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Research article

A decision-making strategy to combat CO_2 emissions using sine trigonometric aggregation operators with cubic bipolar fuzzy input

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Abstract: A cubic bipolar fuzzy set (CBFS) is by far the most efficient model for handling bipolar fuzziness because it carries both single-valued (SV) and interval-valued (IV) bipolar fuzzy numbers at the same time. The sine trigonometric function possesses two consequential qualities, namely, periodicity and symmetry, both of which are helpful tools for matching decision makers' conjectures. This article aims to integrate the sine function and cubic bipolar fuzzy data. As a result, sine trigonometric operational laws (STOLs) for cubic bipolar fuzzy numbers (CBFNs) are defined in this article. Premised on these laws, a substantial range of aggregation operators (AOs) are introduced. Certain features of these operators, including monotonicity, idempotency, and boundedness, are explored as well. Using the proffered AOs, a novel multi-criteria group decision-making (MCGDM) strategy is developed. An extensive case study of carbon capture and storage (CCS) technology has been provided to show the viability of the suggested method. A numerical example is provided to manifest the feasibility of the developed approach. Finally, a comparison study is executed to discuss the efficacy of the novel MCGDM framework.

Keywords: sine trigonometric operational laws; cubic bipolar fuzzy sets; aggregation operators; carbon capture; MCGDM
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1. Introduction

Data imprecision, uncertain human judgments and crisp methodologies lead to ambiguities and vagueness in an MCGDM process. To address the aforementioned issues, Zadeh [1] put forth the conception of a fuzzy set (FS). This theory was further revolutionized by different researchers [2–4]. The conception of intuitionistic fuzzy sets (IFSs) was developed by Atanassov [5]. Yager [6, 7] developed Pythagorean fuzzy sets (PFSs) and further extended this concept to q-rung orthopair fuzzy sets (q-ROFSs). In a similar way, several new approaches for addressing ambiguity in decision-making have been developed. Examples of these models include neutrosophic sets (NSs), picture fuzzy sets (PiFSs), and spherical fuzzy sets (SFSs), all of these may be found in the works [8–10]. Gou et al. [11] presented the characteristics of continuous PF information. Gou and Xu [12] introduced the fundamental operational rules for linguistic terms (LTs), hesitant fuzzy LTSs and probabilistic LTSs. The probabilistic double hierarchy LTS was proposed by Gou et al. [13] and used in the formulation of an enhanced VIKOR method.

The frameworks discussed above have all shown to be very helpful in dealing with uncertainty and are frequently employed by decision-makers (DMs); however, they focus on the existence or absence of a single attribute at a time. They are incapable of handling the counter-property of any property. While performing a decision analysis, it is highly common to have to contemplate both the favorable and adverse aspects of a particular object. Action and reaction, gain and loss, well-being and illness, and other polar opposites are examples of well-known conflicting aspects in decision analysis. Zhang [14, 15] developed the notion of bipolar fuzzy sets (BFSs) to tackle contradictory qualities. A cubic set (CS) [16] combines both fuzzy set (FS) as well as IV fuzzy set (IVFS). Due to the inclusion of both SV and IV fuzzy data, this hybrid model is particularly effective at handling MCGDM issues. Garg and Kaur [17] introduced cubic intuitionistic fuzzy set (CIFS) and discussed its properties. Garg and Kaur [18] proposed CIFS technique for the order preference by similarity to ideal solution (TOPSIS) on the basis of nonlinear-programming methodology. Garg and Kaur [19] defined correlation coefficients for CIFSs and developed an algorithm based on these correlation coefficients to decipher decision-making challenges. Abbas et al. [20] developed the idea of cubic Pythagorean fuzzy sets (CPFSs). Wang and Zhao [21] developed prospect theory and geometric distance measure based MCDM method for CPFS.

The paradigm of a cubic bipolar fuzzy set (CBFS), which incorporates both an IV bipolar fuzzy set (IVBFS) and a bipolar fuzzy set (BFS) concurrently, was put out by Riaz and Tehrim [22]. In this model, a DM assigns an IV positive membership grade (IVPMG), an IV negative membership grade (IVNMG), an SV positive membership grade (SVPMG) and an SV negative membership grade (SVNMG) to an object. Some major contributions in CBF theory are presented in Table 1.

Table 1. Some contributions in the field of CBF theory.

Authors	Research work
Riaz and Tehrim [23]	CBF averaging AOs.
Riaz and Tehrim [24]	CBF geometric AOs.
Jan et al. [25]	CBF graphs.
Jamil and Riaz [26]	CBF TOPSIS and ELECTRE-I methods.
Riaz and Jamil [27]	CBF topological spaces.

AOs are essential for aggregating information and are governed by a number of operational rules. Xu [28] and Xu and Yager [29] presented AOs for IFSs using algebraic operational rules. Senapati et al. [30] suggested Aczel-Alsina AOs for IVIFS. Senapati [31] introduced picture fuzzy Aczel-Alsina averaging AOs. Tian et al. [32] suggested a picture fuzzy MULTIMOORA approach based on Schweizer-Sklar AOs. Xiao et al. [33] proposed a generalized divergence-based decision making mechanism for pattern classification. Xiao [34] proposed a generalized smart quality-based framework for merging multi-source data. Xiao [35] discovered a distance measure for IFSs, as well as its use in pattern categorization tasks. Riaz and Hashmi [36] introduced the linear Diophantine fuzzy set (LDFS) and applied this model in MADM. Using linguistic LDF information, Kamaci et al. [37] defined a distance-measures driven modified TOPSIS method. Another good option for information fusion is the sine trigonometric function. During the object evaluation process, this function's periodicity and symmetry about the origin help to meet the aspirations of the DMs. Akram et al. [38–40] introduced novel MCDM methods for bipolar fuzzy information. Garg [41, 42] proposed sine trigonometric operational laws (STOLs) for PFSs and q-ROFSs. The literature review on STOLs of various models is summarized in Table 2.

Table 2. Literature review on sine trigonometric AOs.

Authors	Research work
Garg [41]	Pythagorean fuzzy sine trigonometric AOs
Garg [42]	Sine trigonometric based q-rung orthopair fuzzy AOs
Qiyas and Shahzaib [43]	Sine trigonometric spherical fuzzy AOs
Ashraf et al. [44]	Single-valued neutrosophic sine trigonometric AOs

Lin et al. [45–47] developed an innovative probabilistic linguistic MCDM, a Pythagorean fuzzy MULTIMOORA MCDM method and a TOPSIS method based on entropy measure and correlation coefficient. Riaz et al. [48] suggested a medical tourism supply chain based on bipolar fuzzy sine trigonometric AOs and superiority and inferiority ranking (SIR) method.

The objectives of this article are mentioned below.

(1) To manage bipolarity and vagueness using CBFS.

(2) To introduce and analyze STOLs for CBFNs in P and R order.

(3) To propose averaging AOs based on CBF-STOLs including "P-cubic bipolar fuzzy sine trigonometric weighted averaging (P-CBFSTWA) operator," "P-cubic bipolar fuzzy sine trigonometric ordered weighted averaging (P-CBFSTOWA) operator," "P-cubic bipolar fuzzy sine trigonometric hybrid weighted averaging (P-CBFSTHWA) operator," "R-cubic bipolar fuzzy sine trigonometric weighted averaging (R-CBFSTWA) operator," "R-cubic bipolar fuzzy sine trigonometric ordered weighted averaging (R-CBFSTOWA) operator," and "R-cubic bipolar fuzzy sine trigonometric hybrid weighted averaging (R-CBFSTHWA) operator."

(4) To propose geometric AOs based on CBF-STOLs including "P-cubic bipolar fuzzy sine trigonometric weighted geometric (P-CBFSTWG) operator," "P-cubic bipolar fuzzy sine trigonometric ordered weighted geometric (P-CBFSTOWG) operator," "P-cubic bipolar fuzzy sine trigonometric hybrid weighted geometric (P-CBFSTHWG) operator," "R-cubic bipolar fuzzy sine trigonometric weighted geometric (R-CBFSTWG) operator," "R-cubic bipolar fuzzy sine trigonometric ordered weighted geometric (R-CBFSTOWG) operator," and "R-cubic bipolar fuzzy sine trigonometric hybrid weighted geometric (R-CBFSTHWG) operator."

(5) Certain key features of these operators, including monotonicity, idempotency, and boundedness, are explored as well.

(6) To construct an MCGDM algorithm using the specified AOs and apply it to an MCGDM problem involving CCS technology.

The rest of this article is broken down into the sections that are listed below. Section 2 goes over some fundamental ideas. The STOLs for CBFNs are set forth in Section 3. We suggest novel averaging AOs based on CBF-STOLs and investigate their properties in Section 4. Geometric AOs are covered along with their characteristics in Section 5. In Section 6, we design an algorithm that is based on the suggested AOs and apply it to a challenging MCGDM problem related to CCS. To emphasize how well our proposed method works, we have provided a numerical illustration in this section. In addition, a comparison is performed in order to determine the usefulness and efficaciousness of the technique that we have proposed. The concluding perspective is put forward in Section 7 of the article.

2. Preliminaries

The following section will review certain fundamental concepts, such as CBFS, its various operations, and the comparability of two CBFNs.

Definition 2.1. [22] A CBFS $\mathfrak{W}$ in Θ can be represented as

$$\mathfrak{W} = \{(\kappa, [\mathcal{A}_{l\mathfrak{W}}(\kappa), \mathcal{A}_{u\mathfrak{W}}(\kappa)], [\mathcal{B}_{l\mathfrak{W}}(\kappa), \mathcal{B}_{u\mathfrak{W}}(\kappa)], \{\mathcal{A}_{\mathfrak{W}}(\kappa), \mathcal{B}_{\mathfrak{W}}(\kappa)\}) | \kappa \in \Theta\}$$

where $[\mathcal{A}_{l\mathfrak{W}}(\kappa), \mathcal{A}_{u\mathfrak{W}}(\kappa)] \subseteq [0, 1]$ and $[\mathcal{B}_{l\mathfrak{W}}(\kappa), \mathcal{B}_{u\mathfrak{W}}(\kappa)] \subseteq [-1, 0]$ denote the IVPMG and IVNMG, respectively, and $\mathcal{A}_{\mathfrak{W}}(\kappa) \in [0, 1]$ and $\mathcal{B}_{\mathfrak{W}}(\kappa) \in [-1, 0]$ indicate the SVPMG and SVNMG, respectively, of an object $\kappa \in \Theta$. A CBFN is written as $\mathfrak{W} = \langle [\mathcal{A}_{l\mathfrak{W}}, \mathcal{A}_{u\mathfrak{W}}], [\mathcal{B}_{l\mathfrak{W}}, \mathcal{B}_{u\mathfrak{W}}], \{\mathcal{A}_{\mathfrak{W}}, \mathcal{B}_{\mathfrak{W}}\}\rangle$.

Definition 2.2. [22] Consider $\mathfrak{W}_i = \langle [\mathcal{A}_{l\mathfrak{W}_i}, \mathcal{A}_{u\mathfrak{W}_i}], [\mathcal{B}_{l\mathfrak{W}_i}, \mathcal{B}_{u\mathfrak{W}_i}], \{\mathcal{A}_{\mathfrak{W}_i}, \mathcal{B}_{\mathfrak{W}_i}\}\rangle$, $(i = 1, 2)$, and $\gamma > 0$, and then P-order algebraic laws are specified as

(i) $\mathfrak{W}_1 \oplus_{\mathrm{P}} \mathfrak{W}_2 = \Big\langle \Big[\mathcal{A}_{l\mathfrak{W}_1} + \mathcal{A}_{l\mathfrak{W}_2} - \mathcal{A}_{l\mathfrak{W}_1}\mathcal{A}_{l\mathfrak{W}_2}, \mathcal{A}_{u\mathfrak{W}_1} + \mathcal{A}_{u\mathfrak{W}_2} - \mathcal{A}_{u\mathfrak{W}_1}\mathcal{A}_{u\mathfrak{W}_2}\Big], \Big[-(\mathcal{B}_{l\mathfrak{W}_1}\mathcal{B}_{l\mathfrak{W}_2}), -(\mathcal{B}_{u\mathfrak{W}_1}\mathcal{B}_{u\mathfrak{W}_2})\Big], \Big\{\mathcal{A}_{\mathfrak{W}_1} + \mathcal{A}_{\mathfrak{W}_2} - \mathcal{A}_{\mathfrak{W}_1}\mathcal{A}_{\mathfrak{W}_2}, -(\mathcal{B}_{\mathfrak{W}_1}\mathcal{B}_{\mathfrak{W}_2})\Big\}\Big\rangle$

(ii) $\mathfrak{W}_1 \otimes_{\mathrm{P}} \mathfrak{W}_2 = \Big\langle \Big[\mathcal{A}_{l\mathfrak{W}_1}\mathcal{A}_{l\mathfrak{W}_2}, \mathcal{A}_{u\mathfrak{W}_1}\mathcal{A}_{u\mathfrak{W}_2}\Big], \Big[-\Big(-\mathcal{B}_{l\mathfrak{W}_1} - \mathcal{B}_{l\mathfrak{W}_2} - (\mathcal{B}_{l\mathfrak{W}_1}\mathcal{B}_{l\mathfrak{W}_2})\Big), -\Big(-\mathcal{B}_{u\mathfrak{W}_1} - \mathcal{B}_{u\mathfrak{W}_2} - (\mathcal{B}_{u\mathfrak{W}_1}\mathcal{B}_{u\mathfrak{W}_2})\Big)\Big], \Big\{\mathcal{A}_{\mathfrak{W}_1}\mathcal{A}_{\mathfrak{W}_2}, -\Big(-\mathcal{B}_{\mathfrak{W}_1} - \mathcal{B}_{\mathfrak{W}_2} - (\mathcal{B}_{\mathfrak{W}_1}\mathcal{B}_{\mathfrak{W}_2})\Big)\Big\}\Big\rangle$

(iii) $\mathfrak{W}_1^{\gamma} = \Big\langle \Big[(\mathcal{A}_{l\mathfrak{W}_1})^{\gamma}, (\mathcal{A}_{u\mathfrak{W}_1})^{\gamma}\Big], \Big[-\Big(1 - (1 - (-\mathcal{B}_{l\mathfrak{W}_1}))^{\gamma}\Big), -\Big(1 - (1 - (-\mathcal{B}_{u\mathfrak{W}_1}))^{\gamma}\Big)\Big], \Big\{(\mathcal{A}_{\mathfrak{W}_1})^{\gamma}, -\Big(1 - (1 - (-\mathcal{B}_{\mathfrak{W}_1}))^{\gamma}\Big)\Big\}\Big\rangle$

(iv) $\gamma\mathfrak{W}_1 = \Big\langle \Big[1 - (1 - \mathcal{A}_{l\mathfrak{W}_1})^{\gamma}, 1 - (1 - \mathcal{A}_{u\mathfrak{W}_1})^{\gamma}\Big], \Big[-(-\mathcal{B}_{l\mathfrak{W}_1})^{\gamma}, -(-\mathcal{B}_{u\mathfrak{W}_1})^{\gamma}\Big], \Big\{1 - (1 - \mathcal{A}_{\mathfrak{W}_1})^{\gamma}, -(-\mathcal{B}_{\mathfrak{W}_1})^{\gamma}\Big\}\Big\rangle$.

Definition 2.3. [22] Consider $\mathfrak{W}_i = \langle [\mathcal{A}_{l\mathfrak{W}_i}, \mathcal{A}_{u\mathfrak{W}_i}], [\mathcal{B}_{l\mathfrak{W}_i}, \mathcal{B}_{u\mathfrak{W}_i}], \{\mathcal{A}_{\mathfrak{W}_i}, \mathcal{B}_{\mathfrak{W}_i}\}\rangle$, $(i = 1, 2)$, and $\gamma > 0$, and then R-order algebraic operations are specified as

(i) $\mathfrak{W}_1 \oplus_{\mathrm{R}} \mathfrak{W}_2 = \Big\langle \Big[\mathcal{A}_{l\mathfrak{W}_1} + \mathcal{A}_{l\mathfrak{W}_2} - \mathcal{A}_{l\mathfrak{W}_1}\mathcal{A}_{l\mathfrak{W}_2}, \mathcal{A}_{u\mathfrak{W}_1} + \mathcal{A}_{u\mathfrak{W}_2} - \mathcal{A}_{u\mathfrak{W}_1}\mathcal{A}_{u\mathfrak{W}_2}\Big], \Big[-(\mathcal{B}_{l\mathfrak{W}_1}\mathcal{B}_{l\mathfrak{W}_2}), -(\mathcal{B}_{u\mathfrak{W}_1}\mathcal{B}_{u\mathfrak{W}_2})\Big], \Big\{\mathcal{A}_{\mathfrak{W}_1}\mathcal{A}_{\mathfrak{W}_2}, -\Big(-\mathcal{B}_{\mathfrak{W}_1} - \mathcal{B}_{\mathfrak{W}_2} - (\mathcal{B}_{\mathfrak{W}_1}\mathcal{B}_{\mathfrak{W}_2})\Big)\Big\}\Big\rangle$

(ii) $\mathfrak{W}_1 \otimes_R \mathfrak{W}_2 = \Big\langle\Big[\mathcal{A}_{l\mathfrak{W}_1}\mathcal{A}_{l\mathfrak{W}_2}, \mathcal{A}_{u\mathfrak{W}_1}\mathcal{A}_{u\mathfrak{W}_2}\Big], \Big[-\Big(-\mathcal{B}_{l\mathfrak{W}_1} - \mathcal{B}_{l\mathfrak{W}_2} - (\mathcal{B}_{l\mathfrak{W}_1}\mathcal{B}_{l\mathfrak{W}_2})\Big), -\Big(-\mathcal{B}_{u\mathfrak{W}_1} - \mathcal{B}_{u\mathfrak{W}_2} - (\mathcal{B}_{u\mathfrak{W}_1}\mathcal{B}_{u\mathfrak{W}_2})\Big)\Big], \Big\{\mathcal{A}_{\mathfrak{W}_1} + \mathcal{A}_{\mathfrak{W}_2} - \mathcal{A}_{\mathfrak{W}_1}\mathcal{A}_{\mathfrak{W}_2}, -(\mathcal{B}_{\mathfrak{W}_1}\mathcal{B}_{\mathfrak{W}_2})\Big\}\Big\rangle$

(iii) $\mathfrak{W}_1^\gamma = \Big\langle\Big[(\mathcal{A}_{l\mathfrak{W}_1})^\gamma, (\mathcal{A}_{u\mathfrak{W}_1})^\gamma\Big], \Big[-\Big(1 - (1 - (-\mathcal{B}_{l\mathfrak{W}_1}))^\gamma\Big), -\Big(1 - (1 - (-\mathcal{B}_{u\mathfrak{W}_1}))^\gamma\Big)\Big], \Big\{1 - (1 - \mathcal{A}_{\mathfrak{W}_1})^\gamma, -(-\mathcal{B}_{\mathfrak{W}_1})^\gamma\Big\}\Big\rangle$

(iv) $\gamma\mathfrak{W}_1 = \Big\langle\Big[1-(1-\mathcal{A}_{l\mathfrak{W}_1})^\gamma, 1-(1-\mathcal{A}_{u\mathfrak{W}_1})^\gamma\Big], \Big[-(-\mathcal{B}_{l\mathfrak{W}_1})^\gamma, -(-\mathcal{B}_{u\mathfrak{W}_1})^\gamma\Big], \Big\{(\mathcal{A}_{l\mathfrak{W}_1})^\gamma, -\Big(1-(1-(-\mathcal{B}_{\mathfrak{W}_1}))^\gamma\Big)\Big\}\Big\rangle$.

Definition 2.4. [22] For two CBFNs $\mathfrak{W}_i = \langle[\mathcal{A}_{l\mathfrak{W}_i}, \mathcal{A}_{u\mathfrak{W}_i}], [\mathcal{B}_{l\mathfrak{W}_i}, \mathcal{B}_{u\mathfrak{W}_i}], \{\mathcal{A}_{\mathfrak{W}_i}, \mathcal{B}_{\mathfrak{W}_i}\}\rangle$, $(i = 1, 2)$,

(i) $\mathfrak{W}_1 = \mathfrak{W}_2$ if
$[\mathcal{A}_{l\mathfrak{W}_1}, \mathcal{A}_{u\mathfrak{W}_1}] = [\mathcal{A}_{l\mathfrak{W}_2}, \mathcal{A}_{u\mathfrak{W}_2}]$ and $[\mathcal{B}_{l\mathfrak{W}_1}, \mathcal{B}_{u\mathfrak{W}_1}] = [\mathcal{B}_{l\mathfrak{W}_2}, \mathcal{B}_{u\mathfrak{W}_2}]$
$\mathcal{A}_{\mathfrak{W}_1} = \mathcal{A}_{\mathfrak{W}_2}$ and $\mathcal{B}_{\mathfrak{W}_1} = \mathcal{B}_{\mathfrak{W}_2}$.

(ii) $\mathfrak{W}_1 \leq_P \mathfrak{W}_2$ if
$[\mathcal{A}_{l\mathfrak{W}_1}, \mathcal{A}_{u\mathfrak{W}_1}] \leq [\mathcal{A}_{l\mathfrak{W}_2}, \mathcal{A}_{u\mathfrak{W}_2}]$ and $[\mathcal{B}_{l\mathfrak{W}_1}, \mathcal{B}_{u\mathfrak{W}_1}] \geq [\mathcal{B}_{l\mathfrak{W}_2}, \mathcal{B}_{u\mathfrak{W}_2}]$
$\mathcal{A}_{\mathfrak{W}_1} \leq \mathcal{A}_{\mathfrak{W}_2}$ and $\mathcal{B}_{\mathfrak{W}_1} \geq \mathcal{B}_{\mathfrak{W}_2}$.

