Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 15232–15254.
DOI: 10.3934/math.2023778
Received: 27 December 2022
Revised: 17 March 2023
Accepted: 03 April 2023
Published: 25 April 2023

Research article

Qualitative results for a relativistic wave equation with multiplicative noise and damping terms

Hatice Taskesen*

Department of Mathematics, Van Yuzuncu Yil University, 65080, Van, Turkey

* **Correspondence:** Email: haticetaskesen@yyu.edu.tr.

Abstract: Wave equations describing a wide variety of wave phenomena are commonly seen in mathematical physics. The inclusion of a noise term in a deterministic wave equation allows neglected degrees of freedom or fluctuations of external fields describing the environment to be considered in the equation. Moreover, adding a noise term to the deterministic equation reveals remarkable new features in the qualitative behavior of the solution. For example, noise can lead to singularities in some equations and prevent singularities in others. Taking into account the effects of the fluctuations along with a space-time white noise, we consider a relativistic wave equation with weak and strong damping terms and investigate the effect of multiplicative noise on the behavior of solutions. The existence of local and global solutions is provided, and some qualitative properties of solutions, such as continuous dependence of solutions on initial data, and blow up of solutions, are given. Moreover, an upper bound is provided for the blow up time.

Keywords: relativistic wave equation; local existence; blow up; energy inequality; noise
Mathematics Subject Classification: 35A01, 35B44, 35Q40, 60H15

1. Introduction

Stochastic evolution equations (SEE) have been used as an effective tool for spatio-temporal biological, chemical and physical systems under random effects. Interest in such equations has increased recently due to their success in describing systems that include uncertainty. The randomness in these equations can appear in the form of uncertain parameters, random sources, fluctuating forces and random boundary conditions and take the form of additive or multiplicative noise. The multiplicative noise, in which the noise intensity depends on the variables of the equation, can result from adiabatically eliminating the fast variables of a system or allowing a parameter in a phenomenological equation of motion to be a random variable with predetermined statistics when modeling an undulating environment or superimposed external fluctuation. The multiplicative noise has been considered experimentally in modeling some physical phenomena, such as liquid crystals, electrical circuits and chemical reactions [1]. It is worth examining SEE with multiplicative noise since they can exhibit a rich dynamical behavior, although multiplicative noise leads to analytical and statistical difficulties. Inspired by this fact, we investigate the qualitative behavior of the following equation with space-time multiplicative white noise:

$$du_t + [\alpha u_t + \lambda u - \Delta u - \Delta u_t]\, dt = \kappa(u)\, dt + h(u, u_t, \nabla u)\, dW(x,t), \quad x \in \mathcal{D}, \quad t > 0, \tag{1.1}$$

and initial and boundary conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathcal{D}, \tag{1.2}$$

$$u(x,t) = 0, \quad x \in \partial\mathcal{D}, \quad t > 0. \tag{1.3}$$

Here, $\mathcal{D} \subset \mathbb{R}^d$, $d \geq 1$ is a bounded domain, $W(x,t)$ is a Wiener process, u_t and Δu_t are damping terms, and $\kappa(u)$ is a nonlinear source term. The term $-\Delta u_t$ (strong damping) arises in modeling the motion of viscoelastic materials, such as transverse vibrations of a homogeneous string and longitudinal vibrations of a homogeneous rod with viscous effects. The weak damping term αu_t, with $\alpha > 0$, occurs when there exists dynamical friction (for more detailed information on physical applications, see [2]).

When $h = 0$, in the absence of damping terms, Eq (1.1) is a relativistic wave equation describing propagation of spinless particles (also known as a Klein-Gordon equation). It was proposed as a relativistic generalization of the Schrdinger equation. In 1966, Nelson derived the nonrelativistic Schrdinger equation with a stochastic model including Newton's second law. In [3], Lehr and Park extended Nelson's work to the relativistic case and derived a relativistic wave equation using elements of stochastic mechanics. Arbab [4] showed that fluctuations of the charge densities behave as a quantum particle governed by the Klein-Gordon equation. The Klein-Gordon equation also arises in the modeling of dislocations in crystals and has been considered from different perspectives in numerous papers [5–9].

For the deterministic case, in the absence of damping terms, the Cauchy problem for Eq (1.1) is known to have solutions locally in time if κ satisfies some local Lipschitz conditions. If $\int_0^u \kappa(s)ds < 0$, then all solutions associated with the Cauchy problem of the deterministic form of Eq (1.1) without damping terms exist globally, but in some cases (for example, if $\kappa(s) = |s|^{p-1} s$, $p > 1$) there exist solutions blowing up in a finite time [10]. A deterministic form of Eq (1.1) with damping terms is also investigated in [5, 7, 11, 12].

The solution's behavior for stochastic wave equations with or without damping terms has received much attention from a mathematical viewpoint [13–23]. Chow [13] investigated local and global solutions of the following stochastic wave equation

$$\partial_t^2 v = \nabla^2 v + f(v) + \sigma(v)\partial_t W(t,x), \quad x \in \mathbb{R}^d, t > 0,$$

with a polynomial nonlinearity in some Sobolev spaces for $d \leq 3$. The same author also studied global nonexistence of solutions for stochastic wave equations that have no damping terms [16,24]. Chow also considered in [15] a stochastic weakly damped wave equation. The solution's global existence and the equilibrium solution's exponential stability were proved in [15]. Moreover, the conditions providing the existence of a unique invariant measure were given. Wang and Zhou [25, 26] and Wang et al. [27] considered the asymptotic behavior of solutions for the following strongly damped wave equations:

$$v_{tt} - \alpha\Delta v_t + v_t + f(v) - \Delta v = g + cv \circ \frac{\mathrm{d}W}{\mathrm{d}t}, \tag{1.4}$$

$$v_{tt} - \Delta v_t + \alpha v_t - \Delta v + \lambda v + f(x, v) = g(x) + \sum_{j=1}^{m} h_j(x)\frac{\mathrm{d}W_j}{\mathrm{d}t}, \tag{1.5}$$

$$dv_t + dv + (f(v) - \Delta v - \alpha\Delta v_t)\, dt = gdt + \sum_{j=1}^{m} h_j dW_j, \tag{1.6}$$

respectively. Equation (1.4) was considered on a bounded domain with multiplicative noise of the Stratonovich sense and Dirichlet boundary condition. The asymptotic behavior of Eq (1.5) was studied on an unbounded domain with additive noise. To tackle the lack of compact Sobolev embedding on unbounded domains, the authors used some uniform estimates together with a splitting technique when investigating the global attractor for Eq (1.5). Equation (1.6) was studied on a bounded domain with Neumann boundary condition and additive noise. Jones and Wang [28] investigated the wave equation

$$u_{tt} + \mu u_t + \beta u - \Delta u - \Delta u_t - \lambda\Delta u_{tt} + h(x, u) = \varphi(x) + \sigma(x)\frac{dW(t)}{dt} \quad x \in \mathbb{R}^d, \quad t > 0, \tag{1.7}$$

with both weak and strong damping. They converted the equation into a deterministic system with random parameters. Using uniform estimates and a cut-off method, the asymptotic behavior of solutions for the above wave equation was studied in [28]. Equation (1.7) was also studied in [29–31]. Existence, uniqueness and asymptotic robustness of pullback random attractors for (1.7) with operator type noise was studied in [29], where the stochastic integral was taken in the sense of Stratonovich. In [30], global existence and random dynamics of (1.7) with infinite-dimensional nonlinear noise were handled. Some time-related features of the random attractor were investigated in [31] on an unbounded domain. Bo et al. [22] investigated explosive solutions of the damped wave equation

$$v_{tt} - \Delta v + Av_t = \mu\, |v|^p\, v + f(v, v_t, Du)\frac{\partial}{\partial t}W(t, x), \tag{1.8}$$

where a particular case of nonlinear term ($\alpha|v|^p v$) was taken. The Cauchy problem of (1.8) was considered separately for weak and strong damping terms. First, the local existence results were provided, and then the conditions ensuring the explosion of solutions were given for weak and strong damping terms, respectively. Although it was stated that a multiplicative noise was considered, the noise was taken as additive while examining the explosion of the solutions. Parshad et al. [19] studied global existence and blow up of solutions for a class of nonlinearly damped stochastic wave equation. They showed that in the case of large initial data and the domination of the source term on the damping term, solutions of their problem blow up in finite time in the mean L^p norm. Some numerical simulations were also performed in [19] to verify the theoretical results.

In this paper, we are concerned with blow up of solutions of (1.1) and (1.2). As in the deterministic case, blow up means for the stochastic case that trajectories may tend to infinity when time approaches a finite time T^* which mostly depends on some certain sample path [32]. This work differs from [22] in two aspects. First, unlike [22], a general source term is used, and both strong and weak damping terms are handled together. Second, the noise term is taken as a multiplicative noise. To the author's knowledge, the blow up of solutions for the stochastic relativistic wave Eq (1.1) with a general nonlinear term $f(u)$ and double dissipative (damping) terms has not been investigated so far. The proof of our main result is based on a "concavity method" which mainly defines a nonnegative functional $\theta(t)$ that includes the L^2-norms of the solution with some additional terms and does not need the positivity of the solution.

The paper is organized as follows: In the second section, we give the notations, spaces and lemmas to be used throughout the paper. In Section 3, we impose some conditions on nonlinear terms κ and h to guarantee the existence of solutions in a finite time interval, and then we extend the existence to an infinite time interval by an energy inequality. We also prove the continuous dependence of solutions on initial data. In the fourth section, we state our main result about the blow up of the solutions for the problems (1.1) and (1.2) via a differential inequality and provide an upper bound for the finite blow up time.

