 4.4, the novelty here is when b is odd. The proof of the proposition is based on an adaptation of the desingularization procedure introduced in [GMW1]. However, this adaptation only works for surfaces.

Before proving the proposition, we note that Corollary 4.5 and 4.6 as before hold under the conditions that we consider here. However, we will see that in the case of surfaces, using Lemma 4.8, we will be able to prove a stronger result.

Proof of Proposition 4.9. The sketch of the proof is as in Proposition 4.4: we will first construct a smooth symplectic form $\widetilde{\omega}$ from the b^m-symplectic one, without distinguishing whether the surface is cyclic or not. In the case that the b^m-surface is acyclic, we will also construct a smooth function $\widetilde{H}_\varepsilon$ such that its Hamiltonian vector field with respect to $\widetilde{\omega}_\epsilon$ coincides with the initial b^m-Hamiltonian vector field everywhere. In the cyclic case, it will not be possible in general to construct the function $\widetilde{H}$ globally, but it will exist locally, thus showing that the vector field X_H is still symplectic with respect to the smooth symplectic structure $\widetilde{\omega}$.

As Z is compact and M is a surface, a connected component of it is diffeomorphic to S^1 and any tubular neighbourhood of a connected component of the critical set is diffeomorphic to $(-\varepsilon, \varepsilon) \times S^1$ and the b^m-symplectic structure in this neighbourhood writes down as $\omega = \sum_{i=1}^m \frac{dz}{z^i} \wedge \pi^*(\alpha_i)$ where $\alpha_i \in \Omega^1(Z)$ are closed 1-forms and z is a defining function for Z. As Z is one dimensional, $\alpha_i = a_i d\theta$, where a_i are smooth functions on S^1, and we therefore rewrite

$$\omega = \sum_{i=1}^m \left(z^{m-i} a_i\right) \frac{dz}{z^m} \wedge d\theta. \tag{4.4}$$

We assume that $a_m \neq 0$. If it is zero, then denote by m' the maximum integer such that $a_{m'} \neq 0$. Such a m' exists as otherwise this connected component of Z would not be a singular locus. In this case, the integer m has to be replaced by m' in the rest of the proof.

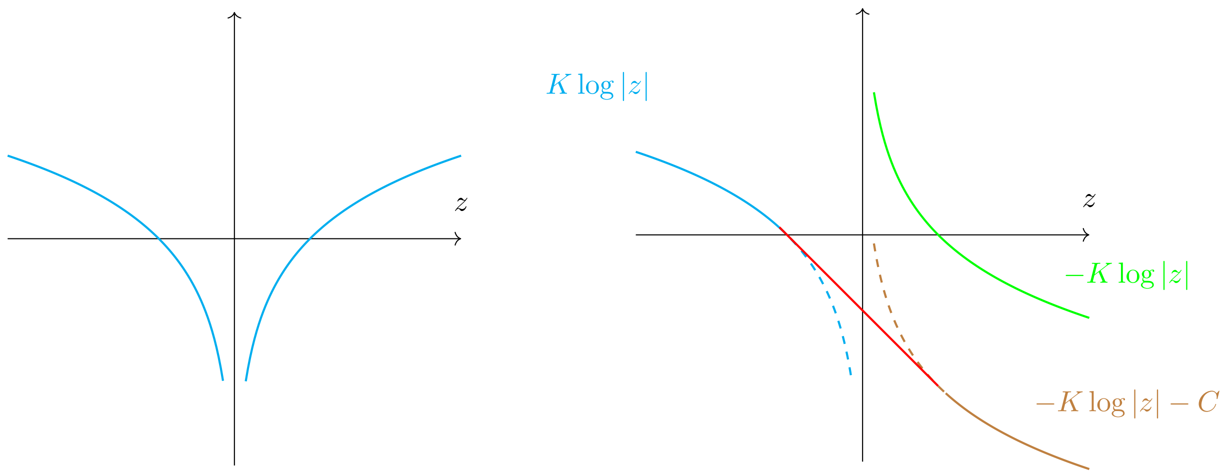

FIGURE 2. An illustration of the regularization function g_ε used to build a symplectic form and a smooth Hamiltonian out of the initial b-symplectic form and the b-Hamiltonian.

