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Research article

Optimal investment and reinsurance for the insurer and reinsurer with the joint exponential utility under the CEV model

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Abstract: This paper considers the problem of optimal investment-reinsurance for the insurer and reinsurer under the constant elasticity of variance (CEV) model. It is assumed that the net claims process is approximated by a diffusion process, both the insurer and reinsurer can invest in risk-free assets and risky assets. We use the variance premium principle to calculate the premiums of the insurer and reinsurer, and the reinsurance proportion is constrained by the net profit condition. Our objective is to maximize the joint exponential utility of the insurer and reinsurer's terminal wealth for a fixed time. By solving the HJB equation, we obtain the explicit expressions of the optimal investment-reinsurance strategy and value function. We find that the optimal reinsurance strategy can be divided into many cases and is related to the risk aversion coefficient of the insurer and reinsurer, but independent of the price of risky assets. Furthermore, we give the proof of the verification theorem. Finally, we demonstrate a numerical analysis to explain the results.

Keywords: investment; reinsurance; variance premium principle; net profit condition; CEV model; joint exponential utility
Mathematics Subject Classification: 91B05, 91G05

1. Introduction

In recent decades, the insurance industry has developed rapidly and played an extremely important role in the financial market. Stochastic control theory is widely used in insurance related business. Scholars have done a lot of research on optimal investment, optimal reinsurance, optimal dividend and so on for various risk models. For example, Browne [1] assumed that the risky stock price follows a geometric Brownian motion (GBM) and presented the optimal investment strategy under the two criteria of maximizing the expected utility of wealth at terminal time and minimizing the ruin probability. Cox et al. [2] proposed the CEV Model which is an extension of the GBM model, and they gave the solutions of the limit diffusion case and the one-stage form of several alternative jump and diffusion processes. Asmussen and Taksar [3] studied the optimal dividend problem under restricted and unrestricted dividend rates, they found that the optimal dividend strategy in the unrestricted case is singular. Taksar [4] considered the optimal reinsurance and dividend problem for an insurance company, and gave the explicit expressions for the optimal strategies and value function. Bai and Guo [5] studied the optimal investment and reinsurance strategy with no-short selling under the criteria of maximizing terminal wealth utility and minimizing the ruin probability. Sun et al. [6] studied the optimal reinsurance and investment strategy for an insurer with two dependent classes of insurance business and no-shorting constraint. Gu et al. [7] studied the optimal excess-of-loss reinsurance and investment problem under the CEV model, they found that the parameters of risky asset price have no effect on the optimal reinsurance, and the optimal value function with reinsurance is larger than that without reinsurance under exponential utility. Recently, Cao et al. [8] studied the optimal reinsurance problem when the insurer has mean-variance risk preference and the total claim process is a compound dynamic contagion process. Jiang et al. [9] studied the optimal investment reinsurance strategy with premium control, which modeled the claim arrival rate as a function of t and expressed the claim arrival rate as a decreasing bounded concave function of the safety loading. Xu et al. [10] investigated the optimal investment and dividend problem for an insurer under a Markov regime switching market with high gain tax. Zhang et al. [11] studied the optimal excess-of-loss reinsurance and investment with thinning dependent risks under Heston model. Sun et al. [12] considered an optimal asset-liability management problem for an insurer under the mean-variance criterion, which financial market consists of one risk-free asset and n risky assets with the risk premium relying on an affine diffusion factor process. Chen et al. [13] presented the optimal excess-of-loss reinsurance and dividends strategy for the model with thinning-dependence structure.

The expected value premium principle is commonly used as the reinsurance premium principle due to its simplicity and popularity in practice. However, the variance of the risk with the same expectation is not necessarily the same, so the fluctuation of claims need to be taken into account in stipulating premiums. In recent years, the variance or mean-variance premium principle has gained more and more attention in the literature. For example, Kaluszka ([14, 15]) studied several optimal reinsurance problem under the mean-variance premium principle. Under the criterion of maximizing the expected exponential utility, Liang et al. [16] considered the optimal proportional reinsurance strategy in a risk model with dependent risks and variance premium principle. Zhang et al. [17] assumed that the reinsurance premium is calculated by generalized mean-variance principle, and they obtained the optimal investment-reinsurance strategy under the criteria of maximizing the expected utility of terminal wealth and minimizing the ruin probability. Liang et al. [18] derived the explicit expression of reinsurance strategy for minimizing the ruin probability in a diffusion approximation model where the reinsurance premium is given by mean-variance premium principle.