2. Preliminaries

Let $L^p(\mathcal{D})$ indicate the class of all measurable functions u on $\mathcal{D}$ for which $\int_{\mathcal{D}} |u(x)|^p \, dx < \infty$, endowed with the norm $\|\cdot\|_{L^p}$. In the case of $p = 2$, $L^2(\mathcal{D})$ corresponds to a Hilbert space with the inner product $(\cdot,\cdot)$, and we denote its norm by $\|\cdot\|$. Let $W^{m,p}(\mathcal{D})$ be the $m-th$ order Sobolev space,

$$W^{m,p}(\mathcal{D}) = \{u \in L^p(\mathcal{D}) : D^\alpha u \in L^p(\mathcal{D}), \ \ 0 \le |\alpha| \le m\},$$

furnished with the inner product and the norm given by

$$(u,v)_m = \sum_{0\le|\alpha|\le m} (D^\alpha u, D^\alpha v)$$

and

$$\|u\|_{W^{m,p}(\mathcal{D})} = \left(\sum_{0\le|\alpha|\le m} \|D^\alpha u\|_{L^p}^p \right)^{1/p},$$

respectively, where $m = 1,2,3,...$. Here, $D^\alpha u$ denotes the weak derivative. For $p = 2$, the Sobolev space $W^{m,2}(\mathcal{D})$ is demonstrated by $H^m(\mathcal{D})$. It is obvious that $H^0 = L^2$, and it is denoted by H. The norm of $H_0^1(\mathcal{D})$ is denoted by $\|\cdot\|_1$.

Throughout the paper, we use the space $\mathcal{H} := H_0^1 \times H$ that is equipped with the norm

$$\|\phi\|_{\mathcal{H}} = \left\{ \|u\|_1^2 + \|v\|^2 \right\}^{1/2}$$

for any $\phi = (u,v) \in \mathcal{H}$.

Assume that $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t, \ t \ge 0\}$ of increasing sub $\sigma-$fields is a complete probability space. Let $W(x,t)$, $x \in \mathcal{D}$, $t \ge 0$ be an $H-$valued $R-$Wiener process, where R is a covariance operator. $W(x,t)$ has mean zero, and its covariance function is given by $EW(x,t)\,W(y,s) = (t \wedge s) r(x,y)$, where $(t \wedge s) = min(t,s)$, $r(x,y)$, $x, y \in G$, is a spatial correlation function that is assumed to be bounded and continuous for bounded domains, and $\int_G r(x,x) < \infty$ for unbounded domains G. The operator R is self-adjoint, compact and of trace class with $TrR < \infty$. For problems (1.1) and (1.2), $TrR = \int_G r(x,x)h^2(u,v,\nabla u)dx$. Note also that R may be written in the following form:

$$Re_k = \lambda_k e_k, \tag{2.1}$$

where $\{\lambda_k\}$ is a bounded sequence of nonnegative eigenvalues of R satisfying $\sum_{k=1}^{\infty} \lambda_k < \infty$, and $\{e_k\}$ are the corresponding eigenfunctions with $c_* := \sup_{k\geq 1} \|e_k\|_\infty < \infty$ forming a complete orthonormal base in H [33]. In that case, $W(x,t)$ may be expanded as in the following form:

$$W(x,t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} B_k(t) e_k,$$

where $\{B_k(t)\}$ is a sequence of independent real-valued Brownian motions. Let us indicate that the above series is convergent in L^2 [33] since

$$E\|W\|^2 = E\left(\left\|\sum_{k=1}^{\infty} \sqrt{\lambda_k} B_k(t) e_k\right\|^2\right) = \sum_{k=1}^{\infty} \lambda_k E(B_k(t))^2 = t\sum_{k=1}^{\infty} \lambda_k = tTrR < \infty. \tag{2.2}$$

Problems (1.1) and (1.3) may be treated as a system as in the following form:

$$\begin{cases} du = vdt, \\ dv = ((-\lambda I + \Delta) u + (-\alpha I + \Delta) v + \kappa(u))dt + h(u,v,\nabla u)dW(x,t), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\ u(x,t) = 0. \end{cases} \tag{2.3}$$

We notify that $-\Delta$ generates a semigroup $S(t)$ in $\mathcal{H}$ that is strongly continuous. For the deterministic case, Bahuguna [34] handled the problem (1.1), (1.2) as a particular case of the following abstract equation:

$$u''(t) + (\alpha I + A) u'(t) + (\lambda I + A) u(t) = \kappa(t, u(t), u'(t)),$$

where $A = -\Delta$. As mentioned in [35] replacing the operator A by $\bar{A} = A + \gamma I$ for a proper constant $\gamma \in \mathbb{R}$ and rearranging the constants λ, α accordingly, it may be assumed that every complex number $\mu \in \mathbb{C}$ with $Re\mu > -r, \quad r > 0$, is included in the resolvent $\rho(-A)$ of the generator $-A$. For the local existence and uniqueness of mild solutions, the authors of [34] realized that, inspired by the work of Engler et al. [35], absorbing the terms $(\lambda I + A) u$ and $\alpha u'(t)$ into the source term κ does not change the character of the problem, and that it is adequate to study the following problem:

$$u''(t) + Au'(t) = \kappa(t, u(t), u'(t)),$$

where $-A$ generates a strongly continuous semigroup $S(t)$ on a Banach space E.

Without loss of generality, one may take $\alpha = \lambda = 1$ for the sake of simplicity. Then system (2.3) can be reduced to an Ito equation:

$$\begin{cases} d\mathcal{Z}(t) = (\Lambda\mathcal{Z}(t) + K(\mathcal{Z}(t)))\,dt + \Theta(\mathcal{Z}(t))dW(t), \\ \mathcal{Z}(0) = \mathcal{Z}_0 = (u_0, u_1)^T, \end{cases} \tag{2.4}$$

where

$$\mathcal{Z}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & I \\ -I + \Delta & -I + \Delta \end{pmatrix},$$

$$K(\mathcal{Z}(t)) = \begin{pmatrix} 0 \\ \kappa(u) \end{pmatrix}, \quad \Theta(\mathcal{Z}(t)) = \begin{pmatrix} 0 \\ h\,(u, v, \nabla u) \end{pmatrix}.$$

It is more appropriate to look for the existence of mild solutions for problem (2.4) rather than strong solutions that require too much regularity of solution. So we define a mild solution in the following.

Remark 2.1. *A mild solution is an $\mathcal{F}_t$ adapted $\mathcal{H} = H_0^1 \times H-$ valued process $\mathcal{Z}(t)$, $t \in [0, T]$ to (2.4) that is predictable and satisfies the following integral equation*

$$\mathcal{Z}(t) = e^{t\Lambda}\mathcal{Z}(0) + \int_0^t e^{(t-s)\Lambda}K(\mathcal{Z}(s))ds + \int_0^t e^{(t-s)\Lambda}\Theta(\mathcal{Z}(s))dW(s), \tag{2.5}$$

where Λ is the infinitesimal generator of the semigroup $\left\{e^{t\Lambda},\ t \geq 0\right\}$ in $\mathcal{H}$.

Lemma 2.1. *[36] For every $(u, v) \in \mathcal{H}$ and $1 < p \leq \frac{d}{(d-2)^+}$ there exist constants C, C_0 such that*

$$\|u\|_{L^{2p}} \leq C\,\|u\|_1\,,$$
$$\left\|u^{p-1}v\right\|_{L^2} \leq C_0\,\|u\|_1^{p-1}\,\|v\|_1\,,$$

where $(s)^+ = max\{0, s\}$.

Lemma 2.2. *[37] Assume that a twice-differentiable, positive function $\Upsilon(t)$ satisfies the inequality*

$$\Upsilon''(t)\Upsilon(t) - (1 + \varsigma)\left(\Upsilon'(t)\right)^2 \geq 0,$$

where $\varsigma > 0$. If $\Upsilon(0) > 0, \Upsilon'(0) > 0$, then $\Upsilon(t) \to \infty$ as $t \to t_* \leq t^* = \frac{\Upsilon(0)}{\varsigma\Upsilon'(0)}$.

3. Local and global existence of solutions

Let us define the fuctional $e(\cdot) : \mathcal{H} \to \mathbb{R}^+ = [0, \infty)$

$$e(t) = \|u\|_1^2 + \|v\|^2\,, \tag{3.1}$$

where $(u, v) \in \mathcal{H}$. Throughout the paper, "E" stands for expectation. In the following, by imposing some conditions on nonlinear terms, we obtain local and global in time existence of solutions for problem (2.4). We expand the local solution to the global solution, provided that the energy is bounded.

3.1. Existence of local solution

In this subsection, we will prove with the aid of a truncation technique that the solution exists locally if f and h satisfy certain conditions. While the Ito formula performs well for parabolic equations, it fails for the mild solutions of hyperbolic equations due to the required smoothness of solutions, so a truncation technique is used for the proof of the following theorem.

Theorem 3.1. *Assume that κ and h satisfies the following conditions:*

L1. *For* $u^1, u^2 \in H_0^1$

$$|\kappa(u)|^2 \leq C_1\left(1 + |u|^{2(p-1)}\right)|u|^2,$$

$$\left|\kappa\left(u^1\right) - \kappa\left(u^2\right)\right|^2 \leq C_2\left(1 + \left|u^1\right|^{2(p-1)} + \left|u^2\right|^{2(p-1)}\right)\left|u^1 - u^2\right|^2,$$

where $C_1, C_2 > 0$ *and* $p > 1$.

