CVAR-BASED ROBUST MODELS FOR PORTFOLIO SELECTION

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(Communicated by Ken Siu)

ABSTRACT. This study relaxes the distributional assumption of the return of
the risky asset, to arrive at the optimal portfolio. Studies of portfolio selec-
tion models have typically assumed that stock returns conform to the normal
distribution. The application of robust optimization techniques means that
only the historical mean and variance of asset returns are required instead
of distributional information. We show that the method results in an opti-
mal portfolio that has comparable return and yet equivalent risk, to one that
assumes normality of asset returns.

1. **Introduction.** Portfolio selection models are of great practical significance to investors around the world. The optimal portfolio is chosen such that expected return is maximized, and risk is minimized. The way risk is defined and measured will lead to different optimal portfolios. Markowitz laid the foundation for this line of research with the well-known mean-variance (M-V) model in a single period case [14]. Since then, research has developed to include multi-period cases and alternate risk measures in place of variance.

Examples of risk measures used include the semi-variance [13, 15] and the mean absolute deviation [5, 7]. Some researchers combined two measures of risk, for example, Konno and Suzuki who considered both variance and skewness as the measure of risk [6]. Yet other researchers explored minimax type of risk functions which include minimizing the maximum of individual risk [1, 17, 18]; minimizing the average of maximum individual risks over a number of time periods [19]; and maximizing the minimum possible expected rates of returns on portfolio [4]. Of the minimax type of risk functions, an analytical solution was obtained for the single-period case [17] and for the multi-period case [18]. However, in both [17] and [18], the log returns were assumed to be normally distributed. However, this assumption is clearly not realistic.

The main contribution of this paper is adding to literature of portfolio selection models by relaxing the normality assumption of log returns. Our proposed method only requires the mean and variance of the asset returns, to obtain the optimal portfolio. Full knowledge of the distribution of asset returns is not required. By applying robust optimization techniques to portfolio selection problem, this study shows that the optimal results improve with additional constraints, and are consistent with reality.

Our paper has two additional contributions. First, we add to the literature by comparing the results of the portfolio obtained via robust optimization technique to the optimal portfolio obtained by [17] using the assumption of normal distribution of log returns. In both cases, the same risk measure was used. We find that, when only basic information of upper and lower bounds of the returns is available, the robust optimization technique is unable to arrive at a solution for the efficient frontier due to inadequate information. When more information on the mean and variance of the return is provided, the optimal portfolio using robust optimization is similar to the optimal portfolio obtained by using the analytical formula obtained in [17]. Our study illustrates that even without the knowledge of the distribution of asset returns, the results under robust optimization is not compromised.

Second, we obtain a new risk measure by applying a distributionally robust optimization technique [8, 9, 10, 11, 12, 20], and the exact corresponding computationally tractable reformulation is derived. The optimal portfolio obtained under this new risk measure outperforms the portfolio obtained using the risk measure in [17]. Particularly, the computation of the new risk measure and that of the old one are both tackled by the CVaR based schemes.

The remainder of this paper is organized as follows. Section 2 formulates the single period portfolio selection problem as a bi-criteria optimization problem. Section 3 is divided into Section 3.1 and Section 3.2, where the problem in transformed into a robust linear programming problem and a second-order cone programming problem, respectively. Section 4 introduces a new risk measure. Section 5 provides some numerical simulations and Section 6 concludes the paper.

2. **Problem formulation.** The portfolio selection problem is based on a single investment period. Assume that there are N risky assets in the market. An investor allocates the capital at the investment time by assigning a share to each selected available asset. The share is expressed as a percentage of the capital, i.e., x_j, $j = 1, \ldots, N$. To avoid ambiguity, we further assume that the investor will invest all available capital, and short selling of the risky assets is not allowed. To express the assumptions mathematically, we define

$$\mathcal{X} = \left\{ \boldsymbol{x} = [x_1, \ldots, x_N]^\top \in \mathbb{R}^N : \sum_{j=1}^N x_j = 1,\ x_j \geq 0,\ j = 1, \ldots, N \right\}$$

The return of asset j is denoted as R_j. Here, unlike the normality assumption of the return of the risky asset imposed in previous works, R_j is a random variable of unknown distribution. The expected value of R_j is represented by r_j, which is calculated by averaging the historical returns of asset j over T periods.

$$r_j = \frac{1}{T} \sum_{i=1}^T R_{ji}$$

where R_{ji} denotes the actual return of asset j for the ith time period.