By the assumptions on admissibility, the Hamiltonian is given by

$$H = \begin{cases} k(t)\log|z| \text{ when } m = 1 \\ -k(t)\frac{1}{m-1}\frac{1}{z^{m-1}} \text{ when } m > 1. \end{cases}$$

We denote by M^+, respectively M^-, the set of connected components where the globally defined defining function z is positive, respectively negative, i.e.

$$M^{\pm} = \{x \in M | \pm z > 0\}.$$

When $m = 1$, let $g : (-\varepsilon, \varepsilon) \to \mathbb{R}$ be a smooth function such that $g_\varepsilon(z) = \log|z|$ in $\left(\frac{\varepsilon}{2}, \varepsilon\right)$ and $g_\varepsilon(z) = -\log|z| - C(\varepsilon, m)$ in $\left(-\varepsilon, -\frac{\varepsilon}{2}\right)$, where $|C(\varepsilon, m)| > 0$ is a constant large enough so that g_ε can be defined without any critical points in the neighbourhood $(-\varepsilon, \varepsilon)$ (see Figure 2).

When $m > 1$, instead of the above function we consider $g_\varepsilon : (-\varepsilon, \varepsilon) \to \mathbb{R}$ a smooth function such that $g_\varepsilon(z) = -\frac{1}{m-1}\frac{1}{z^{m-1}}$ in $\left(\frac{\varepsilon}{2}, \varepsilon\right)$ and $g_\varepsilon(z) = (-1)^m\frac{1}{m-1}\frac{1}{z^{m-1}} - C(\varepsilon, m)$ in $\left(-\varepsilon, -\frac{\varepsilon}{2}\right)$, where $|C(\varepsilon, m)| > 0$ is a constant large enough so that g_ε can be defined without any critical points in the neighbourhood $(-\varepsilon, \varepsilon)$.

We define a smooth 2-form on M by

$$\widetilde{\omega}_\varepsilon = \begin{cases} \omega \text{ on } M^+ \\ (-1)^m\omega \text{ on } M^- \\ \sum_{i=1}^m \left(z^{m-i}a_i\right) dg_\varepsilon \wedge d\theta \text{ in } (-\varepsilon, \varepsilon) \times S^1. \end{cases}$$

Note that the function ω_ε is well-defined (independent on whether the b-manifold is acyclic or not) as the differential of g_ε is well-defined. The function g_ε however may not be well-defined, as will be the case for cyclic b-manifolds as described below.

The proof now follows closely the proof of Proposition 4.4: in the case where the b-manifold is cyclic, we define a smooth Hamiltonian in the tubular neighbourhoods $N_\varepsilon(Z)$ by $\widehat{H}_\varepsilon = k(t)g_\varepsilon$. In these tubular neighbourhoods the Hamiltonian vector field $X_{\widehat{H}_\varepsilon}$ with respect to $\widetilde{\omega}_\varepsilon$ is the initial b^m-Hamiltonian vector field X_H. Hence X_H is locally a smooth Hamiltonian vector field (as in any open component of $M \setminus Z$, ω is a *smooth* symplectic form), thus it is a symplectic vector field with respect to $\widetilde{\omega}_\varepsilon$.

In the case where the b-manifold is acyclic, we globally define a smooth function $\widetilde{H}_\varepsilon$. As in the proof of Proposition 4.4, we use the notation of the associated graph.

We start to define a smooth function $\widetilde{H}_\varepsilon$ on a connected component of $M^+ \setminus N_\varepsilon(Z)$ and we denote the corresponding vertex in the associated graph by $v_{initial}$. On this connected component, we set $\widetilde{H}_\varepsilon = H$. We extend the definition of the function $\widetilde{H}_\varepsilon$ by defining it on the connected components of $M \setminus Z$ corresponding to the adjacent vertices of $v_{initial}$. We then reiterate this step to continue to extend $\widetilde{H}_\varepsilon$ to the connected components of $M \setminus Z$ corresponding to the adjacent vertices. The condition on the associated graph being acyclic implies that the graph does not close up and therefore this gives rise to a globally defined function $\widetilde{H}_\varepsilon$. In order to simplify the wording, we use interchangeably the connected components of $M \setminus Z$ (respectively Z) and the vertices (respectively edges) of the associated graph V. For instance, when we write $v_{initial}$, we mean the connected component of $M \setminus Z$ corresponding to $v_{initial}$ and vice versa.

Having already defined $\widetilde{H}_\varepsilon$ on $v_{initial}$, we then define $\widetilde{H}_\varepsilon$ on the edges connected to $v_{initial}$ by $\widetilde{H}_\varepsilon = k(t)g_\varepsilon$ and on the vertices adjacent to it given by $\widetilde{H}_\varepsilon = (-1)^m H - k(t)C(\varepsilon, m)$.

By the choices of the constant $C(\varepsilon, m)$ and the function g_ε, the function $\widetilde{H}_\varepsilon$ defines a smooth function on the domain of M corresponding to the vertices $v_{initial}$, v_i and edges e_i. Furthermore, $\widetilde{H}_\varepsilon$ does not admit any critical points on any tubular neighbourhood $N_\varepsilon(Z)$.