L2. $p \in (1, \infty)$, *if* $d = 1, 2$, *and* $1 < p \leq \frac{d}{(d-2)^+}$ *otherwise. Let* $h(\cdot) : [0, T] \to L(H)$ *be a continuous map for any* $h : \mathbb{R}^{d+2} \to \mathbb{R}$. *For any* $u^1, v^1, u^2, v^2 \in \mathbb{R}$, *there exist* $C_3, C_4 > 0$ *such that*

$$|h(u, v, \nabla u)|^2 \leq C_3\left(1 + |u|^{2p} + |v|^2 + |\nabla u|^2\right),$$

and

$$\left|h\left(u^1, v^1, \nabla u^1\right) - h\left(u^2, v^2, \nabla u^2\right)\right|^2 \leq C_4[(1+\left|u^1\right|^{2(p-1)}+\left|u^2\right|^{2(p-1)})\left|u^1 - u^2\right|^2+\left|v^1 - v^2\right|^2+\left|\nabla u^1 - \nabla u^2\right|^2].$$

L3. *W is a Wiener process with value in H-and with covariance operator R satisfying* $TrR < \infty$.

In that case for $u_0 \in H_0^1$ *and* $u_1 \in H$, *problem (2.4) has a unique local mild solution* (u, v) *in the energy space* $\mathcal{Z} = (u, v)^T \in C([0, T], \mathcal{H})$.

Proof. As we stated above, the proof of the theorem relies on a truncation technique. To this end, for $N > 0$ let us define a mollifier function $\eta_N(\cdot) : \mathbb{R}^+ = [0, \infty) \to [0, 1]$, which is a C^∞ function such that

$$\eta_N(s) = \begin{cases} 1, & |s| \leq N, \\ \in (0, 1), & N < |s| < N + 1, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.2}$$

and assume that $\|\eta_N\|_\infty \leq 2$. Introducing

$$K_N(\mathcal{Z}(t)) = \begin{pmatrix} 0 \\ \kappa_N(u) \end{pmatrix}, \quad \Theta_N(\mathcal{Z}(t)) = \begin{pmatrix} 0 \\ h_N(u, v, \nabla u) \end{pmatrix},$$

one can write the system (2.4) as

$$\begin{cases} d\mathcal{Z}(t) = \Lambda\mathcal{Z}(t)dt + K_N(\mathcal{Z}(t))dt + \Theta_N(\mathcal{Z}(t))dW(t), \\ \mathcal{Z}(0) = \mathcal{Z}_0 = (u_0, u_1)^T, \end{cases} \tag{3.3}$$

where $\kappa_N(u(t)) = \eta_N(\|u\|_1)\kappa(u)$ and $h_N(u, v, \nabla u, x, t) = \eta_N(\|u\|_1)h(u, v, \nabla u, x, t)$. Without any loss of generality, let us assume that $\left\|u^1\right\|_1 \leq \left\|u^2\right\|_1$. Using Hlder inequality and Lemma 2.1, we obtain

$$\begin{aligned} \left\|\kappa_N(u^1) - \kappa_N(u^2)\right\|^2 &= \left\|\eta_N(\left\|u^1\right\|_1)\kappa(u^1) - \eta_N(\left\|u^2\right\|_1)\kappa(u^2)\right\|^2 \\ &\leq \left\|\left(\eta_N\left(\left\|u^1\right\|_1\right) - \eta_N(\left\|u^2\right\|_1\right)\kappa(u^1)\right\|^2 \\ &+ \eta_N(\left\|u^1\right\|_1)\left\|\kappa(u^1) - \kappa(u^2)\right\|^2 \\ &\leq C_5\|\eta_N\|_\infty^2\left|\left\|u^1\right\|_1 - \left\|u^2\right\|_1\right|^2\left(1 + \left\|u^1\right\|_{2p}^{2p}\right)\chi_{\{\left\|u^2\right\|_1 \leq N+1\}} \end{aligned}$$

$$
\begin{aligned}
&+ C_6\eta_N\left(\left\|u^2\right\|_1\right)\left\|\left|u^1-u^2\right|(1+\left|u^1\right|^{p-1}+\left|u^2\right|^{p-1})\right\|^2\\
&\le C_5\left\|\eta_N\right\|_\infty^2\left|\left\|u^1\right\|_1-\left\|u^2\right\|_1\right|^2\left(1+\left\|u^1\right\|_{2p}^{2p}\right)\chi_{\{\|u^2\|_1\le N+1\}}\\
&+ C_6\eta_N(\left\|u^2\right\|_1)\left\|u^1-u^2\right\|_{L^{2p}}^2\left(\left\|u^1\right\|_{L^{2p}}^{2(p-1)}+\left\|u^2\right\|_{L^{2p}}^{2(p-1)}\right)\\
&\le 2C_5C^{2p}\left\|u^1-u^2\right\|_1^2\left\|1+u^1\right\|_1^{2p}\chi_{\{\|u^2\|_1\le N+1\}}\\
&+ 2C_6C^{2p}\eta_N(\left\|u^2\right\|_1)\left\|u^1-u^2\right\|_1^2\left(\left\|u^1\right\|_1^{2(p-1)}+\left\|u^2\right\|_1^{2(p-1)}\right)\\
&\le C_7(N,p)\left\|u^1-u^2\right\|_1^2,
\end{aligned}
\tag{3.4}
$$

where χ_B is the indicator function of B. The above inequality yields

$$
\|K_N(\mathcal{Z})-K_N(\mathcal{Z}')\|^2\le C_8(N,p)\,\|\mathcal{Z}-\mathcal{Z}'\|_{\mathcal{H}}^2\,. \tag{3.5}
$$

Similar to (3.4) and (3.5), we get

$$
\|K_N(\mathcal{Z})\|^2\le C_9(N,p)\left(1+\|\mathcal{Z}\|_{\mathcal{H}}^2\right). \tag{3.6}
$$

For $u,v\in\mathcal{H}$, using Lemma 2.1 and the assumptions L2 of Theorem 3.1, we have

$$
\begin{aligned}
Tr[h_N(u,v,\nabla u)Rh_N^*(u)] &= \sum_{i=1}^{\infty}(h_N(u,v,\nabla u)Re_i,h_N(u,v,\nabla u)e_i)\\
&= \sum_{i=1}^{\infty}\lambda_i\,(h_N(u,v,\nabla u)e_i,h_N(u,v,\nabla u)e_i)\\
&\le C_9\sum_{i=1}^{\infty}\lambda_i\eta_N^2(\|u\|_1)\int_{\mathcal{D}}e_i^2(x)[1+|u|^{2p}+v^2+|\nabla u|^2]dx\\
&\le C_9\sum_{i=1}^{\infty}\lambda_i\eta_N^2(\|u\|_1)\left[\|e_i\|^2+\sup_{i\ge1}\|e_i\|_\infty^2\left(\|u\|_{L^{2p}}^{2p}+\|v\|^2+\|\nabla u\|^2\right)\right]\\
&\le C_9\sum_{i=1}^{\infty}\lambda_i\eta_N^2(\|u\|_1)\left[1+\sup_{i\ge1}\|e_i\|_\infty^2\left(C^{2p}\|u\|_1^{2p}+\|v\|^2+\|\nabla u\|^2\right)\right],
\end{aligned}
$$

which yields

$$
Tr[\Theta_N(\mathcal{Z})R\Theta_N^*(\mathcal{Z})]\le C_{10}(N,p)(1+\|\mathcal{Z}\|_{\mathcal{H}}^2). \tag{3.7}
$$

Similar to the above computations, we get

$$
\begin{aligned}
&Tr\left[\left(h_N(u^1,v^1,\nabla u^1)-h_N(u^2,v^2,\nabla u^2)\right)R\left(h_N(u^1,v^1,\nabla u^1)-h_N(u^2,v^2,\nabla u^2)\right)^*\right]\\
&=\sum_{i=1}^{\infty}\lambda_i\left\|\left(h_N(u^1,v^1,\nabla u^1)-h_N(u^2,v^2,\nabla u^2)\right)e_i\right\|^2\\
&\le 2\sum_{i=1}^{\infty}\lambda_i\left\|\left(\eta_N(\left\|u^1\right\|_1)-\eta_N(\left\|u^2\right\|_1)\right)h(u^1,v^1,\nabla u^1)e_i\right\|^2
\end{aligned}
$$

$$+ 2\eta_N(\left\|u^2\right\|_1) \sup_{i\geq 1} \|e_i\|_\infty^2 \, TrR \left\|h(u^1, v^1, \nabla u^1) - h(u^2, v^2, \nabla u^2)\right\|^2$$
$$\leq 8\lambda_i \left\|u^1 - u^2\right\|_1^2 \left\|h(u^1, v^1, \nabla u^1)e_i\right\|^2 \chi_{\{\|u^2\|_1 \leq N+1\}} + C_{11} \sup_{i\geq 1} \|e_i\|_\infty^2 \, TrR\eta_N(\left\|u^2\right\|_1) \left[\|u^1 - u^2\|^2\right.$$
$$\left. + \left\||u^1|^{p-1} \left|u^1 - u^2\right|\right\|^2 + \left\||u^2|^{p-1} \left|u^1 - u^2\right|\right\|^2 + \left\|v^1 - v^2\right\|^2 + \left\|\nabla u^1 - \nabla u^2\right\|^2\right],$$

from which we conclude that

$$Tr\left[(\Theta_N(\mathcal{Z}) - \Theta_N(\mathcal{Z}'))\, R\, (\Theta_N(\mathcal{Z}) - \Theta_N(\mathcal{Z}'))^*\right] \leq C_{12}(N, p) \left\|\mathcal{Z} - \mathcal{Z}'\right\|_{\mathcal{H}}^2 . \tag{3.8}$$