(iii) $\mathfrak{W}_1 \leq_R \mathfrak{W}_2$ if
$[\mathcal{A}_{l\mathfrak{W}_1}, \mathcal{A}_{u\mathfrak{W}_1}] \leq [\mathcal{A}_{l\mathfrak{W}_2}, \mathcal{A}_{u\mathfrak{W}_2}]$ and $[\mathcal{B}_{l\mathfrak{W}_1}, \mathcal{B}_{u\mathfrak{W}_1}] \geq [\mathcal{B}_{l\mathfrak{W}_2}, \mathcal{B}_{u\mathfrak{W}_2}]$
$\mathcal{A}_{\mathfrak{W}_1} \geq \mathcal{A}_{\mathfrak{W}_2}$ and $\mathcal{B}_{\mathfrak{W}_1} \leq \mathcal{B}_{\mathfrak{W}_2}$.

Definition 2.5. [24] For any CBFN $\mathfrak{W}$, the score function Ψ_P is calculated as

$$\Psi_P(\mathfrak{W}) = \frac{3 + \mathcal{A}_{l\mathfrak{W}} + \mathcal{A}_{u\mathfrak{W}} + \mathcal{B}_{l\mathfrak{W}} + \mathcal{B}_{u\mathfrak{W}} + \mathcal{A}_{\mathfrak{W}} + \mathcal{B}_{\mathfrak{W}}}{6}. \tag{2.1}$$

Similarly, we take the score function Ψ_R as follows:

$$\Psi_R(\mathfrak{W}) = \frac{3 + \mathcal{A}_{l\mathfrak{W}} + \mathcal{A}_{u\mathfrak{W}} + \mathcal{B}_{l\mathfrak{W}} + \mathcal{B}_{u\mathfrak{W}} - \mathcal{A}_{\mathfrak{W}} - \mathcal{B}_{\mathfrak{W}}}{6} \tag{2.2}$$

where $\Psi_{P,R}(\mathfrak{W}) \in [0, 1]$.

Definition 2.6. [24] The accuracy function Φ for a CBFN $\mathfrak{W}$ can be formulated as

$$\Phi(\mathfrak{W}) = \frac{3 + \mathcal{A}_{l\mathfrak{W}} + \mathcal{A}_{u\mathfrak{W}} - \mathcal{B}_{l\mathfrak{W}} - \mathcal{B}_{u\mathfrak{W}} + \mathcal{A}_{\mathfrak{W}} - \mathcal{B}_{\mathfrak{W}}}{6} \tag{2.3}$$

where $\Phi(\mathfrak{W}) \in [0, 1]$.

Score and accuracy functions are used to rank CBFNs in the following manner:
(i) $\mathfrak{W}_1 \prec \mathfrak{W}_2$, if $\Psi_{P,R}(\mathfrak{W}_1) < \Psi_{P,R}(\mathfrak{W}_2)$.
(ii) $\mathfrak{W}_1 \succ \mathfrak{W}_2$, if $\Psi_{P,R}(\mathfrak{W}_1) > \Psi_{P,R}(\mathfrak{W}_2)$.
(iii) $\mathfrak{W}_1 \succ \mathfrak{W}_2$, if $\Phi(\mathfrak{W}_1) > \Phi(\mathfrak{W}_2)$ and $\Psi_{P,R}(\mathfrak{W}_1) = \Psi_{P,R}(\mathfrak{W}_2)$.
(iv) $\mathfrak{W}_1 \sim \mathfrak{W}_2$, if $\Phi(\mathfrak{W}_1) = \Phi(\mathfrak{W}_2)$ and $\Psi_{P,R}(\mathfrak{W}_1) = \Psi_{P,R}(\mathfrak{W}_2)$.

3. Sine trigonometric operational laws for CBFSs

The objective of this section is to set up the STOLs for CBFNs and explore some noteworthy results related to them.

Definition 3.1. Corresponding to a CBFS $\mathfrak{W} = \{(\kappa, [\mathcal{A}_{l\mathfrak{W}}(\kappa), \mathcal{A}_{u\mathfrak{W}}(\kappa)], [\mathcal{B}_{l\mathfrak{W}}(\kappa), \mathcal{B}_{u\mathfrak{W}}(\kappa)], \{\mathcal{A}_{\mathfrak{W}}(\kappa), \mathcal{B}_{\mathfrak{W}}(\kappa)\})$ $|\kappa \in \Theta\}$, a sine trigonometric operator on $\mathfrak{W}$ is expressed as

$$\sin \mathfrak{W} = \left\{ \begin{array}{c} \Big(\kappa, \Big[\sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}}(\kappa)\right), \sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}}(\kappa)\right)\Big], \\ \Big[\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}}(\kappa))\right) - 1, \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}}(\kappa))\right) - 1\Big], \\ \Big\{ \sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}}(\kappa)\right), \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}}(\kappa))\right) - 1\Big\} : \kappa \in \Theta\Big) \end{array} \right\}.$$

The set $\sin \mathfrak{W}$ is called a sine trigonometric-CBFS (ST-CBFS).

Theorem 3.2. *A ST-CBFS is also a CBFS.*

Proof. Since $[\mathcal{A}_{l\mathfrak{W}}(\kappa), \mathcal{A}_{u\mathfrak{W}}(\kappa)] \subseteq [0,1]$ and $[\mathcal{B}_{l\mathfrak{W}}(\kappa), \mathcal{B}_{u\mathfrak{W}}(\kappa)] \subseteq [-1,0]$, so $\Big[\sin(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}}(\kappa)), \sin(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}}(\kappa))\Big] \subseteq [0,1]$ and $\Big[\sin(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}}(\kappa))) - 1, \sin(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}}(\kappa))) - 1\Big] \subseteq [-1,0]$. Similarly, $\mathcal{A}_{\mathfrak{W}}(\kappa) \in [0,1]$ and $\mathcal{B}_{\mathfrak{W}}(\kappa) \in [-1,0]$ imply that $\sin(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}}(\kappa)) \in [0,1]$ and $\sin(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}}(\kappa))) - 1 \in [-1,0]$, respectively. This shows that a ST-CBFS is a CBFS. □

Definition 3.3. Let $\mathfrak{W} = \langle[\mathcal{A}_{l\mathfrak{W}}, \mathcal{A}_{u\mathfrak{W}}], [\mathcal{B}_{l\mathfrak{W}}, \mathcal{B}_{u\mathfrak{W}}], \{\mathcal{A}_{\mathfrak{W}}, \mathcal{B}_{\mathfrak{W}}\}\rangle$ be a CBFN. Then,

$$\sin \mathfrak{W} = \left\{ \begin{array}{c} \Big[\sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}}\right), \sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}}\right)\Big], \\ \Big[\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}})\right) - 1, \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}})\right) - 1\Big], \\ \Big\{ \sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}}\right), \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}})\right) - 1\Big\} \end{array} \right\}$$

is known as a sine trigonometric-CBFN (ST-CBFN).

Definition 3.4. Let $\mathfrak{W}_i = \langle[\mathcal{A}_{l\mathfrak{W}_i}, \mathcal{A}_{u\mathfrak{W}_i}], [\mathcal{B}_{l\mathfrak{W}_i}, \mathcal{B}_{u\mathfrak{W}_i}], (\mathcal{A}_{\mathfrak{W}_i}, \mathcal{B}_{\mathfrak{W}_i})\rangle$, $(i = 1,2)$ be any two CBFNs and let $\gamma > 0$, and then STOLs under P-order are expounded as

$$\text{(i)}\ \sin \mathfrak{W}_1 \oplus_{\mathrm{P}} \sin \mathfrak{W}_2 = \left\{ \begin{array}{c} \Big[1 - \Big(1 - \sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_1}\right)\Big)\Big(1 - \sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_2}\right)\Big), \\ 1 - \Big(1 - \sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_1}\right)\Big)\Big(1 - \sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_2}\right)\Big)\Big] \\ \Big[-\Big(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_1})\right) - 1\Big)\Big(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_2})\right) - 1\Big), \\ -\Big(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_1})\right) - 1\Big)\Big(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_2})\right) - 1\Big)\Big], \\ \Big\{1 - \Big(1 - \sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}_1}\right)\Big)\Big(1 - \sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}_2}\right)\Big), \\ -\Big(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_1})\right) - 1\Big)\Big(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_2})\right) - 1\Big)\Big\} \end{array} \right\},$$

$$\text{(ii)}\ \sin \mathfrak{W}_1 \otimes_{\mathrm{P}} \sin \mathfrak{W}_2 = \left\{ \begin{array}{c} \Big[\sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_1}\right) \sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_2}\right), \sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_1}\right) \sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_2}\right)\Big], \\ \Big[-\Big(1 - \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_1})\right) \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_2})\right)\Big), \\ -\Big(1 - \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_1})\right) \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_2})\right)\Big)\Big], \\ \Big\{ \sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}_1}\right) \sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}_2}\right), -\Big(1 - \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_1})\right) \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_2})\right)\Big)\Big\} \end{array} \right\},$$

$$\text{(iii)}\ \ \gamma \sin \mathfrak{W}_1 = \left\{\begin{array}{c} \left[1-\left(1-\sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_1}\right)\right)^{\gamma}, 1-\left(1-\sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_1}\right)\right)^{\gamma}\right], \\ \left[-\left(-\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_1})\right)-1\right)\right)^{\gamma}, -\left(-\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_1})\right)-1\right)\right)^{\gamma}\right], \\ \left\{1-\left(1-\sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}_1}\right)\right)^{\gamma}, -\left(-\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_1})\right)-1\right)\right)^{\gamma}\right\} \end{array}\right\},$$

$$\text{(iv)}\ \ (\sin \mathfrak{W}_1)^{\gamma} = \left\{\begin{array}{c} \left[\left(\sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_1}\right)\right)^{\gamma}, \left(\sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_1}\right)\right)^{\gamma}\right], \\ \left[-\left(1-\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_1})\right)\right)^{\gamma}\right), -\left(1-\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_1})\right)\right)^{\gamma}\right)\right], \\ \left\{\left(\sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}_1}\right)\right)^{\gamma}, -\left(1-\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_1})\right)\right)^{\gamma}\right)\right\} \end{array}\right\}.$$

Definition 3.5. Let $\mathfrak{W}_i = \langle [\mathcal{A}_{l\mathfrak{W}_i}, \mathcal{A}_{u\mathfrak{W}_i}], [\mathcal{B}_{l\mathfrak{W}_i}, \mathcal{B}_{u\mathfrak{W}_i}], (\mathcal{A}_{\mathfrak{W}_i}, \mathcal{B}_{\mathfrak{W}_i})\rangle$, $(i = 1, 2)$ be any two CBFNs and let $\gamma > 0$, and then STOLs under R-order are delineated as

$$\text{(i)}\ \ \sin \mathfrak{W}_1 \oplus_{\mathrm{R}} \sin \mathfrak{W}_2 = \left\{\begin{array}{c} \left[1-\left(1-\sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_1}\right)\right)\left(1-\sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_2}\right)\right), \right. \\ \left. 1-\left(1-\sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_1}\right)\right)\left(1-\sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_2}\right)\right)\right] \\ \left[-\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_1})\right)-1\right)\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_2})\right)-1\right), \right. \\ \left. -\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_1})\right)-1\right)\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_2})\right)-1\right)\right], \\ \left\{\sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}_1}\right)\sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}_2}\right), -\left(1-\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_1})\right)\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_2})\right)\right)\right\} \end{array}\right\},$$

$$\text{(ii)}\ \ \sin \mathfrak{W}_1 \otimes_{\mathrm{R}} \sin \mathfrak{W}_2 = \left\{\begin{array}{c} \left[\sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_1}\right)\sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_2}\right), \sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_1}\right)\sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_2}\right)\right], \\ \left[-\left(1-\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_1})\right)\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_2})\right)\right), \right. \\ \left. -\left(1-\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_1})\right)\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_2})\right)\right)\right], \\ \left\{1-\left(1-\sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}_1}\right)\right)\left(1-\sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}_2}\right)\right), \right. \\ \left. -\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_1})\right)-1\right)\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_2})\right)-1\right)\right\} \end{array}\right\},$$

$$\text{(iii)}\ \ \gamma \sin \mathfrak{W}_1 = \left\{\begin{array}{c} \left[1-\left(1-\sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_1}\right)\right)^{\gamma}, 1-\left(1-\sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_1}\right)\right)^{\gamma}\right], \\ \left[-\left(-\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_1})\right)-1\right)\right)^{\gamma}, -\left(-\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_1})\right)-1\right)\right)^{\gamma}\right], \\ \left\{\left(\sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}_1}\right)\right)^{\gamma}, -\left(1-\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_1})\right)\right)^{\gamma}\right)\right\} \end{array}\right\},$$

$$\text{(iv)}\ \ (\sin \mathfrak{W}_1)^{\gamma} = \left\{\begin{array}{c} \left[\left(\sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_1}\right)\right)^{\gamma}, \left(\sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_1}\right)\right)^{\gamma}\right], \\ \left[-\left(1-\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_1})\right)\right)^{\gamma}\right), -\left(1-\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_1})\right)\right)^{\gamma}\right)\right], \\ \left\{1-\left(1-\sin\left(\frac{\pi}{2}\mathcal{A}_{\mathfrak{W}_1}\right)\right)^{\gamma}, -\left(-\left(\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_1})\right)-1\right)\right)^{\gamma}\right\} \end{array}\right\}.$$

Theorem 3.6. *Consider two CBFNs $\mathfrak{W}_1$ and $\mathfrak{W}_2$ and three positive real numbers $\gamma, \gamma_1, \gamma_2$; the results stated below are true under* P*-order:*

(i) $\gamma(\sin \mathfrak{W}_1 \oplus_{\mathrm{P}} \sin \mathfrak{W}_2) = \gamma \sin \mathfrak{W}_1 \oplus_{\mathrm{P}} \gamma \sin \mathfrak{W}_2$,
(ii) $(\sin \mathfrak{W}_1 \otimes_{\mathrm{P}} \sin \mathfrak{W}_2)^{\gamma} = (\sin \mathfrak{W}_1)^{\gamma} \otimes_{\mathrm{P}} (\sin \mathfrak{W}_2)^{\gamma}$,
(iii) $\gamma_1 \sin \mathfrak{W}_1 \oplus_{\mathrm{P}} \gamma_2 \sin \mathfrak{W}_1 = (\gamma_1 + \gamma_2) \sin \mathfrak{W}_1$,
(iv) $(\sin \mathfrak{W}_1)^{\gamma_1} \otimes_{\mathrm{P}} (\sin \mathfrak{W}_1)^{\gamma_2} = (\sin \mathfrak{W}_1)^{\gamma_1+\gamma_2}$,
(v) $((\sin \mathfrak{W}_1)^{\gamma_1})^{\gamma_2} = (\sin \mathfrak{W}_1)^{\gamma_1\gamma_2}$.

Proof. Consult Appendix A. □

Theorem 3.7. *For two CBFNs $\mathfrak{W}_1$ and $\mathfrak{W}_2$ and three positive real values $\gamma, \gamma_1, \gamma_2$; the results stated below are true under* R*-order:*

(i) $\gamma(\sin \mathfrak{W}_1 \oplus_{\mathrm{R}} \sin \mathfrak{W}_2) = \gamma \sin \mathfrak{W}_1 \oplus_{\mathrm{R}} \gamma \sin \mathfrak{W}_2$,

(ii) $(\sin \mathfrak{W}_1 \otimes_{\mathrm{R}} \sin \mathfrak{W}_2)^{\gamma} = (\sin \mathfrak{W}_1)^{\gamma} \otimes_{\mathrm{R}} (\sin \mathfrak{W}_2)^{\gamma}$,
(iii) $\gamma_1 \sin \mathfrak{W}_1 \oplus_{\mathrm{R}} \gamma_2 \sin \mathfrak{W}_1 = (\gamma_1 + \gamma_2) \sin \mathfrak{W}_1$,
(iv) $(\sin \mathfrak{W}_1)^{\gamma_1} \otimes_{\mathrm{R}} (\sin \mathfrak{W}_1)^{\gamma_2} = (\sin \mathfrak{W}_1)^{\gamma_1+\gamma_2}$,
(v) $((\sin \mathfrak{W}_1)^{\gamma_1})^{\gamma_2} = (\sin \mathfrak{W}_1)^{\gamma_1\gamma_2}$.

Proof. Straightforward. □

Theorem 3.8. *If* $\mathfrak{W}_1 \leq_{\mathrm{P}} \mathfrak{W}_2$*, then* $\sin \mathfrak{W}_1 \leq_{\mathrm{P}} \sin \mathfrak{W}_2$.

Proof. For $\mathfrak{W}_1$ and $\mathfrak{W}_2$, we have $[\mathcal{A}_{l\mathfrak{W}_1}, \mathcal{A}_{u\mathfrak{W}_1}] \leq [\mathcal{A}_{l\mathfrak{W}_2}, \mathcal{A}_{u\mathfrak{W}_2}]$, which gives $\left[\sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_1}\right), \sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_1}\right)\right] \leq \left[\sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_2}\right), \sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_2}\right)\right]$. Consonantly, $[\mathcal{B}_{l\mathfrak{W}_1}, \mathcal{B}_{u\mathfrak{W}_1}] \geq [\mathcal{B}_{l\mathfrak{W}_2}, \mathcal{B}_{u\mathfrak{W}_2}]$ implies that $\left[\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_1})\right)-1, \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_1})\right)-1\right] \geq \left[\sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_2})\right)-1, \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_2})\right)-1\right]$. Similar arguments can be made for single-valued positive and negative membership grades. By Definition 2.4, it is evident that $\sin \mathfrak{W}_1 \leq_{\mathrm{P}} \sin \mathfrak{W}_2$. □

Theorem 3.9. *If* $\mathfrak{W}_1 \leq_{\mathrm{R}} \mathfrak{W}_2$*, then* $\sin \mathfrak{W}_1 \leq_{\mathrm{R}} \sin \mathfrak{W}_2$.

Proof. Same as above. □

4. CBF sine trigonometric averaging aggregation operators

In this segment, we suggest a variety of averaging AOs derived from CBF-STOLs. These operators are P-CBFSTWAO, P-CBFSTOWAO and P-CBFSTHWAO. The same operators will be set up for R-order as well. Henceforth, we let $\Upsilon_{l_i} = \sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_i}\right)$, $\Upsilon_{u_i} = \sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_i}\right)$, $\Xi_{l_i} = \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_i})\right)$, $\Xi_{u_i} = \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_i})\right)$, $\Upsilon_i = \sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_i}\right)$ and $\Xi_i = \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_i})\right)$.

4.1. CBFSTWAO

Definition 4.1. Let $\{\mathfrak{W}_i : i = 1, 2, ..., n\}$ be a collection of CBFNs with the weight vector (WV) $\upsilon = \{\upsilon_1, \upsilon_2, ..., \upsilon_n\}$ such that $\upsilon_i > 0$ and $\sum\limits_{i=1}^{n} \upsilon_i = 1$. Then,

$$\text{P-}CBFSTWA(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \upsilon_1 \sin \mathfrak{W}_1 \oplus_{\mathrm{P}} \upsilon_2 \sin \mathfrak{W}_2 \oplus_{\mathrm{P}} ... \oplus_{\mathrm{P}} \upsilon_n \sin \mathfrak{W}_n. \tag{4.1}$$

Theorem 4.2. *The* P-*CBFSTWA operator yields a congregated value of n CBFNs, which is a CBFN provided by*

$$\text{P-}CBFSTWA(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \left\{\begin{array}{c} \left[1-\prod\limits_{i=1}^{n}\left(1-\Upsilon_{l_i}\right)^{\upsilon_i}, 1-\prod\limits_{i=1}^{n}\left(1-\Upsilon_{u_i}\right)^{\upsilon_i}\right], \\ \left[-\prod\limits_{i=1}^{n}\left(-\left(\Xi_{l_i}-1\right)\right)^{\upsilon_i}, -\prod\limits_{i=1}^{n}\left(-\left(\Xi_{u_i}-1\right)\right)^{\upsilon_i}\right], \\ \left\{1-\prod\limits_{i=1}^{n}\left(1-\Upsilon_i\right)^{\upsilon_i}, -\prod\limits_{i=1}^{n}\left(-\left(\Xi_i-1\right)\right)^{\upsilon_i}\right\} \end{array}\right\}. \tag{4.2}$$

Proof. See Appendix B. □

Example 4.3. Let $\mathfrak{W}_1 = \langle[0.32, 0.48], [-0.27, -0.18], \{0.57, -0.31\}\rangle$, $\mathfrak{W}_2 = \langle[0.62, 0.71], [-0.44, -0.22], \{0.39, -0.31\}\rangle$, $\mathfrak{W}_3 = \langle[0.49, 0.66], [-0.55, -0.37], \{0.26, -0.46\}\rangle$

and $\mathfrak{W}_4 = \langle [0.21, 0.42], [-0.71, -0.49], \{0.66, -0.52\} \rangle$ be four CBFNs and $\upsilon = \{0.22, 0.31, 0.19, 0.28\}$ be the WV for them. To calculate the congregation of these CBFNs using Eq (4.2), we first take $\Upsilon_{l_i} = \sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_i}\right)$, $\Upsilon_{u_i} = \sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_i}\right)$, $\Xi_{l_i} = \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_i})\right)$, $\Xi_{u_i} = \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_i})\right)$, $\Upsilon_i = \sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_i}\right)$ and $\Xi_i = \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_i})\right)$, where $i = 1, 2, 3, 4$. Their values are given in Table 3.

