TWO-STAGE QUADRATIC GAMES UNDER UNCERTAINTY AND THEIR SOLUTION BY PROGRESSIVE HEDGING ALGORITHMS

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Abstract: A model of two-stage N-person non-cooperative game under uncertainty is studied, in which each player solves a quadratic program parameterized by other players' decisions at the first stage, then at the second stage the player solves a recourse quadratic program parameterized by the realization of a random vector, the second-stage decisions of other players, and the first stage decisions of all players. The problem of finding a Nash equilibrium of this game is shown to be equivalent to a stochastic linear complementarity problem. A linearly convergent progressive hedging algorithm is proposed for finding a Nash equilibrium if the resulting complementarity problem is monotone. For the non-monotone case, it is shown that, as long as the complementarity problem satisfies an additional elicitability condition, the progressive hedging algorithm can be modified to find a local Nash equilibrium at linear rate. The elicitability condition is reminiscent of the sufficient second-order optimality condition in nonlinear programming. Various numerical experiments indicate that the progressive hedging algorithms are efficient for mid-sized problems. In particular, the numerical results include a comparison with the best response method that was commonly adopted in the literature.

Key words. Multi-stage non-cooperative game under uncertainty, progressive hedging algorithm, stochastic linear complementarity problem, stochastic variational inequality.

AMS subject classification. 49J40, 90C15, 90C20, 91A10

1 Introduction

Many problems in non-cooperative game theory come with a structure where each player has to make a decision at a first stage and to make a recourse decision in response to a random event at a second stage, see Pang et al. [6] for a recent development of this topic and a list of references. The complication of finding a Nash equilibrium for such games is that the optimal strategy of each player is dependent not only on a random vector, but also on other players' strategies in both stages. As indicated in Wets [18] and in Pang et al. [6], such "entanglement" often jeopardizes the convexity and smoothness of the Nash equilibrium problem even if for each player the objectives and the constraints in both stages are smooth and convex. Technically, due to the non-smoothness of the recourse function, each player's objective function is at best directionally differentiable, which brings in serious difficulty in the design of efficient algorithms.

A recent development in the theory of stochastic variational inequality (SVI) due to Rockafellar and Wets [13] has brought in a new framework for dealing with multi-stage stochastic optimization and equilibrium problems. Rockafellar and Sun [11] suggested to use the progressive hedging algorithm (PHA) for solving these problems when the SVI is monotone. Their numerical experiments show that the PHA is efficient in general and is

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very efficient in particular, if the SVI reduces to a stochastic linear complementarity (SLC) problem.

It is therefore natural to ask the following question: Could PHA shed some light on solving the difficult game problem mentioned above? We try to answer this question affirmatively in three steps. First, we argue that the two-stage games under uncertainty can be converted to an SLC problem if the players' problems are linear-quadratic in both stages; second, we show that if the "private" quadratic term dominates the bi-linear "public" terms of each player in both stages, then the resulting SLC problem is monotone, therefore it can be efficiently solved by the PHA; third, even if the resulting SLC problem is not monotone, we develop an elicited version of the PHA for solving it. We show that if the SLC satisfies an "elicitability" condition, then an elicited version of the PHA will be locally convergent to the equilibrium at linear rate. This elicitability condition is similar to the second-order sufficient optimality condition in nonlinear programming. We provide numerical evidence to support the usage of PHA for both monotone and non-monotone games.

Let ξ be a random vector defined on the probability space $(\Xi, \mathcal{F}, \mathbb{P})$, where Ξ is a finite sample space, $\mathcal{F}$ is the σ-algebra generated by subsets of Ξ, and $\mathbb{P}$ is a probability measure defined on $\mathcal{F}$. We assume Ξ consists of K possible realizations (scenarios) of ξ. Each realization of ξ has a probability $p(\xi) > 0$ and these probabilities add up to one.

Consider a non-cooperative two-stage generalized Nash game of N players. Let $x_i \in \mathbb{R}^{n_i}$ and $y_i \in \mathbb{R}^{m_i}, i = 1, ..., N$, be the decision vectors of the ith player at first stage and at second stage, respectively. Let

$$x := (x_1, ..., x_N)^T \in \mathbb{R}^n, \quad n = n_1 + \cdots + n_N,$$

be the combined strategy vector of the N players in the first stage, where "T" stands for the transpose. As usual, we use

$$x_{-i} := (x_1, ..., x_{i-1}, x_{i+1}, ..., x_N)^T$$

to represent the combined strategies of all players other than i in the first stage and denote $n_{-i} = n - n_i$. We similarly define y and y_{-i} for the second stage, where

$$y := (y_1, ..., y_N)^T \in \mathbb{R}^m, \quad m = m_1 + \cdots + m_N \text{ and}$$

$$y_{-i} := (y_1, ..., y_{i-1}, y_{i+1}, ..., y_N)^T, \quad m_{-i} = m - m_i.$$

Assume that, in this two-stage generalized Nash game, player i solves the following two-stage stochastic optimization problem:

$$\min_{x_i \in X_i(x_{-i})} \quad \theta_i(x_i, x_{-i}) + \mathbb{E}_\xi[\psi_i(x_i, x_{-i}, \xi)], \tag{1}$$

where for each fixed x_{-i}, $X_i(x_{-i})$ is a closed convex set, and

$$\psi_i(x_i, x_{-i}, \xi) := \min_{y_i \in Y_i(x, y_{-i}, \xi)} \quad \phi_i(y_i, x, y_{-i}, \xi) \tag{2}$$

is the optimal value function of the recourse action y_i of player i at the second stage, and $Y_i(x, y_{-i}, \xi)$ is a certain closed convex set for each fixed (x, y_{-i}, ξ). Here, the name of "generalized Nash game" is due to that the constraints of each player's problem (1) depends on the others' strategies x_{-i}. This problem has a spectrum of applications ranging from production management to power market, see for instances Pang et al. [6] and Shanbhag

[14] and references therein. Since ψ_i is an optimal value function, the objective $\theta_i(x_i, x_{-i}) + \mathbb{E}_\xi[\psi_i(x_i, x_{-i}, \xi)]$ is generally nondifferentiable, and usually at most piece-wise smooth, hence it is difficult to solve this generalized Nash game by the exiting methods [1, 2, 5, 6].

Let $\mathscr{L}_{n+m}$ be the Hilbert space consisting of all "response functions" from Ξ to $\mathbb{R}^{n+m}$, equipped with the inner product

$$\langle z(\cdot), w(\cdot)\rangle := \mathbb{E}_\xi[z(\xi)^T w(\xi)] := \sum_{\xi\in\Xi} p(\xi) z(\xi)^T w(\xi),$$

where $\mathbb{E}_\xi$ stands for the expectation with respect to ξ. We designate $z(\cdot) := (x(\cdot), y(\cdot)) \in \mathscr{L}_{n+m}$, where $x(\cdot) \in \mathscr{L}_n$ and $y(\cdot) \in \mathscr{L}_m$ are respectively the x-part and the y-part of $z(\cdot)$. A solution, or a generalized Nash equilibrium, to the two-stage game (1)-(2) is defined as such a response function $z(\cdot)$ that, for fixed $z_{-i}(\cdot)$, $z_i(\cdot) := (x_i(\cdot), y_i(\cdot))$ is optimal to problem (1)-(2) for $i = 1, \ldots, N$.

A notational difference ought to be emphasized. By $z(\cdot)$ we mean a function from Ξ to $\mathbb{R}^{n+m}$, but by $z(\xi)$ we mean the image of ξ under the mapping $z(\cdot)$, where ξ is a certain scenario in Ξ. Namely,

$$z(\xi) := \begin{pmatrix} x(\xi) \\ y(\xi) \end{pmatrix}, \text{ with } x(\xi) = \begin{pmatrix} x_1(\xi) \\ \vdots \\ x_N(\xi) \end{pmatrix} \text{ and } y(\xi) = \begin{pmatrix} y_1(\xi) \\ \vdots \\ y_N(\xi) \end{pmatrix}. \tag{3}$$

This paper concentrates on the quadratic case of game (1)-(2), its reformulation as a stochastic linear complementarity problem in the form of Rockafellar and Sun [11], and its solution via progressive hedging algorithms. We assume that the first stage and the second stage problems are convex quadratic and the feasible sets $X_i(x_{-i})$ and $Y_i(x, y_{-i}, \xi)$ are convex polyhedra. Due to the decomposability of the objective functions and constraints with respect to ξ, the problem can be solved efficiently by the PHA. At each iteration, it first solves (1)-(2) for each individual scenario, regardless of the requirement that the first-stage decision should be independent of ξ (nonanticipativity). The resulted solutions $\hat{z}(\xi)$ are then projected to the nonanticipative subspace that results in the next iterate. More details of this PHA and its numerical behavior will be given in Sections 3 and 4.

The main contributions of this paper are as follows.

1. The model under study might be the most general one in the linear-quadratic category of stochastic Nash equilibrium problems (SNEPs) since it allows uncertainty and $z_{-i}(\cdot)$ to show up both in objective level and constraints level of all stages.

2. This paper clearly establishes an equivalence between a basic class of SNEPs and a stochastic complementarity problem. Therefore, it open a door for a different solution approach based on the new notion of SVI, which gets around the nonsmoothness of the players' objective functions and accommodates a decomposition scheme that may speed up the solution time.