The results obtained in (3.5)–(3.8) shows that the truncated system (3.3) fulfills the linear growth and global Lipschitz condition. Thus, we deduce from Theorem 7.2 of [33] that the truncated system (3.3) has a unique mild solution $\mathcal{Z}_N = (u_N, v_N) \in \mathcal{H}$ ($u_N := \eta_N(\|u\|_1)u$, $v_N := \eta_N(\|u\|_1)v$) for $t \in [0, T]$. If there exists $t > 0$, such that $\|u_N\|_1 > N$ for fixed number N, then we define a stopping time τ_N as

$$\tau_N = \inf\{t \geq 0 : \|u_N\|_1 \geq N\}.$$

Subsequently, for $t < \tau_N$, $\mathcal{Z} = \mathcal{Z}_N$ is the solution of (3.3). Since τ_N is nondecreasing in N, we can define $\zeta = \lim_{N\to\infty} \tau_N$ a.s. For $t < \zeta$, we have $t < \tau_N$ for some $N > 0$ and define the solution $u = u_N$. Moreover, $\lim_{t\to\zeta} \|u\|_1 = \infty$ if $\zeta < \tau$ and therefore u is the unique local solution.

3.2. Extending the local solution to the global solution

In this subsection, we construct a uniform bound on the functional defined as

$$e_k(t) = e(t) + 2k \|u\|_{p+1}^{p+1},$$

to prevent the unlimited growth of a solution.

Theorem 3.2. *Suppose that the conditions of Theorem 3.1 holds, and let us define $\Upsilon(u) = -\int_0^u \kappa(s)ds$ that satisfies for any $\beta > 0$*

$$\Upsilon(u) \geq \left(\beta + k|u|^{2p}\right)|u|^2 . \tag{3.9}$$

Then, under the above assumptions for any $T > 0$, problem (2.4) has a unique continuous solution $\mathcal{Z}$ that satisfies

$$E \sup_{0\leq t\leq T} e_k(t) < \infty. \tag{3.10}$$

Proof. To extend the local solution to a global one, we should first prove that for any finite time $T > 0$, there is a constant C_T depending on T, so that

$$Ee_k(u(t \wedge \tau_N)) \leq C_T, \tag{3.11}$$

where $u(t \wedge \tau_N)$ denotes the value of $u = u_N$ at time $t \wedge \tau_N$. Applying the Ito formula (Theorem 4.32 of [33]) gives

$$e(u(t \wedge \tau_N)) + 2\int_0^{t\wedge\tau_N} \|v(s)\|^2 ds + 2\int_0^{t\wedge\tau_N} \|\nabla v(s)\|^2 ds$$

$$= 2\int_{\mathcal{D}} \Upsilon(u(t\wedge\tau_N))dx - 2\int_{\mathcal{D}} \Upsilon(u(0))dx + e(u_0)$$
$$+ 2\int_0^{t\wedge\tau_N} (v, hdW)ds + \int_0^{t\wedge\tau_N} Tr[hRh^*]ds. \quad (3.12)$$

Employing $\Upsilon(u) \geq \left(\beta + k|u|^{2p}\right)|u|^2$, we have

$$e_k(u(t\wedge\tau_N))+\beta\|u(t\wedge\tau_N)\|^2 \leq e(u_0)-2\int_{\mathcal{D}} \Upsilon(u(0))dx+2\int_0^{t\wedge\tau_N}(v, hdW)ds+\int_0^{t\wedge\tau_N} Tr[hRh^*]ds. \quad (3.13)$$

Using the properties of the covariance operator R, we get

$$\int_0^t Tr[hRh^*]ds = \sum_{i=1}^{\infty}\int_0^t (hRe_i, he_i)\,ds \leq c_*^2 TrR\int_0^t \|h\|^2\,ds. \quad (3.14)$$

The above inequality together with (3.13) yields

$$e_k(u(t\wedge\tau_N)) + \beta\|u(t\wedge\tau_N)\|^2 \leq e(u_0) - 2\int_{\mathcal{D}} \Upsilon(u(0))dx$$
$$+ 2\int_0^{t\wedge\tau_N}(v, hdW)ds + Ct + M\int_0^{t\wedge\tau_N} e_k(u(t\wedge\tau_N))ds, \quad (3.15)$$

where C and M are constants. Considering positivity of β and taking the expected value of both sides of (3.15), we have

$$Ee_k(u(t\wedge\tau_N)) \leq Ee(u_0) - 2E\int_{\mathcal{D}} \Upsilon(u(0))dx + Ct + ME\int_0^{t\wedge\tau_N} e_k(u(t\wedge\tau_N))ds. \quad (3.16)$$

Now, using L1 of Theorem 3.1, we have

$$\left|\int_{\mathcal{D}} \Upsilon(u(0))dx\right| = \left|\int_{\mathcal{D}}\int_0^u \kappa(s)dsdx\right| \leq C\|u\|^{p+1} \leq \|u\|_1^{p+1} < \infty.$$

Taking into account the above inequality and applying the integral version of Gronwall's lemma to (3.16), we get

$$Ee_k(u(t\wedge\tau_N)) \leq \left(Ee_k(u_0) + 2E\int_{\mathcal{D}} \Upsilon(u(0))dx + CT\right)e^{MT} \leq C_T. \quad (3.17)$$

Now, we should prove that as $N \to \infty$, $\mathcal{Z}_N(t) = \mathcal{Z}(t\wedge\tau_N) \to \mathcal{Z}(t)$ a.s. for any $t \leq T$. To accomplish this, it is sufficient to demonstrate that $\tau_N \to \infty$ whenever $N \to \infty$ with a probability of one. To this end, we will make use of the Borel-Cantelli lemma. By definition of τ_N, one can write

$$Ee_k(u(t\wedge\tau_N)) \geq E\{\ell(\tau_N \leq T)e_k(u(t\wedge\tau_N))\}$$
$$\geq E\left\{\|u\|_1^2\,\ell(\tau_N \leq T)\right\}$$
$$\geq N^2 P\{\tau_N \leq T\},$$

where ℓ stands for indicator function. If the inequality (3.11) holds, then we obtain

$$P\{\tau_N \leq T\} \leq Ee_k(u(t \wedge \tau_N))/N^2 \leq C_T/N^2.$$

Making use of the Borel-Cantelli lemma, we come to the following conclusion

$$P\{\zeta \leq T\} = 0,$$

and, consequently, $P\{\zeta > T\} = 1$, i.e., $\zeta = \lim_{N\to\infty} \tau_N = \infty$ a.s. In this way, $\lim_{N\to\infty} u_N = u$ is the global solution on $[\zeta \wedge T]$, as claimed.

To confirm the bound (3.10) for the energy, for $N \to \infty$ we take the limit in (3.12)

$$e(u(t)) + 2\int_0^t \|v(s)\|^2 ds + 2\int_0^t \|\nabla v(s)\|^2 ds = e(u_0) + 2\int_{\mathcal{D}} \Upsilon(u(t))dx + 2\int_{\mathcal{D}} \Upsilon(u(0))dx$$
$$+ 2\int_0^t (v, hdW)ds + \int_0^t Tr[h(u,v)Rh^*(u,v)]ds.$$

Taking the expectation in the above inequality and using (3.9) gives

$$E\sup_{0\leq t\leq T} e_k(u(t) \leq Ee(u_0) + 2E\int_{\mathcal{D}} \Upsilon(u_0)dx + 2E\sup_{0\leq t\leq T}\int_0^t (v, hdW)ds + E\int_0^t Tr[h(u,v)Rh^*(u,v)]ds. \tag{3.18}$$

The first and second terms of r.h.s. of (3.18) are finite, i.e., $Ee(u_0) < \infty$, $2E\int \Upsilon(u_0)dx < \infty$. The term $\int_0^t (v, hdW)ds$ on the r.h.s. of (3.18) is a local martingale, for an estimation of its supremum, we use Burkholder-Davis-Gundy inequality. For any constants $M_0, M_1 > 0$, we have

$$E\sup_{0\leq t\leq T}\left|\int_0^t (v, hdW)ds\right| \leq M_0 E\left[\sup_{0\leq t\leq T}\|v\|^2\left(\int_0^t Tr[h(u,v)Rh^*(u,v)]ds\right)^{1/2}\right]$$
$$\leq \frac{1}{2}E\left[\sup_{0\leq t\leq T}\|v\|^2\right] + M_0 E\int_0^t Tr[h(u,v)Rh^*(u,v)]ds$$
$$\leq E\left[\sup_{0\leq t\leq T}\frac{1}{2}\|v\|^2\right] + M_1.$$

where use was made of (3.14). Using the above inequality in (3.18), we get

$$E\sup_{0\leq t\leq T} e_k(u(t)) \leq M_2 + M_3\int_0^t E\sup_{0\leq t\leq T} e_k(u(t))ds, \tag{3.19}$$

which, together with the Gronwall inequality, yields

$$E \sup_{0\le t\le T} e_k(u(t)) \le M_2 e^{M_3 T},$$

where $M_2, M_3 > 0$. From the above inequality, we conclude that

$$E \sup_{0\le t\le T} e_k(t) < \infty.$$

This finishes the proof.