Most of the existing literature only considered the optimization problem from the insurer's point of view. However, there is always an interest relationship between the insurer and the reinsurer in reality, so the reinsurer can not be ignored. And both of the insurer and reinsurer want to maximize their own interests, so it is necessary to maintain a good dynamic balance between the insurer and reinsurer at the same time. From Borch [19], we know that the two companies should negotiate to maximize their common interests and they have to reach a compromise. Kaishev [20] derived the expectation formula of the common survival profit of the insurer and reinsurer in a fixed time. Kaishev and Dimitrova [21] obtained the optimal division of total premium income to maximize the joint survival probability in the case of fixed retention level, ceiling and continuous claim size. Furthermore, Cai et al. [22] studied the joint survival and profit probability of insurer and reinsurer under the expectation criterion. For maximizing the expected product of exponential utilities, Li et al. [23] studied the optimal proportional reinsurance-investment strategy. They made a further study in Li et al. [24] and reconsidered the time-consistent investment strategy of the insurer that can be decomposed into two parts under the CEV model. Zhao et al. [25] also considered the common interests of insurers and reinsurers. In order to maximize the joint survival probability, Zhang et al. [26] obtained the optimal quota-share reinsurance under five criteria. Bai et al. [27] presented the optimal investment and proportional reinsurance under the optimization criterion of maximizing the expectation of the weighted sum of the wealth process of insurers and reinsurers in discrete time.

In this paper, adopting the idea of Huang et al. [28], we choose the joint exponential utility instead of the product exponential utility in Li et al. [23], which can better reflect the concavity properties of exponential utility function, i.e., $U_x > 0, U_{xx} < 0$. Under the criterion of maximizing the terminal joint exponential utility, we study the optimal investment-reinsurance strategies for both insurer and reinsurer under the constant elasticity of variance (CEV) model. Besides, we present the explicit expression of the value function, and give the proof of the verification theorem which is not considered in Li et al. [23]. Compared with Li et al. [23] and Huang et al. [28], we use the variance premium criterion to calculate the premiums of the insurer and reinsurer, and the reinsurance proportion in this paper is constrained by the net profit condition. Moreover, we consider the price of risky assets conforms to the CEV Model which will be more general than the GBM model to solve the HJB equation and give the proof of verification theorem.

To the best of our knowledge, there are only a few literature on the utility maximization of the joint terminal wealth of the insurer and reinsurer. In this paper, we mainly study the optimal investment-reinsurance problem of insurers and reinsurers under the joint exponential utility. We assume that the claims process conforms to a diffusion approximation process, and the premium of the insurer and reinsurer are stipulated by the variance premium principle. Furthermore, both the insurer and reinsurer are allowed to invest in risk-free assets and risky assets, and the price of the risky assets are described by the CEV model. We permit borrowing and short selling, but limit the proportional reinsurance under the net profit condition. Then, we obtain the HJB equation under the optimization criterion of maximizing the terminal joint exponential utility. By solving the HJB equation, we obtain the optimal investment-reinsurance strategies, and present the proof of the verification theorem. Finally, we demonstrate a numerical analysis and explain the results for better understanding in the economic sense.

The remainder of this paper proceeds as follows. In Section 2, we introduce our model in three aspects. In Section 3, we describe the HJB equation under the objective of maximizing the joint exponential utility of terminal wealth and present the optimal strategy and value function along with a verification theorem. In Section 4, we demonstrate numerical simulations to illustrate our results. Section 5 concludes the whole paper. And Appendix contains the proof of all the theorems and lemma.

2. Model formulation

In this paper, we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{0\le t\le T}$, T is a positive constant representing the fixed terminal time. In addition, we assumed that continuous investment is allowed and all securities are infinitely divisible, and the claim process can be approximated by a diffusion process.

2.1. Surplus process

Firstly, we consider that the insurer have no investment, and the surplus process $R(t)$ is given by

$$dR(t) = cdt - dC(t) = cdt - d\sum_{i=1}^{N(t)} Z_i,$$

where c is the premium rate, $N(t)$ is a homogeneous Poission process with intensity $\lambda > 0$, representing the number of claims up to time t. The claim sizes Z_i is a sequence of i.i.d. (independent and identically distributed) nonnegative random variables. In addition, $N(t)$ and Z_i are mutually independent. Here, premium rate $c = \lambda E(Z_1) + \alpha_1\lambda E(Z_1^2)$ is obtained under the variance premium principle, and $\alpha_1 > 0$ is the safety loading of the insurer. Furthermore, we denote the first two moments of Z_1 as $E(Z_1) = \mu$, $E(Z_1^2) = \mu_2$. Refer to Grandell [29], the net claims process can be approximated by a diffusion process $\tilde{C}(t)$:

$$\tilde{C}(t) = adt - \sigma_0 dW_t^{(0)},$$

where $a = \lambda\mu$, $\sigma_0 = \sqrt{\lambda\mu_2}$ are positive constants, $W_t^{(0)}$ is a standard Brownian motion on the complete probability space $(\Omega, \mathcal{F}, P)$. Then the surplus process of the insurer becomes

$$d\tilde{R}(t) = cdt - d\tilde{C}(t) = \lambda\alpha_1\mu_2 dt + \sigma_0 dW_t^{(0)}.$$