3. The progressive hedging algorithm, originally designed for convex stochastic optimization [12] and monotone stochastic variational inequalities [11], is extended to non-monotone SLC problems. Theoretical results on convergence are established based on an elicitability condition.

4. A numerical comparison is presented between the best response approach and the progressive hedging approach for two-stage game problems under uncertainty. These numerical results, in particular the results on nonmonotone Nash games appears to be new in the literature.

The paper is organized as follows. We describe the special quadratic case of model (1) – (2) in Section 2 and formulate it as an SLC problem. We describe the PHA for problem (1)–(2) in Section 3 for both monotone and non-monotone cases, with a convergence analysis for the non-monotone PHA. Our numerical experiments are reported in Section 4. The paper is concluded in Section 5.

2 Problem formulation and its reduction to an SLC problem

2.1 Problem formulation

Consider a special case of (1)-(2), where

$$\theta_i(x_i, x_{-i}) \;:=\; \frac{1}{2}x_i^T Q_i x_i + c_i^T x_i + x_i^T R_{-i} x_{-i}, \tag{4}$$

with $Q_i \in \mathbb{R}^{n_i \times n_i}$, $R_{-i} \in \mathbb{R}^{n_i \times n_{-i}}$ and $c_i \in \mathbb{R}^{n_i}$, and

$$X_i(x_{-i}) := \{x_i \in \mathbb{R}_+^{n_i} : A_i x_i + A_{-i} x_{-i} \geq a_i\}, \tag{5}$$

with $A_i \in \mathbb{R}^{r_i \times n_i}$, $A_{-i} \in \mathbb{R}^{r_i \times n_{-i}}$ and $a_i \in \mathbb{R}^{n_i}$.

The objective function ϕ_i of the recourse problem is defined as

(6)

$$\phi_i(y_i, x, y_{-i}, \xi) := \frac{1}{2}y_i^T T_i(\xi) y_i + d_i(\xi)^T y_i + x_i^T S_i(\xi) y_i + y_i^T P_{-i}(\xi) x_{-i} + y_i^T O_{-i}(\xi) y_{-i},$$

and the feasible set of the recourse problem is

$$Y_i(x, y_{-i}, \xi) := \left\{ y_i \in \mathbb{R}_+^{m_i} : D_i(\xi) x_i + D_{-i}(\xi) x_{-i} + B_i(\xi) y_i + B_{-i}(\xi) y_{-i} \geq b_i(\xi) \right\}, \tag{7}$$

with $T_i : \Xi \to \mathbb{R}^{m_i \times m_i}$, $S_i : \Xi \to \mathbb{R}^{n_i \times m_i}$, $P_{-i} : \Xi \to \mathbb{R}^{m_i \times n_{-i}}$, $O_{-i} : \Xi \to \mathbb{R}^{m_i \times m_{-i}}$, $D_i : \Xi \to \mathbb{R}^{s_i \times n_i}$, $D_{-i} : \Xi \to \mathbb{R}^{s_i \times n_{-i}}$, $B_i : \Xi \to \mathbb{R}^{s_i \times m_i}$ and $B_{-i} : \Xi \to \mathbb{R}^{s_i \times m_{-i}}$ being random matrix functions, and $d_i : \Xi \to \mathbb{R}^{m_i}$ being a random vector function for all $i = 1, \cdots, N$.

2.2 Reformulation of the two-stage game into an SLC problem

It is important to note that the requirement of the x-part of $z(\cdot)$ being independent of ξ induces a constraint on any feasible solution $z(\cdot)$ to (1)-(2), which is called the nonanticipativity constraint. Nonanticipativity comes from the physical requirement that the decision x has to be made before ξ is realized. All $z(\cdot) \in \mathscr{L}_{n+m}$ satisfying the nonanticipativity constraint form a linear subspace $\mathcal{N}$ in $\mathscr{L}_{n+m}$. The orthogonal complement of $\mathcal{N}$ is then also important in theory and computation.

The nonanticipativity constraint also helps to normalize our notations, For example, from now on we can write $Y_i(x, y_{-i}, \xi)$ and $\phi_i(y_i, x, y_{-i}, \xi)$ respectively as $Y_i(x(\xi), y_{-i}(\xi), \xi)$ and $\phi_i(y_i(\xi), x(\xi), y_{-i}(\xi), \xi)$ in (7).

In addition to nonanticipativity, a feasible $z(\cdot)$ to (1)-(2) must satisfy that $\forall \xi \in \Xi$, $z_i(\xi)$ belongs to

$$C_i(z_{-i}(\xi),\xi) := \left\{ z_i(\xi) = \begin{pmatrix} x_i(\xi) \\ y_i(\xi) \end{pmatrix} : x_i(\xi) \in X_i(x_{-i}(\xi)), y_i(\xi) \in Y_i(x(\xi), y_{-i}(\xi), \xi) \right\},$$

which we call admissibility.

By the definitions of $X_i(x_{-i}(\xi))$ and $Y_i(x(\xi), y_{-i}(\xi), \xi)$ in (5) and (7), each of such $C_i(z_{-i}(\xi),\xi)$ is a convex polyhedron of $z_i(\xi)$ for fixed $z_{-i}(\xi)$. Define

$$\mathcal{C}_i(z_{-i}(\cdot)) := \{z_i(\cdot) \in \mathscr{L}_{n_i+m_i} : z_i(\xi) \in C_i(z_{-i}(\xi),\xi) \ \forall \ \xi\}. \tag{8}$$

Then $\mathcal{C}_i(z_{-i}(\cdot))$ is a convex polyhedron in $\mathscr{L}_{n_i+m_i}$ and one can re-write the problem of player i as an optimization problem in $\mathscr{L}_{n_i+m_i}$ as follows.

The objective function of player i is

$$\begin{aligned} & \mathbb{E}_\xi[\theta_i(x_i(\xi), x_{-i}(\xi), \xi) + \phi_i(y_i(\xi), x(\xi), y_{-i}(\xi), \xi)] \\ = \quad & \mathbb{E}_\xi \left[\frac{1}{2} z_i(\xi)^T \bar{Q}_i(\xi) z_i(\xi) + (\bar{c}_i(\xi) + \bar{R}_{-i}(\xi) z_{-i}(\xi))^T z_i(\xi) \right] \\ =: \quad & \mathbb{E}_\xi[f_i(z_i(\xi), z_{-i}(\xi), \xi)] \ (\text{“=:” means “denote it by”}) \\ =: \quad & \mathcal{G}_i(z_i(\cdot), z_{-i}(\cdot)), \end{aligned}$$

where

$$\bar{Q}_i(\xi) = \begin{pmatrix} Q_i & S_i(\xi) \\ S_i^T(\xi) & T_i(\xi) \end{pmatrix}, \quad \bar{c}_i(\xi) = \begin{pmatrix} c_i \\ d_i(\xi) \end{pmatrix}, \text{ and } \bar{R}_{-i}(\xi) = \begin{pmatrix} R_{-i} & 0 \\ P_{-i}(\xi) & O_{-i}(\xi) \end{pmatrix}.$$

The constraints for player i are $z_i(\cdot) \in \mathcal{N}_i \cap \mathcal{C}_i(z_{-i}(\cdot))$, where $\mathcal{N}_i$ is the nonanticipativity subspace of $z_i(\cdot)$, and $\mathcal{C}_i(z_{-i}(\cdot))$ is defined as in (8) with the following specific $C_i(z_{-i}(\xi),\xi)$

$$C_i(z_{-i}(\xi),\xi) = \left\{ z_i(\xi) : \bar{A}_i(\xi) z_i(\xi) \geq \bar{b}_i(\xi) - \bar{A}_{-i}(\xi) z_{-i}(\xi) \text{ and } z_i(\xi) \geq 0 \right\},$$

where

$$\bar{A}_i(\xi) = \begin{pmatrix} A_i & 0 \\ D_i(\xi) & B_i(\xi) \end{pmatrix}, \bar{A}_{-i}(\xi) = \begin{pmatrix} A_{-i} & 0 \\ D_{-i}(\xi) & B_{-i}(\xi) \end{pmatrix}, \text{ and } \bar{b}_i(\xi) = \begin{pmatrix} a_i \\ b_i(\xi) \end{pmatrix}.$$

Finally, let $\delta_{\mathcal{N}_i}(z_i(\cdot))$ be the indicator function of $\mathcal{N}_i$. Then the problem of player i can be written as

$$\min_{z_i(\cdot) \in \mathcal{C}_i(z_{-i}(\cdot))} \quad \mathcal{G}_i(z_i(\cdot), z_{-i}(\cdot)) + \delta_{\mathcal{N}_i}(z_i(\cdot)). \tag{9}$$

Assuming constraint qualification

$$\mathcal{C}_i(z_{-i}(\cdot)) \cap \mathcal{N}_i \neq \emptyset, \tag{10}$$

then a necessary condition for optimality of (9) is that $\exists \lambda_i(\cdot) \in \mathcal{M}_i := \mathcal{N}_i^\perp$ and a dual vector $\eta_i(\cdot)$ such that for each $\xi \in \Xi$, the following KKT condition holds

(11)

$$0 \leq \begin{pmatrix} z_i(\xi) \\ \eta_i(\xi) \end{pmatrix} \perp \begin{pmatrix} \bar{Q}_i(\xi) & -\bar{A}_i^T(\xi) \\ \bar{A}_i(\xi) & 0 \end{pmatrix} \begin{pmatrix} z_i(\xi) \\ \eta_i(\xi) \end{pmatrix} + \begin{pmatrix} \bar{c}_i(\xi) + \lambda_i(\xi) \\ -\bar{b}_i(\xi) \end{pmatrix} + \begin{pmatrix} \bar{R}_{-i}(\xi) & 0 \\ \bar{A}_{-i}(\xi) & 0 \end{pmatrix} \begin{pmatrix} z_{-i}(\xi) \\ \eta_{-i}(\xi) \end{pmatrix} \geq 0.$$

Moreover, if $\bar{Q}_i(\xi)$ is positive semidefinite for all ξ and the optimal value of (9) is finite, then the KKT condition (11) is also sufficient for the existence of optimal $z_i(\cdot), \lambda_i(\cdot)$, and $\eta_i(\cdot)$. However, we don't assume $\bar{Q}_i$ to be positive semi-definite in the following analysis.