Table 3. The values of Υ_{l_i}, Υ_{u_i}, Ξ_{l_i}, Ξ_{u_i}, Υ_i and Ξ_i, for $i = 1, 2, 3, 4$.

Υ_{l_1}	0.4818	Υ_{u_1}	0.6845	Ξ_{l_1}	0.9114	Ξ_{u_1}	0.9603	Υ_1	0.7804	Ξ_1	0.8838
Υ_{l_2}	0.8217	Υ_{u_2}	0.8980	Ξ_{l_2}	0.7705	Ξ_{u_2}	0.9409	Υ_2	0.5750	Ξ_2	0.8838
Υ_{l_3}	0.6959	Υ_{u_3}	0.8607	Ξ_{l_3}	0.6494	Ξ_{u_3}	0.8358	Υ_3	0.3971	Ξ_3	0.7501
Υ_{l_4}	0.3239	Υ_{u_4}	0.6129	Ξ_{l_4}	0.4399	Ξ_{u_4}	0.7181	Υ_4	0.8607	Ξ_4	0.6845

Now, we have

$$\begin{aligned}\prod_{i=1}^{4}\left(1-\Upsilon_{l_i}\right)^{\upsilon_i} =& \left(1-\Upsilon_{l_1}\right)^{\upsilon_1}\times\left(1-\Upsilon_{l_2}\right)^{\upsilon_2}\times\left(1-\Upsilon_{l_3}\right)^{\upsilon_3}\times\left(1-\Upsilon_{l_4}\right)^{\upsilon_4}\\ =&(1-0.4818)^{0.22}\times(1-0.8271)^{0.31}\times(1-0.6959)^{0.19}\times(1-0.3239)^{0.28}\\ =&0.3590\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4}\left(1-\Upsilon_{u_i}\right)^{\upsilon_i} =& \left(1-\Upsilon_{u_1}\right)^{\upsilon_1}\times\left(1-\Upsilon_{u_2}\right)^{\upsilon_2}\times\left(1-\Upsilon_{u_3}\right)^{\upsilon_3}\times\left(1-\Upsilon_{u_4}\right)^{\upsilon_4}\\ =&(1-0.6845)^{0.22}\times(1-0.8980)^{0.31}\times(1-0.8607)^{0.19}\times(1-0.6129)^{0.28}\\ =&0.2016\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4}\left(-\left(\Xi_{l_i}-1\right)\right)^{\upsilon_i} =& \left(-\left(\Xi_{l_1}-1\right)\right)^{\upsilon_1}\times\left(-\left(\Xi_{l_2}-1\right)\right)^{\upsilon_2}\times\left(-\left(\Xi_{l_3}-1\right)\right)^{\upsilon_3}\times\left(-\left(\Xi_{l_4}-1\right)\right)^{\upsilon_4}\\ =&(-(0.9114-1))^{0.22}\times(-(0.7705-1))^{0.31}\times(-(0.6494-1))^{0.19}\times(-(0.4399-1))^{0.28}\\ =&0.2590\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4}\left(-\left(\Xi_{u_i}-1\right)\right)^{\upsilon_i} =& \left(-\left(\Xi_{u_1}-1\right)\right)^{\upsilon_1}\times\left(-\left(\Xi_{u_2}-1\right)\right)^{\upsilon_2}\times\left(-\left(\Xi_{u_3}-1\right)\right)^{\upsilon_3}\times\left(-\left(\Xi_{u_4}-1\right)\right)^{\upsilon_4}\\ =&(-(0.9603-1))^{0.22}\times(-(0.9409-1))^{0.31}\times(-(0.8358-1))^{0.19}\times(-(0.7181-1))^{0.28}\\ =&0.1018\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4}\left(1-\Upsilon_{i}\right)^{\upsilon_i} =& \left(1-\Upsilon_{1}\right)^{\upsilon_1}\times\left(1-\Upsilon_{2}\right)^{\upsilon_2}\times\left(1-\Upsilon_{3}\right)^{\upsilon_3}\times\left(1-\Upsilon_{4}\right)^{\upsilon_4}\\ =&(1-0.7804)^{0.22}\times(1-0.5750)^{0.31}\times(1-0.3971)^{0.19}\times(1-0.8607)^{0.28}\\ =&0.2874\end{aligned}$$

$$\prod_{i=1}^{4}\Big(-\big(\Xi_i-1\big)\Big)^{\upsilon_i} =\Big(-\big(\Xi_1-1\big)\Big)^{\upsilon_1}\times\Big(-\big(\Xi_2-1\big)\Big)^{\upsilon_2}\times\Big(-\big(\Xi_3-1\big)\Big)^{\upsilon_3}\times\Big(-\big(\Xi_4-1\big)\Big)^{\upsilon_4}$$
$$=(-(0.8838-1))^{0.22}\times(-(0.8838-1))^{0.31}\times(-(0.7501-1))^{0.19}\times(-(0.6845-1))^{0.28}$$
$$=0.1778.$$

Now, using the previously obtained values, we obtain

$$\text{P-}CBFSTWA(\mathfrak{W}_1,\mathfrak{W}_2,\mathfrak{W}_3,\mathfrak{W}_4)=\left\{\begin{array}{c}\Big[1-\prod_{i=1}^{4}\big(1-\Upsilon_{l_i}\big)^{\upsilon_i},1-\prod_{i=1}^{4}\big(1-\Upsilon_{u_i}\big)^{\upsilon_i}\Big],\\ \Big[-\prod_{i=1}^{4}\Big(-\big(\Xi_{l_i}-1\big)\Big)^{\upsilon_i},-\prod_{i=1}^{4}\Big(-\big(\Xi_{u_i}-1\big)\Big)^{\upsilon_i}\Big],\\ \Big\{1-\prod_{i=1}^{4}\big(1-\Upsilon_i\big)^{\upsilon_i},-\prod_{i=1}^{4}\Big(-\big(\Xi_i-1\big)\Big)^{\upsilon_i}\Big\}\end{array}\right\}$$
$$=\langle[1-0.3590,1-0.2016],[-0.2590,-0.1018],\{1-0.2874,-0.1778\}\rangle$$
$$=\langle[0.6410,0.7984],[-0.2590,-0.1918],\{0.7126,-0.1778\}\rangle.$$

Theorem 4.4. *The* P-*CBFSTWAO exhibits the following features:*
(i) (Idempotency) If $\mathfrak{W}_i=\mathfrak{W}$, $\forall i=1,2,...,n$. Then

$$\text{P-}CBFSTWA(\mathfrak{W}_1,\mathfrak{W}_2,...,\mathfrak{W}_n)=\sin\mathfrak{W}.$$

(ii) (Monotonicity) If $\mathfrak{W}_i$ and $\mathfrak{W}_i^*$ are two stockpiles of n CBFNs such that $\mathfrak{W}_i\leq_{\text{P}}\mathfrak{W}_i^*$, $\forall i=1,2,...,n$, and then

$$\text{P-}CBFSTWA(\mathfrak{W}_1,\mathfrak{W}_2,...,\mathfrak{W}_n)\leq_{\text{P}}\text{P-}CBFSTWA(\mathfrak{W}_1^*,\mathfrak{W}_2^*,...,\mathfrak{W}_n^*).$$

(iii) (Boundedness) If $\mathfrak{W}_i$, $i=1,2,...,n$ is a stockpile of CBFNs with $\mathfrak{W}_{\min}=\langle[\min_i(\mathcal{A}_{l\mathfrak{W}_i}),\min_i(\mathcal{A}_{u\mathfrak{W}_i})]$ $[\max_i(\mathcal{B}_{l\mathfrak{W}_i}),\max_i(\mathcal{B}_{u\mathfrak{W}_i})],\{\min_i(\mathcal{A}_{\mathfrak{W}_i}),\max_i(\mathcal{B}_{\mathfrak{W}_i})\}\rangle$ and $\mathfrak{W}_{\max}=\langle[\max_i(\mathcal{A}_{l\mathfrak{W}_i}),\max_i(\mathcal{A}_{u\mathfrak{W}_i})],$ $[\min_i(\mathcal{B}_{l\mathfrak{W}_i}),\min_i(\mathcal{B}_{u\mathfrak{W}_i})],\{\max_i(\mathcal{A}_{\mathfrak{W}_i}),\min_i(\mathcal{B}_{\mathfrak{W}_i})\}\rangle$, and then

$$\sin\mathfrak{W}_{\min}\leq_{\text{P}}\text{P-}CBFSTWA(\mathfrak{W}_1,\mathfrak{W}_2,...,\mathfrak{W}_n)\leq_{\text{P}}\sin\mathfrak{W}_{\max}.$$

Proof. See Appendix C. □

The identical aggregate will now be computed in R-order as depicted below.

Definition 4.5. Let $\{\mathfrak{W}_i:i=1,2,...,n\}$ be a stockpile of CBFNs. Then,

$$\text{R-}CBFSTWA(\mathfrak{W}_1,\mathfrak{W}_2,...,\mathfrak{W}_n)=\upsilon_1\sin\mathfrak{W}_1\oplus_{\text{R}}\upsilon_2\sin\mathfrak{W}_2\oplus_{\text{R}}...\oplus_{\text{R}}\upsilon_n\sin\mathfrak{W}_n.$$

Theorem 4.6. *Using the* R-*CBFSTWA operator, the total value of n CBFNs is a CBFN and is provided by*

$$\text{R-}CBFSTWA(\mathfrak{W}_1,\mathfrak{W}_2,...,\mathfrak{W}_n)=\left\{\begin{array}{c}\Big[1-\prod_{i=1}^{n}\big(1-\Upsilon_{l_i}\big)^{\upsilon_i},1-\prod_{i=1}^{n}\big(1-\Upsilon_{u_i}\big)^{\upsilon_i}\Big],\\ \Big[-\prod_{i=1}^{n}\Big(-\big(\Xi_{l_i}-1\big)\Big)^{\upsilon_i},-\prod_{i=1}^{n}\Big(-\big(\Xi_{u_i}-1\big)\Big)^{\upsilon_i}\Big],\\ \Big\{\prod_{i=1}^{n}\big(\Upsilon_i\big)^{\upsilon_i},-\big(1-\prod_{i=1}^{n}\Xi_i\big)^{\upsilon_i}\Big\}\end{array}\right\}.$$

4.2. CBFSTOWAO

Definition 4.7. A P-CBFSTOWA operator is characterized as follows:

$$\text{P-}CBFSTOWA(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \upsilon_1 \sin \mathfrak{W}_{\clubsuit(1)} \oplus_{\text{P}} \upsilon_2 \sin \mathfrak{W}_{\clubsuit(2)} \oplus_{\text{P}} ... \oplus_{\text{P}} \upsilon_n \sin \mathfrak{W}_{\clubsuit(n)}$$

where $\{\clubsuit(1), \clubsuit(2), ..., \clubsuit(n)\}$ is a configuration of $\{1, 2, ..., n\}$ such that $\mathfrak{W}_{\clubsuit(i-1)} \geq \mathfrak{W}_{\clubsuit(i)}$, $\forall i = 2, 3, ..., n$.

Theorem 4.8. *The aggregation of n CBFNs procured by employing the* P-*CBFSTOWA operator is also a CBFN and is provided by*

$$\text{P-}CBFSTOWA(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \left\{ \begin{array}{c} \left[1 - \prod_{i=1}^{n}\left(1 - \Upsilon_{l_{\clubsuit(i)}}\right)^{\upsilon_i}, 1 - \prod_{i=1}^{n}\left(1 - \Upsilon_{u_{\clubsuit(i)}}\right)^{\upsilon_i}\right], \\ \left[-\prod_{i=1}^{n}\left(-\left(\Xi_{l_{\clubsuit(i)}} - 1\right)\right)^{\upsilon_i}, -\prod_{i=1}^{n}\left(-\left(\Xi_{u_{\clubsuit(i)}} - 1\right)\right)^{\upsilon_i}\right], \\ \left\{1 - \prod_{i=1}^{n}\left(1 - \Upsilon_{\clubsuit(i)}\right)^{\upsilon_i}, -\prod_{i=1}^{n}\left(-\left(\Xi_{\clubsuit(i)} - 1\right)\right)^{\upsilon_i}\right\} \end{array} \right\}$$

where $\Upsilon_{l_{\clubsuit(i)}} = \sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_{\clubsuit(i)}}\right)$, $\Upsilon_{u_{\clubsuit(i)}} = \sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_{\clubsuit(i)}}\right)$, $\Xi_{l_{\clubsuit(i)}} = \sin\left(\frac{\pi}{2}(1 + \mathcal{B}_{l\mathfrak{W}_{\clubsuit(i)}})\right)$, $\Xi_{u_{\clubsuit(i)}} = \sin\left(\frac{\pi}{2}(1 + \mathcal{B}_{u\mathfrak{W}_{\clubsuit(i)}})\right)$, $\Upsilon_{\clubsuit(i)} = \sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_{\clubsuit(i)}}\right)$ *and* $\Xi_{\clubsuit(i)} = \sin\left(\frac{\pi}{2}(1 + \mathcal{B}_{\mathfrak{W}_{\clubsuit(i)}})\right)$.

Example 4.9. Take the CBFNs which were given in Example 4.3, and $\upsilon = \{0.22, 0.31, 0.19, 0.28\}$ is the WV for their ordered positions. To calculate the congregation of these CBFNs using P-CBFSTOWA operator, we begin by calculating the score functions of these CBFNs via Eq (2.1), as shown below.

$$\Psi_{\text{P}}(\mathfrak{W}_1) = 0.6017, \Psi_{\text{P}}(\mathfrak{W}_2) = 0.6250, \Psi_{\text{P}}(\mathfrak{W}_3) = 0.5050, \textit{ and } \Psi_{\text{P}}(\mathfrak{W}_4) = 0.4283.$$

On the basis of these score values, we get $\mathfrak{W}_2 \succ \mathfrak{W}_1 \succ \mathfrak{W}_3 \succ \mathfrak{W}_4$. This gives

$$\mathfrak{W}_{\clubsuit(1)} = \mathfrak{W}_2 = \langle[0.62, 0.71], [-0.44, -0.22], \{0.39, -0.31\}\rangle$$

$$\mathfrak{W}_{\clubsuit(2)} = \mathfrak{W}_1 = \langle[0.32, 0.48], [-0.27, -0.18], \{0.57, -0.31\}\rangle$$

$$\mathfrak{W}_{\clubsuit(3)} = \mathfrak{W}_3 = \langle[0.49, 0.66], [-0.55, -0.37], \{0.26, -0.46\}\rangle$$

$$\mathfrak{W}_{\clubsuit(4)} = \mathfrak{W}_4 = \langle[0.21, 0.42], [-0.71, -0.49], \{0.66, -0.52\}\rangle.$$

Now, the values of $\Upsilon_{l_{\clubsuit(i)}}$, $\Upsilon_{u_{\clubsuit(i)}}$, $\Xi_{l_{\clubsuit(i)}}$, $\Xi_{u_{\clubsuit(i)}}$, $\Upsilon_{\clubsuit(i)}$ and $\Xi_{\clubsuit(i)}$ are listed in Table 4.

Table 4. The values of $\Upsilon_{l_{\clubsuit(i)}}$, $\Upsilon_{u_{\clubsuit(i)}}$, $\Xi_{l_{\clubsuit(i)}}$, $\Xi_{u_{\clubsuit(i)}}$, $\Upsilon_{\clubsuit(i)}$ and $\Xi_{\clubsuit(i)}$, for $i = 1, 2, 3, 4$.

$\Upsilon_{l_{\clubsuit(1)}}$	0.8271	$\Upsilon_{u_{\clubsuit(1)}}$	0.8980	$\Xi_{l_{\clubsuit(1)}}$	0.7705	$\Xi_{u_{\clubsuit(1)}}$	0.9409	$\Upsilon_{\clubsuit(1)}$	0.5750	$\Xi_{\clubsuit(1)}$	0.8838
$\Upsilon_{l_{\clubsuit(2)}}$	0.4818	$\Upsilon_{u_{\clubsuit(2)}}$	0.6845	$\Xi_{l_{\clubsuit(2)}}$	0.9114	$\Xi_{u_{\clubsuit(2)}}$	0.9603	$\Upsilon_{\clubsuit(2)}$	0.7804	$\Xi_{\clubsuit(2)}$	0.8838
$\Upsilon_{l_{\clubsuit(3)}}$	0.6959	$\Upsilon_{u_{\clubsuit(3)}}$	0.8607	$\Xi_{l_{\clubsuit(3)}}$	0.6494	$\Xi_{u_{\clubsuit(3)}}$	0.8358	$\Upsilon_{\clubsuit(3)}$	0.3971	$\Xi_{\clubsuit(3)}$	0.7501
$\Upsilon_{l_{\clubsuit(4)}}$	0.3239	$\Upsilon_{u_{\clubsuit(4)}}$	0.6129	$\Xi_{l_{\clubsuit(4)}}$	0.4399	$\Xi_{u_{\clubsuit(4)}}$	0.7181	$\Upsilon_{\clubsuit(4)}$	0.8607	$\Xi_{\clubsuit(4)}$	0.6845

Now, we have

$$\prod_{i=1}^{4}\left(1 - \Upsilon_{l_{\clubsuit(i)}}\right)^{\upsilon_i} = \left(1 - \Upsilon_{l_{\clubsuit(1)}}\right)^{\upsilon_1} \times \left(1 - \Upsilon_{l_{\clubsuit(2)}}\right)^{\upsilon_2} \times \left(1 - \Upsilon_{l_{\clubsuit(3)}}\right)^{\upsilon_3} \times \left(1 - \Upsilon_{l_{\clubsuit(4)}}\right)^{\upsilon_4}$$

$$=(1-0.8272)^{0.22}\times(1-0.4818)^{0.31}\times(1-0.6959)^{0.19}\times(1-0.3239)^{0.28}$$
$$=0.3962$$

$$\prod_{i=1}^{4}\left(1-\Upsilon_{u_{\clubsuit(i)}}\right)^{\upsilon_i}=\left(1-\Upsilon_{u_{\clubsuit(1)}}\right)^{\upsilon_1}\times\left(1-\Upsilon_{u_{\clubsuit(2)}}\right)^{\upsilon_2}\times\left(1-\Upsilon_{u_{\clubsuit(3)}}\right)^{\upsilon_3}\times\left(1-\Upsilon_{u_{\clubsuit(4)}}\right)^{\upsilon_4}$$
$$=(1-0.8980)^{0.22}\times(1-0.6845)^{0.31}\times(1-0.8607)^{0.19}\times(1-0.6129)^{0.28}$$
$$=0.2231$$

$$\prod_{i=1}^{4}\left(-\left(\Xi_{l_{\clubsuit(i)}}-1\right)\right)^{\upsilon_i}=\left(-\left(\Xi_{l_{\clubsuit(1)}}-1\right)\right)^{\upsilon_1}\times\left(-\left(\Xi_{l_{\clubsuit(2)}}-1\right)\right)^{\upsilon_2}\times\left(-\left(\Xi_{l_{\clubsuit(3)}}-1\right)\right)^{\upsilon_3}\times\left(-\left(\Xi_{l_{\clubsuit(4)}}-1\right)\right)^{\upsilon_4}$$
$$=(-(0.7705-1))^{0.22}\times(-(0.9114-1))^{0.31}\times(-(0.6494-1))^{0.19}\times(-(0.4399-1))^{0.28}$$
$$=0.2377$$

$$\prod_{i=1}^{4}\left(-\left(\Xi_{u_{\clubsuit(i)}}-1\right)\right)^{\upsilon_i}=\left(-\left(\Xi_{u_{\clubsuit(1)}}-1\right)\right)^{\upsilon_1}\times\left(-\left(\Xi_{u_{\clubsuit(2)}}-1\right)\right)^{\upsilon_2}\times\left(-\left(\Xi_{u_{\clubsuit(3)}}-1\right)\right)^{\upsilon_3}\times\left(-\left(\Xi_{u_{\clubsuit(4)}}-1\right)\right)^{\upsilon_4}$$
$$=(-(0.9409-1))^{0.22}\times(-(0.9603-1))^{0.31}\times(-(0.8358-1))^{0.19}\times(-(0.7181-1))^{0.28}$$
$$=0.0982$$

$$\prod_{i=1}^{4}\left(1-\Upsilon_{\clubsuit(i)}\right)^{\upsilon_i}=\left(1-\Upsilon_{\clubsuit(1)}\right)^{\upsilon_1}\times\left(1-\Upsilon_{\clubsuit(2)}\right)^{\upsilon_2}\times\left(1-\Upsilon_{\clubsuit(3)}\right)^{\upsilon_3}\times\left(1-\Upsilon_{\clubsuit(4)}\right)^{\upsilon_4}$$
$$=(1-0.5750)^{0.22}\times(1-0.7804)^{0.31}\times(1-0.3971)^{0.19}\times(1-0.8607)^{0.28}$$
$$=0.2708$$

$$\prod_{i=1}^{4}\left(-\left(\Xi_{\clubsuit(i)}-1\right)\right)^{\upsilon_i}=\left(-\left(\Xi_{\clubsuit(1)}-1\right)\right)^{\upsilon_1}\times\left(-\left(\Xi_{\clubsuit(2)}-1\right)\right)^{\upsilon_2}\times\left(-\left(\Xi_{\clubsuit(3)}-1\right)\right)^{\upsilon_3}\times\left(-\left(\Xi_{\clubsuit(4)}-1\right)\right)^{\upsilon_4}$$
$$=(-(0.8838-1))^{0.22}\times(-(0.8838-1))^{0.31}\times(-(0.7501-1))^{0.19}\times(-(0.6845-1))^{0.28}$$
$$=0.1778.$$