We iterate this construction over all the vertices. As the graph is acyclic, the function $\widetilde{H}$ is a well-defined smooth function on M.

The reader can check that $\widetilde{\omega}_\varepsilon$ is a symplectic 2-form and that the Hamiltonian vector field of $\widetilde{H}_\varepsilon$ associated to $(\Sigma, \widetilde{\omega}_\varepsilon)$ equals to X_H, i.e. $X_H^\omega = X_{\widetilde{H}_\varepsilon}^{\widetilde{\omega}_\varepsilon}$. □

Remark 4.10. Proposition 4.9 can be phrased in terms of the one form associated to the contraction of the b-Hamiltonian vector field with a desingularized symplectic form, namely

$$\eta := \iota_{X_H^\omega} \widetilde{\omega}_\varepsilon.$$

The symplectic form $\widetilde{\omega}_\varepsilon$ is constructed in such a way that η is closed. Moreover, in the particular case that (M, Z) is acyclic, η is exact.

Next we remark that the proof of Proposition 4.9 does not generalize to higher dimensions.

Remark 4.11. The construction from Proposition 4.9 does not necessarily work in dimensions higher than 2 for m odd. We reason here for $m = 1$, the case $m > 1$ odd being similar. Nearby a connected component of Z the b-symplectic form has the form $\omega = \frac{dz}{z} \wedge \alpha + \beta$. In contrast to dimension 2, in higher dimensions there is a similar interpolation to glue ω on M^+ to $-\frac{dz}{z} \wedge \alpha + \beta$ on $\mathcal{N}(Z) \cap M^-$ as in the proof, however, a priori the latter does not extend to M^-.

While the argument does not generalize directly to higher dimensions, it applies to products of b^m-symplectic surfaces with symplectic surfaces (with the product b^m-symplectic structure).

Remark 4.12. The extension (as raised in Remark 4.11) holds for products of b^m-symplectic surface with symplectic manifolds (always with the product b^m-symplectic structure). In fact, as the b^m-symplectic structure on the product manifold is given by $\omega := \omega_1 + \omega_2$, where ω_1 is the b^m-symplectic structure on the surface and ω_2 is the symplectic structure on a symplectic manifold (with no restriction on the dimension). To conclude the proof, it is sufficient to realize that ω can be interpolated to $-\omega_1 + \omega_2$ on M^- and that the latter one extends trivially over M^- By the admissibility conditions, the Hamiltonian dynamics is just the product dynamics. This also holds for c-symplectic manifolds that arise from products of b^m-symplectic surfaces.

Proposition 4.9 clears the way to prove a better lower bound on the number of 1-periodic orbits of the Hamiltonian flow. Indeed, under the same hypothesis, the flow on each connected component of $M\backslash Z$ is Hamiltonian (for a smooth symplectic form) and we therefore can use the known result about the dynamics on symplectic manifolds. In particular, we can state the same corollaries as for general b^{2m}-symplectic manifolds (Corollary 4.5 and 4.6), however, we will improve the lower bound in the following proposition.

Associated to a vertex v of the associated graph of a b-manifold, recall that g_v denotes the genus of the corresponding connected component of $M \setminus Z$. Furthermore, we denote by $\deg(v)$ the degree of the vertex v, that is, in terms of graph theory, the number of edges attached to it. In the b-manifold this number corresponds to the number of connected components of Z that are the boundary of the connected component of $M \setminus Z$, represented by v. With this notation in mind, we obtain the following theorem.

Theorem 4.13. *Let (Σ, Z, ω) a compact b^m-symplectic orientable surface. Let H_t be an admissible Hamiltonian in (Σ, Z, ω) whose periodic orbits are all non-degenerate. Then the number of 1-periodic orbits of X_H is bounded below by*

$$\sum_{v\in V} \max\{2 + 2g_v - \deg(v), 0\}.$$

Proof. We will prove that for each connected component of $M\backslash Z$, there are at least $\max\{2+2g_v - \deg(v), 0\}$ critical points.

Let $v \in V$ be the vertex associated to a connected component of $M \setminus Z$. Seen in isolation, the associated connected component of $M \setminus Z$ is diffeomorphic to a surface of genus g_v punctured at $\deg(v)$ different points. We will now complete it to a b^m-symplectic surface, diffeomorphic to a surface of genus g_v without punctures by attaching 2-disks to each puncture.