Now suppose that

$$R_{-i}x_{-i} = \sum_{j\neq i} R_{ij}x_j, \quad P_{-i}(\xi)x_{-i} = \sum_{j\neq i} P_{ij}(\xi)x_j, \quad O_{-i}(\xi)y_{-i} = \sum_{j\neq i} O_{ij}(\xi)y_j,$$
$$A_{-i}x_{-i} = \sum_{j\neq i} A_{ij}x_j, \quad D_{-i}(\xi)x_{-i} = \sum_{j\neq i} D_{ij}(\xi)x_j, \quad B_{-i}(\xi)y_{-i} = \sum_{j\neq i} B_{ij}(\xi)y_j,$$

and denote

$$u_i(\xi) = \begin{pmatrix} z_i(\xi) \\ \eta_i(\xi) \end{pmatrix}, \; u(\xi) = \begin{pmatrix} u_1(\xi) \\ \vdots \\ u_N(\xi) \end{pmatrix}, \text{ and } \; q_i(\xi) = \begin{pmatrix} \bar{c}_i(\xi) \\ -\bar{b}_i(\xi) \end{pmatrix}, \text{ respectively,}$$

then the condition (11) can be written as

$$0 \le u_i(\xi) \perp M_i(\xi)u(\xi) + q_i(\xi) + \begin{pmatrix} \lambda_i(\xi) \\ 0 \end{pmatrix} \ge 0. \tag{12}$$

where $M_i(\xi) = (U_{i1}(\xi) \cdots U_{iN}(\xi))$, with

$$U_{ii}(\xi) = \begin{pmatrix} \bar{Q}_i(\xi) & -\bar{A}_i(\xi)^T \\ \bar{A}_i(\xi) & 0 \end{pmatrix} \text{ and } U_{ij}(\xi) = \begin{pmatrix} \bar{R}_{ij}(\xi) & 0 \\ \bar{A}_{ij}(\xi) & 0 \end{pmatrix} \; \forall\, j \neq i,$$

where

$$\bar{R}_{ij}(\xi) = \begin{pmatrix} R_{ij} & 0 \\ P_{ij}(\xi) & O_{ij}(\xi) \end{pmatrix}, \; \bar{A}_{ij}(\xi) = \begin{pmatrix} A_{ij} & 0 \\ D_{ij}(\xi) & B_{ij}(\xi) \end{pmatrix} \; \forall\, j \neq i.$$

The Nash equilibrium of the game requires condition (12) to hold for all players, writing all such conditions together, the necessary conditions of the Nash equilibrium of the quadratic game under uncertainty (4)-(7) is

$$u(\cdot) \in \hat{\mathcal{N}}, \; \lambda(\cdot) \in \hat{\mathcal{M}} \text{ such that } 0 \le u(\xi) \perp M(\xi)u(\xi) + q(\xi) + \lambda(\xi) \ge 0, \; \forall \xi \in \Xi,$$

where $\hat{\mathcal{N}} = \{u(\cdot) :$ The x-part of $u(\cdot)$ is independent of $\xi\}$, $\hat{\mathcal{M}} = \hat{\mathcal{N}}^\perp$,

$$M(\xi) = \begin{pmatrix} M_1(\xi) \\ \vdots \\ M_N(\xi) \end{pmatrix}, \text{ and } q(\xi) = \begin{pmatrix} q_1(\xi) \\ \vdots \\ q_N(\xi) \end{pmatrix}.$$

To obtain a matrix with an easier-understood structure, we re-arrange the order of the variables as follows. Put all players' first-stage decision variables together as x-part and second-stage variables together as y-part, meanwhile put all dual variables corresponding to the first-stage constraints, followed by the dual variables corresponding to the second-stage constraints and denote the entire dual vector as $\zeta(\cdot)$. Besides, let

$$\omega(\xi) = (\lambda_1(\xi), \cdots, \lambda_N(\xi), 0, \cdots, 0)^T.$$

Then (12) becomes

$$\begin{aligned} &\exists\, z(\cdot) \in \mathcal{N} \text{ and } \omega(\cdot) \in \mathcal{M} \text{ such that} \\ &0 \le \begin{pmatrix} z(\xi) \\ \zeta(\xi) \end{pmatrix} \perp \begin{pmatrix} H_{11}(\xi) & H_{12}(\xi) \\ H_{21}(\xi) & 0 \end{pmatrix} \begin{pmatrix} z(\xi) \\ \zeta(\xi) \end{pmatrix} + \begin{pmatrix} \bar{c}(\xi) \\ -\bar{b}(\xi) \end{pmatrix} + \begin{pmatrix} \omega(\xi) \\ 0 \end{pmatrix} \ge 0, \; \forall \xi \in \Xi, \end{aligned} \tag{13}$$

where

$$
(14) \qquad H_{11}(\xi) = \begin{pmatrix}
Q_1 & R_{12} & \cdots & R_{1N} & S_1(\xi) & & & \\
R_{21} & Q_2 & \cdots & R_{2N} & & S_2(\xi) & & \\
\vdots & \vdots & \vdots & \vdots & & & \ddots & \\
R_{N1} & R_{N2} & \cdots & Q_N & & & & S_N(\xi) \\
S_1(\xi)^T & P_{12}(\xi) & \cdots & P_{1N}(\xi) & T_1(\xi) & O_{12}(\xi) & \cdots & O_{1N}(\xi) \\
P_{21}(\xi) & S_2(\xi)^T & \cdots & P_{2N}(\xi) & O_{21}(\xi) & T_2(\xi) & \cdots & O_{2N}(\xi) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
P_{N1}(\xi) & P_{N2}(\xi) & \cdots & S_N(\xi)^T & O_{N1}(\xi) & O_{N2}(\xi) & \cdots & T_N(\xi)
\end{pmatrix},
$$

$$
H_{12}(\xi) = \begin{pmatrix}
-A_1^T & & & & -D_1(\xi)^T & & & \\
& -A_2^T & & & & -D_2(\xi)^T & & \\
& & \ddots & & & & \ddots & \\
& & & -A_N^T & & & & -D_N(\xi)^T \\
& & & & -B_1(\xi)^T & & & \\
& & & & & -B_2(\xi)^T & & \\
& & & & & & \ddots & \\
& & & & & & & -B_N(\xi)^T
\end{pmatrix},
$$

and

$$
H_{21}(\xi) = \begin{pmatrix}
A_1 & A_{12} & \cdots & A_{1N} & & & & \\
A_{21} & A_2 & \cdots & A_{2N} & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & \\
A_{N1} & A_{N2} & \cdots & A_N & & & & \\
D_1(\xi) & D_{12}(\xi) & \cdots & D_{1N}(\xi) & B_1(\xi) & B_{12}(\xi) & \cdots & B_{1N}(\xi) \\
D_{21}(\xi) & D_2(\xi) & \cdots & D_{2N}(\xi) & B_{21}(\xi) & B_2(\xi) & \cdots & B_{2N}(\xi) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
D_{N1}(\xi) & D_{N2}(\xi) & \cdots & D_N(\xi) & B_{N1}(\xi) & B_{N2}(\xi) & \cdots & B_N(\xi)
\end{pmatrix}.
$$

The blank parts of the matrices are all zeros.

In summary, we have shown the following result. Let $\mathcal{C} = \{z(\cdot) \in \mathscr{L}_{n+m} : z_i(\cdot) \in \mathcal{C}_i(z_{-i}), \forall i\}$ and $\mathcal{N} = \mathcal{N}_1 \times \cdots \times \mathcal{N}_N$.

Theorem 2.1 *Under the constraint qualification that $\mathcal{C} \cap \mathcal{N} \neq \emptyset$, the problem of finding a Nash equilibrium of (4)-(7) can be converted to a stochastic linear complementarity problem. More specifically, suppose in addition the optimal value of (9) is finite for every i, then a necessary condition for $z^*(\cdot)$ being a Nash equilibrium of the two-stage stochastic game (4)-(7) is that $z^*(\cdot) \in \mathcal{N}$ and there exist $\omega^*(\cdot) \in \mathcal{M}$ and $\zeta^*(\cdot)$ such that the stochastic linear complementarity problem (13) holds at $z^*(\cdot), \omega^*(\cdot)$, and $\zeta^*(\cdot)$.*

Conversely, if $\bar{Q}_i(\xi)$ is positive semidefinite for all ξ and all i, then the solution to (13) is a global Nash equilibrium of (4)-(7).