Thus, the expected return of the portfolio $\boldsymbol{x} = [x_1, \ldots, x_N]^\top$ is given by

$$r(x_1, \ldots, x_N) = E\left\{ \sum_{j=1}^N R_j x_j \right\} = \sum_{j=1}^N E\{R_j\} x_j$$
$$= \sum_{j=1}^N r_j x_j \tag{1}$$

Here, we assume that the investor is sensitive to downside losses, relative to upside gains. The portfolio is only considered risky when it is more sensitive to the downward market movement than to the upward market movement. We adjust the probabilistic risk measure introduced in [17] to cater for the risk preference of the investor. The portfolio risk is measured by $\underline{w}_p(\boldsymbol{x})$, which is defined as the largest individual downside risk shown below.

$$\underline{w}_p(\boldsymbol{x}) = \min_{1 \leq j \leq N} \Pr\{ R_j x_j - r_j x_j \geq -\theta \delta \} \tag{2}$$

As defined in [17], θ is a constant to adjust the risk level, while δ denotes the average risk of the entire portfolio, which is obtained by averaging the standard deviation of all single risky assets in the portfolio,

$$\delta = \frac{1}{N} \sum_{j=1}^N \sigma_j \tag{3}$$

where σ_j indicates the standard deviation of asset j's return, which is evaluated using historical data.

To find such a portfolio that maximizes the expected return while minimizing the risk, the portfolio selection problem can be formulated as a bi-criteria optimization problem as follows.

$$\max \left(\min_{1 \leq j \leq N} \underline{f}(x_j),\ \sum_{j=1}^N r_j x_j \right) \tag{4a}$$

$$\text{s.t. } \boldsymbol{x} \in \mathcal{X} \tag{4b}$$

where $\underline{f}(x_j) = \Pr\{\, R_j x_j - r_j x_j \geq -\theta\delta \,\}$.

The maximin objective in (4a) can be tackled by adding another variable y, and N constraints, so that problem (4) is converted into an equivalent bi-criteria optimization problem.

$$\max\left(\, y, \sum_{j=1}^{N} r_j x_j \,\right) \tag{5a}$$

$$\text{s.t. } y \leq \underline{f}(x_j), \quad j = 1, \ldots, N \tag{5b}$$

$$\boldsymbol{x} \in \mathcal{X} \tag{5c}$$

where $y \leq \underline{f}(x_j)$ is the $j^{(th)}$ probabilistic constraint. Since the optimization process will push the value of y to be equal to $\min_{1 \leq j \leq N} \underline{f}(x_j)$, it is clear that the optimization problem (5) is equivalent to the optimization problem (4).

3. **Robust transformation of the problem.** As mentioned in Section 2, y will eventually be equal to $\min_{1 \leq j \leq N} \underline{f}(x_j)$. By definition, $\underline{f}(x_j)$ is a probability, so y should range from 0 to 1. If we choose y to be an arbitrary but fixed real number within the range, problem (5) becomes the following single objective optimization problem.

$$\max_{\boldsymbol{x}} \sum_{j=1}^{N} r_j x_j \tag{6a}$$

$$\text{s.t. } \underline{f}(x_j) \geq y, \quad j = 1, \ldots, N \tag{6b}$$

$$\boldsymbol{x} \in \mathcal{X} \tag{6c}$$

Now, the focus lies in (6b), and thus, the evaluation of $\Pr\{\, R_j x_j - r_j x_j \geq -\theta\delta \,\}$. The probabilistic downside risk cannot be derived directly without distributional assumption of R_j. However, inspired by the connections between chance constraints and the bounds on the CVaR measure in [2], we can successfully approximate the chance constraints (6b) with different sets of conventional inequality constraints.