For each edge $Z_i \cong S^1$, we attach a 2-disk. The initial b^m-symplectic form in the annulus around the connected component of Z_i has the expression $\sum_{i=1}^m a_i \frac{dr}{(1-r)^i} \wedge d\theta$ and using Lemma 4.8 it can be glued to a globally defined the b^m-symplectic form in D^2. We also choose an extension $\overline{H}_v$ of the given Hamiltonian H to this disk as in Lemma 4.8 in such a way that

(1) near the boundary ∂D^2 the Hamiltonian vector field has the expression $X_{\overline{H}} = k\frac{\partial}{\partial\theta}$, where k is the function associated to H_t near the boundary Z_i.
(2) $X_{\overline{H}}$ vanishes exactly at one point in the interior of the disk.

With these choices, we can define $(\overline{\Sigma}_v, \overline{\omega}_v, \overline{H}_v)$, where $\overline{\Sigma}_v$, as described before, is diffeomorphic to a closed surface of genus g_v and $\overline{\omega}_v$ and $\overline{H}_v$ are constructed by extending (ω, H) as explained earlier. This is a b^m-symplectic surface whose associated graph is given by the initial vertex v with the same degree as in the original graph, with one vertex of degree 1 for each disk attached in this proof. In particular, this graph is acyclic. We can therefore apply Proposition 4.9 to this new surface, so the time-1-flow of the Hamiltonian vector field $X_{\overline{H}_v}$ must have at least $2 + 2g_v$ fixed points by Corollary 4.5. However, by construction we know that $\deg(v)$ of these fixed points must lie in the interior of the corresponding attached disks, so the lower bound for the number of fixed points in the interior of v must be $2 + 2g_v - \deg(v)$ if $\deg(v) \leq 2 + 2g_v$ and 0 otherwise. □

A few remarks are in order. First, we note that the lower bound obtained in Theorem 4.13 is indeed an improvement of the lower bound obtained in Corollary 4.5 in the acyclic case because of the properties of acyclic graphs as is shown in the next remark.

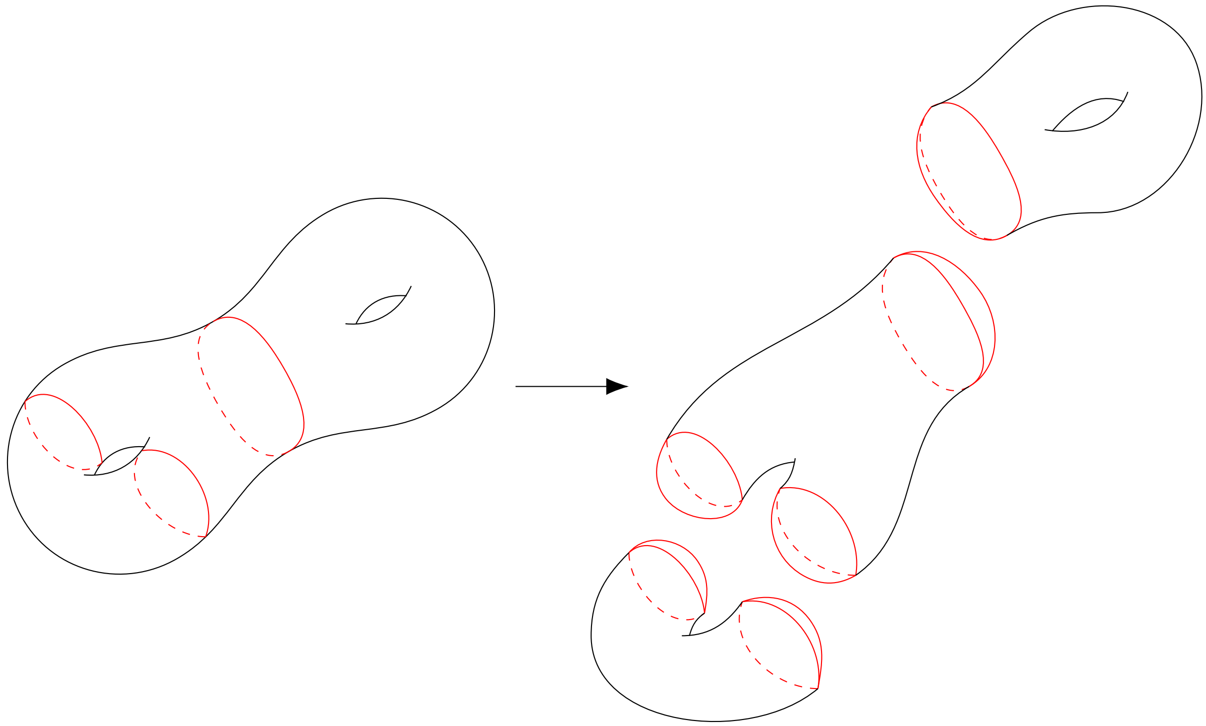

FIGURE 3. Completion of a b^m-symplectic surface with disks at the punctures

Remark 4.14. It is well-known that in an acyclic and connected graph the number of vertices is precisely the number of edges plus one. Moreover, if a b^m-symplectic surface is acyclic then its genus is equal to $\sum_{v\in V} g_v$. Also, take into account that $\sum_{v\in V}\deg(v)$ is exactly the number of edges times two. Combining these properties, we conclude that