2.2. Reinsurance and investment

Note that typically we allow the insurer to continuously reinsure part of the claim to reducing the underlying claims risk. Let $q(t)$ be the reinsurance retention level at time t, usually called the risk exposure, $1 - q(t)$ represents the proportional reinsurance level. Then, the surplus process of the reinsurer can be described by

$$dR_2(t) = c_2 dt - [1 - q(t)]d\tilde{C}(t) = \lambda\alpha_2[1 - q(t)]^2\mu_2 dt + [1 - q(t)]\sigma_0 dW_t^{(0)},$$

where $c_2 = \lambda\mu[1-q(t)] + \alpha_2\lambda\mu_2[1 - q(t)]^2$ is obtained by the variance premium principle. Furthermore, we assume that $\alpha_2 > \alpha_1$ which means the reinsurance is non-cheap. Then, the surplus process of the insurer $R_1(t)$ becomes

$$dR_1(t) = d\tilde{R}(t) - dR_2(t) = [\lambda\alpha_1\mu_2 - \lambda\alpha_2(1 - q(t))^2\mu_2]dt + q(t)\sigma_0 dW_t^{(0)}.$$

Remark 2.1. In this paper, we require that the risk exposure $q(t)$ needs to meet the net profit condition, so we obtain $0 < 1 - \sqrt{\frac{\alpha_1}{\alpha_2}} \le q(t) \le 1$ from $\lambda\alpha_1\mu_2 - \lambda\alpha_2(1 - q(t))^2\mu_2 \ge 0$.

Furthermore, we assume that there is a financial market consist of a risk-free asset and two risky assets. Both the insurer and reinsurer are allowed to invest in the financial market. Suppose that $B(t)$ is the price process of the risk-free asset

$$dB(t) = r_0 B(t)dt,\ B(0) = 1, \tag{2.1}$$

where $r_0 > 0$ is the risk-free interest rate. The price of the risky assets invested by the insurer and reinsurer are described by constant elasticity of variance (CEV) model, which are given by

$$\begin{aligned} dS_1(t) &= S_1(t)(r_1 dt + S_1^{\beta_1}(t)\sigma_1 dW_t^{(1)}), S_1(0) = s_1, \\ dS_2(t) &= S_2(t)(r_2 dt + S_2^{\beta_2}(t)\sigma_2 dW_t^{(2)}), S_2(0) = s_2, \end{aligned} \tag{2.2}$$

where $r_1 > 0$, $r_2 > 0$ are expected instantaneous rates of return of the risky assets. Without any loss of generality, we assume that $r_1 > r_0$, $r_2 > r_0$. $S_1^{\beta_1}(t)\sigma_1$, $S_2^{\beta_2}(t)\sigma_2$ are instantaneous volatilities, β_1, β_2 are elasticity parameters which satisfy the general condition $\beta_1 \leq 0, \beta_2 \leq 0$. $W_t^{(0)}$, $W_t^{(1)}$ and $W_t^{(2)}$ are mutually independent Brownian motions defined on the complete probability space $(\Omega, \mathcal{F}, P)$, i.e., $E[W_t^{(0)}W_t^{(1)}] = 0$, $E[W_t^{(0)}W_t^{(2)}] = 0$ and $E[W_t^{(1)}W_t^{(2)}] = 0$.

2.3. Wealth process

In this paper, we suppose that the insurer can invest in the risk-free asset and risky asset 1, while the reinsurer can invest in the risk-free asset and risky asset 2.

We denote by $A_1(t)$ the amount of investment in risky asset 1 at time t when the insurer's wealth is $X(t)$, $X(t) - A_1(t)$ represents the amount of investment in risk-free asset. Meanwhile, $A_2(t)$ is the amount of investment in risky asset 2 at time t when the reinsurer's wealth is $Y(t)$, $Y(t) - A_2(t)$ represents the amount of investment in risk-free asset. We allow $A_1(t) < 0, X(t) < A_1(t)$ and $A_2(t) < 0, Y(t) < A_2(t)$, in other words, suppose that both the insurer and reinsurer can oversell risky assets and borrow risk-free asset. Then, the insurer's wealth process is given by

$$\begin{aligned} dX(t) &= A_1(t)\frac{dS_1(t)}{S_1(t)} + (X(t) - A_1(t))\frac{dB(t)}{B(t)} + dR_1 \\ &= \left\{A_1(t)(r_1 - r_0) + r_0 X(t) + \lambda\alpha_1\mu_2 - \lambda\alpha_2[1 - q(t)]^2\mu_2\right\}dt + q(t)\sigma_0 W_t^{(0)} + A_1(t)S_1^{\beta_1}(t)\sigma_1 dW_t^{(1)} \end{aligned} \tag{2.3}$$

with the initial condition $X(0) = x$. Similarly, the reinsurer's wealth process is given by

$$\begin{aligned} dY(t) &= A_2(t)\frac{dS_2(t)}{S_2(t)} + (Y(t) - A_2(t))\frac{dB(t)}{B(t)} + dR_2 \\ &= \left\{A_2(t)(r_2 - r_0) + r_0 Y(t) + \lambda\alpha_2[1 - q(t)]^2\mu_2\right\}dt + [1 - q(t)]\sigma_0 W_t^{(0)} + A_2(t)S_2^{\beta_2}(t)\sigma_2 dW_t^{(2)} \end{aligned} \tag{2.4}$$

with the initial condition $Y(0) = y$.