3.1. **A LP transformation.** For the sake of uniformity in CVaR measure, we rewrite constraints (6b) as

$$\Pr\{\, g_j \leq 0 \,\} \geq 1 - \varepsilon, \quad j = 1, \ldots, N \tag{7}$$

where $g_j = (r_j - R_j)x_j - d_j$, $\varepsilon = 1 - y$, and $d_j = \theta\delta$. For each $j = 1, \ldots, N$, the constraint in (7) is a typical VaR measure, which is known to be non-convex. As a result, we replace it by a CVaR constraint, which is the best convex approximation of (7) [16]. For $j = 1, \ldots, N$,

$$\text{CVaR}_{1-\varepsilon}(g_j) \leq 0$$

or equivalently,

$$\min\left\{\, \beta_j + \frac{1}{\varepsilon}\big[\, E(g_j - \beta_j)^+ \,\big] \,\right\} \leq 0 \tag{8}$$

$\beta_j \in \mathbb{R}$ is an introduced variable. Then, we define set

$$W_j = \{\, g_j : g_j = (r_j - R_j)x_j - d_j, \boldsymbol{x} \in \mathcal{X} \,\}$$

By Theorem 2.3 in [2], an upper bound for $E(g_j - \beta_j)^+$ is

$$E(g_j - \beta_j)^+ \leq [\,-\beta_j + \max_{g_j \in W_j} g_j\,]^+$$

Therefore, (8) is valid if the following constraint is satisfied.

$$\min_{\beta_j} \left\{ \beta_j + \frac{1}{\varepsilon}(-\beta_j + \max_{g_j \in W_j} g_j)^+ \right\} \leq 0 \tag{9}$$

Again, to eliminate the minimax components in (9), we add another variable α_j so that constraint (9) is relaxed to the following set of constraints.

$$\min_{\alpha_j, \beta_j} \beta_j + \frac{\alpha_j}{\varepsilon} \leq 0 \tag{10a}$$

$$\text{s.t. } \alpha_j + \beta_j \geq \max\{\, g_j : g_j \in W_j \,\} \tag{10b}$$

$$\alpha_j \geq 0. \tag{10c}$$

Suppose that R_j is bounded. We have

$$l_j \leq R_j \leq u_j$$

It is obvious that

$$\max\{\, g_j : g_j \in W_j \,\} \leq \max\{\, -l_j x_j + r_j x_j - d_j, -u_j x_j + r_j x_j - d_j \,\}$$

Thus, constraint (7) is valid if the following set of constraints is satisfied.

$$\beta_j + \frac{\alpha_j}{\varepsilon} \leq 0, \quad j = 1, \ldots, N \tag{11}$$

$$\alpha_j + \beta_j + l_j x_j - r_j x_j \geq -d_j, \quad j = 1, \ldots, N \tag{12}$$

$$\alpha_j + \beta_j + u_j x_j - r_j x_j \geq -d_j, \quad j = 1, \ldots, N \tag{13}$$

$$\alpha_j \geq 0, \quad j = 1, \ldots, N \tag{14}$$

With constraints (6b) replaced by multiple sets of constraints in (11) - (14), there are new decision variables α_j, β_j, $j = 1, \ldots, N$, introduced into problem (6). Now, problem (6) can be written as the following linear programming problem:

$$\min_{\boldsymbol{\xi}} \boldsymbol{c}^\top \boldsymbol{\xi} \tag{15a}$$

$$\text{s.t. } \beta_j + \frac{\alpha_j}{\varepsilon} \leq 0, \quad j = 1, \ldots, N \tag{15b}$$

$$\alpha_j + \beta_j + l_j x_j - r_j x_j \geq d_j, \quad j = 1, \ldots, N \tag{15c}$$

$$\alpha_j + \beta_j + u_j x_j - r_j x_j \geq d_j, \quad j = 1, \ldots, N \tag{15d}$$

$$\alpha_j \geq 0, \quad j = 1, \ldots, N \tag{15e}$$

$$\boldsymbol{x} \in \mathcal{X} \tag{15f}$$

where

$$\boldsymbol{c} = [\,-r_1, \ldots, -r_N, 0, \ldots, 0\,]^\top \in \mathbb{R}^{3N}$$

and

$$\boldsymbol{\xi} = [\,x_1, \ldots, x_N, \alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N\,]^\top \in \mathbb{R}^{3N}$$

Clearly, problem (15) is in the form of a linear programming problem. It is convenient for us to solve this problem using an interior-point algorithm.