Remark 3.1. *We should point out that the inequality (3.10) given in the global existence theorem is the key point to avoid the unlimited growth, that is, in the absence of this inequality, the solution will cease to exist, i.e., blows up in a finite time.*

3.3. Continuous dependence on initial data

The quality of mathematical models is measured by how well they fit the physical phenomena they model. When a physical process is described (modeled) with the initial and/or boundary value problem of a PDE, it is desirable that any errors made in the measurement of the initial data do not affect the solution too much. This is because the solution with incorrect initial data may not be close enough to the real solution to predict the behavior of the problem. Mathematically, this is known as the continuous dependence of the solution of an initial and/or boundary value problem on the data.

In this section, we investigate the continuous dependence of the solution for problems (1.1) and (1.2). To this end, we pick up two solutions u and w with any two initial data (u_0^1, u_1^1) and (u_0^2, u_1^2) for problems (1.1) and (1.2). Let $V = u - w$, and $V_0(x) = u_0^1(x) - u_0^2(x)$, $V_1(x) = u_1^1(x) - u_1^2(x)$ and $\tilde{V} = (V_0, V_1)^T$. Let us also define $\kappa^*(V) = \kappa(u) - \kappa(w)$, $\quad h^*(V, V_t, \nabla V) = h(u, u_t, \nabla u) - h(w, w_t, \nabla w)$. Then, the following theorem can be stated.

Theorem 3.3. *Let the initial datum $(V(0), V_t(0)) \in \mathcal{H}$. Assume that for κ and h, (3.4)–(3.8) are satisfied with the same constants. Let $V(t)$ be the unique mild solution of*

$$dV_t + [\alpha V_t + \lambda V - \Delta V - \Delta V_t]\, dt = \kappa^*(V)\, dt + h^*(V, V_t, \nabla V)\, dW(x,t) \quad x \in \mathbb{R}^d, \quad t > 0, \tag{3.20}$$

$$V(x,0) = V_0(x), \quad V_t(x,0) = V_1(x). \tag{3.21}$$

Then,

$$E\, \|V(t\wedge\tau)\|_{\mathcal{H}}^2 \le CE \left\|\tilde{V}\right\|_{\mathcal{H}}^2, \tag{3.22}$$

where τ is the stopping time defined by

$$\tau(\cdot) = \inf\left\{t > 0; \|u(t,\cdot)\|_1^2 \ge 2\,\|u_0\|_1^2 \ or\ \|v(t,\cdot)\|_1^2 \ge 2\,\|u_1\|_1^2\right\}. \tag{3.23}$$

Proof. Let u and w be two different solutions of problems (1.1) and (1.2) with $V = u - w$. Without loss of generality we may take $\alpha = \lambda = 1$. Multiplying (3.20) by V_t, using Ito's formula and taking expectation, we get

$$d\left[E\,\|V_t(t\wedge\tau)\|^2 + E\,\|V(t\wedge\tau)\|_1^2\right] = 2E\left(\kappa^*(V(t\wedge\tau)), V_t(t\wedge\tau)\right) dt$$

$$+2\left[E\left\|h^*\left(V(t\wedge\tau),V_t(t\wedge\tau),\nabla V(t\wedge\tau)\right)\right\|^2-E\left\|V_t(t\wedge\tau)\right\|_1^2\right]dt. \tag{3.24}$$

Conditions (3.4)–(3.8) imposed on nonlinear terms yield estimates for the right hand side of (3.24)

$$\left(\kappa^*\left(V(t\wedge\tau)\right),V_t(t\wedge\tau)\right)\leq C\left(1+|u(t\wedge\tau)|^{p-1}+|w(t\wedge\tau)|^{p-1}\right)\|V(t\wedge\tau)\|^2\|V_t(t\wedge\tau)\| \tag{3.25}$$

$$\leq C\left(\|V(t\wedge\tau)\|_1+\|V_t(t\wedge\tau)\|\right), \tag{3.26}$$

and

$$\left\|h^*\left(V(t\wedge\tau),V_t(t\wedge\tau),\nabla V(t\wedge\tau)\right)\right\|^2\leq C\left(\|V(t\wedge\tau)\|_1^2+\|V_t(t\wedge\tau)\|^2\right). \tag{3.27}$$

which yields

$$\frac{d}{dt}\left[E\|V_t(t\wedge\tau)\|^2+E\|V(t\wedge\tau)\|_1^2\right]+2E\|V_t(t\wedge\tau)\|_1^2\leq C\left(E\|V(t\wedge\tau)\|_1^2+E\|V_t(t\wedge\tau)\|^2\right), \tag{3.28}$$

and hence

$$\frac{d}{dt}\left[E\|V_t(t\wedge\tau)\|^2+E\|V(t\wedge\tau)\|_1^2\right]\leq C\left(E\|V(t\wedge\tau)\|_1^2+E\|V_t(t\wedge\tau)\|^2\right). \tag{3.29}$$

The required result is obtained by integrating the above inequality, which completes the proof.

4. Blow up of solutions

In this section, we discuss the solutions growing with no boundary in a finite time interval to the problems (1.1)–(1.3) or equivalently problem (2.4) (i.e., as $t\uparrow T_*,\quad E\|u\|\to\infty$), which is a type of singularity. Such behaviors have physical meaning in thermal runaway problems, accumulation of shock waves, etc. One of the methods used to prove the blow up of solutions is Levine's concavity method, which mainly depends on concavity of a nonnegative, twice differentiable function $\Psi=[\theta]^{-\alpha}$. Demonstrating the concavity of Ψ is actually equivalent to demonstrating the existence of the following differential inequality

$$\theta(t)\,\theta''(t)-\beta\left[\theta'(t)\right]^2\geq 0,\quad t\geq 0,\quad \beta>1.$$

Then, under the conditions, $\theta(0)>0$ and $\theta'(0)>0$, Ψ decays to zero in a finite time $T_*=\frac{\theta(0)}{(\beta-1)\theta'(0)}$, i.e., $\theta\to\infty$. The function θ usually contains the L^2 norms of the solution and some additional terms depending on the equation studied.

In the following, will give some assumptions concerning initial data and noise intensity to ensure the blow up of solutions. The proof of the blow up theorem will be performed with an energy bound and the concavity method.

The energy equation related with (1.1) is

$$\mathcal{E}(t)=\frac{1}{2}\|v(t)\|^2+\frac{1}{2}\|u(t)\|^2+\frac{1}{2}\|\nabla u(t)\|^2-\int_{\mathcal{D}}\Psi(u(t))\,dx. \tag{4.1}$$

where $\Psi(u)=\int_0^u\kappa(s)\,ds$.

The following lemma provides an energy bound that is indispensable for the proof of the main result.

Lemma 4.1. *Assume that (u, v) is the unique pair of mild solution of problem (2.4) with values in H^1 and H. Then the followings are satisfied:*

$$E\mathcal{E}(t) \leq E\mathcal{E}(0) - E\int_0^t \|v\|_1^2\, ds + \frac{1}{2}c_*^2 TrR \int_0^t \int_{\mathcal{D}} h^2 dxds, \tag{4.2}$$

and

$$E(u, v) = E(u_0, u_1) + E\int_0^t \|v\|^2\, ds - \frac{1}{2}E\|u\|_1^2 - E\int_0^t \|u\|_1^2\, ds + \frac{1}{2}E\|u_0\|_1^2 + E\int_0^t (u, \kappa(u))\, ds. \tag{4.3}$$

Proof. We make use of the Ito formula for $\|v\|^2$ to get control of $E\mathcal{E}(t)$

$$\begin{aligned}\|v\|^2 &= \|u_1\|^2 + 2\int_0^t (v, dv) + \int_0^t (dv, dv)\\ &= \|u_1\|^2 - 2\int_0^t (\nabla u, \nabla v)\, ds - 2\int_0^t \|v\|^2\, ds - 2\int_0^t (u, v)\, ds - 2\int_0^t \|\nabla v\|^2\, ds\\ &\quad + 2\int_0^t (v, \kappa)\, ds + 2\int_0^t (v, hdW) + \sum_{k=1}^{\infty}\int_0^t (hRe_k, he_k)\, ds.\end{aligned} \tag{4.4}$$

By direct computation, we get

$$2\int_0^t (u, v)\, ds = \|u\|^2 - \|u_0\|^2, \tag{4.5}$$

$$2\int_0^t (\nabla u, \nabla v)\, ds = \|\nabla u\|^2 - \|\nabla u_0\|^2, \tag{4.6}$$

and

$$2\int_0^t (v, \kappa)\, ds = 2\int_{R^d} (\Psi(u) - \Psi(u_0))\, dx. \tag{4.7}$$

Employing (4.5)–(4.7) and (2.1) in (4.4) yield

$$\begin{aligned}\|v\|^2 &= 2\mathcal{E}(0) - \|\nabla u\|^2 - 2\int_0^t \|v\|^2\, ds - \|u\|^2 + 2\int_{\mathcal{D}} \Psi(u)\, dx - 2\int_0^t \|\nabla v\|^2\, ds\\ &\quad + 2\int_0^t (v, hdW) + \sum_{k=1}^{\infty}\lambda_k \int_0^t \int_{\mathcal{D}} h^2 e_k^2(x)\, dxds.\end{aligned} \tag{4.8}$$

By introducing $c_* := \sup_{k\geq 1}\|e_k\|_\infty < \infty$ (where $\|.\|_\infty$ stands for the super-norm), keeping in mind that $TrR = \sum_{k=1}^{\infty}\lambda_k$ and taking the expectation in (4.8), (4.2) can be deduced.