$$\sum_{v\in V}(2+2g_v-\deg(v)) = 2\#V + 2g_\Sigma - 2\#E = 2\#V + 2g_\Sigma - 2(\#V-1) = 2+2g_\Sigma,$$

and therefore

$$\sum_{v\in V}\max(2+2g_v-\deg(v),0) \geq 2+2g_\Sigma.$$

Remark 4.15. All conditions on the admissibility of Hamiltonians are used in Proposition 4.9. This means, if H is not linear along X^σ or if it is not preserved along the modular vector field, this proof does not work anymore.

As an example for the case that H is not preserved along the modular vector field, consider for instance the case of a b-sphere (as one would obtain by gluing for instance two 2-disks described in Lemma 4.8) with Hamiltonian $H=\sin\theta$. As we saw in Proposition 3.9, the flow of X_H cannot preserve the volume form, and therefore it cannot be symplectic with respect to any symplectic form.

As discussed in Remark 4.12, the extension method of Proposition 4.9 holds for products of b^m-symplectic surfaces and symplectic manifolds. The resulting dynamics is the product dynamics. Under the same hypothesis, we therefore obtain the following corollary (where we adopt the same notation as in Theorem 4.13).

Corollary 4.16 (Arnold conjecture for b^m-symplectic product manifolds)**.** *Let $(M = \Sigma \times W, Z = Z_\Sigma \times W, \omega = \omega_1 + \omega_2)$ be the product of a compact orientable b^m-symplectic surface $(\Sigma, Z_\Sigma, \omega_1)$ and a compact symplectic manifold (W, ω_2). Let H_t be an admissible Hamiltonian in (M, Z, ω) whose periodic orbits are all non-degenerate. Then the number of 1-periodic orbits of X_H is bounded below by*

$$\#\mathcal{P}(H) \geq \left(\sum_i \dim HM^i(W; \mathbb{Z}_2)\right) \cdot \left(\sum_{v \in V} \max\{2 + 2g_v - \deg(v), 0\}\right).$$

Similarly, the analog corollary holds for products of b^m-symplectic manifolds.

5. A Floer complex on b^m-symplectic manifolds

In the preceding section a proof of the Arnold conjecture could be attained in the surface case for b^m-symplectic manifold. Floer developed a whole toolbox of Floer theory in order to address the Arnold conjecture (see [F1] for the surface case). This motivates us to introduce the Floer complex for higher dimensional b^m-symplectic manifolds.

5.1. The Floer equation.

In order to control the behavior of the J-holomorphic curves approaching the critical set Z, we need the following proposition.

Proposition 5.1. *Let (M, Z, ω) a b^m-symplectic manifold, and let $\mathcal{N} = \mathcal{N}(Z) \cong (0, \varepsilon) \times Z$ a collar neighbourhood of the singular hypersurface (not including Z) with a normal symplectic vector field X^σ. Let $\Omega \subset \mathbb{C}$ with coordinates $\eta = s + it$, and take H an admissible Hamiltonian in $\mathcal{C}^\infty(\Omega \times \mathcal{N})$, so that locally it has the form*

$$H(s,t,x,z) = \begin{cases} k(s,t)\log|z| + h(s,t,x) & \text{if } m = 1 \\ -k(s,t)\frac{1}{(m-1)z^{m-1}} + h(s,t,x) & \text{if } m > 1. \end{cases}$$

Take also $J \in \Gamma(\Omega \times \mathcal{N}, T^*M \otimes TM)$ a compatible almost complex structure adapted to ω and H.

Let $u : \Omega \to \mathcal{N} \subset M$ a solution to the Floer equation,

$$\frac{\partial u}{\partial s} + J(\eta, u(\eta))\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0, \tag{5.1}$$

and take $f : \mathcal{N} \to \mathbb{R}$ given by $f(z,p) = \log|z|$ if $m = 1$ and $f(z,p) = -\frac{1}{(m-1)z^{m-1}}$ if $m > 1$. Then

$$\Delta(f \circ u) = -\frac{\partial k}{\partial s}.$$

Proof. Let $d^c(v) := dv \circ i = \frac{\partial v}{\partial t}ds - \frac{\partial v}{\partial s}dt$. Then