3. Optimal strategy for the joint exponential utility of terminal wealth

According to Ferguson [30], the goal of investor is to maximize the utility of wealth at a fixed terminal time when the investor has an exponential utility function. Gerber [31] mentioned that the exponential utility function plays an important role in insurance mathematics and it is the only utility function under the principle of "zero utility". Inspired by Huang et al. [28], we suppose that the insurer and reinsurer have the joint exponential utility function

$$U(x,y) = -\frac{\eta_1\eta_2}{m_1 m_2} e^{-m_1 x - m_2 y}, \quad m_1 \neq m_2,$$

where m_1, m_2, η_1, η_2 are positive contants, m_1, m_2 are the risk aversion coefficients of the insurer and reinsurer. This utility function is not simply a utility function obtained by multiplying two exponential utilities. It is formed by the fusion of two functions and contains the properties of exponential utility function, which satisfies $U_x > 0, U_y > 0$ and $U_{xx} < 0, U_{yy} < 0$.

Definition 3.1. Let $\Gamma = (t, s_1, s_2, x, y)$, $\Lambda := \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, $\Theta := [0,T]\times\Lambda$. Then, a strategy $\pi(t) = (A_1(t), A_2(t), q(t))$, $t \in [0,T]$ is said to be admissible if it satisfies the following conditions:

(i) $\forall t \in [0,T], q(t) \in [1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, 1]$;

(ii) $\forall\Gamma \in \Theta$, both the Eqs (2.3) and (2.4) have a strong solution $\{X^{t,x}(s), s \in [t,T]\}$ and $\{Y^{t,y}(s), s \in [t,T]\}$ with the initial condition $S_1(t) = s_1, S_2(t) = s_2, X(t) = x, Y(t) = y$;

(iii) $E[\int_0^T A_i^2(t) S_i^{2\beta_i}(t)dt] < \infty, i = 1,2$;

(iv) $E^\pi\{U[X(T), Y(T)] | S_1(t) = s_1, S_2(t) = s_2, X(t) = x, Y(t) = y\} < \infty$, where $\pi(t) = (A_1(t), A_2(t), q(t)) \in \Pi$, $t \in [0,T]$ is the proportional reinsurance-investment strategy and Π is the set of admissible strategies π. We suppose that $\pi(t)$ is $\mathcal{F}_t$−predictable with $\mathcal{F}_t = \sigma(X(s), Y(s), S_1(s), S_2(s), s \le t)$.

Assume that we are interested in maximizing the joint exponential utility of terminal wealth at a fixed time T. In order to apply the classical tools of stochastic optimal control, now we introduce the associated value function

$$V(t, s_1, s_2, x, y) = \sup_{(A_1, A_2, q)\in\Pi} E\{U[X(T), Y(T)] | S_1(t) = s_1, S_2(t) = s_2, X(t) = x, Y(t) = y\},\ t \in [0,T],$$

with boundary condition $V(T, s_1, s_2, x, y) = U(x,y)$.

To solve the above problem, we use the dynamic programming approach described in Fleming and Soner [32]. Suppose that $C^{1,2,2,2,2}([0,T)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$ is the space of $V(t, s_1, s_2, x, y)$, which are first-order continuously differentiable in $t \in [0,T]$, second-order continuously differentiable in $x \in \mathbb{R}$, $y \in \mathbb{R}$, $s_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}$. Denote V_t, V_x, V_y, V_{s_1}, V_{s_2}, V_{xx}, V_{yy}, $V_{s_1s_1}$, $V_{s_2s_2}$, V_{xs_1}, V_{ys_2} and V_{xy} as the first and second partial derivatives of V, which are continuous on $C^{1,2,2,2,2}$. Then V satisfies the following Hamilton-Jacobi-Bellman equation

$$\sup_{(A_1,A_2,q)\in\Pi} \mathcal{A}^{A_1,A_2,q} V(t, s_1, s_2, x, y) = 0, \tag{3.1}$$

where

$$\begin{aligned}
&\mathcal{A}^{A_1,A_2,q} V(t, s_1, s_2, x, y) \\
&= V_t + r_1 s_1 V_{s_1} + r_2 s_2 V_{s_2} + [A_1(r_1 - r_0) + r_0 x + \lambda\alpha_1\mu_2 - \lambda\alpha_2(1-q)^2\mu_2]V_x + [A_2(r_2 - r_0) + r_0 y \\
&\quad + \lambda\alpha_2(1-q)^2\mu_2]V_y + \frac{1}{2}(A_1^2 s_1^{2\beta_1}\sigma_1^2 + q^2\sigma_0^2)V_{xx} + \frac{1}{2}[A_2^2 s_2^{2\beta_2}\sigma_2^2 + (1-q)^2\sigma_0^2]V_{yy} + \frac{1}{2}s_1^{2\beta_1+2}\sigma_1^2 V_{s_1s_1} \\
&\quad + \frac{1}{2}s_2^{2\beta_2+2}\sigma_2^2 V_{s_2s_2} + \sigma_0^2 q(1-q)V_{xy} + A_1 s_1^{2\beta_1+1}\sigma_1^2 V_{xs_1} + A_2 s_2^{2\beta_2+1}\sigma_2^2 V_{ys_2}.
\end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1), we have the following HJB equation