Remark 1. For this method, we need the bounds of the return l_j and u_j. They can be simply obtained by taking the lowest value and the largest value the return from the historical data. However, with assumption we cannot capture the extreme

losses of the portfolio. Therefore, in the next section and Section 4, we present two method without making this assumption.

3.2. **A second-order cone formulation.** Our focus in this paper is on relaxing the distributional assumption of the return of the risky asset in regards of reality. In the last section, it is assumed that the rate of return is bounded within a certain finite range. The problem is then approximated into a linear programming problem (15). This transformation only requires the bounds of each risky asset. The resulting linear programming problem is computationally economical. However, a deficiency is that a large feasible region of the original problem is being cutoff. Thus, although the feasible region of problem (15) is a convex set, the difference between the two feasible regions may be large. Therefore, the optimal solution of problem (15) may be far away from the optimal solution of the original problem. One way to achieve a better solution to the problem is to enlarge the convex feasible region so that this region could possibly get nearer to the global optimal solution, or even cover it. In this section, we will show that problem (6) can be converted into a second-order cone programming (SOCP) problem after more information is fed into the system.

As stated in Section 3.1, the best convex approximation of constraints (6b) is

$$\min\left\{\beta_j + \frac{1}{\varepsilon}\left[E(g_j - \beta_j)^+\right]\right\} \le 0, \quad j = 1\ldots, N \tag{16}$$

where β_j, ε and g_j are defined as in Section 3.1.

Suppose that z is a random variable with zero mean and positive variance. Then an upper bound is presented in [2]. That is,

$$E(\mathcal{Y}_0 + \mathcal{Y}z)^+ \le \frac{1}{2}\mathcal{Y}_0 + \frac{1}{2}\sqrt{\mathcal{Y}_0^2 + \mathcal{Y}^2\mathrm{Var}z} \tag{17}$$

We set $z = r_j - R_j$, $\mathcal{Y} = x_j$ and $\mathcal{Y}_0 = -d_j - \beta_j$, and obtain

$$E(-d_j - \beta_j + (r_j - R_j)x_j)^+ \le \frac{1}{2}(-d_j - \beta_j) + \frac{1}{2}\sqrt{(-d_j - \beta_j)^2 + x_j^2\mathrm{Var}R_j}$$

where $\mathrm{Var}R_j$ is the variance of R_j. Hence, (16) is valid if

$$\beta_j + \frac{1}{\varepsilon}\left[\frac{1}{2}(-d_j - \beta_j) + \frac{1}{2}\sqrt{(-d_j - \beta_j)^2 + x_j^2\mathrm{Var}R_j}\right] \le 0$$

is valid.

Consequently, the approximate problem can be stated formally as

$$\min_{\tilde{\boldsymbol{\xi}}} \ \tilde{\boldsymbol{c}}^\top\tilde{\boldsymbol{\xi}} \tag{18a}$$

$$\text{s.t.} \ \sqrt{(d_j + \beta_j)^2 + x_j^2\mathrm{Var}R_j} \le (1 - 2\varepsilon)\beta_j + d_j, \quad j = 1\ldots, N \tag{18b}$$

$$\boldsymbol{x} \in \mathcal{X} \tag{18c}$$

where $\tilde{\boldsymbol{c}} = [r_1, \ldots, r_N, 0, \ldots, 0]^\top \in \mathbb{R}^{2N}$ and $\tilde{\boldsymbol{\xi}} = [x_1, \ldots, x_N, \beta_1, \ldots, \beta_N]^\top \in \mathbb{R}^{2N}$.

