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New generalization of cubic partition of n

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Abstract. Let $c^*_{(1,r,a)}(n)$ be the generalization of the cubic partition function $c(n)$. In this paper, we prove some new congruences modulo odd prime p by taking $r = 3, 4, 5, 7, 11$ and 13 using q-series identities. We study congruence properties of generalization of cubic partition function for different values of a and give some particular cases as examples.

Keywords: Partitions; k-colors; Partition Congruences

MSC 2022 classification: Primary 11P83; Secondary 05A15, 05A17.

1 Introduction

In a paper [1], Chan started the study of cubic partitions by exhibiting a close relation between a certain type of partition function and Ramanujan's cubic continued fraction. For example, there are four cubic partitions of 3, namely $3, 2_1 + 1, 2_2 + 1$ and $1 + 1 + 1$, where the subscripts 1 and 2 denote the colours. Cubic partition function $c(n)$ is defined by

$$\sum_{n=0}^{\infty} c(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty} = \frac{1}{E(q)E(q^2)}, \tag{1.1}$$

where $E(q)$ is Euler's product,

$$E(q) = (q;q)_\infty := \prod_{n=1}^{\infty}(1 - q^n), \quad | q |< 1.$$

The function $c(n)$ satisfies many Ramanujan type congruences, for example $c(3n + 2) \equiv 0 \ (\ mod\ 3), \ \ \forall\ n \geq 0$. Motivated by his works in [2, 3], many partition congruences for analogous partition functions have been investigated. For example, Chen and Lin [4] found four new congruences modulo 7 by using modular forms, whereas Xiong [11] established sets of congruences modulo powers of 5. In [6], Chern and Dastidar have presented two new congruences modulo 11 for $c(n)$. Furthermore, they have established a recursion for $c(n)$, which is a special case of a broader class of recursions. Recently Hirschhorn [8] gave an elementary proof of

$$c(5^{\alpha}n+\delta_{\alpha}) \equiv 0 \pmod{5^{\lfloor(\alpha/2)\rfloor}},$$

where $\alpha \geq 2$, $n \geq 0$ and δ_{α} is the reciprocal of 8 modulo 5^{α}.

Zhao and Zhang [12] explored congruences for the following function:

$$\sum_{n=0}^{\infty} cp(n)q^n = \frac{1}{(q;q)_{\infty}^2(q^2;q^2)_{\infty}^2} = \frac{1}{E^2(q)E^2(q^2)} \tag{1.2}$$

and proved that $cp(5n+4) \equiv 0 \ (\ mod\ 5), \quad \forall\ n \geq 0$. Since $cp(n)$ counts a pair of cubic partitions, it is the number of cubic partition pairs. We can interpret $cp(n)$ as the number of 4-colour partitions of n with colours r, y, o and b subject to the restriction that the colours o and b appear only in even parts. Recently Lin [9] studied the arithmetic properties of $cp(n)$ modulo 27 and conjectured the following four congruences:

$$cp(49n+37) \equiv 0 \pmod{49},$$

$$cp(81n+61) \equiv 0 \pmod{243},$$

$$\sum_{n=0}^{\infty} cp(81n+7)q^n \equiv \frac{q(q^2;q^2)_{\infty}(q^3;q^3)_{\infty}^2}{(q^6;q^6)_{\infty}} \pmod{81},$$

$$\sum_{n=0}^{\infty} cp(81n+34)q^n \equiv \frac{36(q;q)_{\infty}(q^6;q^6)_{\infty}^2}{(q^3;q^3)_{\infty}} \pmod{81}$$

In two recent papers, Chern [5] and Lin, Wang and Xia [10] independently proved all the above four congruences.

Let $c^*_{(1,r,a)}(n)$ be defined by

$$\sum_{n=0}^{\infty} c^*_{(1,r,a)}(n)q^n = \frac{1}{[E(q)E(q^r)]^a} \tag{1.3}$$

where $a, r \geq 1$ are positive integers. $c^*_{(1,r,a)}(n)$ is the generalization of the cubic partition function $c(n)$.