Remark 2.1 *Consider two special cases:*

- *Player i's constraints are independent of other players' strategies (only the objective involves other players' strategies), which we call the* autonomously constrained *case . In our two-stage stochastic game (4)-(7), this means for all $j \neq i$ $(i = 1, \cdots, N)$*

$$A_{ij} = 0, \; B_{ij}(\xi) = 0, \; D_{ij}(\xi) = 0 \; \forall \; \xi,$$

thus $H_{21}(\xi) = -H_{12}(\xi)^T$.

- *Let us call the case of* $\bar{Q}_i(\xi)$ *being positive semidefinite for all* ξ *and all* i *the* privately convex *case for the game. Various sufficient conditions can be deduced for the solvability of (11) via the theory of linear complementarity [3] and we do not go further in that direction. We just point out that the condition of private convexity alone cannot guarantee the monotonicity of problem (13). That is, even every player's problem is convex, the Nash equilibrium problem may still be non-monotone.*

A possible approach to solving Problem (13) is the progressive hedging algorithm, which will be discussed in the next section.

3 Finding an equilibrium via progressive hedging

3.1 The Monotone Case

The progressive hedging algorithm (PHA) was originally designed for multi-stage stochastic minimization problems by Rockafellar and Wets [12] and it has been recently extended in [11] to the monotone SVI problems of the form

$$z(\cdot) \in \mathcal{N},\ \omega(\cdot) \in \mathcal{M},\ 0 \in [\mathcal{F} + N_{\mathcal{C}}](z(\cdot)) + \omega(\cdot) \tag{15}$$

where $\mathcal{F}$ is a continuous mapping. As a special case, monotone stochastic complementarity problems can be solved via PHA, and numerical results in [11] showed its efficiency.

According to Spingarn [16, 17], the PHA is a special version of the proximal point algorithm (PPA) developed by Rockafellar [9] applied to a set-valued mapping $A\mathcal{T}_{\mathcal{N}}A$, where A is a certain symmetric nonsingular matrix and $\mathcal{T}_{\mathcal{N}}$ is the partial inverse of mapping $\mathcal{F}+N_{\mathcal{C}}$ with respect to subspace $\mathcal{N}$. When applied to convex linear-quadratic stochastic multistage optimization, the convergence of PHA is guaranteed at a q-linear rate if an optimal solution exists.

The PHA for monotone stochastic complementarity problem (13) developed in [11] could be stated as follows.

Algorithm 1. PHA for two-stage quadratic games under uncertainty

Initiation. Set $z^0(\xi) = 0$, $\zeta^0(\xi) = 0$, $\omega^0(\xi) = 0$ for all ξ, and $k = 0$.

Iterations.

Step 1. For each $\xi \in \Xi$, obtain $\hat{z}^k(\xi)$ and $\hat{\zeta}^k(\xi)$ via the following LCP

$$0 \le \begin{pmatrix} z(\xi) \\ \zeta(\xi) \end{pmatrix} \perp \begin{pmatrix} H_{11}(\xi) & H_{12}(\xi) \\ H_{21}(\xi) & 0 \end{pmatrix} \begin{pmatrix} z(\xi) \\ \zeta(\xi) \end{pmatrix} + \begin{pmatrix} \bar{c} \\ -\bar{b}(\xi) \end{pmatrix} + \begin{pmatrix} \omega^k(\xi) \\ 0 \end{pmatrix} + r \begin{pmatrix} z(\xi) - z^k(\xi) \\ 0 \end{pmatrix} \ge 0. \tag{16}$$

Step 2. (Primal Update)

$$x^{k+1} = \mathbb{E}_\xi(\hat{x}^k(\xi)),\ z^{k+1}(\xi) = \begin{pmatrix} x^{k+1} \\ \hat{y}^k(\xi) \end{pmatrix},\ \zeta^{k+1}(\xi) = \hat{\zeta}^k(\xi).$$

Step 3. (Dual Update) $\omega^{k+1}(\xi) = \omega^k(\xi) + r(\hat{z}^k(\xi) - z^{k+1}(\xi))$.

Set $k := k+1$, **repeat** until a stopping criterion is met.

Observe that Step 1 of Algorithm 1 is to find a solution to a linear complementarity problem for every scenario. Putting all scenario solutions together, we obtain $\hat{z}^k(\cdot)$. Since the solution $\hat{z}^k(\cdot)$ may not satisfy the nonanticipativity constraint, the primal update makes a projection on $\mathcal{N}$ and the dual update makes a move in $\mathcal{M}$ because $z^{k+1}(\cdot) = P_{\mathcal{N}}(\hat{z}^k(\cdot))$ which yields $\hat{z}^k(\cdot) - z^{k+1}(\cdot) \in \mathcal{M}$.

Algorithm 1 is a gradient-based method, so it is not surprising that the convergence rate is at best linear, However, since it is also a proximal point based method, the rate θ_k in the estimate

$$\|(z^{k+1}(\cdot), \omega^{k+1}(\cdot)) - (z^*(\cdot), \omega^*(\cdot))\|_r \le \theta_k \|(z^k(\cdot), \omega^k(\cdot)) - (z^*(\cdot), \omega^*(\cdot))\|_r$$

can be made arbitrarily close to zero if a certain strong regularity assumption is satisfied (Rockafellar [9, p.886]. Thus, by taking a carefully chosen large r, the algorithm could converge reasonably fast.

The spirit of Algorithm 1 is to find a collective solution $z(\cdot)$ for all players by an interactive procedure, which is different from the idea of the best response method (BRM) in the literature. Stochastic versions of the BRM can be found, for examples, as the Sampled Best-Response Algorithms (BRM) for $\mathcal{G}^{\text{prMD}}$ in Pang et al. [6] and the inexact best response methods for SNEPs in Shanbhag et al. [15]. In principle, the best response methods are based on a special reformulation of the game (4)-(7), which requires convexity and differentiability. Thus, the model in [6] has no $x_i^T S_i(\xi) y_i(\xi)$ in the objective of ψ_i, the recourse ψ_i is not involved the other players' second-stage strategies y_{-i}, and the matrix before y_i in the objective and constraint are independent of ξ, which is actually the so-called fixed recourse. As long as the best-response mapping is continuous and contractive, the Nash equilibrium may be achieved successfully. Notice that the optimization involved in the best-response mapping turns to be a two-stage stochastic programming problem. In other words, when applying BRM to solve a two-stage stochastic game, at every iteration, each player needs to solve a two-stage stochastic optimization problem based on the information of others' strategies of previous iteration.

The convergence of Algorithm 1 requires that the game (4)-(7) has a solution and the corresponding SLC problem is monotone [11], which requires positive semidefiniteness of the following matrix for all $\xi \in \Xi$,

$$H(\xi) = \begin{pmatrix} H_{11}(\xi) & H_{12}(\xi) \\ H_{21}(\xi) & 0 \end{pmatrix}. \tag{17}$$

Note that to check the positive semidefiniteness of $H(\xi)$ is a challenging job. We therefore turn to the autonomously constrained case, in which $H_{12}(\xi) = -H_{21}(\xi)^T$, so we only have to determine the positive semidefiniteness of smaller-sized matrix $H_{11}(\xi)$. In practice it is quite common for the players to have interaction with other players in their objective functions. To simplify the analysis, we denote $H_{11}(\xi)$ as four blocks, i.e.,

$$H_{11}(\xi) = \begin{pmatrix} \bar{H}_{11} & \bar{H}_{12}(\xi) \\ \bar{H}_{21}(\xi) & \bar{H}_{22}(\xi) \end{pmatrix}, \tag{18}$$

which is correspondingly defined in (14). It should be pointed out that $H_{11}(\xi)$ is generally non-symmetric due to the existence of R_{ij}, $P_{ij}(\xi)$ and $O_{ij}(\xi)$. However, the positive semidefiniteness of matrix $H_{11}(\xi)$ can be guaranteed by the diagonal dominance of $H_{11}(\xi)$, namely, for $i = 1, ..., N$, if one has

$$|q_{ii}| \geq \sum_{j \neq i} |q_{ij}| + \sum_{j \neq i} \sum_{k} |(R_{ij})_{ik}| + \sum_{j} |s_{ij}(\xi)|, \ \forall \ \xi,$$

and

$$|t_{ii}(\xi)| \geq \sum_{j \neq i} |t_{ji}(\xi)| + \sum_{j} |s_{ji}(\xi)| + \sum_{j \neq i} \sum_{k} (|(P_{ij}(\xi))_{ik}| + |(O_{ij}(\xi))_{ik}|), \ \forall \ \xi.$$

These diagonal dominance conditions are strong, but they guarantee Algorithm 1 to converge to a global Nash equilibrium if such a point exists.

3.2 The Non-monotone Case

Next we investigate the possibility of a "non-monotone version" of PHA in this subsection and apply it to two-stage quadratic game with uncertainty. The non-monotone version of PHA was inspired by Rockafellar [10]. The word "elicited monotonicity" is also due to him.