$$-dd^c(v) = (\Delta v) ds \wedge dt.$$

Computing,

$$\begin{aligned}
-d^c(f\circ u) =& \frac{\partial}{\partial s}(f\circ u)dt - \frac{\partial}{\partial t}(f\circ u)ds = \left(df(u)\frac{\partial u}{\partial s}\right)dt - \left(df(u)\frac{\partial u}{\partial t}\right)ds\\
=& \left(df(u)\left(\frac{\partial u}{\partial s} + J(\eta,u)\frac{\partial u}{\partial t}\right)\right)dt - \left(df(u)\left(J(\eta,u)\frac{\partial u}{\partial t}\right)\right)dt+\\
&+\left(df(u)\left(J(\eta,u)\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t}\right)\right)ds - \left(df(u)\left(J(\eta,u)\frac{\partial u}{\partial s}\right)\right)ds\\
=& -\omega(\nabla f(u), X_H(u))dt + \omega\left(\nabla f(u), \frac{\partial u}{\partial t}\right)dt-\\
&-\omega\left(\nabla f(u), J(\eta,u)X_H(u)\right)ds + \omega\left(\nabla f(u), \frac{\partial u}{\partial s}\right)ds
\end{aligned} \tag{5.2}$$

Now, we apply the fact that with our choice of J we have $\nabla f = X^\sigma$, so, for the first term

$$\omega(X^\sigma, X_H(u))dt = \left(\mathcal{L}_{X^\sigma}H\right)dt = k(s,t)dt.$$

For the second and fourth terms

$$\omega\left(X^\sigma, \frac{\partial u}{\partial t}\right)dt + \omega\left(X^\sigma, \frac{\partial u}{\partial s}\right)ds = u^* i_{X^\sigma}\omega.$$

Finally, for the third term

$$\begin{gathered}
\omega(X^\sigma, J(\eta,u)X_H(u))ds = \omega(X_H(u), J(\eta,u)X^\sigma(u))ds =\\
\omega(X_H(u), v_{\mathrm{mod}}(u))ds = \left(\mathcal{L}_{v_{\mathrm{mod}}}H\right)ds = 0.
\end{gathered}$$

Collecting everything, we deduce that Equation (5.2) yields that

$$-d^c(f\circ u) = -k(s,t)dt - u^* i_{X^\sigma}\omega.$$

If we apply the differential, it is clear that $d(k(s,t)dt) = \frac{\partial k}{\partial s}ds\wedge dt$, and

$$d\left(u^* i_{X^\sigma}\omega\right) = u^*(di_{X^\sigma}\omega) = u^*(\mathcal{L}_{X^\sigma}\omega) = 0.$$

Therefore,

$$(\Delta(f\circ u))ds\wedge dt = -dd^c(f\circ u) = -\frac{\partial k}{\partial s}ds\wedge dt.$$

□

As a corollary of Proposition 5.1, we obtain the following result.

Theorem 5.2 (Minimum Principle). *Let $u \in \mathcal{C}^\infty(\Omega, \mathcal{N})$ satisfying one of the following conditions:*

(1) u is a solution to the Floer equation 5.1 for an admissible Hamiltonian $H \in \mathcal{C}^\infty(S^1\times\mathcal{N})$. If $f\circ u$ attains its maximum or minimum on Ω, then $f\circ u$ is constant.
(2) u is a solution to the Floer equation 5.1 for a parameter-dependent admissible Hamiltonian $H \in \mathcal{C}^\infty(\Omega\times\mathcal{N})$ such that $\frac{\partial k}{\partial s}(s,t) \geq 0\ \forall (s,t)\in\Omega$. If $f\circ u$ attains its minimum on Ω, then $f\circ u$ is constant.

Proof. In the first case this is trivially the case because by Proposition 5.1 we deduce that $f\circ u$ is harmonic, and therefore satisfies the Maximum (respectively Minimum) Principle.

In the second case, it is a consequence of the minimum principle, because $\Delta(f\circ u) = -\frac{\partial k}{\partial s} \leq 0$ by the choice of the admissible Hamiltonian. □

5.2. **The Floer complex.** For the remaining of this section, we assume that (M,ω) is aspherical, this means, that $[\omega]$ vanishes on $\pi_2(M)$. Recall from Definition 3.23 that we denote by $\mathcal{P}(H)$ the set of 1-periodic orbits of X_H for an admissible and regular Hamiltonian $H : M \times S^1 \to \mathbb{R}$. Since M is compact and the image of all elements of $\mathcal{P}(H)$ lies outside of an open neighbourhood of Z, $\mathcal{P}(H)$ is finite.