Clearly, problem (18) is a second order programming problem which can be solved using interior-point algorithm. In the next section, some numerical examples using real stock data are presented and solved to validate the models.

4. **A distributionally robust risk measure.** In this section, we consider a new risk measure, in which the distribution information is not fully known. More specifically, we only know the the mean and the variance of the distribution, and the new portfolio risk measure $\underline{w}_p(\boldsymbol{x})$ is defined as below.

$$\underline{w}_p(\boldsymbol{x}) = \min_{1 \leq j \leq N} \inf_{\mathbb{P}_j \in \mathcal{P}_j} \Pr_{[\mathbb{P}_j]} \{ R_j x_j - r_j x_j \geq -\theta\delta \} \tag{19}$$

where $\mathbb{P}_j$ denotes the distribution of R_j and

$$\mathcal{P}_j = \left\{ \mathbb{P}_j : \mathbb{E}_{[\mathbb{P}_j]}[R_j] = r_j, \ \mathbb{E}_{[\mathbb{P}_j]}[R_j - r_j]^2 = \sigma_j^2 \right\} \tag{20}$$

$\mathcal{P}_j$ is called ambiguity set. Thus, this risk measure is more practical in real world applications and more general in formulation.

Under this risk measure, the single objective optimization problem (6a) - (6c) becomes the following problem.

$$\max_{\boldsymbol{x}} \sum_{j=1}^{N} r_j x_j \tag{21a}$$

$$\text{s.t. } \tilde{f}(x_j) \geq y, \quad j = 1, \dots, N \tag{21b}$$

$$\boldsymbol{x} \in \mathcal{X} \tag{21c}$$

where

$$\tilde{f}(x_j) = \inf_{\mathbb{P}_j \in \mathcal{P}_j} \Pr_{[\mathbb{P}_j]} \{ R_j x_j - r_j x_j \geq -\theta\delta \}$$

To derive the computationally tractable reformulation of (21a) - (21c), we need the following results.

Lemma 1 [Theorem 2.2 in [20]]. *Let $L : \mathbb{R}^k \to \mathbb{R}$ be a continuous loss function that is either concave in $\boldsymbol{\xi}$ or quadratic in $\boldsymbol{\xi}$. Then, the following equivalence holds.*

$$\inf_{\mathbb{P} \in \mathcal{P}} \Pr_{[\mathbb{P}]}[L(\boldsymbol{\xi}) \leq 0] \geq 1 - \epsilon \iff \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P} - \text{CVaR}_\epsilon[L(\boldsymbol{\xi})] \leq 0 \tag{22}$$

where $\mathcal{P}$ is defined in (20).

Lemma 2 [Theorem 21 in [20]]. *The feasible set*

$$\left\{ \boldsymbol{x} \in \mathbb{R}^n : \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P} - \text{CVaR}_\epsilon \left[y^0(\boldsymbol{x}) + \boldsymbol{y}^T(\boldsymbol{x})\boldsymbol{\xi} \right] \leq 0 \right\} \tag{23}$$

can be written as

$$\left\{ \boldsymbol{x} \in \mathbb{R}^n : \begin{array}{c} \boldsymbol{M} \succeq 0, \ \beta + \frac{1}{\epsilon}\text{Tr}(\boldsymbol{\Omega M}) \leq 0, \\ \boldsymbol{M} - \begin{bmatrix} \boldsymbol{0} & \frac{1}{2}\boldsymbol{y}(\boldsymbol{x}) \\ \frac{1}{2}\boldsymbol{y}^T(\boldsymbol{x}) & y^0(\boldsymbol{x}) - \beta \end{bmatrix} \succeq 0 \end{array} \right\} \tag{24}$$

where

$$\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^T & \boldsymbol{\mu} \\ \boldsymbol{\mu}^T & 1 \end{bmatrix} \tag{25}$$

$\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the mean and the variance matrix of $\boldsymbol{\xi}$, $y^0(\boldsymbol{x})$ and $\boldsymbol{y}(\boldsymbol{x})$ depend affinely on $\boldsymbol{x}$, and $\text{Tr}(\cdot)$ denotes the matrix trace.