Definition 3.1 *Monotonicity of $\mathcal{F} + N_{\mathcal{C}}$ is said to be elicitable (or elicited) at level $s > 0$:*

- *globally if $\mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}}$ is maximal monotone globally, and*
- *locally around $(z, y) \in \operatorname{graph}[\mathcal{F} + N_{\mathcal{C}}]$ with $z \in \mathcal{N}, y \in \mathcal{M}$, if $\mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}}$ is maximal monotone locally around (z, y), where $P_{\mathcal{M}}$ is the projection operator on subspace $\mathcal{M}$.*

Definition 3.2 *Let $\mathcal{T}$ be set-valued mapping. The partial inverse of $\mathcal{T}$ with respect to $\mathcal{N}$ is the set-valued mapping $\mathcal{T}_{\mathcal{N}} : \mathscr{L}_{n+m} \to \mathscr{L}_{n+m}$ defined by*

$$v(\cdot) \in \mathcal{T}_{\mathcal{N}}(u(\cdot)) \iff P_{\mathcal{M}}(u(\cdot)) + P_{\mathcal{N}}(v(\cdot)) \in \mathcal{T}(P_{\mathcal{N}}(u(\cdot)) + P_{\mathcal{M}}(v(\cdot))).$$

Spingarn [16] showed that

- $\mathcal{T}_{\mathcal{N}}$ is (maximal) monotone iff $\mathcal{T}$ is (maximal) monotone.
- The following two problems are equivalent

  (19) Find $u(\cdot) \in \mathcal{N}$ and $v(\cdot) \in \mathcal{M}$ such that $v(\cdot) \in \mathcal{T}(u(\cdot))$,

  (20) Find $u(\cdot) \in \mathcal{N}$ and $v(\cdot) \in \mathcal{M}$ such that $0 \in \mathcal{T}_{\mathcal{N}}(u(\cdot) + v(\cdot))$.

The elicited PHA is based on the fact that, although $\mathcal{F} + N_{\mathcal{C}}$ is not monotone, the mapping $\mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}}$ may be maximal monotone for large $s > 0$. Moreover, it is easy to show that

$$(\mathcal{F} + N_{\mathcal{C}})^{-1}_{\mathcal{N}}(0) = (\mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}})^{-1}_{\mathcal{N}}(0) \text{ for any } s > 0.$$

Then one can apply the proximal point algorithm to $(\mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}})_{\mathcal{N}}$, instead of $(\mathcal{F} + N_{\mathcal{C}})_{\mathcal{N}}$, to obtain a solution to the SVI, which results in the following algorithm.

Algorithm 2. Elicited PHA for elicitable SVI

Initiation. Let parameter $r > s \geq 0$. Set $z^0(\xi) = 0$, $\omega^0(\xi) = 0$ for all ξ, and $k = 0$.
Iterations.
Step 1. For each $\xi \in \Xi$, $\hat{z}^k(\xi) :=$ the unique $z(\xi)$ such that
$-\mathcal{F}(z(\xi)) - \omega^k(\xi) - r[z(\xi) - z^k(\xi)] \in N_{C(\xi)}(z(\xi))$.
Step 2. (Primal Update) $z^{k+1}(\cdot) = P_{\mathcal{N}}(\hat{z}^k)$.
Step 3. (Dual Update) $\omega^{k+1}(\xi) = \omega^k(\xi) + (r - s)(\hat{z}^k(\xi) - z^{k+1}(\xi))$.
Set $k := k + 1$, **repeat** until a stopping criterion is met.

It is interesting to note that the only difference between Algorithm 2 and Algorithm 1 is that r in the dual update step of Algorithm 1 is replaced by $r - s$, although the idea and convergence proof are not that simple. In the following theorem, we show that Algorithm 2 is in fact an application of PPA to the partial inverse of $\mathcal{T} = \mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}}$ for some $s \in [0, r)$. Hence, the convergence rate is q-linear in the special case of elicitable SLC.

Theorem 3.1 *Suppose that $\mathcal{F} + N_{\mathcal{N}}$ is globally elicitable at level s. Then Algorithm 2 is equivalent to PPA for $A\mathcal{T}_{\mathcal{N}}A$, where $\mathcal{T}_{\mathcal{N}}$ is the partial inverse of $\mathcal{F} + N_{\mathcal{C}} + sP_{\mathcal{M}}$, and A is a non-singular linear operator defined as $A : u(\cdot) \mapsto P_{\mathcal{N}}(u(\cdot)) + \sqrt{r(r-s)}P_{\mathcal{M}}(u(\cdot))$.*

Moreover, in the special case that $\mathcal{F}$ is linear and $\mathcal{C}$ is polyhedral, if $\mathcal{N} \cap \mathcal{C} \neq \emptyset$ and the SVI problem has a solution, then the sequence $\{z^k(\cdot), \omega^k(\cdot)\}$ generated by Algorithm 2 will globally converge to some pair $\{z^*(\cdot), \omega^*(\cdot)\}$ with $z^*(\cdot)$ being a solution to (15) at linear rate with respect to the norm

$$\|(z(\cdot), \omega(\cdot))\|_{r,s}^2 = \|z(\cdot)\|^2 + \frac{1}{r(r-s)}\|\omega(\cdot)\|^2.$$

Proof. It is known that the iterates of PHA is

$$u^{k+1}(\cdot) = (I + r^{-1}A\mathcal{T}_{\mathcal{N}}A)^{-1}(u^k(\cdot)),$$

which is equivalent to

$$rA^{-2}(Au^k(\cdot) - Au^{k+1}(\cdot)) \in \mathcal{T}_{\mathcal{N}}(Au^{k+1}(\cdot)).$$

Let $v(\cdot) := Au(\cdot)$. Since $rA^{-2} : u(\cdot) \mapsto rP_{\mathcal{N}}(u(\cdot)) + \frac{1}{r-s}P_{\mathcal{M}}(u(\cdot))$, we have

$$rP_{\mathcal{N}}(v^k(\cdot) - v^{k+1}(\cdot)) + \frac{1}{r-s}P_{\mathcal{M}}(v^k(\cdot) - v^{k+1}(\cdot)) \in \mathcal{T}_{\mathcal{N}}(v^{k+1}(\cdot)).$$

From the definition of $\mathcal{T}_{\mathcal{N}}$, one can obtain that

$$\begin{aligned} & rP_{\mathcal{N}}(v^k(\cdot) - v^{k+1}(\cdot)) + P_{\mathcal{M}}(v^{k+1}(\cdot)) - \tfrac{s}{r-s}P_{\mathcal{M}}(v^k(\cdot) - v^{k+1}(\cdot)) \\ \in\ & (\mathcal{F} + N_{\mathcal{C}})[P_{\mathcal{N}}(v^{k+1}(\cdot)) + \tfrac{1}{r-s}P_{\mathcal{M}}(v^k(\cdot) - v^{k+1}(\cdot))]. \end{aligned} \tag{21}$$

Set $z^k(\cdot) := P_{\mathcal{N}}(v^k(\cdot))$, $\omega^k(\cdot) := -P_{\mathcal{M}}(v^k(\cdot))$ and

$$\hat{z}^k(\cdot) := P_{\mathcal{N}}(v^{k+1}(\cdot)) + \frac{1}{r-s}P_{\mathcal{M}}(v^k(\cdot) - v^{k+1}(\cdot)) = z^{k+1}(\cdot) + \frac{1}{r-s}(\omega^{k+1}(\cdot) - \omega^k(\cdot)),$$

Then, $z^{k+1}(\cdot) = P_{\mathcal{N}}(\hat{z}^k(\cdot))$ and $\omega^{k+1}(\cdot) = \omega^k(\cdot) + (r-s)(\hat{z}^k(\cdot) - z^{k+1}(\cdot))$, which coincide with Step 2 and Step 3 of Algorithm 2, based on the definition of $\mathcal{N}$ and $\mathcal{M}$, and we have that (21) is equivalent to

$$r(z^k(\cdot) - \hat{z}^k(\cdot)) + \omega^k(\cdot) + r(\hat{z}^k(\cdot) - z^{k+1}(\cdot) + \frac{1}{r-s}(\omega^{k+1}(\cdot) - \lambda^k(\cdot))) \in (\mathcal{F} + N_{\mathcal{C}})(\hat{z}^k(\cdot)),$$

which is just the Step 1 of Algorithm 2:

$$-\mathcal{F}(\hat{z}^k(\cdot)) - \omega^k(\cdot) - r(\hat{z}^k(\cdot) - z^k(\cdot)) \in N_{\mathcal{C}}(\hat{z}^k(\cdot)).$$

Therefore, the equivalence of Algorithm 2 and PPA for $A\mathcal{T}_{\mathcal{N}}A$ is established.

The second part comes directly by the convergence results of PPA (Theorem 2 in Rockafellar [9]). ■

Theorem 3.1 was first shown by Rockafellar in [10] and our proof above is different and simplified compared to the one in [10].

Back to the elicitability of the SLC problem (16), we next present a result of Rockafellar and use it to derive a sufficient condition for elicitability of the SLC problem.