Now it's time to prove (4.3). Let us assume that (u, v) is a global mild solution of problem (2.4), then for each $k \geq 1$, both of $\{(u(t), \widetilde{e}_k); t \geq 0\}$ and $\{(v(t), \widetilde{e}_k); t \geq 0\}$ are $\{\mathcal{F}_t\}_{t\geq 0}$ adapted and the first one is a continuous process of finite variation while the second one is a continuous semi-martingale, where $\{\widetilde{e}_k\}_{k\geq 1}$ is an orthonormal base of L^2. Then, by the Ito formula

$$(u(t), \widetilde{e}_k)(v(t), \widetilde{e}_k) = (u_0, \widetilde{e}_k)(u_1, \widetilde{e}_k) + \int_0^t (u(t), \widetilde{e}_k)\, d(v(t), \widetilde{e}_k) + \int_0^t (v(t), \widetilde{e}_k)\, d(u(t), \widetilde{e}_k),$$

from which we have

$$
\begin{aligned}
(u,v) &= (u_0,u_1) + \int_0^t (u,dv) + \int_0^t (v,du) \\
&= (u_0,u_1) - \frac{1}{2}\|u\|_1^2 + \frac{1}{2}\|u_0\|_1^2 - \int_0^t \|u\|^2 ds - \int_0^t \|\nabla u\|^2 ds \\
&+ \int_0^t \|v\|^2 ds + \int_0^t (u,\kappa)\, ds + \int_0^t (u, hdW). \qquad (4.9)
\end{aligned}
$$

Taking the expectation in (4.9) gives the desired identity.

For the proof of the main theorem, we impose the following condition on h instead of condition $(L2)$.

L4. Let $h : R^{d+2} \to R$ be a continuous function and satisfies

$$
\|h(u,v,\nabla u)\|^2 \le \frac{\eta}{(2\eta+1)\, c_*^2 TrR} \|v\|_1^2, \qquad (4.10)
$$

where η is a positive constant.

Let us define for any $t < T$,

$$
\Xi(t) := E\|u\|^2 + E\int_0^t \|u\|_1^2 ds + (T-t) E\|u_0\|_1^2 + \gamma (t+\mu)^2, \qquad (4.11)
$$

where γ and μ are positive parameters that are appropriately chosen in the proof of the blow up theorem. Then by Lemmas 4.1, 4.5 and 4.6, we get

$$
\begin{aligned}
\Xi'(t) &= 2E(u,v) + E\|u\|_1^2 - E\|u_0\|_1^2 + 2\gamma(t+\mu) \\
&= 2E(u,v) + 2E\int_0^t (u,v)_1 ds + 2\gamma(t+\mu) \\
&= 2E(u_0,u_1) - 2E\int_0^t \|u\|_1^2 ds + 2E\int_0^t (u,\kappa(u))\, ds + 2E\int_0^t \|v\|^2 ds + 2\gamma(t+\mu), \qquad (4.12)
\end{aligned}
$$

and taking the inner product of (1.1) by $u(t)$, we obtain

$$
\begin{aligned}
\Xi''(t) &= 2E(u,v_t) + 2E\|v\|^2 + 2E(u,v)_1 + 2\gamma \\
&= -2E\|u\|_1^2 + 2E(u,\kappa(u)) + 2E\|v\|^2 + 2\gamma.
\end{aligned}
$$

Theorem 4.1. *Assume that conditions $(L1)$, $(L2)$ and $(L4)$ are fulfilled, and that for $(u_0,u_1) \in \mathcal{H}$ problems (1.1)–(1.3) or (2.4) have a unique local mild solution $u \in H^1$ with $u_t \in H$. Then, under the following conditions,*

(i) $(u,\kappa(u)) \ge 2(2\eta+1)(\Psi(u),1)$*, where* $\eta > 0$ *and* $(\Psi(u),1) = \int_{\mathcal{D}} \Psi(u(x))\, dx$.

(ii) $2\left(\Psi\left(u_0\right),1\right)-\gamma \geq \left(u_0,u_0\right)_1+\left(u_1,u_1\right)+\left(1+\frac{1}{2\eta+1}\right)\int\limits_0^t\left(u_1,u_1\right)_1 ds.$

The solution u exists only in a finite time interval $(0,T^*)$ such that

$$\lim_{t\to T^{*-}}\left(E\left\|u\right\|^2+E\int\limits_0^t\left\|u\right\|_1^2 ds\right)=+\infty.$$

Furthermore, the finite time T^* can be estimated from above by

$$T^*\leq 4\eta\gamma^{-1}\left[S+\left(S^2+4\eta^2\gamma E\left\|u_0\right\|^2\right)^{1/2}\right],$$

where S and γ are given in the proof.

Proof. For the proof of the theorem, let us define

$$\Phi\left(t\right)=\Xi^{-\eta}\left(t\right),$$

for any $\eta>0$. By direct computation, we have

$$\begin{aligned}\Phi'\left(t\right)&=-\eta\Xi^{-(\eta+1)}\Xi'\left(t\right),\\ \Phi''\left(t\right)&=-\eta\Xi^{-(\eta+2)}\left(t\right)\left\{\Xi\left(t\right)\Xi''\left(t\right)-\left(\eta+1\right)\left[\Xi'\left(t\right)\right]^2\right\}.\end{aligned}$$

By aid of the above computations on $\Xi\left(t\right)$, we can write the bracketed statement on the right hand side of the above equality as follows:

$$\begin{aligned}\Xi\left(t\right)\Xi''\left(t\right)-\left(\eta+1\right)\left[\Xi'\left(t\right)\right]^2&=2\Xi\left(t\right)\left[E\left\|v\right\|^2-E\left\|u\right\|_1^2+E\int_{\mathcal{D}}u\kappa\left(u\right)dx+\gamma\right]\\ &\quad-4\left(\eta+1\right)\left[E\left(u,v\right)+E\int\limits_0^t\left(u,v\right)_1 ds+\gamma\left(t+\mu\right)^2\right]^2.\end{aligned}\tag{4.13}$$

Employing Schwarz, Hölder and Young inequalities yield

$$E\left(u,v\right)\leq E\left\|u\right\|E\left\|v\right\|,$$

$$\left(E\int\limits_0^t\left(u,v\right)_1 ds\right)^2\leq\left(E\int\limits_0^t\left\|u\right\|_1^2 ds\right)\left(E\int\limits_0^t\left\|v\right\|_1^2 ds\right),$$

$$\begin{aligned}2E\left(u,v\right)\left(E\int\limits_0^t\left(u,v\right)_1 ds\right)&\leq 2E\left\|u\right\|E\left\|v\right\|\left(E\int\limits_0^t\left\|u\right\|_1^2 ds\right)^{1/2}\left(E\int\limits_0^t\left\|v\right\|_1^2 ds\right)^{1/2},\\ &\leq E\left\|u\right\|^2E\int\limits_0^t\left\|v\right\|_1^2 ds+E\left\|v\right\|^2E\int\limits_0^t\left\|u\right\|_1^2 ds,\end{aligned}$$

from which we deduce that

$$\begin{aligned}\Xi(t)\Xi''(t)-(\eta+1)\left[\Xi'(t)\right]^2 &\geq 2\Xi(t)\Xi''(t)\\ &\quad-4(\eta+1)\left(E\|u\|^2+E\int_0^t\|u\|_1^2\,ds+\gamma(t+\mu)^2\right)\left(E\|v\|^2+E\int_0^t\|v\|_1^2\,ds+\gamma\right)\\ &=2\Xi(t)\Xi''(t)-4(\eta+1)\left(\Xi(t)-(T-t)E\|u_0\|_1^2\right)\left(E\|v\|^2+E\int_0^t\|v\|_1^2\,ds+\gamma\right).\end{aligned}\tag{4.14}$$

By the positivity of the term $-\left[-(T-t)E\|u_0\|_1^2\right]\left(E\|v\|^2+E\int_0^t\|v\|_1^2\,ds\right)$, inequality (4.14) turns into

$$\begin{aligned}\Xi(t)\Xi''(t)-(\eta+1)\left[\Xi'(t)\right]^2\geq 2\Xi(t)&\Bigg[E\int_{\mathcal{D}}u\kappa(u)\,dx-E\|u\|_1^2-(2\eta+1)E\|v\|^2\\ &-2(\eta+1)E\int_0^t\|v\|_1^2\,ds-\gamma(2\eta+1)\Bigg].\end{aligned}$$

Let

$$J(t):=E(u,\kappa(u))-E\|u\|_1^2-(2\eta+1)E\|v\|^2-2(\eta+1)E\int_0^t\|v\|_1^2\,ds-\gamma(2\eta+1).\tag{4.15}$$