$$\begin{aligned}&V_t + \sup_{(A_1,A_2,q)\in\Pi}\Big\{r_1s_1V_{s_1} + r_2s_2V_{s_2} + [A_1(r_1-r_0) + r_0x + \lambda\alpha_1\mu_2 - \lambda\alpha_2(1-q)^2\mu_2]V_x + [A_2(r_2-r_0)\\&\quad + r_0y + \lambda\alpha_2(1-q)^2\mu_2]V_y + \frac{1}{2}(A_1^2s_1^{2\beta_1}\sigma_1^2 + q^2\sigma_0^2)V_{xx} + \frac{1}{2}[A_2^2s_2^{2\beta_2}\sigma_2^2 + (1-q)^2\sigma_0^2]V_{yy}\\&\quad + \frac{1}{2}s_1^{2\beta_1+2}\sigma_1^2V_{s_1s_1} + \frac{1}{2}s_2^{2\beta_2+2}\sigma_2^2V_{s_2s_2} + \sigma_0^2q(1-q)V_{xy} + A_1s_1^{2\beta_1+1}\sigma_1^2V_{xs_1} + A_2s_2^{2\beta_2+1}\sigma_2^2V_{ys_2}\Big\} = 0.\end{aligned} \tag{3.3}$$

Inspired by Li et al. [23], we try a solution to (3.3) by

$$V(t,s_1,s_2,x,y) = -\frac{\eta_1\eta_2}{m_1m_2}e^{[-m_1x-m_2y-d(t)]e^{r_0(T-t)+g(t,s_1,s_2)}}, \tag{3.4}$$

with the boundary condition $g(T,s_1,s_2)=0$ and $d(T)=0$. Let g_t, g_{s_1}, g_{s_2}, $g_{s_1s_1}$ and $g_{s_2s_2}$ be the first and second partial derivatives of g with respect to (w.r.t) t, s_1 and s_2, which are given by

$$\begin{aligned}&V_t = \{-r_0e^{r_0(T-t)}[-m_1x-m_2y-d(t)] - d_te^{r_0(T-t)} + g_t\}V,\ \ V_x = -m_1e^{r_0(T-t)}V,\ \ V_y = -m_2e^{r_0(T-t)}V,\\&V_{s_1} = g_{s_1}V,\ \ V_{s_2} = g_{s_2}V,\ \ V_{xx} = m_1^2e^{2r_0(T-t)}V,\ \ V_{yy} = m_2^2e^{2r_0(T-t)}V,\ \ V_{s_1s_1} = (g_{s_1s_1} + g_{s_1}^2)V,\\&V_{s_2s_2} = (g_{s_2s_2} + g_{s_2}^2)V,\ V_{xs_1} = -m_1e^{r_0(T-t)}g_{s_1}V,\ V_{ys_2} = -m_2e^{r_0(T-t)}g_{s_2}V,\ V_{xy} = m_1m_2e^{2r_0(T-t)}V.\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.3), after simplification, we obtain

$$\begin{aligned}&[r_0d(t) - d_t - m_1\lambda\alpha_1\mu_2]e^{r_0(T-t)} + g_t + r_1s_1g_{s_1} + r_2s_2g_{s_2} + \frac{1}{2}s_1^{2\beta_1+2}\sigma_1^2(g_{s_1s_1} + g_{s_1}^2)\\&+ \frac{1}{2}s_2^{2\beta_2+2}\sigma_2^2(g_{s_2s_2} + g_{s_2}^2) + \inf_{A_1}\Big[-A_1m_1e^{r_0(T-t)}(r_1 - r_0 + s_1^{2\beta_1+1}\sigma_1^2g_{s_1}) + \frac{1}{2}A_1^2s_1^{2\beta_1}\sigma_1^2m_1^2e^{2r_0(T-t)}\Big]\\&+ \inf_{A_2}\Big[-A_2m_2e^{r_0(T-t)}(r_2 - r_0 + s_2^{2\beta_2+1}\sigma_2^2g_{s_2}) + \frac{1}{2}A_2^2s_2^{2\beta_2}\sigma_2^2m_2^2e^{2r_0(T-t)}\Big]\\&+ \inf_{q}\Big\{\lambda\alpha_2(1-q)^2\mu_2e^{r_0(T-t)}(m_1 - m_2) + \frac{1}{2}\sigma_0^2e^{2r_0(T-t)}[q^2m_1^2 + (1-q)^2m_2^2]\\&+ \sigma_0^2q(1-q)m_1m_2e^{2r_0(T-t)}\Big\} = 0,\end{aligned} \tag{3.6}$$