Lemma 3.1 (Theorem 5 of [10]) *Let S be a symmetric matrix in $\mathbb{R}^{p\times p}$. Suppose L is a linear subspace in $\mathbb{R}^p$ and $M = L^\perp$. Let P_L and P_M be the projection matrices from $\mathbb{R}^p$ to L and M, respectively. Suppose that*

$$\exists \alpha > 0 : \quad \langle x, Ax\rangle > \alpha \|x\|^2 \quad \forall 0 \neq x \in L.$$

Let

$$\beta = \|P_L S P_L\| \text{ and } \gamma = \|P_M S P_M\|.$$

Then $G = S + sP_M \succ 0$ for all $s > \alpha^{-1}\beta^2 + \gamma$.

Theorem 3.2 *Let $diag(H(\xi))$ be the block-diagonal matrix, consisting of blocks $H(\xi)$ for all ξ and let $diag(\mathbf{H}(\xi))$ be its symmetric part, i.e.,*

$$diag(\mathbf{H}(\xi)) := [\,diag(H(\xi)) + \, diag(H(\xi))^T]/2.$$

If $diag(\mathbf{H}(\xi))$ is positive definite on $\mathcal{N}$, then $\mathcal{F} + sP_{\mathcal{M}} + N_{\mathcal{C}}$ is maximal monotone for some large $s > 0$, where $\mathcal{F} : \mathscr{L}_{n+m} \to \mathscr{L}_{n+m}$ is the mapping defined by $\mathcal{F}(z(\xi)) = H(\xi)z(\xi) + q(\xi)$.

Proof. Let the cardinality of Ξ be K and let z be the vector made by stacking up the vectors $z(\xi)$ for all ξ. So $z \in \mathbb{R}^{K(n+m)}$. Let $v \in \mathbb{R}^{K(n+m)}$ be the scaled z such that $v(\xi) = \sqrt{p(\xi)}z(\xi)$, where $p(\xi)$ is the probability of scenario ξ. Then we have

$$v^T v = \sum_{\xi\in\Xi} v(\xi)^T v(\xi) = \sum_{\xi\in\Xi} p(\xi) z(\xi)^T z(\xi) = \langle z(\cdot), z(\cdot)\rangle. \tag{22}$$

There exists an idempotent matrix $P_M \in \mathbb{R}^{K(n+m)\times K(n+m)}$ such that

$$v^T P_M v = \langle P_{\mathcal{M}}(z(\cdot)), z(\cdot)\rangle.$$

Therefore P_M can be regarded as the projection matrix onto subspace M in $\mathbb{R}^{K(n+m)}$ where M is identified with the linear subspace $\mathcal{M}$ in $\mathscr{L}_{n+m}$ under the isomorphic relationship $z \leftrightarrow z(\cdot)$.

We next show that $\mathcal{F} + sP_{\mathcal{M}}$ is monotone, that is

$$\langle [\mathcal{F} + sP_{\mathcal{M}}](z(\cdot) - z'(\cdot)), z(\cdot) - z'(\cdot)\rangle \geq 0,$$

which, according to (22), is equivalent to

$$(v - v')^T(\,\mathrm{diag}\,(\mathbf{H}(\xi)) + sP_M)(v - v') \geq 0, \quad \forall v, v'. \tag{23}$$

Since $\mathrm{diag}\,(\mathbf{H}(\xi))$ is positive definite on $\mathcal{N}$, it follows from Theorem 3.1 that (23) is true. Thus $\mathcal{F} + sP_{\mathcal{M}}$ is monotone.

Given that $\mathcal{F} + sP_{\mathcal{M}}$ is monotone, it follows from Rockafellar [8, Theorem 3] that $\mathcal{F} + sP_{\mathcal{M}} + N_{\mathcal{C}}$ is maximal monotone, which complete the proof. ∎

Combining Theorem 3.1 and Theorem 3.2, we have the following result.

Corollary 3.1 *If $diag(H(\xi))$ is positive semidefinite on $\mathcal{N}$, then the two-stage quadratic game under uncertainty is globally elicitable and Algorithm 2 will produce series $\{z^k(\cdot), \omega^k(\cdot)\}$ that converges q-linearly to $(z^*(\cdot), \omega^*(\cdot))$ with respect to the $(r-s)$-norm defined in Theorem 3.1, if the game has a solution and satisfies the constraint qualification.*

Remark 3.1 *The requirement for $H(\xi)$ is similar to the requirement of the second-order optimality condition in nonlinear programming that requires the Hessian of the objective function to be positive semidefinite on a certain subspace. Overall, Corollary 3.1 indicates that the three steps of analysis planned in Section 1 are accomplished at least for the class of games that satisfies the condition of Theorem 3.2.*

4 Numerical Experiments

In this section, we conduct some numerical experiments to test the efficiency of Algorithm 1 and Algorithm 2. The linear complementarity problem in Step 1 is solved by the semismooth Newton method of Qi and Sun [7]. The semismooth Newton method is especially fast in solving a linear complementarity problem because it reduces to finding a root of a simple semismooth equation and it uses the solution with respect to last scenario as the initial point to find the solution corresponding to the current scenario. More details are discussed in [11].

All numerical experiments are coded in Matlab R2015b and run on a PC with an Intel(R) Core(TM) i7-7500U 2.90 GHz CPU and 16 GB of RAM under WINDOWS 10 operating system.

4.1 Test on a production problem

To clarify in which applications the two-stage games can be applied, let us consider two factories, competing to sell similar products, say products 1 and 2, in an open market. Each factory arranges the production of the two products in two stages. At stage 1, they purchase a raw material, say steel, at \$5 per unit without knowing the demands and prices of the products. At stage 2, the demands and prices are disclosed and each of the factory has to decide the amount of each product they produce to maximize their respective revenue subject to the amount of steel they bought at stage 1 and a market saturation bound of the products. Below is the consumption of steel for the factories to producing 1 piece of each product.

	Product 1	Product 2
Factory 1	1.4	1.1
Factory 2	1.3	1.2

There are two scenarios of the uncertainties at the second stage with probability .4 and .6, respectively, as follows.

	Scenario 1	Scenario 2
Price of product 1	17	18
Price of product 2	15	16
Market limit of product 1	1000	1900
Market limit of product 2	2000	2000

Let the amount of steel to purchase be x_1 and x_2, respectively for factories 1 and 2. Let the amount of products to produce be (y_1, y_2) and (y_3, y_4), respectively for factories 1 and 2. The two scenarios are

$$\xi^1 = \begin{pmatrix} \xi_1^1 \\ \xi_2^1 \\ \xi_3^1 \end{pmatrix} = \begin{pmatrix} -17 \\ -15 \\ 1000 \end{pmatrix}, \xi^2 = \begin{pmatrix} \xi_1^2 \\ \xi_2^2 \\ \xi_3^2 \end{pmatrix} = \begin{pmatrix} -18 \\ -16 \\ 1900 \end{pmatrix} \text{ and } \Xi = \{\xi^1, \xi^2\}.$$

Note that the market limit of product 2 is deterministic, so it is not included in the definition of ξ^1 and ξ^2. Following the notations in Section 2, we denote for $\xi \in \Xi$

$$x(\xi) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y(\xi) = \begin{pmatrix} y_1(\xi) \\ y_2(\xi) \\ y_3(\xi) \\ y_4(\xi) \end{pmatrix}, z(\xi) = \begin{pmatrix} x(\xi) \\ y(\xi) \end{pmatrix}.$$

Then Player 1's problem is

$$\begin{aligned} \text{(24)} \qquad \min \quad & 5x_1 + \mathbb{E}_\xi[\xi_1 y_1 + \xi_2 y_2] \\ \text{s.t.} \quad & x_1 - 1.4y_1(\xi) - 1.1y_2(\xi) \geq 0, \\ & -y_1(\xi) - y_3(\xi) \geq -\xi_3, \\ & -y_2(\xi) - y_4(\xi) \geq -2000, \\ & \text{All variables are } \geq 0. \end{aligned}$$

By using the fact that the linear program

$$\min c^T z \text{ s.t. } Az \geq b, z \geq 0$$

is equivalent to the linear complementarity problem

$$0 \leq \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix} + \begin{pmatrix} c \\ -b \end{pmatrix} \perp \begin{pmatrix} z \\ u \end{pmatrix} \geq 0,$$

where u is the dual variable, problem (24) can be equivalently written as a stochastic linear complementarity problem as follows

$$\text{(25)} \qquad 0 \leq \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix} + \begin{pmatrix} c_1 \\ 0 \\ \xi_3 \\ 2000 \end{pmatrix} \perp \begin{pmatrix} z \\ u \end{pmatrix} \geq 0$$

where

$$A = \begin{pmatrix} 1 & 0 & -1.4 & -1.1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}, c_1 = (5, 0, \xi_1, \xi_2, 0, 0)^T.$$

Similarly, Player 2's problem is equivalent to

$$0 \le \begin{pmatrix} 0 & -B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} z \\ v \end{pmatrix} + \begin{pmatrix} c_2 \\ 0 \\ \xi_3 \\ 2000 \end{pmatrix} \perp \begin{pmatrix} z \\ v \end{pmatrix} \ge 0 \tag{26}$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & -1.3 & -1.2 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}, c_2 = (0, 5, 0, 0, \xi_1, \xi_2)^T.$$

At each iteration of PHA, we solve for fixed ξ the system (25)-(26) at Step 1. Since the objectives are linear and the constraints are polyhedral and bounded, the game has a solution and PHA ends up with $x_1 = 2660, y_1(\xi^1) = 1000, y_1(\xi^2) = 1900, x_2 = 2200, y_4(\xi^1) = 2000, y_4(\xi^2) = 2000$ and all other variables are zero.