Next, we prove that $J(t)\geq 0$. For this purpose, we differentiate (4.15)

$$\begin{aligned}\frac{dJ(t)}{dt}&=\frac{d}{dt}E(u,\kappa(u))-2E(u,v)_1-2(2\eta+1)E(v,v_t)-2(\eta+1)E\|v\|_1^2\\ &=\frac{d}{dt}E(u,\kappa(u))-2E(u,v)_1-2(2\eta+1)\Big[-E(u,v)_1-E\|v\|_1^2+E(v,\kappa(u))\\ &\quad+E\sum_{k=1}^{\infty}\lambda_k\int_{\mathcal{D}}h^2e_k^2dxds\Big]-2(\eta+1)E\|v\|_1^2\\ &=\frac{d}{dt}E(u,\kappa(u))+4\eta E(u,v)_1+2\eta E\|v\|_1^2\\ &\quad-2(2\eta+1)E(v,\kappa(u))-2(2\eta+1)E\sum_{k=1}^{\infty}\lambda_k\int_{\mathcal{D}}h^2e_k^2dxds,\end{aligned}\tag{4.16}$$

where use was made of

$$E\int_0^t(v,v_t)\,ds=-E\int_0^t\|v\|^2\,ds-E\int_0^t(u,v)_1\,ds-E\int_0^t\|\nabla v\|^2\,ds$$

$$+E\int_0^t (v,\kappa(u))\,ds + E\sum_{k=1}^{\infty}\lambda_k\int_0^t\int_{\mathcal{D}} h^2 e_k^2 dxds.$$

Integrating (4.16) from 0 to t yields

$$J(t) = J(0) + E(u,\kappa(u)) - E(u_0,\kappa(u_0)) + 2\eta\left(E\|u\|_1^2 - E\|u_0\|_1^2\right)$$
$$-2(2\eta+1)E\int_{\mathcal{D}}(\Psi(u)-\Psi(u_0))\,dx - 2(2\eta+1)E\sum_{k=1}^{\infty}\lambda_k\int_0^t\int_{\mathcal{D}} h^2 e_k^2 dxds$$
$$+2\eta E\int_0^t \|v\|_1^2\,ds.$$

Using the assumption (i) of the theorem, $c_* := \sup_{k\geq 1}\|e_k\|_\infty < \infty$, and $TrR = \sum_{k=1}^{\infty}\lambda_k$, we have

$$J(t) \geq J(0) - E(u_0,\kappa(u_0)) + 2(2\eta+1)\int_{\mathcal{D}}\Psi(u_0)\,dx - 2\eta E\|u_0\|_1^2 + 2\eta E\int_0^t\|v\|_1^2\,ds$$
$$-2(2\eta+1)c_*^2 TrR\int_0^t\int_{\mathcal{D}} h^2(s,x)\,dxds$$
$$= -(2\eta+1)E\|u_0\|_1^2 - (2\eta+1)E\|u_1\|^2 + 2\eta E\int_0^t\|v\|_1^2\,ds + 2(2\eta+1)\int_{\mathcal{D}}\Psi(u_0)\,dx$$
$$-2(\eta+1)E\int_0^t\|u_1\|_1^2\,ds - 2(2\eta+1)c_*^2 TrR\int_0^t\int_{\mathcal{D}} h^2(s,x)\,dxds - \gamma(2\eta+1).$$

Then by (4.10) and presumption (ii) of the theorem, we get $J(t)\geq 0$, for all $t\geq 0$, which results in

$$\Xi(t)\,\Xi''(t) - (\Xi+1)\left[\Xi'(t)\right]^2 \geq 0,$$

for any

$$\gamma \in \left(0, 2E\int_{\mathcal{D}}\Psi(u_0)\,dx - E\|u_1\|^2 - E\|u_0\|_1^2 - \left(1+\frac{1}{2\eta+1}\right)E\int_0^t\|u_1\|_1^2\,ds\right]. \tag{4.17}$$

Moreover,

$$\Xi(0) = E\|u_0\|^2 + TE\|u_0\|_1^2 + \gamma\mu^2 \geq 0,$$

and

$$\Xi'(0) = 2E(u_0,u_1) + 2\gamma\mu \geq 0,$$

where μ is chosen as

$$\mu > \max\left\{0, \frac{E(u_0,u_1)}{\gamma}\right\}, \tag{4.18}$$

and γ satisfies (4.17). Then since

$$(\Phi(t))'' = -\eta\Xi^{-\eta-2}(t)\left[\Xi(t)\Xi''(t) - (\eta+1)\left[\Xi'(t)\right]^2\right] \leq 0,$$

one conclude that Φ would be concave for $\Xi(t) \neq 0$. In this way, if $\Xi(0) \neq 0$, then as a consequence of concavity $\Xi^{-\eta}(t) \leq \Xi^{-\eta}(0) - \eta t\Xi'(0)\Xi^{-\eta-1}(0)$. Thus, we have

$$\Xi^{\eta}(t) \geq \Xi^{\eta+1}(0)\left[\Xi(0) - \eta t\Xi'(0)\right]^{-1},$$

which shows that all the requirements of Theorem 2.2 are satisfied and as $t \uparrow T^{*-}(\leq \Xi(0)/(\eta\Xi'(0)))$, $\Xi(t) \to \infty$.

Now, we will estimate the upper bound for the blow up time. To this end, we should choose appropriate μ and T. Let μ be any number satisfying (4.18). As a consequence of blow up result T can be chosen as

$$T = \frac{E\|u_0\|^2 + \gamma\mu^2}{2\eta E(u_0, u_1) - E\|u_0\|_1^2 + 2\eta\gamma\mu} = T(\gamma, \mu). \tag{4.19}$$

By a simple computation, it can be shown that $T(\gamma, \mu)$ has a minimum value of

$$T = \min_{\mu>0} T(\gamma, \mu) = T(\gamma, \mu_0) = 4\eta\gamma^{-1}\left[S + \left(S^2 + 4\eta^2\gamma E\|u_0\|^2\right)^{1/2}\right], \tag{4.20}$$

for

$$\mu_0 = (2\eta\gamma)^{-1}\left[S + \left(S^2 + 4\eta^2\gamma E\|u_0\|^2\right)^{1/2}\right],$$

on the interval $(\frac{E\|u_0\|_1^2 - 2\eta E(u_0,u_1)}{2\eta\gamma}, +\infty)$, where $S = E\|u_0\|_1^2 - 2\eta E(u_0, u_1)$. So, the lifespan T^* is bounded by

$$T^* \leq 4\eta\gamma^{-1}\left[S + \left(S^2 + 4\eta^2\gamma E\|u_0\|^2\right)^{1/2}\right].$$

4.1. An example

In this subsection, we take some particular functions to check our assumptions. Without loss of generality one may take $\mathcal{D} = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0\right\}$. Let us consider the following equation

$$du_t + \left[\alpha u_t + \lambda u - \Delta u - \Delta u_t\right]dt = au^p dt + h_0 \tan^{-1}\left(1 + |u|^{2p} + |\nabla u|^2 + |v|^2\right)e^{-bt}dW(x,t), \quad t > 0, x \in \mathcal{D}, \tag{4.21}$$

with initial and boundary conditions

$$u(x,0) = \frac{\delta}{1+|x|^2}, \quad u_t(x,0) = \frac{1}{1+|x|^2}, \quad x \in \mathcal{D}, \tag{4.22}$$

$$u(x,t)|_{x_1=0} = 0, \quad t > 0. \tag{4.23}$$

where $W(x,t)$ stands for a Wiener random field that has covariance function $r(x,y) = r_0 \exp\{-\sigma(x\cdot y)\}$ for $x, y \in \mathcal{D}$, p is an integer, $x \cdot y = x_1y_1 + x_2y_2$, and $a, b, \delta, \sigma, h_0, r_0$ are positive constants.

Now, it's time to meet the conditions (i) and (ii) of Theorem 4.1. A direct computation yields

$$(u, \kappa(u)) = a\int_{\mathcal{D}} u^{p+1}dx, \tag{4.24}$$

$$\Psi(u) = a\int_0^u s^{p+1}ds = \frac{a}{p+1}u^{p+1}, \tag{4.25}$$

and

$$(\Psi(u), 1) = \frac{a}{p+1}\int_{\mathcal{D}} u^{p+1}dx. \tag{4.26}$$

Combining (4.24) and (4.26), one can conclude that the condition (i) is satisfied for appropriately chosen p and η. For (ii), let us compute a number of integrals for H^1 norms of initial data.

$$\|u_0\|^2 = \delta\int_{\mathcal{D}}\frac{dx}{(1+|x|^2)} = \frac{\delta\pi}{2}, \quad \|u_1\|^2 = \int_{\mathcal{D}}\frac{dx}{(1+|x|^2)} = \frac{\pi}{2}, \tag{4.27}$$

$$\|\nabla u_0\|^2 = 4\delta^2\int_{\mathcal{D}}\frac{|x|^2}{(1+|x|^2)^4}dx = \frac{\pi}{3}\delta^2, \quad \|\nabla u_1\|^2 = 4\int_{\mathcal{D}}\frac{|x|^2}{(1+|x|^2)^4}dx = \frac{\pi}{3}, \tag{4.28}$$

$$(\Psi(u_0), 1) = \frac{a}{p+1}\int_{\mathcal{D}}\left(\frac{\delta}{1+|x|^2}\right)^{p+1}dx = \frac{a\pi\delta^{p+1}}{2p(p+1)}. \tag{4.29}$$

Inserting (4.27)–(4.29) in (ii), one can see that condition (ii) is also satisfied for an appropriate a, and the solution of problems (1.1)–(1.3) blow up in a finite time.