Based on the results above, we can show that the problem (21a) - (21c) can be represented as an semi-definite program (SDP) problem and hence is computationally tractable.

Theorem 1. *The problem* (21a) - (21c) *can be reformulated as the following conic optimization problem.*

$$\max_{x_j, \beta_j, \boldsymbol{M}_j} \quad \sum_{j=1}^{N} r_j x_j \tag{26}$$

$$\text{s.t.} \quad \beta_j + \frac{1}{\epsilon} \text{Tr} \left(\boldsymbol{\Omega}_j \boldsymbol{M}_j \right) \leq 0, \ j = 1, 2, \ldots, N \tag{27}$$

$$\boldsymbol{M}_j - \begin{bmatrix} 0 & -\frac{1}{2} x_j \\ -\frac{1}{2} x_j & r_j x_j - \theta \delta - \beta_j \end{bmatrix} \succeq 0, \ j = 1, 2, \ldots, N \tag{28}$$

$$\boldsymbol{M}_j \succeq 0, \ j = 1, 2, \ldots, N \tag{29}$$

where

$$\boldsymbol{\Omega}_j = \begin{bmatrix} \sigma_j^2 + r_j^2 & r_j \\ r_j & 1 \end{bmatrix}$$

and $\boldsymbol{M}_j \in \mathbb{S}^2$ *denotes all the* 2×2 *symmetric matrices.*

Proof. For each j, in view of (21b), we can see that $L(R_j) = -R_j x_j + r_j x_j - \theta \delta$ depends linearly on R_j, which also implies that $L(R_j)$ is concave in R_j. Thus, from LEMMA 1, we know that (21b) is equivalent to

$$\sup_{\mathbb{P}_j \in \mathcal{P}_j} \mathbb{P}_j - \text{CVaR}_\epsilon \left[-R_j x_j + r_j x_j - \theta \delta \right] \leq 0 \tag{30}$$

Let $y^0(x_j) = r_j x_j - \theta \delta$ and $y(x_j) = -x_j$. Clearly, $y^0(x_j)$ and $y(x_j)$ affinely depend on x_j. Then, from LEMMA 2, (30) can be rewritten as the constraints (27)-(29). This completes the proof. □

Remark 2. Comparing with the risk measure (2), the new risk measure (19) optimizes the worst-case distribution in the ambiguity set. Although the formulation of (19) seems more conservative than that of (2), it yields an exact solution. In contrast, only a convex approximation can be offered for (2).

5. **Numerical simulations.** In this section, we present some numerical simulations using 50 real stocks' daily price data for 2480 days. The two curves plotted in Figure 1 illustrate the resulting efficient frontiers obtained respectively from the SOCP transformation of Problem 6 and distributionally robust (DB) risk measure formulation of Problem 21. Both curves are drawn with the x−axis defined as the probability level y ranging from 0 to 1; whereas the y−axis defined as the optimal return of the portfolio.

From Figure 1, for every value of y, the portfolio under DB risk measure provides a higher expected return than one under the SOCP formulation. The results under the LP transformation of Problem 6 are not illustrated because the lack of information provided prevents us from arriving at an optimal portfolio, much less an efficient frontier. This is because the optimization problem is ill-possed if the return is not upwardly bounded according to [3].