4.2 Test on randomly generated non-monotone problems

In this subsection, we test Algorithm 1 and Algorithm 2 on randomly generated problems. Considering that Problem (13) and Problem (4)-(7) in the privately convex case with $X_i(x_{-i}(\xi), \xi) = \mathbb{R}^{n_i}_+$ and $Y_i(x(\xi), y_{-i}(\xi), \xi) = \mathbb{R}^{m_i}_+$ have the same structure (they differ only by adding ζ in the primal vector), we only test problems of the latter type. Note that even in the privately convex case, the equivalent SLC problem (13) may not be monotone.

We generate the symmetric positive semi-definite $\bar{Q}_i(\xi) \in \mathbb{R}^{(n_i+m_i)\times(n_i+m_i)}$ for player i in scenario ξ as

$$\bar{Q}_i(\xi) = \lambda I - \frac{A + A^T}{2},$$

where A is a matrix composed of entries being uniformly distributed in the interval $(-1, 1)$, and $\lambda \ge \lambda_{\max}(\frac{A+A^T}{2})$. Matrices R_{ij}, $P_{ij}(\xi)$ and $O_{ij}(\xi)$ are composed of random numbers uniformly distributed in the interval $(-1, 1)$. $c_i = Q_i(\xi^1)u$ and $d_i(\xi) = T_i(\xi^1)v$, where $u \in \mathbb{R}^{n_i \times n_i}$ $v \in \mathbb{R}^{m_i \times m_i}$ are random vectors with entries being uniformly distributed in $(-1, 1)$. Then, set $\bar{H}_{11}$ in (18) as $\mathbb{E}_\xi[\bar{H}_{11}(\xi)]$. The probability of each scenario is randomly generated as well.

Note that, with $x(\xi)$ being constant as x for all ξ, a sufficient condition that $z(\cdot)$ is the solution to problem (13) is

$$\begin{cases} 0 \le x \;\perp\; \bar{H}_{11}x + E_\xi[\bar{H}_{12}(\xi)y(\xi)] + c \ge 0, \\ 0 \le y(\xi) \;\perp\; \bar{H}_{21}(\xi)x + \bar{H}_{22}(\xi)y(\xi) + d(\xi) \ge 0, \;\forall\, \xi \in \Xi. \end{cases}$$

Therefore, we adopt the following measurement to construct a stopping criteria:

$$\text{rel.err} = \max\{\text{rel.err}_1, \text{rel.err}_2\},$$

where

$$\text{rel.err}_1 = \frac{\|x - \prod_{\ge 0}(x - (\bar{H}_{11}x + E_\xi[\bar{H}_{12}(\xi)y(\xi)] + c))\|}{1 + \|x\|},$$

$$\text{rel.err}_2 = \max_{\xi} \left\{ \frac{\|y(\xi) - \prod_{\geq 0}(y(\xi) - (\bar{H}_{21}(\xi)x + \bar{H}_{22}(\xi)y(\xi) + d(\xi)))\|}{1 + \|y(\xi)\|} \right\},$$

with $(\prod_{\geq 0}(a))_j = \max\{a_j, 0\}$. Set the tolerance to be 10^{-5}, and the maximal iterations to be 1000, i.e., if rel.err $\leq 10^{-5}$ or iteration number ≥ 1000, the algorithm stops.

As stated in [11], the choice of parameter r has great impacts for the performance of PHA, and choosing r as the square root of the dimension of $m + n$ has shown to be an efficient heuristic in solving stochastic linear complementary problems. Besides, since PHA is an application of Spingarn's partial inverse algorithm, which is a specialized form of Douglas-Rachford splitting in [4], we adopt a step length ρ in the dual update step, namely

$$\omega^{k+1}(\xi) = \omega^k(\xi) + \rho r(\hat{z}^k(\xi) - z^{k+1}(\xi)). \tag{27}$$

It can be seen that when $\rho = 1$ it is exactly the original PHA. In our numerical experiments, we set $\rho = 1.618$, which is successfully used in Douglas-Rachford splitting methods.

4.2.1 Numerical results

We design three experiments to show the performance of Algorithm 1 and Algorithm 2 with dual update being (27) for solving the two-stage game in their linear complementarity formulation. Besides, we also adopt BRM for comparison, with subproblems of BRM being treated as a large-scaled LCP and solved by the same solver for subproblems of PHA, and use $\|x^{\nu+1} - x^{\nu}\| \leq 10^{-5}$ or the max.iter$\geq$ 1000 as the stopping criteria for BRM. First two experiments focus on the stochastic game with two players, while the third one is an N-player game.

For two-player game, we conduct two groups of test samples:

- One group is to fix the dimension of the first and second player's decision variable as $[15, 20]$ and $[25, 10]$, respectively, and increase the number of scenario from 5 to 500. For each setting, there are 10 examples being randomly generated by the rules stated above. For every problem we run PHAorg ($s = 0$), PHAelc ($s = r/2$) and BRM to test, and record the convergence iteration number and time. Besides, for PHA we set the parameter $r = \sqrt{n+m}$ and the step length $\rho = 1.618$, while for BRM $\mu = \sqrt{n+m}$. The numerical results are listed in Table 1 and Figure 1.

- The other group is to fix the number of scenario as sn=50 and increase the dimension of each player's decision variable from [50,50] to [300,300]. For each setting, 10 examples are randomly generated and tested by PHAorg ($s = 0$), PHAelc ($s = r/2$) and BRM. Table 2 and Figure 2 show the numerical results.

First, it should be pointed that privately convexity can not guarantee the monotonicity of matrix $H_{11}(\xi)$ for each ξ, which means the equivalent SLC (13) handled by PHA may be non-monotone. However, the original PHA without being elicited still works for the problem, and the convergence is faster than the elicited PHA. From Table 1, one can find that the iteration number for converging of BRM, PHAorg and PHAelc is stable around 30, 30, 40, respectively, when the number of scenarios rising. Since we directly solve the subproblems of BRM by LCPsolver, the LCP of each subproblem is getting larger rapidly when the number of scenario grows, which yields more time consuming for converging. But for PHAorg and PHAelc, the convergence time increases slowly when the scenario number

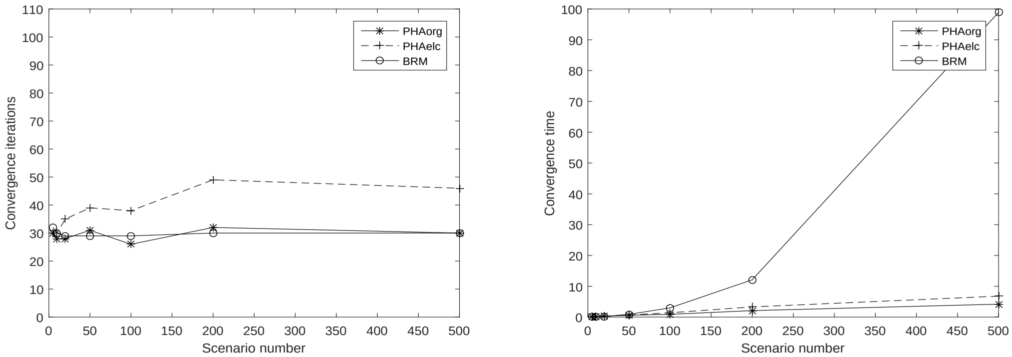

Figure 1: Convergence results when scenario number increases

Table 1: Numerical results while scenario number increases
(dim=[15,20], [25,10])

K	PHAorg (s=0)		PHAelc (s=r/2)		BRM	
	iter	time(s)	iter	time(s)	iter	time(s)
5	30	0.1	30	0.1	32	0.05
10	28	0.2	30	0.2	30	0.1
20	28	0.3	35	0.3	29	0.2
50	31	0.6	39	0.7	29	0.9
100	26	0.9	38	1.4	29	3.0
200	32	2.1	49	3.3	30	12.1
500	30	4.2	46	6.8	30	98.9

rises. Besides, in this case it shows that BRM is faster than PHA for small-size problem, e.g. scenario number less than 50, but behaves worse for problems with large scenario number.

From Table 2 and Figure 2, we can see that the number of iterations for convergence of PHAorg and PHAelc grows steadily when the problem dimension is increasing, while the number of iterations for convergence of BRM is still stable around 30. However, the convergence time of these three algorithms appear to grow much fast in this group, with BRM being much slower than the PHAs when the dimension gets larger.