Now, using the previously obtained values, we obtain

$$\text{P-}CBFSTOWA(\mathfrak{W}_1,\mathfrak{W}_2,\mathfrak{W}_3,\mathfrak{W}_4)=\left\{\begin{array}{c}\left[1-\prod_{i=1}^{4}\left(1-\Upsilon_{l_{\clubsuit(i)}}\right)^{\upsilon_i},1-\prod_{i=1}^{4}\left(1-\Upsilon_{u_{\clubsuit(i)}}\right)^{\upsilon_i}\right],\\ \left[-\prod_{i=1}^{4}\left(-\left(\Xi_{l_{\clubsuit(i)}}-1\right)\right)^{\upsilon_i},-\prod_{i=1}^{4}\left(-\left(\Xi_{u_{\clubsuit(i)}}-1\right)\right)^{\upsilon_i}\right],\\ \left\{1-\prod_{i=1}^{4}\left(1-\Upsilon_{\clubsuit(i)}\right)^{\upsilon_i},-\prod_{i=1}^{4}\left(-\left(\Xi_{\clubsuit(i)}-1\right)\right)^{\upsilon_i}\right\}\end{array}\right\}$$
$$=\langle[1-0.3962,1-0.2231],[-0.2377,-0.0982],\{1-0.2708,-0.1778\}\rangle$$
$$=\langle[0.6038,0.7769],[-0.2377,-0.0982],\{0.7292,-0.1778\}\rangle.$$

Definition 4.10. An R-CBFSTOWA operator assumes the following form:

$$\text{R-}CBFSTOWA(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \upsilon_1 \sin \mathfrak{W}_{\clubsuit(1)} \oplus_{\text{R}} \upsilon_2 \sin \mathfrak{W}_{\clubsuit(2)} \oplus_{\text{R}} ... \oplus_{\text{R}} \upsilon_n \sin \mathfrak{W}_{\clubsuit(n)}.$$

Theorem 4.11. *The combined value of n CBFNs obtained by applying the* R-*CBFSTOWA operator is given as*

$$\text{R-}CBFSTOWA(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \left\{ \begin{array}{c} \left[1 - \prod_{i=1}^{n}\left(1 - \Upsilon_{l_{\clubsuit(i)}}\right)^{\upsilon_i}, 1 - \prod_{i=1}^{n}\left(1 - \Upsilon_{u_{\clubsuit(i)}}\right)^{\upsilon_i}\right], \\ \left[-\prod_{i=1}^{n}\left(-\left(\Xi_{l_{\clubsuit(i)}} - 1\right)\right)^{\upsilon_i}, -\prod_{i=1}^{n}\left(-\left(\Xi_{u_{\clubsuit(i)}} - 1\right)\right)^{\upsilon_i}\right], \\ \left\{\prod_{i=1}^{n}\left(\Upsilon_{\clubsuit(i)}\right)^{\upsilon_i}, -\left(1 - \prod_{i=1}^{n}\Xi_{\clubsuit(i)}\right)^{\upsilon_i}\right\} \end{array} \right\}.$$

4.3. CBFSTHWAO

Definition 4.12. A P-CBFSTHWA operator with associated WV $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ with $\varepsilon_i > 0$ and $\sum_{i=1}^{n} \varepsilon_i = 1$ can be described as

$$\text{P-}CBFSTHWA(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \varepsilon_1 \sin \dot{\mathfrak{W}}_{\clubsuit(1)} \oplus_{\text{P}} \varepsilon_2 \sin \dot{\mathfrak{W}}_{\clubsuit(2)} \oplus_{\text{P}} ... \oplus_{\text{P}} \varepsilon_n \sin \dot{\mathfrak{W}}_{\clubsuit(n)}$$

where $\dot{\mathfrak{W}}_i = n\upsilon_i \mathfrak{W}_i$.

Theorem 4.13. *The average of the n CBFNs generated by employing the* P-*CBFSTHWA operator is given by*

$$\text{P-}CBFSTHWA(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \left\{ \begin{array}{c} \left[1 - \prod_{i=1}^{n}\left(1 - \dot{\Upsilon}_{l_{\clubsuit(i)}}\right)^{\varepsilon_i}, 1 - \prod_{i=1}^{n}\left(1 - \dot{\Upsilon}_{u_{\clubsuit(i)}}\right)^{\varepsilon_i}\right], \\ \left[-\prod_{i=1}^{n}\left(-\left(\dot{\Xi}_{l_{\clubsuit(i)}} - 1\right)\right)^{\varepsilon_i}, -\prod_{i=1}^{n}\left(-\left(\dot{\Xi}_{u_{\clubsuit(i)}} - 1\right)\right)^{\varepsilon_i}\right], \\ \left\{1 - \prod_{i=1}^{n}\left(1 - \dot{\Upsilon}_{\clubsuit(i)}\right)^{\varepsilon_i}, -\prod_{i=1}^{n}\left(-\left(\dot{\Xi}_{\clubsuit(i)} - 1\right)\right)^{\varepsilon_i}\right\} \end{array} \right\}$$

where $\dot{\Upsilon}_{l_{\clubsuit(i)}} = \sin\left(\frac{\pi}{2}\mathcal{A}_{l\dot{\mathfrak{W}}_{\clubsuit(i)}}\right)$, $\dot{\Upsilon}_{u_{\clubsuit(i)}} = \sin\left(\frac{\pi}{2}\mathcal{A}_{u\dot{\mathfrak{W}}_{\clubsuit(i)}}\right)$, $\dot{\Xi}_{l_{\clubsuit(i)}} = \sin\left(\frac{\pi}{2}(1 + \mathcal{B}_{l\dot{\mathfrak{W}}_{\clubsuit(i)}})\right)$, $\dot{\Xi}_{u_{\clubsuit(i)}} = \sin\left(\frac{\pi}{2}(1 + \mathcal{B}_{u\dot{\mathfrak{W}}_{\clubsuit(i)}})\right)$, $\dot{\Upsilon}_{\clubsuit(i)} = \sin\left(\frac{\pi}{2}\mathcal{A}_{l\dot{\mathfrak{W}}_{\clubsuit(i)}}\right)$ *and* $\dot{\Xi}_{\clubsuit(i)} = \sin\left(\frac{\pi}{2}(1 + \mathcal{B}_{\dot{\mathfrak{W}}_{\clubsuit(i)}})\right)$.

Example 4.14. Take the CBFNs which were given in Example 4.3, and $\upsilon = \{0.22, 0.31, 0.19, 0.28\}$ is the WV for them. Let $\varepsilon = \{0.15, 0.26, 0.32, 0.27\}$ be the associated WV. Now,

$$\dot{\mathfrak{W}}_1 = \langle[0.44, 0.64], [-0.12, -0.06], \{0.74, -0.15\}\rangle$$

$$\dot{\mathfrak{W}}_2 = \langle[0.89, 0.94], [-0.16, -0.03], \{0.65, -0.07\}\rangle$$

$$\dot{\mathfrak{W}}_3 = \langle[0.60, 0.78], [-0.45, -0.25], \{0.32, -0.35\}\rangle$$

$$\dot{\mathfrak{W}}_4 = \langle[0.35, 0.65], [-0.52, -0.24], \{0.89, -0.27\}\rangle.$$

We calculate the score functions of these CBFNs using Eq (2.1) as follows.

$$\Psi_{\text{P}}(\dot{\mathfrak{W}}_1) = 0.7483, \Psi_{\text{P}}(\dot{\mathfrak{W}}_2) = 0.8700, \Psi_{\text{P}}(\dot{\mathfrak{W}}_3) = 0.6083, \ \textit{and}\ \Psi_{\text{P}}(\dot{\mathfrak{W}}_4) = 0.6433.$$

According to these score values, we get $\dot{\mathfrak{W}}_2 \succ \dot{\mathfrak{W}}_1 \succ \dot{\mathfrak{W}}_4 \succ \dot{\mathfrak{W}}_3$. This gives

$$\dot{\mathfrak{W}}_{\clubsuit(1)} = \dot{\mathfrak{W}}_2 = \langle [0.89, 0.94], [-0.16, -0.03], \{0.65, -0.07\} \rangle$$

$$\dot{\mathfrak{W}}_{\clubsuit(2)} = \dot{\mathfrak{W}}_1 = \langle [0.44, 0.64], [-0.12, -0.06], \{0.74, -0.15\} \rangle$$

$$\dot{\mathfrak{W}}_{\clubsuit(3)} = \dot{\mathfrak{W}}_4 = \langle [0.35, 0.65], [-0.52, -0.24], \{0.89, -0.27\} \rangle$$

$$\dot{\mathfrak{W}}_{\clubsuit(4)} = \dot{\mathfrak{W}}_3 = \langle [0.60, 0.78], [-0.45, -0.25], \{0.32, -0.35\} \rangle.$$

Now, the values of $\dot{\Upsilon}_{l_{\clubsuit(i)}}$, $\dot{\Upsilon}_{u_{\clubsuit(i)}}$, $\dot{\Xi}_{l_{\clubsuit(i)}}$, $\dot{\Xi}_{u_{\clubsuit(i)}}$, $\dot{\Upsilon}_{\clubsuit(i)}$ and $\dot{\Xi}_{\clubsuit(i)}$ are listed in Table 5.

Table 5. The values of $\dot{\Upsilon}_{l_{\clubsuit(i)}}$, $\dot{\Upsilon}_{u_{\clubsuit(i)}}$, $\dot{\Xi}_{l_{\clubsuit(i)}}$, $\dot{\Xi}_{u_{\clubsuit(i)}}$, $\dot{\Upsilon}_{\clubsuit(i)}$ and $\dot{\Xi}_{\clubsuit(i)}$, for $i = 1, 2, 3, 4$.

$\dot{\Upsilon}_{l_{\clubsuit(1)}}$	0.9851	$\dot{\Upsilon}_{u_{\clubsuit(1)}}$	0.9956	$\dot{\Xi}_{l_{\clubsuit(1)}}$	0.9686	$\dot{\Xi}_{u_{\clubsuit(1)}}$	0.9989	$\dot{\Upsilon}_{\clubsuit(1)}$	0.8526	$\dot{\Xi}_{\clubsuit(1)}$	0.9940
$\dot{\Upsilon}_{l_{\clubsuit(2)}}$	0.6374	$\dot{\Upsilon}_{u_{\clubsuit(2)}}$	0.8443	$\dot{\Xi}_{l_{\clubsuit(2)}}$	0.9823	$\dot{\Xi}_{u_{\clubsuit(2)}}$	0.9956	$\dot{\Upsilon}_{\clubsuit(2)}$	0.9178	$\dot{\Xi}_{\clubsuit(2)}$	0.9724
$\dot{\Upsilon}_{l_{\clubsuit(3)}}$	0.5225	$\dot{\Upsilon}_{u_{\clubsuit(3)}}$	0.8526	$\dot{\Xi}_{l_{\clubsuit(3)}}$	0.6845	$\dot{\Xi}_{u_{\clubsuit(3)}}$	0.9298	$\dot{\Upsilon}_{\clubsuit(3)}$	0.9851	$\dot{\Xi}_{\clubsuit(3)}$	0.9114
$\dot{\Upsilon}_{l_{\clubsuit(4)}}$	0.8090	$\dot{\Upsilon}_{u_{\clubsuit(4)}}$	0.9409	$\dot{\Xi}_{l_{\clubsuit(4)}}$	0.7604	$\dot{\Xi}_{u_{\clubsuit(4)}}$	0.9239	$\dot{\Upsilon}_{\clubsuit(4)}$	0.4818	$\dot{\Xi}_{\clubsuit(4)}$	0.8526

Now, we have

$$\begin{aligned}\prod_{i=1}^{4}\left(1-\dot{\Upsilon}_{l_{\clubsuit(i)}}\right)^{\varepsilon_i} =&\left(1-\dot{\Upsilon}_{l_{\clubsuit(1)}}\right)^{\varepsilon_1}\times\left(1-\dot{\Upsilon}_{l_{\clubsuit(2)}}\right)^{\varepsilon_2}\times\left(1-\dot{\Upsilon}_{l_{\clubsuit(3)}}\right)^{\varepsilon_3}\times\left(1-\dot{\Upsilon}_{l_{\clubsuit(4)}}\right)^{\varepsilon_4}\\ =&(1-0.9851)^{0.15}\times(1-0.6374)^{0.26}\times(1-0.5225)^{0.32}\times(1-0.8090)^{0.27}\\ =&0.2063\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4}\left(1-\dot{\Upsilon}_{u_{\clubsuit(i)}}\right)^{\varepsilon_i} =&\left(1-\dot{\Upsilon}_{u_{\clubsuit(1)}}\right)^{\varepsilon_1}\times\left(1-\dot{\Upsilon}_{u_{\clubsuit(2)}}\right)^{\varepsilon_2}\times\left(1-\dot{\Upsilon}_{u_{\clubsuit(3)}}\right)^{\varepsilon_3}\times\left(1-\dot{\Upsilon}_{u_{\clubsuit(4)}}\right)^{\varepsilon_4}\\ =&(1-0.9956)^{0.15}\times(1-0.8443)^{0.26}\times(1-0.8526)^{0.32}\times(1-0.9409)^{0.27}\\ =&0.0690\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4}\left(-\left(\dot{\Xi}_{l_{\clubsuit(i)}}-1\right)\right)^{\varepsilon_i} =&\left(-\left(\dot{\Xi}_{l_{\clubsuit(1)}}-1\right)\right)^{\varepsilon_1}\times\left(-\left(\dot{\Xi}_{l_{\clubsuit(2)}}-1\right)\right)^{\varepsilon_2}\times\left(-\left(\dot{\Xi}_{l_{\clubsuit(3)}}-1\right)\right)^{\varepsilon_3}\times\left(-\left(\dot{\Xi}_{l_{\clubsuit(4)}}-1\right)\right)^{\varepsilon_4}\\ =&(-(0.9686-1))^{0.15}\times(-(0.9823-1))^{0.26}\times(-(0.6845-1))^{0.32}\times(-(0.7604-1))^{0.27}\\ =&0.0980\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4}\left(-\left(\dot{\Xi}_{u_{\clubsuit(i)}}-1\right)\right)^{\varepsilon_i} =&\left(-\left(\dot{\Xi}_{u_{\clubsuit(1)}}-1\right)\right)^{\varepsilon_1}\times\left(-\left(\dot{\Xi}_{u_{\clubsuit(2)}}-1\right)\right)^{\varepsilon_2}\times\left(-\left(\dot{\Xi}_{u_{\clubsuit(3)}}-1\right)\right)^{\varepsilon_3}\times\left(-\left(\dot{\Xi}_{u_{\clubsuit(4)}}-1\right)\right)^{\varepsilon_4}\\ =&(-(0.9989-1))^{0.15}\times(-(0.9956-1))^{0.26}\times(-(0.9298-1))^{0.32}\times(-(0.9239-1))^{0.27}\\ =&0.0187\end{aligned}$$

$$\prod_{i=1}^{4}\left(1-\dot{\Upsilon}_{\clubsuit(i)}\right)^{\varepsilon_i}=\left(1-\dot{\Upsilon}_{\clubsuit(1)}\right)^{\varepsilon_1}\times\left(1-\dot{\Upsilon}_{\clubsuit(2)}\right)^{\varepsilon_2}\times\left(1-\dot{\Upsilon}_{\clubsuit(3)}\right)^{\varepsilon_3}\times\left(1-\dot{\Upsilon}_{\clubsuit(4)}\right)^{\varepsilon_4}$$
$$=(1-0.8526)^{0.15}\times(1-0.9178)^{0.26}\times(1-0.9851)^{0.32}\times(1-0.4818)^{0.27}$$
$$=0.0854$$

$$\prod_{i=1}^{4}\left(-\left(\dot{\Xi}_{\clubsuit(i)}-1\right)\right)^{\varepsilon_i}=\left(-\left(\dot{\Xi}_{\clubsuit(1)}-1\right)\right)^{\varepsilon_1}\times\left(-\left(\dot{\Xi}_{\clubsuit(2)}-1\right)\right)^{\varepsilon_2}\times\left(-\left(\dot{\Xi}_{\clubsuit(3)}-1\right)\right)^{\varepsilon_3}\times\left(-\left(\dot{\Xi}_{\clubsuit(4)}-1\right)\right)^{\varepsilon_4}$$
$$=(-(0.9940-1))^{0.15}\times(-(0.9724-1))^{0.26}\times(-(0.9114-1))^{0.32}\times(-(0.8526-1))^{0.27}$$
$$=0.0501.$$

Now, using the previously obtained values, we obtain

$$P\text{-}CBFSTHWA(\mathfrak{W}_1,\mathfrak{W}_2,\mathfrak{W}_3,\mathfrak{W}_4)=\left\{\begin{array}{c}\left[1-\prod_{i=1}^{4}\left(1-\dot{\Upsilon}_{l_{\clubsuit(i)}}\right)^{\varepsilon_i},1-\prod_{i=1}^{4}\left(1-\dot{\Upsilon}_{u_{\clubsuit(i)}}\right)^{\varepsilon_i}\right],\\ \left[-\prod_{i=1}^{4}\left(-\left(\dot{\Xi}_{l_{\clubsuit(i)}}-1\right)\right)^{\varepsilon_i},-\prod_{i=1}^{4}\left(-\left(\dot{\Xi}_{u_{\clubsuit(i)}}-1\right)\right)^{\varepsilon_i}\right],\\ \left\{1-\prod_{i=1}^{4}\left(1-\dot{\Upsilon}_{\clubsuit(i)}\right)^{\varepsilon_i},-\prod_{i=1}^{4}\left(-\left(\dot{\Xi}_{\clubsuit(i)}-1\right)\right)^{\varepsilon_i}\right\}\end{array}\right\}$$
$$=\langle[1-0.2063,1-0.0690],[-0.0980,-0.0187],\{1-0.0854,-0.0501\}\rangle$$
$$=\langle[0.7937,0.9310],[-0.0980,-0.0187],\{0.9146,-0.0501\}\rangle.$$

Definition 4.15. For n CBFNs, the R-CBFSTHWA operator is expressed as

$$R\text{-}CBFSTHWA(\mathfrak{W}_1,\mathfrak{W}_2,...,\mathfrak{W}_n)=\varepsilon_1\sin\dot{\mathfrak{W}}_{\clubsuit(1)}\oplus_R\varepsilon_2\sin\dot{\mathfrak{W}}_{\clubsuit(2)}\oplus_R...\oplus_R\varepsilon_n\sin\dot{\mathfrak{W}}_{\clubsuit(n)}.$$

Except for order, all of the terms in this definition have the same values as those in the preceding definition. As a result, we leave out their explanation.

Theorem 4.16. *The congregation value of n CBFNs obtained by applying the* R-*CBFSTHWA operator is given by*

$$R\text{-}CBFSTHWA(\mathfrak{W}_1,\mathfrak{W}_2,...,\mathfrak{W}_n)=\left\{\begin{array}{c}\left[1-\prod_{i=1}^{n}\left(1-\dot{\Upsilon}_{l_{\clubsuit(i)}}\right)^{\varepsilon_i},1-\prod_{i=1}^{n}\left(1-\dot{\Upsilon}_{u_{\clubsuit(i)}}\right)^{\varepsilon_i}\right],\\ \left[-\prod_{i=1}^{n}\left(-\left(\dot{\Xi}_{l_{\clubsuit(i)}}-1\right)\right)^{\varepsilon_i},-\prod_{i=1}^{n}\left(-\left(\dot{\Xi}_{u_{\clubsuit(i)}}-1\right)\right)^{\varepsilon_i}\right],\\ \left\{\prod_{i=1}^{n}\left(\dot{\Upsilon}_{\clubsuit(i)}\right)^{\varepsilon_i},-\left(1-\prod_{i=1}^{n}\dot{\Xi}_{\clubsuit(i)}\right)^{\varepsilon_i}\right\}\end{array}\right\}.$$

Remark. Similar to the P-CBFSTWA operator, the R-CBFSTWA, P-CBFSTOWA, and R-CBFSTOWA operators satisfy idempotency, monotonicity and boundedness.

5. CBF sine trigonometric geometric aggregation operators

On the basis of CBF-STOLs, another important family of aggregation operators can be established, namely, geometric aggregation operators. They include P-CBFSTWGO, P-CBFSTOWGO and P-CBFSTHWGO. We also derive R-CBFSTWGO, R-CBFSTOWGO and R-CBFSTHWGO.