In this paper, we prove some quite interesting congruences modulo odd prime p by taking $r = 3, 4, 5, 7, 11$ and 13 using $q-$series identities. We study congruence properties of generalization of cubic partition function for different values of a and give some particular cases as examples.In particular, some of them involve higher powers of the Euler function.

2 New congruences for $c^*_{(1,r,a)}(n)$

In this section, we prove six new congruence modulo an odd prime p. To prove our congruences, we employ the following q-series identity from [7, equation (0.46)]:

$$E^3(q) = \sum_{n=-\infty}^{\infty} (4n+1)q^{[(4n+1)^2-1]/8}. \tag{2.4}$$

We also require the following congruence which follows from the binomial theorem: For prime p and integer $\ell \geq 1$,

$$E_\ell^p \equiv E_{p\ell} \pmod{p}. \tag{2.5}$$

Theorem 1. *Suppose p is an odd prime divisor of $a+3$ and r is an integer with $0 \leq r < p$. Suppose p and r satisfy the condition: $2r+1 \equiv 0 \ (\ mod\ p\ )$ and $p \equiv 5$ or $11 \ (\ mod\ 12\ )$. Then, $\forall n \geq 0$*

$$c^*_{(1,3,a)}(pn+r) \equiv 0 \pmod{p}. \tag{2.6}$$

Proof. Since p divides $a+3$, we can write $a+3 = pm$, for some integer m. Setting $r=3$ in (1.3), we find that

$$\sum_{n=0}^{\infty} c^*_{(1,3,a)}(n)q^n = \frac{[E(q)E(q^3)]^3}{[E(q)E(q^3)]^{pm}}. \tag{2.7}$$

Employing (2.5) in (2.7), we obtain

$$\sum_{n=0}^{\infty} c^*_{(1,3,a)}(n)q^n = \frac{[E(q)E(q^3)]^3}{[E(q^p)E(q^{3p})]^{m}}. \tag{2.8}$$

Using (2.4), we observe that

$$[E(q)E(q^3)]^3 = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} (4n+1)(4m+1)q^{[(4n+1)^2+3(4m+1)^2-4]/8}. \tag{2.9}$$

We note that

$$N = [(4n+1)^2 + 3(4m+1)^2 - 4]/8,$$

which is equivalent to

$$8N + 4 = (4n+1)^2 + 3(4m+1)^2.$$

If $p \equiv 5$ *or* $11 \pmod{12}$, then the Legendre symbol $\left(\frac{-3}{p}\right) = -1$. Therefore, it follows that

$$8N + 4 \equiv 0 \pmod{p}$$

or

$$2N + 1 \equiv 0 \pmod{p}$$

if and only if $4n + 1 \equiv 0 \pmod{p}$ and $4m + 1 \equiv 0 \pmod{p}$. Hence, the congruences (2.6) now follows by employing (2.9) in (2.8) and then comparing the coefficients of q^{pn+r}. QED

Corollary 1. *We have*

$$c^*_{(1,3,2)}(5n+2) \equiv 0 \pmod{5}, \tag{2.10}$$

$$c^*_{(1,3,14)}(17n+8) \equiv 0 \pmod{17}, \tag{2.11}$$

$$c^*_{(1,3,8)}(11n+5) \equiv 0 \pmod{11}, \tag{2.12}$$

$$c^*_{(1,3,20)}(23n+11) \equiv 0 \pmod{23}. \tag{2.13}$$

Proof. Take $p = 5$ and $a = 2$. Then, p is an odd prime, $p \equiv 5 \pmod{12}$ and p divides $a+3$.Therefore, using these in(2.6) we obtain (2.10). Similarly, taking $p = 17$ and $a = 14$ in (2.6) we obtain (2.11), taking $p = 11$ and $a = 8$ in (2.6) we obtain (2.12) and taking $p = 23$ and $a = 20$ in (2.6) we obtain (2.13). QED