for $0 < t < T$. Let

$$f_1(A_1,t) = -A_1m_1e^{r_0(T-t)}(r_1 - r_0 + s_1^{2\beta_1+1}\sigma_1^2g_{s_1}) + \frac{1}{2}A_1^2s_1^{2\beta_1}\sigma_1^2m_1^2e^{2r_0(T-t)}, \tag{3.7}$$

$$f_2(A_2,t) = -A_2m_2e^{r_0(T-t)}(r_2 - r_0 + s_2^{2\beta_2+1}\sigma_2^2g_{s_2}) + \frac{1}{2}A_2^2s_2^{2\beta_2}\sigma_2^2m_2^2e^{2r_0(T-t)}, \tag{3.8}$$

and

$$\begin{aligned}f(q,t) =& \lambda\alpha_2(1-q)^2\mu_2e^{r_0(T-t)}(m_1 - m_2) + \sigma_0^2q(1-q)m_1m_2e^{2r_0(T-t)}\\&+ \frac{1}{2}\sigma_0^2e^{2r_0(T-t)}[q^2m_1^2 + (1-q)^2m_2^2].\end{aligned} \tag{3.9}$$

Differentiating (3.7) with respect to A_1 gives the insurer's optimal investment strategy

$$A_1^*(t) = \frac{r_1 - r_0 + s_1^{2\beta_1+1}\sigma_1^2g_{s_1}}{m_1s_1^{2\beta_1}\sigma_1^2}e^{-r_0(T-t)}. \tag{3.10}$$

Similarly, differentiating (3.8) with respect to A_2 gives the reinsurer's optimal investment strategy

$$A_2^*(t) = \frac{r_2 - r_0 + s_2^{2\beta_2+1}\sigma_2^2 g_{s_2}}{m_2 s_2^{2\beta_2}\sigma_2^2} e^{-r_0(T-t)}. \tag{3.11}$$

To find the value of $q^*(t)$ that minimizes f, we need to take the first and the second derivatives of f w.r.t q. Then, $\frac{\partial f(q,t)}{\partial q}$ and $\frac{\partial^2 f(q,t)}{\partial q^2}$ are given by

$$\begin{aligned}\frac{\partial f(q,t)}{\partial q} =& \sigma_0^2[qm_1^2 + (q-1)m_2^2]e^{2r_0(T-t)} + \sigma_0^2(1-2q)m_1 m_2 e^{2r_0(T-t)} \\ &+ 2(q-1)\lambda\alpha_2\mu_2(m_1 - m_2)e^{r_0(T-t)},\end{aligned} \tag{3.12}$$

and

$$\frac{\partial^2 f(q,t)}{\partial q^2} = 2\lambda\alpha_2\mu_2(m_1 - m_2)e^{r_0(T-t)} + \sigma_0^2(m_1 - m_2)^2 e^{2r_0(T-t)}. \tag{3.13}$$

Let $\frac{\partial f(q,t)}{\partial q} = 0$, we get

$$\bar{q}(t) = \frac{2\lambda\alpha_2\mu_2 - \sigma_0^2 m_2 e^{r_0(T-t)}}{2\lambda\alpha_2\mu_2 + \sigma_0^2(m_1 - m_2)e^{r_0(T-t)}} = 1 - \frac{m_1\sigma_0^2 e^{r_0(T-t)}}{2\lambda\alpha_2\mu_2 + (m_1 - m_2)\sigma_0^2 e^{r_0(T-t)}}. \tag{3.14}$$

Let $\frac{\partial^2 f(q,t)}{\partial q^2} = 0$, we obtain

$$2\lambda\alpha_2\mu_2 + \sigma_0^2(m_1 - m_2)e^{r_0(T-t)} = 0. \tag{3.15}$$

Since $q(t) \in [1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, 1]$, we need to discuss the size relationship of $\bar{q}(t)$, $1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$ and 1, which is closely related to the concavity/convexity of $f(q,t)$.

Note that $\bar{q} = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$ iff

$$\left[m_1\sigma_0^2 e^{r_0(T-t)} - \sqrt{\frac{\alpha_1}{\alpha_2}}\left(2\lambda\alpha_2\mu_2 + (m_1 - m_2)\sigma_0^2 e^{r_0(T-t)}\right)\right] \times \left[2\lambda\alpha_2\mu_2 + (m_1 - m_2)\sigma_0^2 e^{r_0(T-t)}\right] = 0, \tag{3.16}$$

then, from (3.15) and (3.16), we denote

$$\Delta_1 = \frac{2\lambda\alpha_2\mu_2}{(m_2 - m_1)\sigma_0^2}, \quad \Delta_2 = \frac{2\lambda\alpha_2\mu_2}{[m_2 + m_1(\sqrt{\frac{\alpha_2}{\alpha_1}} - 1)]\sigma_0^2}. \tag{3.17}$$

It is easy to see that $\Delta_1 > \Delta_2 > 0$ when $m_1 < m_2$, and $\Delta_2 > 0 > \Delta_1$ when $m_1 > m_2$.