Definition 5.3. Let $H_t \in {}^{b^m}\mathrm{Reg}(M,\omega)$. We define ${}^{b^m}CF(M,\omega,H)$ as the $\mathbb{Z}_2$-vector space generated over $\mathcal{P}(H)$, this means, the set of formal sums of the type

$$v = \sum_{x\in\mathcal{P}(H)} v_x x,\ v_x \in \mathbb{Z}_2.$$

Under the assumption that the first Chern class $c_1 = c_1(\omega) \in H^2(M,\mathbb{Z})$ of the bundle (TM,J) vanishes on $\pi_2(M)$, then the Conley-Zehnder index μ_{CZ} of $x \in \mathcal{P}(H)$ is well-defined (see for instance [Sal]). This index can be normalized in such a way that for any critical points of a $\mathcal{C}^2$-small enough H it is satisfied that

$$\mu_{CZ}(x) = 2n - \mathrm{ind}(x).$$

We can use the Conley-Zehnder index to turn ${}^{b^m}CF(M,H,\omega)$ into a graded vector space.

For $x,y \in \mathcal{P}(H)$ we denote by $\mathcal{M}(x,y)$ the moduli space of Floer trajectories from x to y, this means,

$$\begin{cases} \frac{\partial u}{\partial s} + J_u\frac{\partial u}{\partial t} + \mathrm{grad}_u H = 0 \\ \lim\limits_{s\to-\infty} u(s,t) = x(t),\ \lim\limits_{s\to+\infty} u(s,t) = y(t). \end{cases}$$

As (M,ω) is aspherical, we cannot have bubbles of pseudo-holomorphic spheres (see, for instance, Section 6.6 on [AD]). Moreover, as a consequence of Theorem 5.2 we cannot have solutions of the Floer equation approaching Z in any way. Thus, we conclude that $\mathcal{M}(x,y)$ is compact for any pair $x,y \in \mathcal{P}(M)$ and that it is a manifold of dimension $\mu_{CZ}(x) - \mu_{CZ}(y)$. As in standard Floer theory, we conclude from here that whenever $\mu_{CZ}(x) - \mu_{CZ}(y) = 1$, the quotient $\mathcal{M}(x,y)\,/\mathbb{R}$ by the action along the variable s is a finite set.

Let

$$n(x,y) := \#\left\{\mathcal{M}(x,y)\,/\mathbb{R}\right\} \mod 2.$$

Then, for each index k we can define the boundary operator of the Floer complex,

$$\partial_k : \quad {}^{b^m}CF_k(M,\omega,H,J) \quad\longrightarrow\quad {}^{b^m}CF_{k-1}(M,\omega,H,J)$$

as defined in the generators of ${}^{b^m}CF_k(M,\omega,H,J)$ by

$$\partial_k x := \sum_{\substack{y\in\mathcal{P}(H)\\ \mu(y)=k-1}} n(x,y)y.$$

Definition 5.4. The Floer homology is the one given by

$${}^{b^m}HF_k(M,\omega,H) := \frac{\mathrm{ker}\partial_k}{\mathrm{im}\partial_{k+1}}.$$

Remark 5.5. The homology constructed so far, as the notation implies, depends on the choice of H, J and ω, besides depending on M, Z and the relative topology between them. Being precise, the family of admissible Hamiltonians depends on X^σ and v_{mod}, but we do not include this dependence in the notation as they are accounted for in the choice of a Hamiltonian H.

Showing invariance with respect to these choices is beyond the scope of this paper, but such a proof is the subject of a paper in preparation by the authors.

Remark 5.6. The construction of this complex (and homology) is related to our methods in Section 4 due to the conditions on admissible Hamiltonians, in particular to the fact that the dynamics of X_H are, in a sense, split on the connected components of $M\backslash Z$. Moreover, due to Theorem 5.2 we can deduce that solutions to the Floer equation with finite energy do not cross the singular hypersurface Z, this means, our Floer complex splits on the connected components of $M\backslash Z$,

$$^{b^m}CF_\bullet(M,\omega,H,J) = \bigoplus_{M_i \in M\backslash Z} {}^{b^m}CF_\bullet(M_i,\omega,H,J).$$

6. Conclusions and Open Questions

In this article we were able to give a complete proof of the Arnold conjecture for b-symplectic surfaces.

In view of the b^3-symplectic structure of the restricted three-body problem investigated in [KMS], the first open question would be the following:

Open Question 6.1. Is it possible to find a compact surface on the restricted three-body problem where the Arnold conjecture yields new applications in Celestial mechanics?

As shown in the previous section, by the definition of admissible Hamiltonian functions, there are only finitely many 1-periodic orbits around the critical set and by the minimum principle (Theorem 5.2), there is a Floer complex associated to such a Hamiltonian. We furthermore proved in Theorem 4.13 that for orientable surfaces, the number of 1-periodic orbits of the Hamiltonian vector field is bounded below by the topology of M and the relative position of Z and that in the case of b^{2m}-symplectic manifolds, the dynamics is described by the usual Hamiltonian dynamics and therefore the same lower bound as in the standard Arnold conjecture holds.