Theorem 2. *Suppose p is an odd prime divisor of* $a+3$ *and r is an integer with* $0 \le r < p$. *Suppose p and r satisfy the condition:* $8r + 5 \equiv 0 \pmod{p}$ *and* $p \equiv 3 \pmod{4}$. *Then,* $\forall n \ge 0$,

$$c^*_{(1,4,a)}(pn+r) \equiv 0 \pmod{p}. \tag{2.14}$$

Proof. Since p divides $a+3$, we can write $a+3 = pm$, for some integer m. Setting $r = 4$ in (1.3), we find that

$$\sum_{n=0}^{\infty} c^*_{(1,4,a)}(n)q^n = \frac{[E(q)E(q^4)]^3}{[E(q)E(q^4)]^{pm}}. \tag{2.15}$$

Employing (2.5) in (2.15), we obtain

$$\sum_{n=0}^{\infty} c^*_{(1,4,a)}(n)q^n = \frac{[E(q)E(q^4)]^3}{[E(q^p)E(q^{4p})]^m}. \tag{2.16}$$

Using (2.4), we observe that

$$[E(q)E(q^4)]^3 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (4n+1)(4m+1)q^{[(4n+1)^2+4(4m+1)^2-5]/8}. \quad (2.17)$$

We note that

$$N = [(4n+1)^2 + 4(4m+1)^2 - 5]/8,$$

which is equivalent to

$$8N + 5 = (4n+1)^2 + 4(4m+1)^2.$$

If $p \equiv 3 \pmod 4$, then the Legendre symbol $\left(\frac{-4}{p}\right) = -1$. Therefore, it follows that

$$8N + 5 \equiv 0 \pmod p$$

if and only if $4n+1 \equiv 0 \pmod p$ and $4m+1 \equiv 0 \pmod p$. Hence, the congruences (2.14) now follows by employing (2.17) in (2.16) and then comparing the coefficients of q^{pn+r}. QED

Corollary 2. *We have*

$$c^*_{(1,4,4)}(7n+2) \equiv 0 \pmod 7, \quad (2.18)$$

$$c^*_{(1,4,8)}(11n+9) \equiv 0 \pmod{11}. \quad (2.19)$$

Proof. Taking $p = 7$ and $a = 4$ in (2.14) we obtain (2.18) and taking $p = 11$ and $a = 8$ in (2.14) we obtain (2.19). QED

Theorem 3. *Suppose p is an odd prime divisor of $a+3$ and r is an integer with $0 \le r < p$. Suppose p and r satisfy any of the following two conditions:*

(1) $4r+3 \equiv 0 \pmod p$, $p \equiv 2$ *or* $3 \pmod 5$ *and* $p \equiv 1 \pmod 4$

(2) $4r+3 \equiv 0 \pmod p$, $p \equiv 1$ *or* $4 \pmod 5$ *and* $p \equiv 3 \pmod 4$

Then, $\forall n \ge 0$,

$$c^*_{(1,5,a)}(pn+r) \equiv 0 \pmod p. \quad (2.20)$$

Proof. Since p divides $a+3$, we can write $a+3 = pm$, for some integer m. Setting $r = 5$ in (1.3), we find that

$$\sum_{n=0}^{\infty} c^*_{(1,5,a)}(n)q^n = \frac{[E(q)E(q^5)]^3}{[E(q)E(q^5)]^{pm}}. \quad (2.21)$$

Employing (2.5) in (2.21), we obtain

$$\sum_{n=0}^{\infty} c^*_{(1,5,a)}(n)q^n = \frac{[E(q)E(q^5)]^3}{[E(q^p)E(q^{5p})]^m}. \tag{2.22}$$

Using (2.4), we observe that

$$[E(q)E(q^5)]^3 = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(4n+1)(4m+1)q^{[(4n+1)^2+5(4m+1)^2-6]/8}. \tag{2.23}$$