Remark 3.1. If $m_1 = m_2 = m$, Eq (3.9) becomes $f(q,t) = f(t) = \frac{1}{2}m^2\sigma_0^2 e^{2r_0(T-t)}$, it indicates that there is no reinsurance, the optimal investment-reinsurance problem is transformed into a pure investment problem. If $\beta_1, \beta_2 = 0$, it is equivalent to the price of risky assets described by GBM model, we find that the optimal investment strategy $A_1^*(t), A_2^*(t)$ will change, but the optimal reinsurance strategy is not affected, this shows that the optimal reinsurance strategy is independent of the price of risky assets.

Lemma 3.1. *If the parameters r_0, r_1, r_2 and T satisfy one of the following conditions:*

(1) $r_0 \geq (1 - \frac{1}{\sqrt{6}})r_1$ *and* $r_0 \geq (1 - \frac{1}{\sqrt{6}})r_2$*;*
(2) $r_0 < (1 - \frac{1}{\sqrt{6}})r_1$, $r_0 < (1 - \frac{1}{\sqrt{6}})r_2$ *and* $T < \frac{1}{\gamma_2}\text{arccot}(-\frac{\gamma_1}{\gamma_2})$*, where* $\gamma_1 = -\beta_1 r_1$*, and* $\gamma_2 = -\beta_1 \sqrt{6(r_1 - r_0)^2 - r_1^2}$*,*

then

$$E\Big\{\exp\Big[\int_0^t \frac{3(r_i - r_0)^2}{S_i^{2\beta_i}(u)\sigma_i^2}du\Big]\Big\} < \infty,\ i = 1, 2.$$

The detailed proof of this conclusion can be referred to Theorem 5.1 in Zeng and Taksar [33], we omit it here.

We give Lemma 3.2 for devoting to the proof of the verification theorem.

Lemma 3.2. *If conditions in Lemma 3.1 holds, and $\omega(t, s_1, s_2, x, y)$ is the solution of HJB equation* (3.1) *with the boundary $\omega(T, s_1, s_2, x, y) = U(x, y)$, then we have*

$$E[\omega^2(t, S_1(t), S_2(t), X^{\pi^*}(t), Y^{\pi^*}(t))] < \infty. \tag{3.18}$$

Proof. See Appendix A.

Theorem 3.1 (Verification theorem)**.** *Let $\omega(t, s_1, s_2, x, y) \in C^{1,2,2,2,2}$, and ω satisfies HJB equation* (3.1) *with boundary conditions $\omega(T, s_1, s_2, x, y) = U(x, y)$. Let $\pi^*(t) = \Big(A_1^*(t), A_2^*(t), q^*(t)\Big) \in \Pi$ such that $\mathcal{A}^{\pi^*}V(t, s_1, s_2, x, y) = 0$, then the value function $V(t, s_1, s_2, x, y) = \omega(t, s_1, s_2, x, y)$ and $\pi^*(t)$ is the optimal strategy.*

Proof. See Appendix B.

After giving the verification theorem, we now present the optimal reinsurance-investment strategy and value function in Theorem 3.2.

Theorem 3.2 (The optimal strategy and value function)**.** *The optimal investment strategy is given by*

$$A_1^*(t) = \frac{r_1^2 - r_0^2 - (r_1 - r_0)^2 e^{-2r_0\beta_1(T-t)}}{2r_0 m_1 s_1^{2\beta_1}\sigma_1^2} e^{-r_0(T-t)},$$
$$A_2^*(t) = \frac{r_2^2 - r_0^2 - (r_2 - r_0)^2 e^{-2r_0\beta_2(T-t)}}{2r_0 m_2 s_2^{2\beta_2}\sigma_2^2} e^{-r_0(T-t)}.$$

The optimal reinsurance strategy is given by

(1) If $m_1 > m_2$, $\Delta_2 \geq 1$ *and* $0 \leq t \leq T - \frac{\ln \Delta_2}{r_0}$*, then* $q^* = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$*,*
(2) If $m_1 > m_2$, $\Delta_2 \geq 1$ *and* $T - \frac{\ln \Delta_2}{r_0} < t < T$*, then* $q^*(t) = \bar{q}(t)$*,*
(3) If $m_1 > m_2$ *and* $\Delta_2 < 1$*, then* $q^* = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$*,*
(4) If $m_1 < m_2$, $\Delta_2 \geq 1$ *and* $0 < t < T - \frac{\ln \Delta_2}{r_0}$*, then* $q^* = 1$*,*

(5) If $m_1 < m_2$, $\Delta_2 \geq 1$ *and* $T - \frac{\ln \Delta_2}{r_0} < t < T$, *then*

$$q^* = \begin{cases} 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, & f\left(1 - \sqrt{\frac{\alpha_1}{\alpha_2}}\right) < f(1), \\ 1, & f\left(1 - \sqrt{\frac{\alpha_1}{\alpha_2}}\right) \geq f(1), \end{cases}$$

(6) If $m_1 < m_2$ *and* $\Delta_2 < 1$, *then* $q^* = 1$.