5.1. CBFSTWGO

Definition 5.1. For n CBFNs $\mathfrak{W}_i$, a P-CBFSTWG operator is demonstrated as

$$\text{P-}CBFSTWG(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = (\sin \mathfrak{W}_1)^{\upsilon_1} \otimes_{\mathrm{P}} (\sin \mathfrak{W}_2)^{\upsilon_2} \otimes_{\mathrm{P}} ... \otimes_{\mathrm{P}} (\sin \mathfrak{W}_n)^{\upsilon_n}. \tag{5.1}$$

Theorem 5.2. *Equation 5.1 yields an aggregated value of n CBFNs, which is given by*

$$\text{P-}CBFSTWG(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \left\{ \begin{array}{c} \Big[\prod_{i=1}^{n} \big(\Upsilon_{l_i}\big)^{\upsilon_i}, \prod_{i=1}^{n} \big(\Upsilon_{u_i}\big)^{\upsilon_i} \Big], \\ \Big[-\Big(1 - \prod_{i=1}^{n} \big(\Xi_{l_i}\big)^{\upsilon_i}\Big), -\Big(1 - \prod_{i=1}^{n} \big(\Xi_{u_i}\big)^{\upsilon_i}\Big) \Big], \\ \Big\{ \prod_{i=1}^{n} \big(\Upsilon_i\big)^{\upsilon_i}, -\Big(1 - \prod_{i=1}^{n} \big(\Xi_i\big)^{\upsilon_i}\Big) \Big\} \end{array} \right\}. \tag{5.2}$$

Example 5.3. Take the CBFNs which were presented in Example 4.3, and $\upsilon = \{0.22, 0.31, 0.19, 0.28\}$ is the WV for them. The values of Υ_{l_i}, Υ_{u_i}, Ξ_{l_i}, Ξ_{u_i}, Υ_i and Ξ_i, for $i = 1, 2, 3, 4$, are given in Table 3. Now, we have

$$\begin{aligned} \prod_{i=1}^{4} \big(\Upsilon_{l_i}\big)^{\upsilon_i} &= \big(\Upsilon_{l_1}\big)^{\upsilon_1} \times \big(\Upsilon_{l_2}\big)^{\upsilon_2} \times \big(\Upsilon_{l_3}\big)^{\upsilon_3} \times \big(\Upsilon_{l_4}\big)^{\upsilon_4} \\ &= (0.4818)^{0.22} \times (0.8271)^{0.31} \times (0.6959)^{0.19} \times (0.3239)^{0.28} \\ &= 0.5466 \end{aligned}$$

$$\begin{aligned} \prod_{i=1}^{4} \big(\Upsilon_{u_i}\big)^{\upsilon_i} &= \big(\Upsilon_{u_1}\big)^{\upsilon_1} \times \big(\Upsilon_{u_2}\big)^{\upsilon_2} \times \big(\Upsilon_{u_3}\big)^{\upsilon_3} \times \big(\Upsilon_{u_4}\big)^{\upsilon_4} \\ &= (0.6845)^{0.22} \times (0.8980)^{0.31} \times (0.8607)^{0.19} \times (0.6129)^{0.28} \\ &= 0.7540 \end{aligned}$$

$$\begin{aligned} \prod_{i=1}^{4} \big(\Xi_{l_i}\big)^{\upsilon_i} &= \big(\Xi_{l_1}\big)^{\upsilon_1} \times \big(\Xi_{l_2}\big)^{\upsilon_2} \times \big(\Xi_{l_3}\big)^{\upsilon_3} \times \big(\Xi_{l_4}\big)^{\upsilon_4} \\ &= (0.9114)^{0.22} \times (0.7705)^{0.31} \times (0.6494)^{0.19} \times (0.4399)^{0.28} \\ &= 0.6615 \end{aligned}$$

$$\begin{aligned} \prod_{i=1}^{4} \big(\Xi_{u_i}\big)^{\upsilon_i} &= \big(\Xi_{u_1}\big)^{\upsilon_1} \times \big(\Xi_{u_2}\big)^{\upsilon_2} \times \big(\Xi_{u_3}\big)^{\upsilon_3} \times \big(\Xi_{u_4}\big)^{\upsilon_4} \\ &= (0.9603)^{0.22} \times (0.9409)^{0.31} \times (0.8358))^{0.19} \times (0.7181)^{0.28} \\ &= 0.8568 \end{aligned}$$

$$\prod_{i=1}^{4} \big(\Upsilon_i\big)^{\upsilon_i} = \big(\Upsilon_1\big)^{\upsilon_1} \times \big(\Upsilon_2\big)^{\upsilon_2} \times \big(\Upsilon_3\big)^{\upsilon_3} \times \big(\Upsilon_4\big)^{\upsilon_4}$$

$$=(0.7804)^{0.22}\times(0.5750)^{0.31}\times(0.3971)^{0.19}\times(0.8607)^{0.28}$$
$$=0.6417$$

$$\prod_{i=1}^{4}\left(\Xi_i\right)^{\upsilon_i}=\left(\Xi_1\right)^{\upsilon_1}\times\left(\Xi_2\right)^{\upsilon_2}\times\left(\Xi_3\right)^{\upsilon_3}\times\left(\Xi_4\right)^{\upsilon_4}$$
$$=(0.8838)^{0.22}\times(0.8838)^{0.31}\times(0.7501)^{0.19}\times(0.6845)^{0.28}$$
$$=0.7975.$$

Now, using the previously obtained values, we obtain

$$\text{P-}CBFSTWG(\mathfrak{W}_1,\mathfrak{W}_2,\mathfrak{W}_3,\mathfrak{W}_4)=\left\{\begin{array}{c}\left[\prod_{i=1}^{4}\left(\Upsilon_{l_i}\right)^{\upsilon_i},\prod_{i=1}^{4}\left(\Upsilon_{u_i}\right)^{\upsilon_i}\right],\\ \left[-\left(1-\prod_{i=1}^{4}\left(\Xi_{l_i}\right)^{\upsilon_i}\right),-\left(1-\prod_{i=1}^{4}\left(\Xi_{u_i}\right)^{\upsilon_i}\right)\right],\\ \left\{\prod_{i=1}^{4}\left(\Upsilon_i\right)^{\upsilon_i},-\left(1-\prod_{i=1}^{4}\left(\Xi_i\right)^{\upsilon_i}\right)\right\}\end{array}\right\}$$
$$=\langle[0.5466,0.7540],[-(1-0.6615),-(1-0.8568)],\{0.6417,-(1-0.7975)\}\rangle$$
$$=\langle[0.5466,0.7540],[-0.3385,-0.1432],\{0.6417,-0.2025\}\rangle.$$

Definition 5.4. An R-CBFSTWG operator is defined as

$$\text{R-}CBFSTWG(\mathfrak{W}_1,\mathfrak{W}_2,...,\mathfrak{W}_n)=(\sin\mathfrak{W}_1)^{\upsilon_1}\otimes_{\text{R}}(\sin\mathfrak{W}_2)^{\upsilon_2}\otimes_{\text{R}}...\otimes_{\text{R}}(\sin\mathfrak{W}_n)^{\upsilon_n}.$$

Theorem 5.5. *The aggregated valued obtained by using* R-*CBFSTWG operator is of the form*

$$\text{R-}CBFSTWG(\mathfrak{W}_1,\mathfrak{W}_2,...,\mathfrak{W}_n)=\left\{\begin{array}{c}\left[\prod_{i=1}^{n}\left(\Upsilon_{l_i}\right)^{\upsilon_i},\prod_{i=1}^{n}\left(\Upsilon_{u_i}\right)^{\upsilon_i}\right],\\ \left[-\left(1-\prod_{i=1}^{n}\left(\Xi_{l_i}\right)^{\upsilon_i}\right),-\left(1-\prod_{i=1}^{n}\left(\Xi_{u_i}\right)^{\upsilon_i}\right)\right],\\ \left\{1-\prod_{i=1}^{n}\left(1-\Upsilon_i\right)^{\upsilon_i},-\prod_{i=1}^{n}\left(-\left(\Xi_i-1\right)\right)^{\upsilon_i}\right\}\end{array}\right\}.$$

5.2. CBFSTOWGO

Definition 5.6. A P-CBFSTOWG operator is expounded as

$$\text{P-}CBFSTOWG(\mathfrak{W}_1,\mathfrak{W}_2,...,\mathfrak{W}_n)=(\sin\mathfrak{W}_{\clubsuit(1)})^{\upsilon_1}\otimes_{\text{P}}(\sin\mathfrak{W}_{\clubsuit(2)})^{\upsilon_2}\otimes_{\text{P}}...\otimes_{\text{P}}(\sin\mathfrak{W}_{\clubsuit(n)})^{\upsilon_n}$$

where $\{\clubsuit(1),\clubsuit(2),...,\clubsuit(n)\}$ is a configuration of $\{1,2,...,n\}$ such that $\mathfrak{W}_{\clubsuit(i-1)}\geq\mathfrak{W}_{\clubsuit(i)}$, $\forall i=2,3,...,n$.

Theorem 5.7. *The aggregated valued obtained by using the* P-*CBFSTOWG operator is of the form*

$$\text{P-}CBFSTOWG(\mathfrak{W}_1,\mathfrak{W}_2,...,\mathfrak{W}_n)=\left\{\begin{array}{c}\left[\prod_{i=1}^{n}\left(\Upsilon_{l_{\clubsuit(i)}}\right)^{\upsilon_i},\prod_{i=1}^{n}\left(\Upsilon_{u_{\clubsuit(i)}}\right)^{\upsilon_i}\right],\\ \left[-\left(1-\prod_{i=1}^{n}\left(\Xi_{l_{\clubsuit(i)}}\right)^{\upsilon_i}\right),-\left(1-\prod_{i=1}^{n}\left(\Xi_{u_{\clubsuit(i)}}\right)^{\upsilon_i}\right)\right],\\ \left\{\prod_{i=1}^{n}\left(\Upsilon_{\clubsuit(i)}\right)^{\upsilon_i},-\left(1-\prod_{i=1}^{n}\left(\Xi_{\clubsuit(i)}\right)^{\upsilon_i}\right)\right\}\end{array}\right\}.$$

Example 5.8. Take the CBFNs which were given in Example 4.3, and $\upsilon = \{0.22, 0.31, 0.19, 0.28\}$ is the WV for their ordered positions. Table 4 displays the values of $\Upsilon_{l_{\clubsuit(i)}}$, $\Upsilon_{u_{\clubsuit(i)}}$, $\Xi_{l_{\clubsuit(i)}}$, $\Xi_{u_{\clubsuit(i)}}$, $\Upsilon_{\clubsuit(i)}$ and $\Xi_{\clubsuit(i)}$. Now, we have

$$\begin{aligned}\prod_{i=1}^{4}\left(\Upsilon_{l_{\clubsuit(i)}}\right)^{\upsilon_i} =&\left(\Upsilon_{l_{\clubsuit(1)}}\right)^{\upsilon_1}\times\left(\Upsilon_{l_{\clubsuit(2)}}\right)^{\upsilon_2}\times\left(\Upsilon_{l_{\clubsuit(3)}}\right)^{\upsilon_3}\times\left(\Upsilon_{l_{\clubsuit(4)}}\right)^{\upsilon_4}\\ =&(0.8272)^{0.22}\times(0.4818)^{0.31}\times(0.6959)^{0.19}\times(0.3239)^{0.28}\\ =&0.5207\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4}\left(\Upsilon_{u_{\clubsuit(i)}}\right)^{\upsilon_i} =&\left(\Upsilon_{u_{\clubsuit(1)}}\right)^{\upsilon_1}\times\left(\Upsilon_{u_{\clubsuit(2)}}\right)^{\upsilon_2}\times\left(\Upsilon_{u_{\clubsuit(3)}}\right)^{\upsilon_3}\times\left(\Upsilon_{u_{\clubsuit(4)}}\right)^{\upsilon_4}\\ =&(0.8980)^{0.22}\times(0.6845)^{0.31}\times(0.8607)^{0.19}\times(0.6129)^{0.28}\\ =&0.7358\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4}\left(\Xi_{l_{\clubsuit(i)}}\right)^{\upsilon_i} =&\left(\Xi_{l_{\clubsuit(1)}}\right)^{\upsilon_1}\times\left(\Xi_{l_{\clubsuit(2)}}\right)^{\upsilon_2}\times\left(\Xi_{l_{\clubsuit(3)}}\right)^{\upsilon_3}\times\left(\Xi_{l_{\clubsuit(4)}}\right)^{\upsilon_4}\\ =&(0.7705)^{0.22}\times(0.9114)^{0.31}\times(0.6494)^{0.19}\times(0.4399)^{0.28}\\ =&0.6716\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4}\left(\Xi_{u_{\clubsuit(i)}}\right)^{\upsilon_i} =&\left(\Xi_{u_{\clubsuit(1)}}\right)^{\upsilon_1}\times\left(\Xi_{u_{\clubsuit(2)}}\right)^{\upsilon_2}\times\left(\Xi_{u_{\clubsuit(3)}}\right)^{\upsilon_3}\times\left(\Xi_{u_{\clubsuit(4)}}\right)^{\upsilon_4}\\ =&(0.9409)^{0.22}\times(0.9603)^{0.31}\times(0.8358)^{0.19}\times(0.7181)^{0.28}\\ =&0.8583\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4}\left(\Upsilon_{\clubsuit(i)}\right)^{\upsilon_i} =&\left(\Upsilon_{\clubsuit(1)}\right)^{\upsilon_1}\times\left(\Upsilon_{\clubsuit(2)}\right)^{\upsilon_2}\times\left(\Upsilon_{\clubsuit(3)}\right)^{\upsilon_3}\times\left(\Upsilon_{\clubsuit(4)}\right)^{\upsilon_4}\\ =&(0.5750)^{0.22}\times(0.7804)^{0.31}\times(0.3971)^{0.19}\times(0.8607)^{0.28}\\ =&0.6596\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4}\left(\Xi_{\clubsuit(i)}\right)^{\upsilon_i} =&\left(\Xi_{\clubsuit(1)}\right)^{\upsilon_1}\times\left(\Xi_{\clubsuit(2)}\right)^{\upsilon_2}\times\left(\Xi_{\clubsuit(3)}\right)^{\upsilon_3}\times\left(\Xi_{\clubsuit(4)}\right)^{\upsilon_4}\\ =&(0.8838)^{0.22}\times(0.8838)^{0.31}\times(0.7501)^{0.19}\times(0.6845)^{0.28}\\ =&0.7975.\end{aligned}$$

Now, using the previously obtained values, we obtain

$$\text{P-}CBFSTOWG(\mathfrak{W}_1,\mathfrak{W}_2,\mathfrak{W}_3,\mathfrak{W}_4)=\left\{\begin{array}{c}\left[\prod_{i=1}^{4}\left(\Upsilon_{l_{\clubsuit(i)}}\right)^{\upsilon_i},\prod_{i=1}^{4}\left(\Upsilon_{u_{\clubsuit(i)}}\right)^{\upsilon_i}\right],\\ \left[-\left(1-\prod_{i=1}^{4}\left(\Xi_{l_{\clubsuit(i)}}\right)^{\upsilon_i}\right),-\left(1-\prod_{i=1}^{4}\left(\Xi_{u_{\clubsuit(i)}}\right)^{\upsilon_i}\right)\right],\\ \left\{\prod_{i=1}^{4}\left(\Upsilon_{\clubsuit(i)}\right)^{\upsilon_i},-\left(1-\prod_{i=1}^{4}\left(\Xi_{\clubsuit(i)}\right)^{\upsilon_i}\right)\right\}\end{array}\right\}$$

$$=\langle[0.5207, 7358], [-(1-0.6716), -(1-0.8583)], \{0.6596, -(1-0.7975)\}\rangle$$
$$=\langle[0.5207, 0.7358], [-0.3284, -0.1417], \{0.6596, -0.2025\}\rangle.$$

Definition 5.9. The R-CBFSTOWG operator is defined as

$$\text{R-}CBFSTOWG(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = (\sin \mathfrak{W}_{\clubsuit(1)})^{\upsilon_1} \otimes_{\mathrm{R}} (\sin \mathfrak{W}_{\clubsuit(2)})^{\upsilon_2} \otimes_{\mathrm{R}} ... \otimes_{\mathrm{R}} (\sin \mathfrak{W}_{\clubsuit(n)})^{\upsilon_n}.$$

Theorem 5.10. *The aggregated valued obtained by using the* R-*CBFSTOWG operator is represented by*

$$\text{R-}CBFSTOWG(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \left\{ \begin{array}{c} \Big[\prod_{i=1}^{n}\Big(\Upsilon_{l_{\clubsuit(i)}}\Big)^{\upsilon_i}, \prod_{i=1}^{n}\Big(\Upsilon_{u_{\clubsuit(i)}}\Big)^{\upsilon_i}\Big], \\ \Big[-\Big(1-\prod_{i=1}^{n}\Big(\Xi_{l_{\clubsuit(i)}}\Big)^{\upsilon_i}\Big), -\Big(1-\prod_{i=1}^{n}\Big(\Xi_{u_{\clubsuit(i)}}\Big)^{\upsilon_i}\Big)\Big], \\ \Big\{1-\prod_{i=1}^{n}\Big(1-\Upsilon_{\clubsuit(i)}\Big)^{\upsilon_i}, -\prod_{i=1}^{n}\Big(-\Big(\Xi_{\clubsuit(i)}-1\Big)\Big)^{\upsilon_i}\Big\} \end{array} \right\}.$$

5.3. CBFSTHWGO

Definition 5.11. A P-CBFSTHWG operator assumes the following form:

$$\text{P-}CBFSTHWG(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = (\sin \dot{\mathfrak{W}}_{\clubsuit(1)})^{\varepsilon_1} \otimes_{\mathrm{P}} (\sin \dot{\mathfrak{W}}_{\clubsuit(2)})^{\varepsilon_2} \otimes_{\mathrm{P}} ... \otimes_{\mathrm{P}} (\sin \dot{\mathfrak{W}}_{\clubsuit(n)})^{\varepsilon_n}$$

where $\dot{\mathfrak{W}}_i = \mathfrak{W}_i^{n\upsilon_i}$, $\forall i = 1, 2, ..., n$.

Theorem 5.12. *The congregated valued obtained by utilizing the* P-*CBFSTHWG operator is given by*

$$\text{P-}CBFSTHWG(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \left\{ \begin{array}{c} \Big[\prod_{i=1}^{n}\Big(\dot{\Upsilon}_{l_{\clubsuit(i)}}\Big)^{\varepsilon_i}, \prod_{i=1}^{n}\Big(\dot{\Upsilon}_{u_{\clubsuit(i)}}\Big)^{\varepsilon_i}\Big], \\ \Big[-\Big(1-\prod_{i=1}^{n}\Big(\dot{\Xi}_{l_{\clubsuit(i)}}\Big)^{\varepsilon_i}\Big), -\Big(1-\prod_{i=1}^{n}\Big(\dot{\Xi}_{u_{\clubsuit(i)}}\Big)^{\varepsilon_i}\Big)\Big], \\ \Big\{\prod_{i=1}^{n}\Big(\dot{\Upsilon}_{\clubsuit(i)}\Big)^{\varepsilon_i}, -\Big(1-\prod_{i=1}^{n}\Big(\dot{\Xi}_{\clubsuit(i)}\Big)^{\varepsilon_i}\Big)\Big\} \end{array} \right\}.$$

Example 5.13. Take the CBFNs which were given in Example 4.3, and $\upsilon = \{0.22, 0.31, 0.19, 0.28\}$ is the WV for them. Let $\varepsilon = \{0.15, 0.26, 0.32, 0.27\}$ be the associated WV. Now,

$$\dot{\mathfrak{W}}_1 = \langle[0.52, 0.72], [-0.08, -0.04], \{0.80, -0.10\}\rangle$$
$$\dot{\mathfrak{W}}_2 = \langle[0.79, 0.88], [-0.28, -0.07], \{0.50, -0.14\}\rangle$$
$$\dot{\mathfrak{W}}_3 = \langle[0.76, 0.89], [-0.28, -0.13], \{0.50, -0.20\}\rangle$$
$$\dot{\mathfrak{W}}_4 = \langle[0.28, 0.58], [-0.60, -0.31], \{0.84, -0.34\}\rangle.$$

We calculate the score functions of these CBFNs using Eq (2.1) as follows.

$$\Psi_{\mathrm{P}}(\dot{\mathfrak{W}}_1) = 0.8033, \Psi_{\mathrm{P}}(\dot{\mathfrak{W}}_2) = 0.7800, \Psi_{\mathrm{P}}(\dot{\mathfrak{W}}_3) = 0.7567, \textit{ and } \Psi_{\mathrm{P}}(\dot{\mathfrak{W}}_4) = 0.5750.$$

According to these score values, we obtain $\dot{\mathfrak{W}}_1 \succ \dot{\mathfrak{W}}_2 \succ \dot{\mathfrak{W}}_3 \succ \dot{\mathfrak{W}}_4$. This gives

$$\dot{\mathfrak{W}}_{\clubsuit(1)} = \dot{\mathfrak{W}}_1 = \langle[0.52, 0.72], [-0.08, -0.04], \{0.80, -0.10\}\rangle$$

$$\dot{\mathfrak{W}}_{\clubsuit(2)} = \dot{\mathfrak{W}}_2 = \langle [0.79, 0.88], [-0.28, -0.07], \{0.50, -0.14\} \rangle$$
$$\dot{\mathfrak{W}}_{\clubsuit(3)} = \dot{\mathfrak{W}}_3 = \langle [0.76, 0.89], [-0.28, -0.13], \{0.50, -0.20\} \rangle$$
$$\dot{\mathfrak{W}}_{\clubsuit(4)} = \dot{\mathfrak{W}}_4 = \langle [0.28, 0.58], [-0.60, -0.31], \{0.84, -0.34\} \rangle.$$

Now, the values of $\dot{\Upsilon}_{l_{\clubsuit(i)}}$, $\dot{\Upsilon}_{u_{\clubsuit(i)}}$, $\dot{\Xi}_{l_{\clubsuit(i)}}$, $\dot{\Xi}_{u_{\clubsuit(i)}}$, $\dot{\Upsilon}_{\clubsuit(i)}$ and $\dot{\Xi}_{\clubsuit(i)}$ are provided in Table 6.

Table 6. The values of $\dot{\Upsilon}_{l_{\clubsuit(i)}}$, $\dot{\Upsilon}_{u_{\clubsuit(i)}}$, $\dot{\Xi}_{l_{\clubsuit(i)}}$, $\dot{\Xi}_{u_{\clubsuit(i)}}$, $\dot{\Upsilon}_{\clubsuit(i)}$ and $\dot{\Xi}_{\clubsuit(i)}$, for $i = 1, 2, 3, 4$.