We note that

$$N = [(4n+1)^2 + 5(4m+1)^2 - 6]/8,$$

which is equivalent to

$$8N + 6 = (4n+1)^2 + 5(4m+1)^2.$$

If

$$p \equiv 2 \textit{ or } 3 \pmod 5 \;\&\; p \equiv 1 \pmod 4$$

or

$$p \equiv 1 \textit{ or } 4 \pmod 5 \;\&\; p \equiv 3 \pmod 4,$$

then the Legendre symbol $\left(\frac{-5}{p}\right) = -1$. Therefore, it follows that

$$8N + 6 \equiv 0 \pmod p$$

or

$$4N + 3 \equiv 0 \pmod p$$

if and only if $4n+1 \equiv 0 \pmod p$ and $4m+1 \equiv 0 \pmod p$. Hence, the congruences (2.20) now follows by employing (2.23) in (2.22) and then comparing the coefficients of q^{pn+r}. QED

Corollary 3. *We have*

$$c^*_{(1,5,14)}(17n+12) \equiv 0 \pmod{17}, \tag{2.24}$$

$$c^*_{(1,5,10)}(13n+9) \equiv 0 \pmod{13}, \tag{2.25}$$

$$c^*_{(1,5,8)}(11n+2) \equiv 0 \pmod{11}, \tag{2.26}$$

$$c^*_{(1,5,16)}(19n+4) \equiv 0 \pmod{19}. \tag{2.27}$$

Proof. Setting $p = 17$ and $a = 14$ in (2.20) implies (2.24). For (2.25), we set $p = 13$ and $a = 10$ in (2.20). For (2.26), we put $p = 11$ and $a = 8$ in (2.20). Finally, by setting $p = 19$ and $a = 16$ in (2.20) we obtain (2.27). QED

Theorem 4. *Suppose p is an odd prime divisor of $a+3$ and r is an integer with $0 \leq r < p$. Suppose p and r satisfy any of the following two conditions:*

(1) $r+1 \equiv 0 \pmod{p}$, $p \equiv 3$ *or* 5 *or* $6 \pmod{7}$ *and* $p \equiv 1 \pmod{4}$

(2) $r+1 \equiv 0 \pmod{p}$, $p \equiv 3$ *or* 5 *or* 6 mod 7 *and* $p \equiv 3 \pmod{4}$

Then, $\forall n \geq 0$,

$$c^*_{(1,7,a)}(pn+r) \equiv 0 \pmod{p}. \tag{2.28}$$

Proof. Since p divides $a+3$, we can write $a+3 = pm$, for some integer m. Setting $r = 7$ in (1.3), we find that

$$\sum_{n=0}^{\infty} c^*_{(1,7,a)}(n)q^n = \frac{[E(q)E(q^7)]^3}{[E(q)E(q^7)]^{pm}}. \tag{2.29}$$

Employing (2.5) in (2.29), we obtain

$$\sum_{n=0}^{\infty} c^*_{(1,7,a)}(n)q^n = \frac{[E(q)E(q^7)]^3}{[E(q^p)E(q^{7p})]^m}. \tag{2.30}$$

Using (2.4), we observe that

$$[E(q)E(q^7)]^3 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (4n+1)(4m+1)q^{[(4n+1)^2+7(4m+1)^2-8]/8}. \tag{2.31}$$

We note that

$$N = [(4n+1)^2 + 7(4m+1)^2 - 8]/8,$$

which is equivalent to

$$8N + 8 = (4n+1)^2 + 7(4m+1)^2.$$

If

$$p \equiv 3 \text{ or } 5 \text{ or } 6 \pmod{7} \ \& \ p \equiv 1 \pmod{4}$$

or

$$p \equiv 3 \text{ or } 5 \text{ or } 6 \pmod{7} \ \& \ p \equiv 3 \pmod{4}$$

then the Legendre symbol $\left(\frac{-7}{p}\right) = -1$. Therefore, it follows that

$$8N + 8 \equiv 0 \pmod{p}$$

or

$$N + 1 \equiv 0 \pmod{p}$$

if and only if $4n+1 \equiv 0 \pmod{p}$ and $4m+1 \equiv 0 \pmod{p}$. Hence, the congruences (2.28) now follows by employing (2.31) in (2.30) and then comparing the coefficients of q^{pn+r}. QED