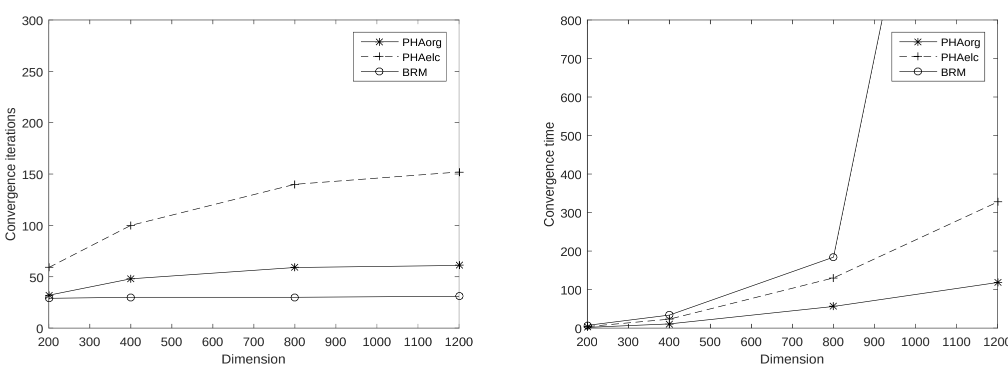

Figure 2: Convergence results when dimension of each player increases

Table 2: Numerical results while dimension increases (K=50)

dim	PHAorg (s=0)		PHAelc (s=r/2)		BRM	
	iter	time(s)	iter	time(s)	iter	time(s)
[50,50]	32	1.9	59	4.0	29	7.1
[100,100]	48	10.9	100	23.2	30	33.8
[200,200]	59	56.2	140	130.3	30	184.2
[300,300]	61	118.4	152	327.4	31	2263.4

In the third experiment, we fix the scenario number as 100 and the dimension of each player's strategy as [15,20], and generate 10 independent monotone problems and 10 independent non-monotone problems for each N (the number of players), which increases from 3 to 15, specifically setting $N = 3, 6, 10$ and 15. The dimension of matrix $H_{11}(\xi)$ is rising rapidly when N increases. PHAorg ($s = 0$), PHAelc ($s = r/2$) and BRM are conducted to solve every problem, and the convergence iteration number and time are recorded.

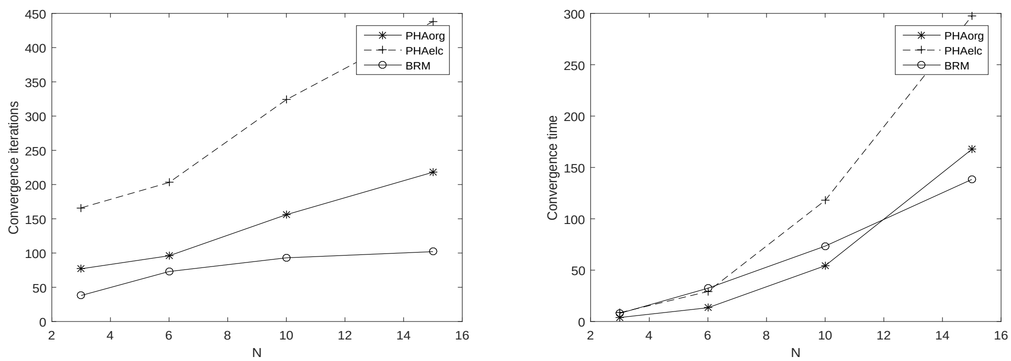

Figure 3: Convergence results when player number increases

From Figure 3 and Table 3, we can find that both the iteration number and time for converging of the three algorithms increase when the number of players grows. Notice that the rise rate of converging time of BRM is lower than that of PHA. More specifically, when the number of players is smaller than 10, PHA costs less time to converge than BRM, but when the game involves more players BRM will perform better than PHA.

Table 3: Numerical results while player number increases
(K=100, dim of each player=[15,20]))

N	PHAorg (s=0)		PHAelc (s=r/2)		BRM	
	iter	time(s)	iter	time(s)	iter	time(s)
3	77	3.9	166	8.6	38	8.0
6	96	13.4	203	29.2	73	32.4
10	156	54.3	324	117.9	93	73.4
15	218	167.6	438	297.3	102	138.3

To sum up, we compared the BRM and PHA and listed the advantages and drawbacks in the following:

- The BRM requires the convexity and twice differentiability of $\theta_i + \mathbb{E}_\xi[\psi_i]$ on x_i and the contractiveness of the best-response mapping. The PHA treats the two-stage optimization problem for each player in a functional framework, thus the model in this paper is more general as long as the problem is decomposable in terms of scenarios.

- In general, the monotonicity of SLC (13) is not satisfied and to check if it is elicitable is not easy. This is similar to what happens in solving nonlinear programs — the user knows that the point generated by his algorithm would be a solution, if that point satisfies certain second-order sufficient condition, but there is no way to check if that condition would be satisfied when the algorithm begins. From the numerical results, it is notable that as long as the game is privately convex, the PHA without being elicited works at least as well as the BRM and moreover, it is applicable to a larger class of problems due to the restriction on the applicability of the BRM approach.
- When the number of players grows, the size of PHA's subproblem increases while there's no difference for the subproblems of BRM, in which case BRM will lead a better performance than PHA. However, for 2-player game with large-scale decision variables and big scenario number, PHA is a better choice.

5 Conclusions

This paper studies the quadratic case of two-stage game models under uncertainty. The model allows entanglement at all levels — the first stage decision is parameterized by the rivals' decisions both at the objective function level and at the constraint level. The second stage decision is parameterized not only by a random vector, but also by the rivals' decisions in two stages and the player's own decision at the first stage, also in both levels — objective function and constraints. It is shown that the problem of finding a Nash equilibrium of this model can be converted to a stochastic linear complementarity problem. This model appears to be a new model in the literature that associates a stochastic variational inequality problems with a stochastic equilibrium problem.

Although under certain strong conditions the resulted stochastic linear complementarity formulation of this game may be of monotone type, it is shown that it can be generally expected that the resulted formulation is of non-monotone type. The progressive hedging algorithm is demonstrated to be able to solve mediate-sized games with efficiency. The tested problems are up to the size of several hundreds of variables and scenarios.

In particular, our study includes theory and computational results on an elicited progressive hedging algorithm for solving class of non-monotone games — the elicitable two-stage quadratic games. Same as the monotone case, it is shown that the elicited progressive hedging algorithms is globally convergent at a linear rate if the problem satisfies an elicitability condition, which requires monotonicity of a certain mapping on the nonanticipativity subspace.

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References

[1] X. Chen, A. Shapiro and H. Sun, *Convergence Analysis of Sample Average Approximation of Two-Stage Stochastic Generalized Equations,* SIAM J. Optim. **29** (2019) 135-161.

[2] X. Chen, R. J-B. Wets and Y. Zhang, *Stochastic Variational Inequalities: Residual Minimization Smoothing/Sample Average approximations*, SIAM J. Optim. **22** (2012) 649-673.

[3] R.W. Cottle, J.-S. Pang and R.E. Stone. *The Linear Complementarity Problem.* Society for Industrial and Applied Mathematics, 2009.

[4] J. Eckstein and D.P. Bertsekas. *On the Douglas-Rachford Splitting Method and the Proximal Point Algorithm for Maximal Monotone Operators*. Math. Program. **55** (1992) 293-318.

[5] J.-S. Pang and M. Fukushima. *Quasi-Variational Inequalities, Generalized Nash Equilibria, and Multi-Leader-Follower Games*. Comput. Manag. Sci. **2** (2005) 21-56.

[6] J.-S. Pang, S. Sen and U. Shanbhag. *Two-Stage Non-Cooperative Games with Risk-Averse Players*. Math. Program. **165** (2017) 235-290.

[7] L. Qi and J. Sun. *A Nonsmooth Version of Newton's Method.* Math. Program. **58** (1993) 353-367.

[8] R. T. Rockafellar. *On the Maximality of Sums of Nonlinear Monotone Operators*. Trans. Amer. Math. Soc. **149** (1970) 75-88.

[9] R. T. Rockafellar. *Monotone Operators and the Proximal Point Algorithms*. SIAM J. Control Optim. **14** (1976) 877-898.

[10] R. T. Rockafellar. *Progressive Decoupling of Linkages in Optimization and Variational Inequalities with Elicitable Convexity or Monotonicity.* Set-Valued Var. Anal (2018). https://doi.org/10.1007/s11228-018-0496-1

[11] R. T. Rockafellar and J. Sun. *Solving Monotone Stochastic Variational Inequalities and Complementarity Problems by Progressive Hedging.* Math. Program. **174** (2019) 453-471.

[12] R. T. Rockafellar and R. J-B. Wets. *Scenarios and Policy Aggregation in Optimization under Uncertainty*. Math. Oper. Res. **16** (1991) 119-147.

[13] R. T. Rockafellar and R. J-B. Wets. *Stochastic Variational Inequalities: Single-Stage to Multistage*. Math. Program. **165** (2017) 331-360.

[14] U. Shanbhag. *Stochastic Variational Inequality Problems: Applications, Analysis, and Algorithms*. Tutorials in Operations Research. INFORMS 2013.

[15] U. Shanbhag, J.S. Pang and S. Sen. *Inexact Best Response Scheme for Stochastic Nash Games: Linear Convergence and Iteration Complexity Analysis.* In: Proceedings of the IEEE Conference on Decision and Control. pp. 3591-3596 (2016).

[16] J.E. Spingarn. *Applications of the Method of Partial Inverse to Convex Programming: Decomposition*. Math. Program. **32** (1985) 247-265.

[17] J.E. Spingarn. *Partial Inverse of a Monotone Operator*. Appl. Math. Optim. **10** (1983) 199-223.

[18] R. J-B. Wets. *Programming under Uncertainty: The Equivalent Convex Program.* SIAM J. Appl. Math. **14** (1966) 89-105.