The second open question that we raise is whether there is a *finer way* to compute the Floer homology associated to an admissible Hamiltonian (both for b-symplectic and for b^{2m}-symplectic manifolds). In view of the Mazzeo-Melrose formula for b-cohomology we might speculate with other decomposition formulas for the Floer complex that take Z into account.

Open Question 6.2. Is it possible to compute $^{b^m}HF_k(M,\omega,H)$ in terms of the topology of M and Z?

Indeed, we suspect that in the b^{2m}-symplectic case, the lower bound obtained in Theorem 4.5 is unsatisfactory. As for the proof of the surface case (Theorem 4.13), the fact that the dynamics is tangent to Z should give a higher lower bound. In that sense, an optimal lower bound should take into account not only the topology of M, but also the one of Z and its relative position in M. Similarly, we expect that a similar result to the one of Theorem 4.13 holds in higher dimensions. Our method of proof in the case of surfaces is a hands-on construction in dimension 2.

Open Question 6.3. Does Theorem 4.13 hold in higher dimension?

In Section 3, we justify the notion of admissible Hamiltonian by the fact that there are only finitely many periodic orbits around the critical set. It would be interesting to construct a Floer-type homology for more general Hamiltonian functions. At this moment, this looks terra incognita to us.

Open Question 6.4. Can we associate a Floer homology to more general Hamiltonian functions?

For instance, for smooth Hamiltonians, the associated Hamiltonian vector field to a b^m-symplectic structure is tangent to the leaves of the foliation. Even compact b-symplectic manifolds can have non-compact leaves, as is shown in the next example.

Consider the 4-torus $M = \mathbb{T}^4$ with coordinates $\theta_1, \theta_2, \theta_3, \theta_4$. Inside this torus, consider Z the union of two 3-tori given by the equation $\sin\theta_4 = 0$. The critical set Z can be endowed with a cosymplectic structure which is also a regular Poisson structure: Denote by $\mathcal{F}$ the codimension one foliation with leaves given by

$$\theta_3 = a\theta_1 + b\theta_2 + k, \quad k \in \mathbb{R},$$

where $a, b, 1 \in \mathbb{R}$ are fixed and independent over $\mathbb{Q}$. Each leaf is diffeomorphic to $\mathbb{R}^2$ [Ma]. The one-form $\alpha = \frac{a}{a^2+b^2+1}\, d\theta_1 + \frac{b}{a^2+b^2+1}\, d\theta_2 - \frac{1}{a^2+b^2+1}\, d\theta_3$ defines a symplectic foliation with symplectic form induced by the form

$$\beta = d\theta_1 \wedge d\theta_2 + b\, d\theta_1 \wedge d\theta_3 - a\, d\theta_2 \wedge d\theta_3$$

The two-form $\omega = \frac{d\theta_4}{\sin\theta_4} \wedge \alpha + \beta$ is a b-symplectic form on M.

Consider now the vector field

$$X = \frac{a}{a^2+b^2+1}\frac{\partial}{\partial\theta_1} + \frac{b}{a^2+b^2+1}\frac{\partial}{\partial\theta_2} - \frac{1}{a^2+b^2+1}\frac{\partial}{\partial\theta_3}.$$

Due to the rational independence of a and b this vector field does not have any periodic orbits. In order to check this, it is enough to consider the projection on the 2-torus with coordinates θ_1 and θ_2 where the vector field projects to a vector field with dense orbits. This vector field is a b-Hamiltonian vector field with Hamiltonian function $H = \log\left|\frac{\sin\theta_4}{1+\cos\theta_4}\right|$. This vector field is in fact the modular vector field associated to the Poisson structure.

It would be interesting to see that, even though the leaves are non-compact, there is a lower bound (depending on the topology of the leaf) on the number of 1-periodic Hamiltonian orbits on each leaf of Z.

Open Question 6.5. Given a b^m-symplectic manifold with a smooth Hamiltonian function H, is the number of non-degenerate periodic time-1 Hamiltonian orbits bounded on each leaf bounded below by the topology of Z?

Given the recent advances in [PiWi], by Theorem 1.3 therein, the b^m-symplectic structure can be regularized (in the sense of [PiWi]) to a symplectic foliation. The last open question can then be reconducted on whether a foliated version of the Arnold conjecture holds. Loosely speaking, the odd-dimensional counterpart of this holds as is proved in [PiPr]. The foliated Arnold conjecture has initially been discussed in the meeting *Workshop on topological aspects of symplectic foliations* held in 2017 at the *Université de Lyon*.

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