Corollary 4. *We have*

$$c^*_{(1,7,14)}(17n+16) \equiv 0 \pmod{17}, \tag{2.32}$$

$$c^*_{(1,7,2)}(5n+4) \equiv 0 \pmod{5}, \tag{2.33}$$

$$c^*_{(1,7,10)}(13n+12) \equiv 0 \pmod{13}, \tag{2.34}$$

$$c^*_{(1,7,28)}(31n+30) \equiv 0 \pmod{31}, \tag{2.35}$$

$$c^*_{(1,7,16)}(19n+18) \equiv 0 \pmod{19}. \tag{2.36}$$

Proof. Setting $p = 17$ and $a = 14$ in (2.28) we obtain (2.32). For (2.33), we set $p = 5$ and $a = 2$ in (2.28).For (2.34), we set $p = 13$ and $a = 10$ in (2.28). For (2.35), we set $p = 31$ and $a = 28$ in (2.28). Finally, by setting $p = 19$ and $a = 16$ in (2.28) we obtain (2.36). QED

Theorem 5. *Suppose p is an odd prime divisor of $a+3$ and r is an integer with $0 \le r < p$. Suppose p and r satisfy any of the following two conditions:*

(1) $2r+3 \equiv 0 \pmod{p}$, $p \equiv 2$ *or* 6 *or* 7 *or* 8 *or* $10 \pmod{11}$ *and* $p \equiv 1 \pmod{4}$

(2) $2r+3 \equiv 0 \pmod{p}$, $p \equiv 2$ *or* 6 *or* 7 *or* 8 *or* $10 \pmod{11}$ *and* $p \equiv 3 \pmod{4}$

Then, $\forall n \ge 0$,

$$c^*_{(1,11,a)}(pn+r) \equiv 0 \pmod{p}. \tag{2.37}$$

Proof. Since p divides $a+3$, we can write $a+3 = pm$, for some integer m. Setting $r = 11$ in (1.3), we find that

$$\sum_{n=0}^{\infty} c^*_{(1,11,a)}(n)q^n = \frac{[E(q)E(q^{11})]^3}{[E(q)E(q^{11})]^{pm}}. \tag{2.38}$$

Employing (2.5) in (2.38), we obtain

$$\sum_{n=0}^{\infty} c^*_{(1,11,a)}(n)q^n = \frac{[E(q)E(q^{11})]^3}{[E(q^p)E(q^{11p})]^{m}}. \tag{2.39}$$

Using (2.4), we observe that

$$[E(q)E(q^{11})]^3 = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(4n+1)(4m+1)q^{[(4n+1)^2+11(4m+1)^2-12]/8}. \tag{2.40}$$

We note that

$$N = [(4n+1)^2 + 11(4m+1)^2 - 12]/8,$$

which is equivalent to

$$8N + 12 = (4n+1)^2 + 11(4m+1)^2.$$

If

$$p \equiv 2 \text{ or } 6 \text{ or } 7 \text{ or } 8 \text{ or } 10 \pmod{11} \;\&\; p \equiv 1 \pmod{4}$$

or

$$p \equiv 2 \text{ or } 6 \text{ or } 7 \text{ or } 8 \text{ or } 10 \pmod{11} \;\&\; p \equiv 3 \pmod{4}$$

then the Legendre symbol $\left(\frac{-11}{p}\right) = -1$. Therefore, it follows that

$$8N + 12 \equiv 0 \pmod{p}$$

or

$$2N + 3 \equiv 0 \pmod{p}$$

if and only if $4n+1 \equiv 0 \pmod{p}$ and $4m+1 \equiv 0 \pmod{p}$. Hence, the congruences (2.37) now follows by employing (2.40) in (2.39) and then comparing the coefficients of q^{pn+r}. QED