$\dot{\Upsilon}_{l_{\clubsuit(1)}}$	0.7290	$\dot{\Upsilon}_{u_{\clubsuit(1)}}$	0.9048	$\dot{\Xi}_{l_{\clubsuit(1)}}$	0.9921	$\dot{\Xi}_{u_{\clubsuit(1)}}$	0.9980	$\dot{\Upsilon}_{\clubsuit(1)}$	0.9510	$\dot{\Xi}_{\clubsuit(1)}$	0.9877
$\dot{\Upsilon}_{l_{\clubsuit(2)}}$	0.9461	$\dot{\Upsilon}_{u_{\clubsuit(2)}}$	0.9823	$\dot{\Xi}_{l_{\clubsuit(2)}}$	0.9048	$\dot{\Xi}_{u_{\clubsuit(2)}}$	0.9940	$\dot{\Upsilon}_{\clubsuit(2)}$	0.7071	$\dot{\Xi}_{\clubsuit(2)}$	0.9759
$\dot{\Upsilon}_{l_{\clubsuit(3)}}$	0.9298	$\dot{\Upsilon}_{u_{\clubsuit(3)}}$	0.9851	$\dot{\Xi}_{l_{\clubsuit(3)}}$	0.9048	$\dot{\Xi}_{u_{\clubsuit(3)}}$	0.9792	$\dot{\Upsilon}_{\clubsuit(3)}$	0.7071	$\dot{\Xi}_{\clubsuit(3)}$	0.9510
$\dot{\Upsilon}_{l_{\clubsuit(4)}}$	0.4258	$\dot{\Upsilon}_{u_{\clubsuit(4)}}$	0.7902	$\dot{\Xi}_{l_{\clubsuit(4)}}$	0.5878	$\dot{\Xi}_{u_{\clubsuit(4)}}$	0.8838	$\dot{\Upsilon}_{\clubsuit(4)}$	0.9686	$\dot{\Xi}_{\clubsuit(4)}$	0.8607

Now, we have

$$\begin{aligned}\prod_{i=1}^{4} \left(\dot{\Upsilon}_{l_{\clubsuit(i)}}\right)^{\varepsilon_i} =& \left(\dot{\Upsilon}_{l_{\clubsuit(1)}}\right)^{\varepsilon_1} \times \left(\dot{\Upsilon}_{l_{\clubsuit(2)}}\right)^{\varepsilon_2} \times \left(\dot{\Upsilon}_{l_{\clubsuit(3)}}\right)^{\varepsilon_3} \times \left(\dot{\Upsilon}_{l_{\clubsuit(4)}}\right)^{\varepsilon_4}\\ =&(0.7290)^{0.15} \times (0.9461)^{0.26} \times (0.9298)^{0.32} \times (0.4258)^{0.27}\\ =&0.7293\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4} \left(\dot{\Upsilon}_{u_{\clubsuit(i)}}\right)^{\varepsilon_i} =& \left(\dot{\Upsilon}_{u_{\clubsuit(1)}}\right)^{\varepsilon_1} \times \left(\dot{\Upsilon}_{u_{\clubsuit(2)}}\right)^{\varepsilon_2} \times \left(\dot{\Upsilon}_{u_{\clubsuit(3)}}\right)^{\varepsilon_3} \times \left(\dot{\Upsilon}_{u_{\clubsuit(4)}}\right)^{\varepsilon_4}\\ =&(0.9048)^{0.15} \times (0.9823)^{0.26} \times (0.9851)^{0.32} \times (0.7902)^{0.27}\\ =&0.9157\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4} \left(\dot{\Xi}_{l_{\clubsuit(i)}}\right)^{\varepsilon_i} =& \left(\dot{\Xi}_{l_{\clubsuit(1)}}\right)^{\varepsilon_1} \times \left(\dot{\Xi}_{l_{\clubsuit(2)}}\right)^{\varepsilon_2} \times \left(\dot{\Xi}_{l_{\clubsuit(3)}}\right)^{\varepsilon_3} \times \left(\dot{\Xi}_{l_{\clubsuit(4)}}\right)^{\varepsilon_4}\\ =&(0.9921)^{0.15} \times (0.9048)^{0.26} \times (0.9048)^{0.32} \times (0.5878)^{0.27}\\ =&0.8165\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4} \left(\dot{\Xi}_{u_{\clubsuit(i)}}\right)^{\varepsilon_i} =& \left(\dot{\Xi}_{u_{\clubsuit(1)}}\right)^{\varepsilon_1} \times \left(\dot{\Xi}_{u_{\clubsuit(2)}}\right)^{\varepsilon_2} \times \left(\dot{\Xi}_{u_{\clubsuit(3)}}\right)^{\varepsilon_3} \times \left(\dot{\Xi}_{u_{\clubsuit(4)}}\right)^{\varepsilon_4}\\ =&(0.9980)^{0.15} \times (0.9940)^{0.26} \times (0.9792)^{0.32} \times (0.8838)^{0.27}\\ =&0.9589\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{4} \left(\dot{\Upsilon}_{\clubsuit(i)}\right)^{\varepsilon_i} =& \left(\dot{\Upsilon}_{\clubsuit(1)}\right)^{\varepsilon_1} \times \left(\dot{\Upsilon}_{\clubsuit(2)}\right)^{\varepsilon_2} \times \left(\dot{\Upsilon}_{\clubsuit(3)}\right)^{\varepsilon_3} \times \left(\dot{\Upsilon}_{\clubsuit(4)}\right)^{\varepsilon_4}\\ =&(0.9510)^{0.15} \times (0.7071)^{0.26} \times (0.7071)^{0.32} \times (0.9686)^{0.27}\\ =&0.8048\end{aligned}$$

$$\prod_{i=1}^{4}\left(\dot{\Xi}_{\clubsuit(i)}\right)^{\varepsilon_i} = \left(\dot{\Xi}_{\clubsuit(1)}\right)^{\varepsilon_1} \times \left(\dot{\Xi}_{\clubsuit(2)}\right)^{\varepsilon_2} \times \left(\dot{\Xi}_{\clubsuit(3)}\right)^{\varepsilon_3} \times \left(\dot{\Xi}_{\clubsuit(4)}\right)^{\varepsilon_4}$$
$$= (0.9877)^{0.15} \times (0.9759)^{0.26} \times (0.9510)^{0.32} \times (0.8607)^{0.27}$$
$$= 0.9373.$$

Now, using the previously obtained values, we obtain

$$\text{P-}CBFSTHWG(\mathfrak{W}_1, \mathfrak{W}_2, \mathfrak{W}_3, \mathfrak{W}_4) = \left\{ \begin{array}{c} \left[\prod_{i=1}^{4}\left(\dot{\Upsilon}_{l_{\clubsuit(i)}}\right)^{\varepsilon_i}, \prod_{i=1}^{4}\left(\dot{\Upsilon}_{u_{\clubsuit(i)}}\right)^{\varepsilon_i}\right], \\ \left[-\left(1-\prod_{i=1}^{4}\left(\dot{\Xi}_{l_{\clubsuit(i)}}\right)^{\varepsilon_i}\right), -\left(1-\prod_{i=1}^{4}\left(\dot{\Xi}_{u_{\clubsuit(i)}}\right)^{\varepsilon_i}\right)\right], \\ \left\{\prod_{i=1}^{4}\left(\dot{\Upsilon}_{\clubsuit(i)}\right)^{\varepsilon_i}, -\left(1-\prod_{i=1}^{4}\left(\dot{\Xi}_{\clubsuit(i)}\right)^{\varepsilon_i}\right)\right\} \end{array} \right\}$$
$$= \langle [0.7293, 0.9157], [-(1-0.8165), -(1-0.9589)], \{0.8048, -(1-0.9373)\}\rangle$$
$$= \langle [0.7293, 0.9157], [-0.1835, -0.0411], \{0.8048, -0.0627\}\rangle.$$

Definition 5.14. An R-CBFSTHWG operator assumes the following form:

$$\text{R-}CBFSTHWG(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = (\sin \dot{\mathfrak{W}}_{\clubsuit(1)})^{\varepsilon_1} \otimes_{\text{R}} (\sin \dot{\mathfrak{W}}_{\clubsuit(2)})^{\varepsilon_2} \otimes_{\text{R}} ... \otimes_{\text{R}} (\sin \dot{\mathfrak{W}}_{\clubsuit(n)})^{\varepsilon_n}.$$

Theorem 5.15. *The aggregated valued obtained by using the* R-*CBFSTHWG operator is given by*

$$\text{R-}CBFSTHWG(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \left\{ \begin{array}{c} \left[\prod_{i=1}^{n}\left(\dot{\Upsilon}_{l_{\clubsuit(i)}}\right)^{\varepsilon_i}, \prod_{i=1}^{n}\left(\dot{\Upsilon}_{u_{\clubsuit(i)}}\right)^{\varepsilon_i}\right], \\ \left[-\left(1-\prod_{i=1}^{n}\left(\dot{\Xi}_{l_{\clubsuit(i)}}\right)^{\varepsilon_i}\right), -\left(1-\prod_{i=1}^{n}\left(\dot{\Xi}_{u_{\clubsuit(i)}}\right)^{\varepsilon_i}\right)\right], \\ \left\{1-\prod_{i=1}^{n}\left(1-\dot{\Upsilon}_{\clubsuit(i)}\right)^{\varepsilon_i}, -\prod_{i=1}^{n}\left(-\left(\dot{\Xi}_{\clubsuit(i)}-1\right)\right)^{\varepsilon_i}\right\} \end{array} \right\}.$$

Remark. The P-STCBFWC, P-CBFSTOWG, R-CBFSTWG and R-CBFSTOWG operators satisfy idempotency, monotonicity and boundedness.

6. MCGDM method

In this part, we tackle an MCGDM problem using novel developed AOs. For this, let $\hat{\beth} = \{\hat{\beth}_i : i = 1, 2, ..., m\}$ represent a collection of alternatives and $\hat{\beth} = \{\hat{\beth}_j : j = 1, 2, ..., n\}$ represent a list of criteria. Suppose that $\hat{\daleth} = \{\hat{\daleth}_k : k = 1, 2, ..., l\}$ is a set of DMs who will make a choice after evaluating each alternative in relation to each criterion. We now give each decision maker and criterion a weight to indicate their importance in the decision-making process. So, let $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ and $\varsigma = \{\varsigma_1, \varsigma_2, ..., \varsigma_l\}$ be the WVs of the criteria and DMs, respectively. Each decision maker records his perceptions as a matrix $\Lambda^{(k)} = (\beta_{ij}^k)_{m\times n}$, where β_{ij}^k represents the CBFN provided by the DM $\hat{\daleth}_k$ as an assessment of the alternative $\hat{\beth}_i$ with regard to the criterion $\hat{\beth}_j$. Now, to tackle this MCGDM problem, the following algorithm is suggested.

Algorithm

Step 1. Set up the individual decision matrices $\Lambda^{(k)} = (\beta_{ij}^{k})_{m\times n}$, $k = 1, 2, ..., l$, in the form of CBFNs.

Step 2. Apply the P-CBFSTWA or P-CBFSTWG operator to obtain the integrated decision matrix $\Delta = (\xi_{ij})_{m\times n}$ as follows.

$$\xi_{ij} = \text{P-}CBFSTWA(\beta_{ij}^{(1)}, \beta_{ij}^{(2)}, ..., \beta_{ij}^{(l)}) = \left\{\begin{array}{c} \Big[1-\prod_{k=1}^{l}\Big(1-\Upsilon_{l_k}\Big)^{\varsigma_k}, 1-\prod_{k=1}^{l}\Big(1-\Upsilon_{u_k}\Big)^{\varsigma_k}\Big], \\ \Big[-\prod_{k=1}^{l}\Big(-\Big(\Xi_{l_k}-1\Big)\Big)^{\varsigma_k}, -\prod_{k=1}^{l}\Big(-\Big(\Xi_{u_k}-1\Big)\Big)^{\varsigma_k}\Big], \\ \Big\{1-\prod_{k=1}^{l}\Big(1-\Upsilon_{k}\Big)^{\varsigma_k}, -\prod_{k=1}^{l}\Big(-\Big(\Xi_{k}-1\Big)\Big)^{\varsigma_k}\Big\} \end{array}\right\} \tag{6.1}$$

or

$$\xi_{ij} = \text{P-}CBFSTWG(\beta_{ij}^{(1)}, \beta_{ij}^{(2)}, ..., \beta_{ij}^{(l)}) = \left\{\begin{array}{c} \Big[\prod_{k=1}^{l}\Big(\Upsilon_{l_k}\Big)^{\varsigma_k}, \prod_{k=1}^{l}\Big(\Upsilon_{u_k}\Big)^{\varsigma_k}\Big], \\ \Big[-\Big(1-\prod_{k=1}^{l}\Big(\Xi_{l_k}\Big)^{\varsigma_k}\Big), -\Big(1-\prod_{k=1}^{l}\Big(\Xi_{u_k}\Big)^{\varsigma_k}\Big)\Big], \\ \Big\{\prod_{k=1}^{l}\Big(\Upsilon_{k}\Big)^{\varsigma_k}, -\Big(1-\prod_{k=1}^{l}\Big(\Xi_{k}\Big)^{\varsigma_k}\Big)\Big\} \end{array}\right\} \tag{6.2}$$

Step 3. Employ the P-CBFSTWA or P-CBFSTWG operator to combine the values ξ_{ij} for each alternative $\hat{\jmath}_i$.

$$\alpha_i = \text{P-}CBFSTWA(\xi_{i1}, \xi_{i2}, ..., \xi_{in}) = \left\{\begin{array}{c} \Big[1-\prod_{j=1}^{n}\Big(1-\Upsilon_{l_j}\Big)^{\lambda_j}, 1-\prod_{j=1}^{n}\Big(1-\Upsilon_{u_j}\Big)^{\lambda_j}\Big], \\ \Big[-\prod_{j=1}^{n}\Big(-\Big(\Xi_{l_j}-1\Big)\Big)^{\lambda_j}, -\prod_{j=1}^{n}\Big(-\Big(\Xi_{u_j}-1\Big)\Big)^{\lambda_j}\Big], \\ \Big\{1-\prod_{j=1}^{n}\Big(1-\Upsilon_{j}\Big)^{\lambda_j}, -\prod_{j=1}^{n}\Big(-\Big(\Xi_{j}-1\Big)\Big)^{\lambda_j}\Big\} \end{array}\right\} \tag{6.3}$$

or

$$\alpha_i = \text{P-}CBFSTWG(\xi_{i1}, \xi_{i2}, ..., \xi_{in}) = \left\{\begin{array}{c} \Big[\prod_{j=1}^{n}\Big(\Upsilon_{l_j}\Big)^{\lambda_j}, \prod_{j=1}^{n}\Big(\Upsilon_{u_j}\Big)^{\lambda_j}\Big], \\ \Big[-\Big(1-\prod_{j=1}^{n}\Big(\Xi_{l_j}\Big)^{\lambda_j}\Big), -\Big(1-\prod_{j=1}^{n}\Big(\Xi_{u_j}\Big)^{\lambda_j}\Big)\Big], \\ \Big\{\prod_{j=1}^{n}\Big(\Upsilon_{j}\Big)^{\lambda_j}, -\Big(1-\prod_{j=1}^{n}\Big(\Xi_{j}\Big)^{\lambda_j}\Big)\Big\} \end{array}\right\} \tag{6.4}$$

Step 4. Find the score function of α_i, $i = 1, 2, ..., m$, by employing Eq (2.1).

Step 5. Sort the alternatives by the ranking of the α_i values.

Step 6. The highest ranked alternative is the required alternative.

Remark. One can also use the R-CBFSTWA or R-CBFSTWG operator in the proposed algorithm. However, we omit this case.

Figure 1 shows a step-by-step diagram of the suggested algorithm.

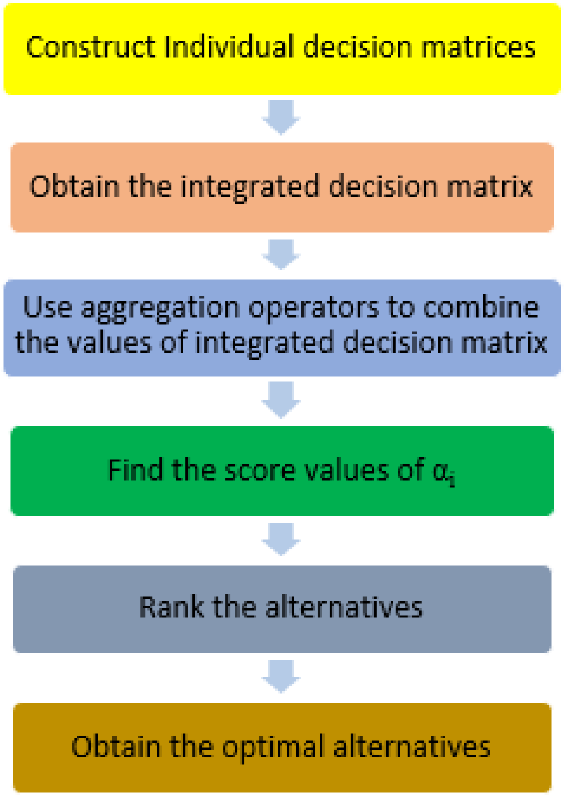

Figure 1. A step-by-step diagram of the suggested algorithm.

6.1. Case study

Ecosystems are now seriously impacted by climate change. The ability to survive has become more challenging due to climate change for both people and wild animals. Extreme temperatures, famines, frequent floods, and melting glaciers are just a few of the primary consequences of climate change. All of these environmental concerns are destroying animal habitats and wreaking havoc on humanity. The primary factor causing global climate change is CO_2 emissions. It is largely agreed that immediate emission cuts are essential for the globe to prevent the catastrophic impacts of climate change. As a result of the industrialization and the massive rise of the manufacturing sector globally, CO_2 concentrations have skyrocketed. The main contributors of CO_2 emissions are transportation, agriculture, and the consumption of fossil fuels. The pie chart (Figure 2) below depicts the amounts of CO_2 emissions generated by various sectors*.

*https://www.statista.com/statistics/1129656/global-share-of-co2-emissions-from-fossil-fuel-and-cement/

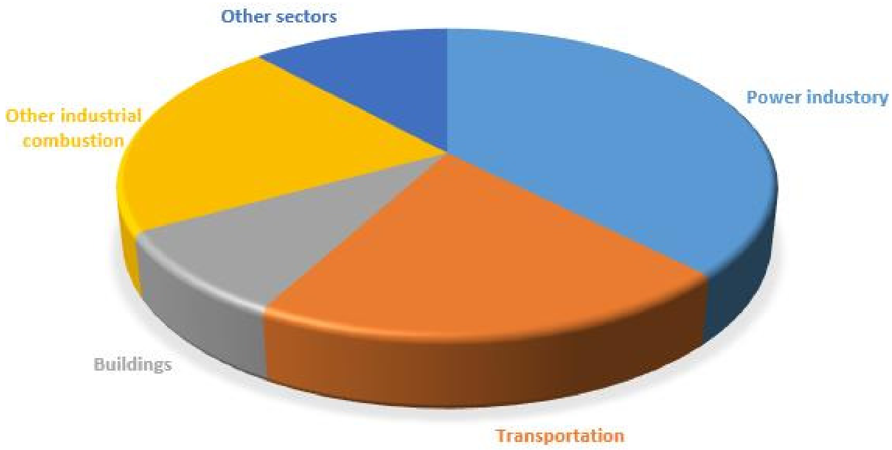

Figure 2. CO_2 Emissions by Sectors, Worldwide (2021).

According to the latest available information from the Global Carbon Atlas, the US, Germany, Russia, China and the UK have released the most CO_2 since industrialization. The countries with the highest emissions in 2020 were China, the US, India, Russia and Japan. The accompanying bar chart (Figure 3) displays the top five countries along with the total amount of CO_2 emissions they generated in 2020[†].

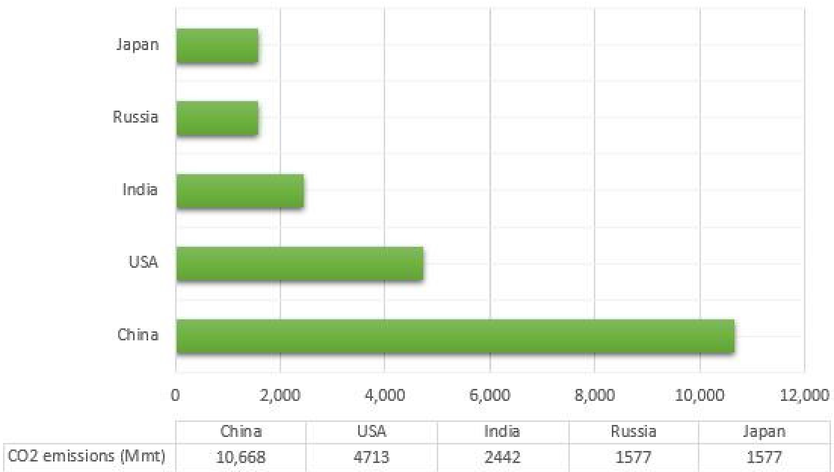

Figure 3. CO_2 emissions (in million metric tons) in 2020.

According to a recent analysis by the International Energy Agency (IEA), roughly 20 percent of all CO_2 emissions worldwide are caused by four industries: Cement, iron, steel and chemicals. The cement making process itself, as well as energy utilization, contributes significantly to industrial emissions, making them exceptionally difficult to mitigate. For instance, about half of the emissions in the cement industry are caused by the disintegration of limestone ($CaCO_3$) into lime (CaO) and CO_2. Although switching to carbon-free energy sources like solar or wind-generated electricity could reduce CO_2 release in the power sector, there are no simple solutions for industries that generate a lot of emissions.