5. Conclusions

In the present work, we examined the qualitative behavior of solutions for a stochastic relativistic wave equation driven by a multiplicative Gaussian white noise. We can summarize our results as follows. We first provided the necessary conditions (Lipschitz continuity of nonlinear terms κ and h and linear growth estimates for these nonlinear terms) for a solution to exist in a small time interval $(0, T]$, and then gave the requirements for extending this solution to an infinite time interval $(0, \infty)$. We also investigated the continuous dependency of solutions on the initial data. Finally, we examined the blow up of solutions in a finite time, which is the main purpose of this study. The interaction of the source term, damping terms and noise intensity specify the behavior of the solutions for problems (1.1)–(1.3). When the noise intensity vanishes ($h = 0$, the deterministic case), the domination of damping terms on source term guarantees the global in-time solutions. For this result to be valid for Eq (1.1), the noise density should be large. As the results reveal, if the source term κ dominates the damping terms, and the noise intensity is less than or equal to H^1 norm of the damping terms (condition (4.10)), then a small noise cannot prevent the explosion.

Conflict of interest

The authors declare no conflict of interest.

Acknowledgement

We would like to thank the anynomous referees for their valuable comments and suggestions which greatly improve the paper.

References

1. J. M. Sancho, M. San Miguel, S. Katz, J. D. Gunton, Multiplicative noise in stochastic differential equations: A numerical study, *Sixth international conference on noise in physical systems*, 1981, 26–29.
2. V. Pata, S. Zelik, Smooth attractors for strongly damped wave equations, *Nonlinearity*, **19** (2006), 1495. http://dx.doi.org/10.1088/0951-7715/19/7/001
3. W. J. Lehr, J. L. Park, A stochastic derivation of the Klein-Gordon equation, *J. Math. Phys.*, **18** (1977), 1235–1240. http://dx.doi.org/10.1063/1.523396
4. A. I. Arbab, Stochastic model related to the Klein-Gordon equation revisited, *EPL*, **96** (2011), 20002. http://dx.doi.org/10.1209/0295-5075/96/20002
5. S. Gur, M. E. Uysal, Continuous dependence of solutions to the strongly damped nonlinear Klein-Gordon equation, *Turk. J. Math.*, **42** (2018), 904–910. http://dx.doi.org/10.3906/mat-1706-30
6. A. I. Komech, E. A. Kopylova, Weighted energy decay for 3D Klein-Gordon equation, *J. Differ. Equations*, **248** (2010), 501–520. http://dx.doi.org/10.1016/j.jde.2009.06.011
7. W. Y. Huang, W. L. Chen, Global existence and blow up of solutions for nonlinear Klein-Gordon equation with damping term and nonnegative potentials, *Abstr. Appl. Anal.*, **2014** (2014), 142892. http://dx.doi.org/10.1155/2014/142892
8. Y. Wang, A sufficient condition for finite time blow up of the nonlinear Klein-Gordon equations with arbitrarily positive initial energy, *Proc. Amer. Math. Soc.*, **136** (2008), 3477–3482. http://dx.doi.org/10.1090/s0002-9939-08-09514-2
9. N. Burq, G. Raugel, W. Schlag, Long time dynamics for damped Klein-Gordon equations, 2017, arXiv: 1505.05981.
10. T. Cazenave, Uniform estimates for solutions of nonlinear Klein-Gordon equations, *J. Funct. Anal.*, **60** (1985), 36–55. http://dx.doi.org/10.1016/0022-1236(85)90057-6
11. T. G. Ha, J. Y. Park, Global existence and uniform decay of a damped Klein-Gordon equation in a noncylindrical domain, *Nonlinear Anal. Theor.*, **74** (2011), 577–584. http://dx.doi.org/10.1016/j.na.2010.09.011
12. R. Xu, Global existence, blow up and asymptotic behaviour of solutions for nonlinear Klein-Gordon equation with dissipative term, *Math. Method. Appl. Sci.*, **33** (2010), 831–844. http://dx.doi.org/10.1002/mma.1196
13. P. L. Chow, Stochastic wave equations with polynomial nonlinearity, *Ann. Appl. Probab.*, **12** (2002), 361–381. http://dx.doi.org/10.1214/aoap/1015961168
14. V. Barb, G. D. Prato, The stochastic nonlinear damped wave equation, *Appl. Math. Optim.*, **46** (2002), 125–141. http://dx.doi.org/10.1007/s00245-002-0744-4
15. P. L. Chow, Asymptotics of solutions to semilinear stochastic wave equations, *Ann. Appl. Probab.*, **16** (2006), 757–789. http://dx.doi.org/10.1214/105051606000000141
16. P. L. Chow, Nonlinear stochastic wave equations: Blow up of second moments in L 2-norm, *Ann. Appl. Probab.*, **19** (2009), 2039–2046. http://dx.doi.org/10.1214/09-AAP602

17. J. U. Kim, On the stochastic wave equation with nonlinear damping, *Appl. Math. Optim.*, **58** (2008), 29–67. http://dx.doi.org/10.1007/s00245-007-9029-2

18. M. Ondrejat, Stochastic nonlinear wave equations in local Sobolev spaces, *Electron. J. Probab.*, **15** (2010), 1041–1091. http://dx.doi.org/10.1214/EJP.v15-789

19. R. D. Parshad, M. Beauregard, A. Kasimov, B. Said-Houari, Global existence and finite time blow up in a class of stochastic nonlinear wave equations, *COSA*, **8** (2014), 7. http://dx.doi.org/10.31390/cosa.8.3.07

20. Z. Brzezniak, M. Ondrejat, J. Seidler, Invariant measures for stochastic nonlinear beam and wave equations, *J. Differ. Equations*, **260** (2016), 4157–4179. http://dx.doi.org/10.1016/j.jde.2015.11.007

21. F. Liang, Explosive solutions of stochastic nonlinear beam equations with damping, *J. Math. Anal. Appl.*, **419** (2014), 849–869. http://dx.doi.org/10.1016/j.jmaa.2014.04.065

22. L. Bo, D. Tang, Y. Wang, Explosive solutions of stochastic wave equations with damping on R^d, *J. Differ. Equations*, **244** (2008), 170–187. http://dx.doi.org/10.1016/j.jde.2007.10.016

23. G. G. Xu, L. B. Wang, G. G. Lin, Global random attractor for the strongly damped stochastic wave equation with white noise and nonlinear term, *Theor. Math. Appl.*, **3** (2013), 111–122.

24. P. L. Chow, Nonexistence of global solutions to nonlinear stochastic wave equations in mean L^p-norm, *Stoch. Anal. Appl.*, **30** (2012), 543–551. http://dx.doi.org/10.1080/07362994.2012.668448

25. Z. Wang, S. Zhou, Asymptotic behavior of stochastic strongly wave equation on unbounded domains, *J. Appl. Math. Phys.*, **3** (2015), 55167. http://dx.doi.org/10.4236/jamp.2015.33046

26. Z. Wang, S. Zhou, Random attractor for non-autonomous stochastic strongly damped wave equation on unbounded domains, *J. Appl. Anal. Comput.*, **5** (2015), 363–387. http://dx.doi.org/10.11948/2015031

27. Z. Wang, S. Zhou, A. Gu, Random attractor of the stochastic strongly damped wave equation, *Commun. Nonlinear Sci.*, **17** (2012), 1649–1658. http://dx.doi.org/10.1016/j.cnsns.2011.09.001

28. R. Jones, B. Wang, Asymptotic behavior of a class of stochastic nonlinear wave equations with dispersive and dissipative terms, *Nonlinear Anal. Real*, **14** (2013), 1308–1322. http://dx.doi.org/10.1016/j.nonrwa.2012.09.019

29. T. Caraballo, B. Guo, N. H. Tuan, R. Wang, Asymptotically autonomous robustness of random attractors for a class of weakly dissipative stochastic wave equations on unbounded domains, *P. Roy. Soc. Edinb. A*, **151** (2021), 1700–1730. http://dx.doi.org/10.1017/prm.2020.77

30. R. Wang, B. Wang, Random dynamics of lattice wave equations driven by infinite-dimensional nonlinear noise, *Discrete Cont. Dyn. B*, **25** (2020), 2461–2493. http://dx.doi.org/10.3934/dcdsb.2020019

31. R. Wang, Y. Li, Backward compactness and periodicity of random attractors for stochastic wave equations with varying coefficients, *Discrete Cont. Dyn. B*, **24** (2019), 4145–4167. http://dx.doi.org/10.3934/dcdsb.2019054

32. P. Groisman, J. D. Rossi, Explosion time in stochastic differential equations with small diffusion, *Electron. J. Differ. Eq.*, **2007** (2007), 1–9.

33. G. D. Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, London: Cambridge University Press, 2014. http://dx.doi.org/10.1017/CBO9781107295513

34. D. Bahuguna, Strongly damped semilinear equations, *J. Appl. Math. Stoch. Anal.*, **8** (1995), 397–404.

35. H. Engler, F. Neubrander, J. Sandefur, Strongly damped semilinear second order equations, In: *Nonlinear semigroups, partial dfferential equations and attractors*, 1987. http://dx.doi.org/10.1007/BFb0077415

36. R. A. Adams, J. J. R. Fournier, *Sobolev spaces*, Netherlands: Academic Press, 2003.

37. H. Yang, Y. Z. Han, Initial boundary value problem for a strongly damped wave equation with a general nonlinearity, *Evol. Equ. Control The.*, **11** (2022), 635–648. http://dx.doi.org/10.3934/eect.2021019

AIMS Press