Corollary 5. *We have*

$$c^*_{(1,11,10)}(13n+5) \equiv 0 \pmod{13}, \tag{2.41}$$

$$c^*_{(1,11,14)}(17n+7) \equiv 0 \pmod{17}, \tag{2.42}$$

$$c^*_{(1,11,26)}(29n+13) \equiv 0 \pmod{29}, \tag{2.43}$$

$$c^*_{(1,11,76)}(79n+38) \equiv 0 \pmod{79}. \tag{2.44}$$

Proof. Setting $p = 13$ and $a = 10$ in (2.37) we obtain (2.41). For (2.42), we set $p = 17$ and $a = 14$ in (2.37). For (2.43), we set $p = 29$ and $a = 26$ in (2.37). Finally, by setting $p = 79$ and $a = 76$ in (2.37) we obtain (2.44). QED

Theorem 6. *Suppose p is an odd prime divisor of $a+3$ and r is an integer with $0 \le r < p$. Suppose p and r satisfy any of the following two conditions:*

(1) $4r+7 \equiv 0 \pmod{p}$, $p \equiv 2$ *or* 5 *or* 6 *or* 7 *or* 8 *or* 11 $\pmod{13}$ *and* $p \equiv 7 \pmod{4}$

(2) $4r+7 \equiv 0 \pmod{p}$, $p \equiv 1$ *or* 3 *or* 4 *or* 9 *or* 10 *or* 12 $\pmod{13}$ *and* $p \equiv 3 \pmod{4}$

Then, $\forall n \geq 0$,

$$c^*_{(1,13,a)}(pn+r) \equiv 0 \pmod{p}. \tag{2.45}$$

Proof. Since p divides $a+3$, we can write $a+3 = pm$, for some integer m. Setting $r = 13$ in (1.3), we find that

$$\sum_{n=0}^{\infty} c^*_{(1,13,a)}(n)q^n = \frac{[E(q)E(q^{13})]^3}{[E(q)E(q^{13})]^{pm}}. \tag{2.46}$$

Employing (2.5) in (2.46), we obtain

$$\sum_{n=0}^{\infty} c^*_{(1,13,a)}(n)q^n = \frac{[E(q)E(q^{13})]^3}{[E(q^p)E(q^{13p})]^{m}}. \tag{2.47}$$

Using (2.4), we observe that

$$[E(q)E(q^{13})]^3 = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(4n+1)(4m+1)q^{[(4n+1)^2+13(4m+1)^2-14]/8}. \tag{2.48}$$

We note that

$$N = [(4n+1)^2 + 13(4m+1)^2 - 14]/8,$$

which is equivalent to

$$8N + 14 = (4n+1)^2 + 13(4m+1)^2.$$

If

$$p \equiv 2 \textit{ or } 5 \textit{ or } 6 \textit{ or } 7 \textit{ or } 8 \textit{ or } 11 \pmod{13} \;\&\; p \equiv 7 \pmod{4}$$

or

$$p \equiv 1 \textit{ or } 3 \textit{ or } 4 \textit{ or } 9 \textit{ or } 10 \textit{ or } 12 \pmod{13} \;\&\; p \equiv 3 \pmod{4}$$

then the Legendre symbol $\left(\frac{-13}{p}\right) = -1$. Therefore, it follows that

$$8N + 14 \equiv 0 \pmod{p}$$

or

$$4N + 7 \equiv 0 \pmod{p}$$

if and only if $4n+1 \equiv 0 \pmod{p}$ and $4m+1 \equiv 0 \pmod{p}$. Hence, the congruences (2.45) now follows by employing (2.48) in (2.47) and then comparing the coefficients of q^{pn+r}. $\boxed{QED}$

Corollary 6. *We have*

$$c^*_{(1,13,38)}(41n+29) \equiv 0 \pmod{41}, \tag{2.49}$$

$$c^*_{(1,13,106)}(109n+80) \equiv 0 \pmod{109}, \tag{2.50}$$

$$c^*_{(1,13,128)}(131n+31) \equiv 0 \pmod{131}. \tag{2.51}$$

Proof. Setting $p = 41$ and $a = 38$ in (2.45) we obtain (2.49). For (2.50), we set $p = 109$ and $a = 106$ in (2.45). Finally, by setting $p = 131$ and $a = 128$ in (2.45) we obtain (2.51). QED

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