Figure 2 shows the results of the second-order cone model and the distributionally robust risk measure model in mean-standard deviation space, compared with Markowitz's efficient frontier (l_2). From the figure, we can see that the relaxation of the normality distribution assumption does not result in an efficient frontier that
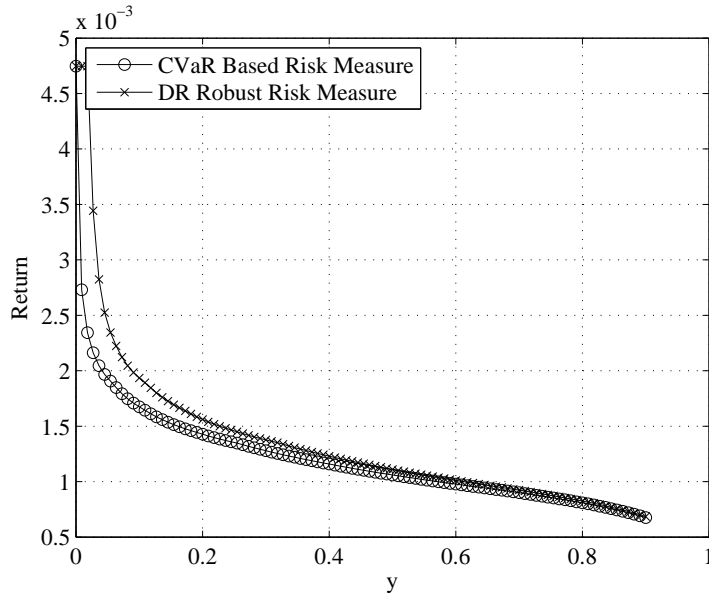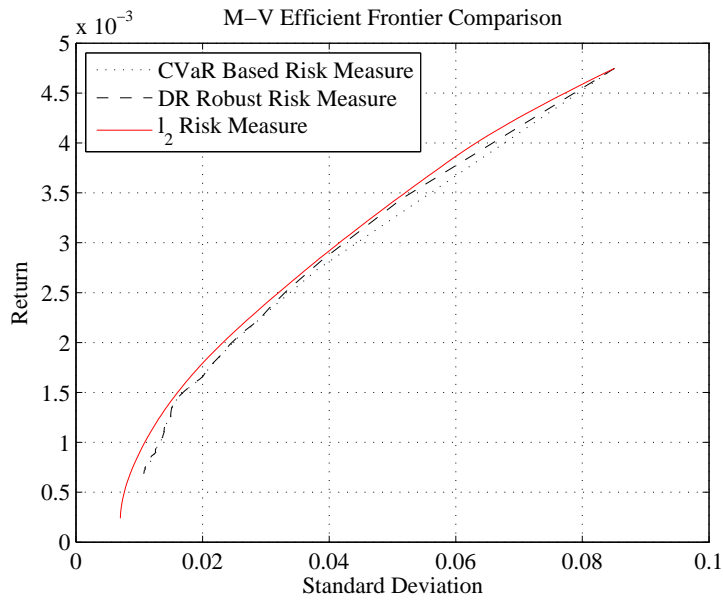
FIGURE 1. Efficient Frontier Comparison

FIGURE 2. Frontier Comparison

deviates much from Markowitz's results, except when return is lower than 0.37%. Again, we can see that the portfolio under the DB risk measure model better replicates the portfolio obtained by Markowitz, but requires much less information than the latter.

6. **Conclusion.** In this paper, we apply the robust optimization method to the study of portfolio selection. Distributionally robust optimization is a paradigm for decision-making under uncertainty where the uncertain problem data is assumed to belong to an ambiguity set comprising all distributions that are compatible with the decision maker's prior information. This method means that the distribution of asset returns is not required in arriving at the optimal portfolio. With just knowledge of the first two moments, the resulting optimal portfolio performs no worse than a comparable portfolio of the same risk but with stronger distributional assumptions. In particular, the second order cone method and the distriubtionally robust method do not assume the return is bounded and hence they are more practical in real world applications.

Markowitz laid the foundation for modern portfolio theory (MPT), or mean-variance analysis. Whilst there have been criticisms of various aspects of the theory, there is no denying its theoretical importance. Amongst the criticisms are the assumption that returns follow a Gaussian distribution, using variance as a risk measure, and the huge amount of data required in arriving at the frontier. Our paper relaxes all these requirements - we do not assume any distribution for the returns, we used two risk measures different from variance, and the data required is much less than under MPT. Nevertheless, we find that our resulting efficient frontier is almost similar to that under MPT.

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