[†]https://www.investopedia.com/articles/investing/092915/5-countries-produce-most-carbon-dioxide-co2.asp

6.1.1. Carbon capture & storage technology

When applied to an industrial site, carbon capture and storage (CCS) technology has the capacity to eliminate 90% to 99% of all CO2 emissions, including process-related and energy-related emissions. With CCS, carbon emissions from discrete sources are hidden away underground. By depositing the CO_2 into appropriate subterranean reservoirs, carbon capture and storage (CCS), a group of technologies, prevents CO_2 emissions from conventional power production and industrial manufacturing processes. In essence, CO_2 capture technology separates CO_2 emissions from the process, and then the compacted CO_2 is moved to and injected in an appropriate geological storage location. Both shipping and pipelines are effective CO_2 delivery options. Abandoned oil and gas fields, severely salty layers, and unmineable coal seams are all potential geological storage options for CO_2. One primary objective of CCS technology is to lessen CO_2 emissions from manufacturing and electricity generation.
Figure 4 displays the carbon capture technologies.

6.1.2. Types of CCS

There are three different types of capture processes, and the one that applies depends on the sort of power plant or industrial operation at hand.

- **Pre-combustion:** An amalgam of carbon monoxide and hydrogen, commonly referred to as a "syngas," is produced when the process's main fuel reacts with steam, air, or oxygen. The conversion of carbon monoxide to CO_2 occurs in a shift reactor. After separating the CO_2, the hydrogen is used to produce electricity or heat. Integrated gasification combined cycle (IGCC) energy plants are a good fit for this approach in particular.
- **Post Combustion:** When a fossil fuel is burned, CO_2 is taken out of the flue gas that results. In post-combustion separation, CO_2 is captured using a solvent. The two most prevalent uses of this technology are in pulverized coal (PC) plants and natural gas combined cycle (NGCC) facilities.
- **Oxy-fuel combustion:** In this case, the basic fuel is ignited in oxygen instead of air, which results in significant CO_2 and water vapor production in the flue gas. The production of a nearly pure stream of CO_2 occurs after the flue gas is cooled to cause the water vapor to precipitate.

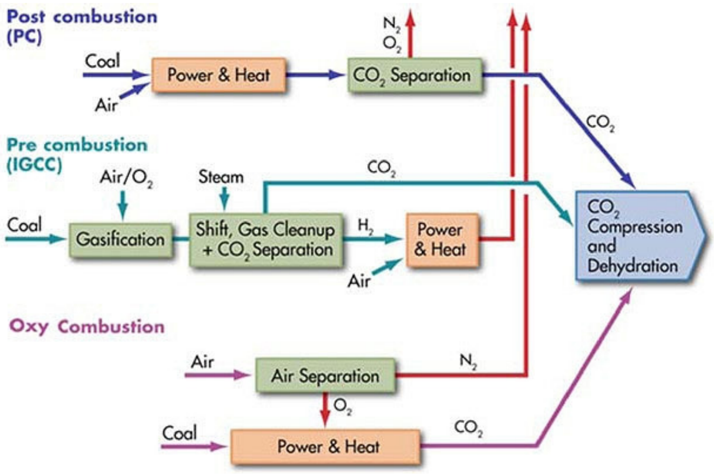

Figure 4. Carbon capture technologies.

6.1.3. Transportation & storage of captured CO_2

It will be necessary to move CO_2 to an appropriate storage site once it has been successfully "captured" from a process. As a result, CO_2 is carried at high pressures in carbon steel tubes that are comparable to standard natural gas pipelines or aboard ships as needed to traverse a significant body of water. Disused oil and gas fields and deep salt deposits with a minimum depth of 800 m and temperatures and pressures high enough to maintain the CO_2 in a liquid or a gaseous state are suitable locations to store CO_2[‡].

6.1.4. Perks of CCS

- More than 90 percent of CO_2 emissions from power stations and other industrial units can be captured using carbon capture, utilization, and storage technology.
- Captured CO2 can be utilized to boost oil recovery and manufacturing of fuels, buildings materials, and other products, or it can be stored in underground geologic formations.
- Capturing carbon dioxide has been identified as the only viable option for significantly reducing industrial greenhouse gas emissions, with the potential to save 14 billion tonnes by 2050[§].

It is obvious that CCS technology can be extremely beneficial in combating climate change. Choosing the best CCS technology, on the other hand, is a complicated MCGDM. Now, in order to solve the MCGDM of selecting the optimum CCS technique, we undertake a numerical analysis to validate the effectiveness of our previously discussed approach.

6.2. Numerical example

Our goal is to find the best CCS technology to reduce CO_2 emissions worldwide. Let $\hat{\gimel} = \{\hat{\gimel}_1, \hat{\gimel}_2, \hat{\gimel}_3\}$ be three alternatives where $\hat{\gimel}_1$ =Pre-combustion, $\hat{\gimel}_2$ =Post combustion and $\hat{\gimel}_3$ =Oxy-fuel combustion. The criteria for selecting the best alternative are displayed in Table 7. The WV of the criteria is $\varsigma = \{0.21, 0.32, 0.27, 0.20\}$. Let $\hat{\daleth} = \{\hat{\daleth}_1, \hat{\daleth}_2, \hat{\daleth}_3\}$ be a group of three DMs and $\lambda = \{0.29, 0.33, 0.38\}$ be their weights. Now, we begin to apply our proposed algorithm to solve this MCGDM.

Step 1. The individual CBF decision matrices $\Lambda^{(1)}$, $\Lambda^{(2)}$ and $\Lambda^{(3)}$ are presented in Tables 8–10, respectively.

Step 2. The integrated decision matrix Δ is obtained by utilizing the P-CBFSTWA operator. Using Eq (6.1), the matrix is displayed in Table 11.

Step 3. Use the P-CBFSTWA operator to combine the entries of the matrix Δ. Get α_i, $i = 1, 2, 3$, by using Eq (6.3).

$$\alpha_1 = \langle [0.9587, 0.9888], [-0.0322, -0.0086], \{0.9754, -0.0468\} \rangle$$

$$\alpha_2 = \langle [0.9409, 0.9843], [-0.1353, -0.0533], \{0.8112, -0.0780\} \rangle$$

$$\alpha_3 = \langle [0.9281, 0.9772], [-0.0603, -0.0190], \{0.9584, -0.0430\} \rangle.$$

Step 4. Find the score function of α_i, $i = 1, 2, 3$ using Eq (2.1). Now, $\Psi_P(\alpha_1) = 0.9726$, $\Psi_P(\alpha_2) = 0.9116$, and $\Psi_P(\alpha_3) = 0.9569$.

[‡]https://www.ctc-n.org/technologies/co2-capture-technologies

[§]https://www.c2es.org/content/carbon-capture/

Step 5. Determine the ranking of α_i according to their score values. The ranking is $\alpha_1 \succ \alpha_3 \succ \alpha_2$. Now, sort the alternatives by the ranking of α_i. The following is the ranking: $\hat{\gimel}_1 \succ \hat{\gimel}_3 \succ \hat{\gimel}_2$.
Step 6. The required alternative is $\hat{\gimel}_1$.
Now, as shown below, we apply the P-CBFSTWG operator to acquire the ranking of alternatives.
Step 2. The integrated decision matrix Δ is obtained by utilizing P-CBFSTWG operator. By using Eq (6.2), the matrix is presented in Table 12.
Step 3. Apply the P-CBFSTWG operator to the entries of the matrix Δ to agglomerate them. By using Eq (6.4), obtain α_i, $i = 1, 2, 3$, as seen below.

$$\alpha_1 = \langle [0.8382, 0.9203], [-0.0946, -0.0400], \{0.9288, -0.1069\} \rangle$$

$$\alpha_2 = \langle [0.8978, 0.9560], [-0.2312, -0.1116], \{0.7456, -0.1400\} \rangle$$

$$\alpha_3 = \langle [0.8789, 0.9491], [-0.1345, -0.0621], \{0.9017, -0.1113\} \rangle.$$

Step 4. Determine the score function of α_i, $i = 1, 2, 3$ by using Eq (2.1). Now, $\Psi_P(\alpha_1) = 0.9076$, $\Psi_P(\alpha_2) = 0.8528$, and $\Psi_P(\alpha_3) = 0.9036$.
Step 5. The ranking of α_i on the basis of their score values is $\alpha_1 \succ \alpha_3 \succ \alpha_2$. The alternatives are now ranked using the α_i ranking: $\hat{\gimel}_1 \succ \hat{\gimel}_3 \succ \hat{\gimel}_2$.
Step 6. The required alternative is $\hat{\gimel}_1$.
It is obvious that the recommended alternative stays the same regardless of the operator chosen.

Table 7. Selection criteria for the best CCS technology.

Criteria	Description
Cost ($\hat{\beth}_1$)	The cost of capturing, transporting, and storing CO2
Efficiency ($\hat{\beth}_2$)	The amount of CO_2 captured by the CCS process
Energy-intensive ($\hat{\beth}_3$)	The amount of energy consumed throughout the CCS process
Risk factors ($\hat{\beth}_4$)	Environmental hazards, public safety and safety of the workers

Table 8. CBF decision matrix $\Lambda^{(1)}$.

	$\hat{\beth}_1$	$\hat{\beth}_2$
$\hat{\gimel}_1$	$\langle [0.47, 0.58], [-0.72, -0.66], \{0.21, -0.44\} \rangle$	$\langle [0.88, 0.94], [-0.18, -0.10], \{0.75, -0.17\} \rangle$
$\hat{\gimel}_2$	$\langle [0.52, 0.71], [-0.37, -0.29], \{0.36, -0.49\} \rangle$	$\langle [0.64, 0.76], [-0.66, -0.56], \{0.35, -0.67\} \rangle$
$\hat{\gimel}_3$	$\langle [0.36, 0.49], [-0.56, -0.43], \{0.67, -0.71\} \rangle$	$\langle [0.64, 0.76], [-0.39, -0.26], (0.59, -0.36) \rangle$
	$\hat{\beth}_3$	$\hat{\beth}_4$
$\hat{\gimel}_1$	$\langle [0.38, 0.47], [-0.25, -0.18], \{0.69, -0.29\} \rangle$	$\langle [0.64, 0.76], [-0.17, -0.11], \{0.48, -0.51\} \rangle$
$\hat{\gimel}_2$	$\langle [0.67, 0.79], [-0.37, -0.22], \{0.33, -0.27\} \rangle$	$\langle [0.51, 0.62], [-0.52, -0.41], \{0.29, -0.36\} \rangle$
$\hat{\gimel}_3$	$\langle [0.51, 0.62], [-0.52, -0.41], \{0.22, -0.36\} \rangle$	$\langle [0.38, 0.49], [-0.66, -0.56], \{0.47, -0.58\} \rangle$

Table 9. CBF decision matrix $\Lambda^{(2)}$.

	$\hat{\beth}_1$	$\hat{\beth}_2$
$\hat{\gimel}_1$	$\langle[0.68, 0.79], [-0.62, -0.51], \{0.66, -0.42\}\rangle$	$\langle[0.28, 0.33], [-0.49, -0.33], \{0.47, -0.62\}\rangle$
$\hat{\gimel}_2$	$\langle[0.46, 0.59], [-0.77, -0.63], \{0.39, -0.47\}\rangle$	$\langle[0.68, 0.81], [-0.73, -0.62], \{0.55, -0.43\}\rangle$
$\hat{\gimel}_3$	$\langle[0.59, 0.67], [-0.44, -0.32], \{0.72, -0.56\}\rangle$	$\langle[0.37, 0.46], [-0.33, -0.25], \{0.44, -0.22\}\rangle$
	$\hat{\beth}_3$	$\hat{\beth}_4$
$\hat{\gimel}_1$	$\langle[0.21, 0.34], [-0.44, -0.31], \{0.86, -0.59\}\rangle$	$\langle[0.29, 0.38], [-0.66, -0.51], \{0.61, -0.42\}\rangle$
$\hat{\gimel}_2$	$\langle[0.49, 0.58], [-0.27, -0.19], \{0.52, -0.41\}\rangle$	$\langle[0.77, 0.88], [-0.62, -0.48], \{0.41, -0.39\}\rangle$
$\hat{\gimel}_3$	$\langle[0.26, 0.39], [-0.41, -0.28], \{0.38, -0.26\}\rangle$	$\langle[0.49, 0.58], [-0.67, -0.54], \{0.81, -0.76\}\rangle$

Table 10. CBF decision matrix $\Lambda^{(3)}$.

	$\hat{\beth}_1$	$\hat{\beth}_2$
$\hat{\gimel}_1$	$\langle[0.36, 0.48], [-0.47, -0.32], \{0.69, -0.73\}\rangle$	$\langle[0.89, 0.95], [-0.17, -0.11], \{0.82, -0.19\}\rangle$
$\hat{\gimel}_2$	$\langle[0.67, 0.78], [-0.61, -0.49], \{0.68, -0.43\}\rangle$	$\langle[0.38, 0.47], [-0.66, -0.56], \{0.35, -0.67\}\rangle$
$\hat{\gimel}_3$	$\langle[0.58, 0.66], [-0.41, -0.29], \{0.73, -0.55\}\rangle$	$\langle[0.77, 0.88], [-0.25, -0.18], \{0.69, -0.29\}\rangle$
	$\hat{\beth}_3$	$\hat{\beth}_4$
$\hat{\gimel}_1$	$\langle[0.51, 0.63], [-0.52, -0.41], \{0.46, -0.56\}\rangle$	$\langle[0.51, 0.62], [-0.52, -0.41], \{0.46, -0.56\}\rangle$
$\hat{\gimel}_2$	$\langle[0.27, 0.36], [-0.79, -0.67], \{0.25, -0.76\}\rangle$	$\langle[0.61, 0.72], [-0.33, -0.22], \{0.18, -0.29\}\rangle$
$\hat{\gimel}_3$	$\langle[0.64, 0.76], [-0.39, -0.26], \{0.59, -0.36\}\rangle$	$\langle[0.45, 0.56], [-0.81, -0.71], \{0.66, -0.52\}\rangle$

Table 11. Integrated decision matrix Δ obtained by applying P-CBFSTWAO.

	$\hat{\beth}_1$	$\hat{\beth}_2$
$\hat{\gimel}_1$	$\langle[0.7290, 0.8435], [-0.3888, -0.2482], \{0.7944, -0.3187\}\rangle$	$\langle[0.9477, 0.9816], [-0.0726, -0.0289], \{0.9038, -0.0884\}\rangle$
$\hat{\gimel}_2$	$\langle[0.7785, 0.8964], [-0.3704, -0.2451], \{0.7272, -0.2497\}\rangle$	$\langle[0.7862, 0.8918], [-0.5212, -0.3859], \{0.6197, -0.3834\}\rangle$
$\hat{\gimel}_3$	$\langle[0.7398, 0.8287], [-0.2488, -0.1358], \{0.8983, -0.4060\}\rangle$	$\langle[0.8417, 0.9301], [-0.1173, -0.0608], \{0.8019, -0.0963\}\rangle$
	$\hat{\beth}_3$	$\hat{\beth}_4$
$\hat{\gimel}_1$	$\langle[0.5725, 0.7122], [-0.1880, -0.1047], \{0.8962, -0.2592\}\rangle$	$\langle[0.7024, 0.8191], [-0.1936, -0.1082], \{0.7298, -0.2876\}\rangle$
$\hat{\gimel}_2$	$\langle[0.6936, 0.8087], [-0.2294, -0.1213], \{0.5562, -0.2446\}\rangle$	$\langle[0.8533, 0.9350], [-0.2520, -0.1392], \{0.4485, -0.1395\}\rangle$
$\hat{\gimel}_3$	$\langle[0.7109, 0.8348], [-0.2203, -0.1117], \{0.6334, -0.1261\}\rangle$	$\langle[0.6432, 0.7582], [-0.5687, -0.4182], \{0.8778, -0.4210\}\rangle$

Table 12. Integrated decision matrix Δ obtained by applying P-CBFSTWGO.

	$\hat{\beth}_1$	$\hat{\beth}_2$
$\hat{\gimel}_1$	$\langle[0.6733, 0.7940], [-0.4244, -0.3062], \{0.6548, -0.3878\}\rangle$	$\langle[0.7463, 0.7912], [-0.1260, -0.0542], \{0.8445, -0.1957\}\rangle$
$\hat{\gimel}_2$	$\langle[0.7545, 0.8789], [-0.4542, -0.2987], \{0.6612, -0.2516\}\rangle$	$\langle[0.7323, 0.8298], [-0.5255, -0.3885], \{0.5914, -0.4244\}\rangle$
$\hat{\gimel}_3$	$\langle[0.7088, 0.8117], [-0.2603, -0.1447], \{0.8966, -0.4235\}\rangle$	$\langle[0.7616, 0.8484], [-0.1261, -0.0642], \{0.7708, -0.1042\}\rangle$
	$\hat{\beth}_3$	$\hat{\beth}_4$
$\hat{\gimel}_1$	$\langle[0.5143, 0.6664], [-0.2235, -0.1284], \{0.8179, -0.3097\}\rangle$	$\langle[0.6403, 0.7532], [-0.3143, -0.1885], \{0.7166, -0.2981\}\rangle$
$\hat{\gimel}_2$	$\langle[0.6078, 0.7183], [-0.4001, -0.2588], \{0.5102, -0.3815\}\rangle$	$\langle[0.8234, 0.9058], [-0.2978, -0.1749], \{0.4100, -0.1446\}\rangle$
$\hat{\gimel}_3$	$\langle[0.6281, 0.7669], [-0.2289, -0.1223], \{0.5549, -0.1321\}\rangle$	$\langle[0.6372, 0.7543], [-0.5904, -0.4396], \{0.8296, -0.4598\}\rangle$

Figure 5 shows the rankings of the alternatives derived with the P-CBFSTWA and P-CBFSTWG operators.

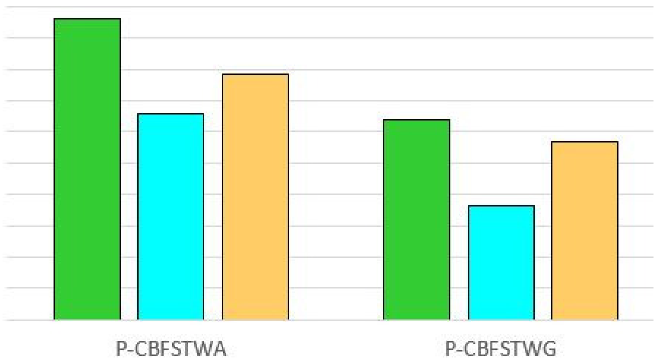

Figure 5. Ranking of alternatives with P-CBFSTWA and P-CBFSTWG operators.

6.3. Comparative analysis

Our proposed methodology is compared to several existing ones in this section. For this purpose, the same numerical data is employed to determine the final ranking of alternatives, and the findings are summarized in Table 13. It has been noticed that different methodologies produce varied rankings; yet, the best option remains the same.

Table 13. Comparison of proposed and incumbent methodologies.

Authors	AOs	Ranking	The recommended alternative
Riaz and Tehrim [23]	CBFWAO	$\hat{\jmath}_1 > \hat{\jmath}_3 > \hat{\jmath}_2$	$\hat{\jmath}_1$
Riaz and Tehrim [24]	CBFWGO	$\hat{\jmath}_1 > \hat{\jmath}_3 > \hat{\jmath}_2$	$\hat{\jmath}_1$
Qiyas and Abdullah [43]	ST-PiFWAO	$\hat{\jmath}_1 > \hat{\jmath}_2 > \hat{\jmath}_3$	$\hat{\jmath}_1$
Jana *et al.* [49]	BF Dombi AOs	$\hat{\jmath}_1 > \hat{\jmath}_3 > \hat{\jmath}_2$	$\hat{\jmath}_1$
Riaz *et al.* [50]	CBF-VIKOR	$\hat{\jmath}_1 > \hat{\jmath}_2 > \hat{\jmath}_3$	$\hat{\jmath}_1$
Kaur and Garg [51]	CIFWAO	$\hat{\jmath}_1 > \hat{\jmath}_3 > \hat{\jmath}_2$	$\hat{\jmath}_1$
Proposed	P-CBFSTWA	$\hat{\jmath}_1 > \hat{\jmath}_3 > \hat{\jmath}_2$	$\hat{\jmath}_1$
Proposed	P-CBFSTWG	$\hat{\jmath}_1 > \hat{\jmath}_3 > \hat{\jmath}_2$	$\hat{\jmath}_1$

The comparison demonstrates that the offered methodology is reliable and efficient for resolving MCGDM problems. Our suggested method outperforms existing ones for the following reasons.

(1) In [49], the BF data was utilized, and this model was unable to handle the cubic environment. Similarly, in [43, 51], PiF and CIF data were employed, respectively. So, these models were inadequate to deal with CBF information.

(2) In [23,24], the simple CBF weighted averaging and geometric AOs were applied to find the results. Nevertheless, these AOs were unable to include the trigonometric analysis into the outputs.

(3) In [50], the CBF-VIKOR method was utilized to find the optimal alternative. This procedure is quite lengthy and requires a lot of calculations.

Our proposed methodology is clearly more effective and superior to current strategies. This procedure is basic and straightforward. Moreover, it provides the accurate results.

7. Conclusions

A CBFS holds both single-valued and interval-valued bipolar fuzzy data, making it an efficient model for dealing with bipolarity and fuzziness simultaneously. The key contributions of this article are listed below.

(1) Because of its periodicity and symmetry about the origin, the employment of the sine trigonometric function in decision-making can be advantageous. As a result, we introduced STOLs for CBFSs. We also looked into several of their properties.

(2) We developed some averaging AOs based on CBF-STOLs, including CBFSTWA, CBFSTOWA, and CBFSTHWA operators, by taking P and R order into consideration independently.

(3) We established geometric AOs based on CBF-STOLs, such as CBFSTWG, CBFSTOWG, and CBFSTHWG operators, by taking P and R order into consideration independently.

(4) We examined some key characteristics of the suggested operators, namely, idempotency, monotonicity, and boundedness.

(5) We developed an MCGDM algorithm based on the suggested AOs to handle MCGDM problems, and we applied this method to solve a challenging MCGDM problem associated with carbon capture and storage (CCS) technology. To illustrate the applicability and reliability of our suggested technique, a numerical study was conducted.

(6) We performed a comparison study to demonstrate that our proposed methodology is consistent with and superior to existing techniques.

In the future, the proposed AOs could be employed in artificial intelligence, bioinformatics, economics, robotics, and seismology. Moreover, we can introduce CBF sine trigonometric Dombi AOs, CBF sine trigonometric Einstein AOs, CBF sine trigonometric Aczel-Alsina AOs, and CBF sine trigonometric Frank AOs.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

Appendix

Appendix A

Proof. We demonstrate parts (i) and (iii), and analogous demonstrations can be made for the other parts.

(i) For convenience, let $\Upsilon_{l_i} = \sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_i}\right)$, $\Upsilon_{u_i} = \sin\left(\frac{\pi}{2}\mathcal{A}_{u\mathfrak{W}_i}\right)$, $\Xi_{l_i} = \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{l\mathfrak{W}_i})\right)$, $\Xi_{u_i} = \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{u\mathfrak{W}_i})\right)$, $\Upsilon_i = \sin\left(\frac{\pi}{2}\mathcal{A}_{l\mathfrak{W}_i}\right)$ and $\Xi_i = \sin\left(\frac{\pi}{2}(1+\mathcal{B}_{\mathfrak{W}_i})\right)$, where $i = 1,2$. Now, utilizing STOLs of CBFNs under P-order, we have

$$
\begin{aligned}
\gamma(\sin\mathfrak{W}_1 \oplus \sin\mathfrak{W}_2) &= \left\{\begin{array}{c}
\left[1-\left(1-\Upsilon_{l_1}\right)^{\gamma}\left(1-\Upsilon_{l_2}\right)^{\gamma}, 1-\left(1-\Upsilon_{u_1}\right)^{\gamma}\left(1-\Upsilon_{u_2}\right)^{\gamma}\right] \\
\left[-\left(-\left(\Xi_{l_1}-1\right)\right)^{\gamma}\left(-\left(\Xi_{l_2}-1\right)\right)^{\gamma}, -\left(-\left(\Xi_{u_1}-1\right)\right)^{\gamma}\left(-\left(\Xi_{u_2}-1\right)\right)^{\gamma}\right], \\
\left\{1-\left(1-\Upsilon_1\right)^{\gamma}\left(1-\Upsilon_2\right)^{\gamma}, -\left(-\left(\Xi_1-1\right)\right)^{\gamma}\left(-\left(\Xi_2-1\right)\right)^{\gamma}\right\}
\end{array}\right\} \\
&= \left\{\begin{array}{c}
\left[1-\left(1-\Upsilon_{l_1}\right)^{\gamma}, 1-\left(1-\Upsilon_{u_1}\right)^{\gamma}\right] \\
\left[-\left(-\left(\Xi_{l_1}-1\right)\right)^{\gamma}, -\left(-\left(\Xi_{u_1}-1\right)\right)^{\gamma}\right], \\
\left\{1-\left(1-\Upsilon_1\right)^{\gamma}, -\left(-\left(\Xi_1-1\right)\right)^{\gamma}\right\}
\end{array}\right\} \oplus_{\mathrm{P}} \\
&\quad \left\{\begin{array}{c}
\left[1-\left(1-\Upsilon_{l_2}\right)^{\gamma}, 1-\left(1-\Upsilon_{u_2}\right)^{\gamma}\right] \\
\left[-\left(-\left(\Xi_{l_2}-1\right)\right)^{\gamma}, -\left(-\left(\Xi_{u_2}-1\right)\right)^{\gamma}\right], \\
\left\{1-\left(1-\Upsilon_2\right)^{\gamma}, -\left(-\left(\Xi_2-1\right)\right)^{\gamma}\right\}
\end{array}\right\} \\
&= \gamma\sin\mathfrak{W}_1 \oplus_{\mathrm{P}} \gamma\sin\mathfrak{W}_2.
\end{aligned}
$$

(iii) For $\gamma_1, \gamma_2 > 0$, we get

$$
\begin{aligned}
\gamma_1\sin\mathfrak{W}_1 \oplus_{\mathrm{P}} \gamma_2\sin\mathfrak{W}_1 &= \left\{\begin{array}{c}
\left[1-\left(1-\Upsilon_{l_1}\right)^{\gamma_1}, 1-\left(1-\Upsilon_{u_1}\right)^{\gamma_1}\right] \\
\left[-\left(-\left(\Xi_{l_1}-1\right)\right)^{\gamma_1}, -\left(-\left(\Xi_{u_1}-1\right)\right)^{\gamma_1}\right], \\
\left\{1-\left(1-\Upsilon_1\right)^{\gamma_1}, -\left(-\left(\Xi_1-1\right)\right)^{\gamma_1}\right\}
\end{array}\right\} \oplus_{\mathrm{P}} \\
&\quad \left\{\begin{array}{c}
\left[1-\left(1-\Upsilon_{l_1}\right)^{\gamma_2}, 1-\left(1-\Upsilon_{u_1}\right)^{\gamma_2}\right] \\
\left[-\left(-\left(\Xi_{l_1}-1\right)\right)^{\gamma_2}, -\left(-\left(\Xi_{u_1}-1\right)\right)^{\gamma_2}\right], \\
\left\{1-\left(1-\Upsilon_1\right)^{\gamma_2}, -\left(-\left(\Xi_1-1\right)\right)^{\gamma_2}\right\}
\end{array}\right\} \\
&= \left\{\begin{array}{c}
\left[1-\left(1-\Upsilon_{l_1}\right)^{\gamma_1+\gamma_2}\left(1-\Upsilon_{l_2}\right)^{\gamma_1+\gamma_2},\right. \\
\left.1-\left(1-\Upsilon_{u_1}\right)^{\gamma_1+\gamma_2}\left(1-\Upsilon_{u_2}\right)^{\gamma_1+\gamma_2}\right] \\
\left[-\left(-\left(\Xi_{l_1}-1\right)\right)^{\gamma_1+\gamma_2}\left(-\left(\Xi_{l_2}-1\right)\right)^{\gamma_1+\gamma_2},\right. \\
\left.-\left(-\left(\Xi_{u_1}-1\right)\right)^{\gamma_1+\gamma_2}\left(-\left(\Xi_{u_2}-1\right)\right)^{\gamma_1+\gamma_2}\right], \\
\left\{1-\left(1-\Upsilon_1\right)^{\gamma_1+\gamma_2}\left(1-\Upsilon_2\right)^{\gamma_1+\gamma_2},\right. \\
\left.-\left(-\left(\Xi_1-1\right)\right)^{\gamma_1+\gamma_2}\left(-\left(\Xi_2-1\right)\right)^{\gamma_1+\gamma_2}\right\}
\end{array}\right\} \\
&= (\gamma_1+\gamma_2)\sin\mathfrak{W}_1.
\end{aligned}
$$

□

Appendix B

Proof. We corroborate it using induction on n. To begin, consider $n = 2$.

$$\begin{aligned}
\text{P-}CBFSTWA(\mathfrak{W}_1, \mathfrak{W}_2) &= \upsilon_1 \sin \mathfrak{W}_1 \oplus_{\text{P}} \upsilon_2 \sin \mathfrak{W}_2 \\
&= \left\{ \begin{array}{c} \left[1 - \left(1 - \Upsilon_{l_1}\right)^{\upsilon_1}, 1 - \left(1 - \Upsilon_{u_1}\right)^{\upsilon_1}\right] \\ \left[- \left(- \left(\Xi_{l_1} - 1\right)\right)^{\upsilon_1}, -\left(- \left(\Xi_{u_1} - 1\right)\right)^{\upsilon_1}\right], \\ \left\{1 - \left(1 - \Upsilon_1\right)^{\upsilon_1}, -\left(- \left(\Xi_1 - 1\right)\right)^{\upsilon_1}\right\} \end{array} \right\} \oplus_{\text{P}} \\
&\quad \left\{ \begin{array}{c} \left[1 - \left(1 - \Upsilon_{l_2}\right)^{\upsilon_2}, 1 - \left(1 - \Upsilon_{u_2}\right)^{\upsilon_2}\right] \\ \left[- \left(- \left(\Xi_{l_2} - 1\right)\right)^{\upsilon_2}, -\left(- \left(\Xi_{u_2} - 1\right)\right)^{\upsilon_2}\right], \\ \left\{1 - \left(1 - \Upsilon_2\right)^{\upsilon_2}, -\left(- \left(\Xi_2 - 1\right)\right)^{\upsilon_2}\right\} \end{array} \right\} \\
&= \left\{ \begin{array}{c} \left[1 - \prod_{i=1}^{2} \left(1 - \Upsilon_{l_i}\right)^{\upsilon_i}, 1 - \prod_{i=1}^{2} \left(1 - \Upsilon_{u_i}\right)^{\upsilon_i}\right], \\ \left[- \prod_{i=1}^{2} \left(- \left(\Xi_{l_i} - 1\right)\right)^{\upsilon_i}, - \prod_{i=1}^{2} \left(- \left(\Xi_{u_i} - 1\right)\right)^{\upsilon_i}\right], \\ \left\{1 - \prod_{i=1}^{2} \left(1 - \Upsilon_i\right)^{\upsilon_i}, - \prod_{i=1}^{2} \left(- \left(\Xi_i - 1\right)\right)^{\upsilon_i}\right\} \end{array} \right\}.
\end{aligned}$$

Thus, the theorem is valid for $n = 2$. Now, suppose that the theorem holds for $n = k, k \in N$.

$$\begin{aligned}
\text{P-}CBFSTWA(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_k) &= \upsilon_1 \sin \mathfrak{W}_1 \oplus_{\text{P}} \upsilon_2 \sin \mathfrak{W}_2 \oplus_{\text{P}} ... \oplus_{\text{P}} \upsilon_k \sin \mathfrak{W}_k \\
&= \left\{ \begin{array}{c} \left[1 - \prod_{i=1}^{k} \left(1 - \Upsilon_{l_i}\right)^{\upsilon_i}, 1 - \prod_{i=1}^{k} \left(1 - \Upsilon_{u_i}\right)^{\upsilon_i}\right], \\ \left[- \prod_{i=1}^{k} \left(- \left(\Xi_{l_i} - 1\right)\right)^{\upsilon_i}, - \prod_{i=1}^{k} \left(- \left(\Xi_{u_i} - 1\right)\right)^{\upsilon_i}\right], \\ \left\{1 - \prod_{i=1}^{k} \left(1 - \Upsilon_i\right)^{\upsilon_i}, - \prod_{i=1}^{k} \left(- \left(\Xi_i - 1\right)\right)^{\upsilon_i}\right\} \end{array} \right\}.
\end{aligned}$$

We demonstrate that the theorem holds valid when $n = k + 1$. That is,

$$\begin{aligned}
\text{P-}CBFSTWA(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_k, \mathfrak{W}_{k+1}) &= \upsilon_1 \sin \mathfrak{W}_1 \oplus_{\text{P}} \upsilon_2 \sin \mathfrak{W}_2 \oplus_{\text{P}} ... \oplus_{\text{P}} \upsilon_k \sin \mathfrak{W}_k \oplus_{\text{P}} \upsilon_{k+1} \sin \mathfrak{W}_{k+1} \\
&= \left\{ \begin{array}{c} \left[1 - \prod_{i=1}^{k} \left(1 - \Upsilon_{l_i}\right)^{\upsilon_i}, 1 - \prod_{i=1}^{k} \left(1 - \Upsilon_{u_i}\right)^{\upsilon_i}\right], \\ \left[- \prod_{i=1}^{k} \left(- \left(\Xi_{l_i} - 1\right)\right)^{\upsilon_i}, - \prod_{i=1}^{k} \left(- \left(\Xi_{u_i} - 1\right)\right)^{\upsilon_i}\right], \\ \left\{1 - \prod_{i=1}^{k} \left(1 - \Upsilon_i\right)^{\upsilon_i}, - \prod_{i=1}^{k} \left(- \left(\Xi_i - 1\right)\right)^{\upsilon_i}\right\} \end{array} \right\} \oplus_{\text{P}} \\
&\quad \left\{ \begin{array}{c} \left[1 - \left(1 - \Upsilon_{l_1}\right)^{\upsilon_{k+1}}, 1 - \left(1 - \Upsilon_{u_1}\right)^{\upsilon_{k+1}}\right] \\ \left[- \left(- \left(\Xi_{l_1} - 1\right)\right)^{\upsilon_{k+1}}, -\left(- \left(\Xi_{u_1} - 1\right)\right)^{\upsilon_{k+1}}\right], \\ \left\{1 - \left(1 - \Upsilon_1\right)^{\upsilon_{k+1}}, -\left(- \left(\Xi_1 - 1\right)\right)^{\upsilon_{k+1}}\right\} \end{array} \right\}
\end{aligned}$$

$$= \left\{ \begin{array}{c} \left[1 - \prod_{i=1}^{k+1}\left(1 - \Upsilon_{l_i}\right)^{\upsilon_i}, 1 - \prod_{i=1}^{k+1}\left(1 - \Upsilon_{u_i}\right)^{\upsilon_i}\right], \\ \left[-\prod_{i=1}^{k+1}\left(-\left(\Xi_{l_i} - 1\right)\right)^{\upsilon_i}, -\prod_{i=1}^{k+1}\left(-\left(\Xi_{u_i} - 1\right)\right)^{\upsilon_i}\right], \\ \left\{1 - \prod_{i=1}^{k+1}\left(1 - \Upsilon_i\right)^{\upsilon_i}, -\prod_{i=1}^{k+1}\left(-\left(\Xi_i - 1\right)\right)^{\upsilon_i}\right\} \end{array} \right\}.$$

Hence, the assumption is true for $n = k + 1$. This indicates that the theorem must be valid $\forall n \in N$. □

Appendix C

Proof. (i) Let $\mathfrak{W}_i = \mathfrak{W} = \langle[\mathcal{A}_{l\mathfrak{W}}, \mathcal{A}_{u\mathfrak{W}}], [\mathcal{B}_{l\mathfrak{W}}, \mathcal{B}_{u\mathfrak{W}}], \{\mathcal{A}_{\mathfrak{W}}, \mathcal{B}_{\mathfrak{W}}\}\rangle$, $\forall i = 1, 2, ..., n$, and then employing Eq (4.2) yields

$$\text{P-}CBFSTWA(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \left\{ \begin{array}{c} \left[1 - \prod_{i=1}^{n}\left(1 - \Upsilon_{l_i}\right)^{\upsilon_i}, 1 - \prod_{i=1}^{n}\left(1 - \Upsilon_{u_i}\right)^{\upsilon_i}\right], \\ \left[-\prod_{i=1}^{n}\left(-\left(\Xi_{l_i} - 1\right)\right)^{\upsilon_i}, -\prod_{i=1}^{n}\left(-\left(\Xi_{u_i} - 1\right)\right)^{\upsilon_i}\right], \\ \left\{1 - \prod_{i=1}^{n}\left(1 - \Upsilon_i\right)^{\upsilon_i}, -\prod_{i=1}^{n}\left(-\left(\Xi_i - 1\right)\right)^{\upsilon_i}\right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left[1 - \left(1 - \Upsilon_l\right)^{\sum_{i=1}^{n}\upsilon_i}, 1 - \left(1 - \Upsilon_u\right)^{\sum_{i=1}^{n}\upsilon_i}\right], \\ \left[-\left(-\left(\Xi_l - 1\right)\right)^{\sum_{i=1}^{n}\upsilon_i}, -\left(-\left(\Xi_u - 1\right)\right)^{\sum_{i=1}^{n}\upsilon_i}\right], \\ \left\{1 - \left(1 - \Upsilon\right)^{\sum_{i=1}^{n}\upsilon_i}, -\left(-\left(\Xi - 1\right)\right)^{\sum_{i=1}^{n}\upsilon_i}\right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left[1 - \left(1 - \Upsilon_l\right), 1 - \left(1 - \Upsilon_u\right)\right], \\ \left[-\left(-\left(\Xi_l - 1\right)\right), -\left(-\left(\Xi_u - 1\right)\right)\right], \\ \left\{1 - \left(1 - \Upsilon\right), -\left(-\left(\Xi - 1\right)\right)\right\} \end{array} \right\}$$

$$= \sin \mathfrak{W}.$$

(ii) If $\mathfrak{W}_i \leq_{\text{P}} \mathfrak{W}_i^*$, then $[\mathcal{A}_{l\mathfrak{W}_i}, \mathcal{A}_{u\mathfrak{W}_i}] \leq [\mathcal{A}_{l\mathfrak{W}_i^*}, \mathcal{A}_{u\mathfrak{W}_i^*}]$, $[\mathcal{B}_{l\mathfrak{W}_i}, \mathcal{B}_{u\mathfrak{W}_i}] \geq [\mathcal{B}_{l\mathfrak{W}_i^*}, \mathcal{B}_{u\mathfrak{W}_i^*}]$, $\mathcal{A}_{\mathfrak{W}_i} \leq \mathcal{A}_{\mathfrak{W}_i^*}$ and $\mathcal{B}_{\mathfrak{W}_i} \geq \mathcal{B}_{\mathfrak{W}_i^*}$, $\forall i = 1, 2, ..., n$. Assume that P-$CBFSTWA(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) = \langle[\tilde{\mathcal{A}}_l, \tilde{\mathcal{A}}_u], [\tilde{\mathcal{B}}_l, \tilde{\mathcal{B}}_u], \{\tilde{\mathcal{A}}, \tilde{\mathcal{B}}\}\rangle$, and P-$CBFSTWA(\mathfrak{W}_1^*, \mathfrak{W}_2^*, ..., \mathfrak{W}_n^*) = \langle[\hat{\mathcal{A}}_l, \hat{\mathcal{A}}_u], [\hat{\mathcal{B}}_l, \hat{\mathcal{B}}_u], \{\hat{\mathcal{A}}, \hat{\mathcal{B}}\}\rangle$. As the sine function is monotonic, we obtain

$$\Upsilon_{l_i} \leq \Upsilon_{l_i}^*$$

$$\Rightarrow 1 - \Upsilon_{l_i} \geq 1 - \Upsilon_{l_i}^*$$

$$\Rightarrow \left(1 - \Upsilon_{l_i}\right)^{\upsilon_i} \geq \left(1 - \Upsilon_{l_i}^*\right)^{\upsilon_i}$$

$$\Rightarrow \prod_{i=1}^{n}\left(1 - \Upsilon_{l_i}\right)^{\upsilon_i} \geq \prod_{i=1}^{n}\left(1 - \Upsilon_{l_i}^*\right)^{\upsilon_i}$$

$$\Rightarrow \tilde{\mathcal{A}}_l = 1 - \prod_{i=1}^{n}\left(1 - \Upsilon_{l_i}\right)^{\upsilon_i} \leq 1 - \prod_{i=1}^{n}\left(1 - \Upsilon_{l_i}^*\right)^{\upsilon_i} = \hat{\mathcal{A}}_l.$$

Also, $\tilde{\mathcal{A}}_u \leq \hat{\mathcal{A}}_u$. This shows that $[\tilde{\mathcal{A}}_l, \tilde{\mathcal{A}}_u] \leq [\hat{\mathcal{A}}_l, \hat{\mathcal{A}}_u]$. Likewise, $\tilde{\mathcal{A}} \leq \hat{\mathcal{A}}$. Now,

$$\Xi_{l_i} \geq \Xi^*_{l_i}$$

$$\Rightarrow \Xi_{l_i} - 1 \geq \Xi^*_{l_i} - 1$$

$$\Rightarrow -\left(\Xi_{l_i} - 1\right) \leq -\left(\Xi^*_{l_i} - 1\right)$$

$$\Rightarrow \left(- \left(\Xi_{l_i} - 1\right)\right)^{\upsilon_i} \leq \left(- \left(\Xi^*_{l_i} - 1\right)\right)^{\upsilon_i}$$

$$\Rightarrow \prod_{i=1}^{n} \left(- \left(\Xi_{l_i} - 1\right)\right)^{\upsilon_i} \leq \prod_{i=1}^{n} \left(- \left(\Xi^*_{l_i} - 1\right)\right)^{\upsilon_i}$$

$$\Rightarrow \tilde{\mathcal{B}}_l = - \prod_{i=1}^{n} \left(- \left(\Xi_{l_i} - 1\right)\right)^{\upsilon_i} \geq - \prod_{i=1}^{n} \left(- \left(\Xi^*_{l_i} - 1\right)\right)^{\upsilon_i} = \hat{\mathcal{B}}_l.$$

In an analogous manner, it can be proven that $\tilde{\mathcal{B}}_u \geq \hat{\mathcal{B}}_u$ which gives $[\tilde{\mathcal{B}}_l, \tilde{\mathcal{B}}_u] \geq [\hat{\mathcal{B}}_l, \hat{\mathcal{B}}_u]$. Moreover, $\tilde{\mathcal{B}} \geq \hat{\mathcal{B}}$. Hence, we conclude that

$$\text{P-}CBFSTWA(\mathfrak{W}_1, \mathfrak{W}_2, ..., \mathfrak{W}_n) \leq_{\text{P}} \text{P-}CBFSTWA(\mathfrak{W}^*_1, \mathfrak{W}^*_2, ..., \mathfrak{W}^*_n).$$

(iii) We omit it. □

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