The explicit expressions of value function when q^* *takes different values are as follows:*

(1) When $q^* = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$, *the value function is given by*

$$V(t, s_1, s_2, x, y) = -\frac{\eta_1\eta_2}{m_1 m_2} e^{[-m_1 x - m_2 y - d(t)]e^{r_0(T-t)+g(t,s_1,s_2)}},$$

where

$$d(t) = -\frac{m_1\lambda\alpha_1\mu_2}{r_0}[e^{-r_0(T-t)} - 1] + \frac{\sigma_0^2}{2r_0}\Big[\Big(\frac{1}{2} - \sqrt{\frac{\alpha_1}{\alpha_2}}\Big){m_1}^2 + \frac{\alpha_1}{2\alpha_2}(m_1 - m_2)^2 + \sqrt{\frac{\alpha_1}{\alpha_2}} m_1 m_2\Big] \times [e^{-r_0(T-t)} - e^{r_0(T-t)}].$$

Let

$$g(t, s_1, s_2) = m(t, k_1, k_2) = I(t) + J_1(t)k_1 + J_2(t)k_2,\ k_1 = s_1^{-2\beta_1},\ k_2 = s_2^{-2\beta_2},$$

where

$$J_1(t) = \frac{(r_1 - r_0)^2}{4r_0\beta_1\sigma_1^2}[e^{-2r_0\beta_1(T-t)} - 1],\quad J_2(t) = \frac{(r_2 - r_0)^2}{4r_0\beta_2\sigma_2^2}[e^{-2r_0\beta_2(T-t)} - 1], \tag{3.19}$$

and

$$I(t) = \frac{(r_1 - r_0)^2(2\beta_1 + 1)}{4r_0}\Big[\frac{1 - e^{-2r_0\beta_1(T-t)}}{2r_0\beta_1} - (T - t)\Big] + \frac{(r_2 - r_0)^2(2\beta_2 + 1)}{4r_0} \times \Big[\frac{1 - e^{-2r_0\beta_2(T-t)}}{2r_0\beta_2} - (T - t)\Big]. \tag{3.20}$$

(2) When $q^*(t) = \bar{q}(t)$, *the value function is given by*

$$V(t, s_1, s_2, x, y) = -\frac{\eta_1\eta_2}{m_1 m_2} e^{[-m_1 x - m_2 y - d(t)]e^{r_0(T-t)+g(t,s_1,s_2)}},$$

where

$$d(t) = -\frac{m_1\lambda\alpha_1\mu_2}{r_0}[e^{-r_0(T-t)} - 1],$$

$$I(t) = \frac{(r_1 - r_0)^2(2\beta_1 + 1)}{4r_0}\Big[\frac{1 - e^{-2r_0\beta_1(T-t)}}{2r_0\beta_1} - (T - t)\Big] + \frac{\lambda\alpha_2\mu_2 m_1^2}{r_0(m_1 - m_2)}[e^{r_0(T-t)} - 1]$$

$$+\frac{(r_2-r_0)^2(2\beta_2+1)}{4r_0}\Big[\frac{1-e^{-2r_0\beta_2(T-t)}}{2r_0\beta_2}-(T-t)\Big]$$
$$+\frac{2\lambda^2\alpha_2^2\mu_2^2m_1^2}{r_0(m_1-m_2)^2\sigma_0^2}\ln\left|\frac{2\lambda\alpha_2\mu_2+(m_1-m_2)\sigma_0^2}{2\lambda\alpha_2\mu_2+(m_1-m_2)\sigma_0^2e^{r_0(T-t)}}\right|,$$

and $J_1(t)$*,* $J_2(t)$ *are given by* (3.19).

(3) When $q^*=1$*, the value function is given by*

$$V(t,s_1,s_2,x,y)=-\frac{\eta_1\eta_2}{m_1m_2}e^{[-m_1x-m_2y-d(t)]e^{r_0(T-t)+g(t,s_1,s_2)}},$$

where

$$d(t)=\frac{m_1^2\sigma_0^2}{4r_0}[e^{-r_0(T-t)}-e^{r_0(T-t)}]+\frac{m_1\lambda\alpha_1\mu_2}{r_0}[1-e^{-r_0(T-t)}],$$

$J_1(t)$ *and* $J_2(t)$ *are given by* (3.19)*,* $I(t)$ *is given by* (3.20).

Proof. See Appendix C.

Remark 3.2. Compared with [23], due to the premium and re-premium are stipulated by the variance premium principle, the extremum of the quadratic function (3.9) related to q in HJB equation becomes uncertain, therefore, we cannot solve the optimal strategy through the first derivative, and we need to solve at the level of second-order derivatives, which makes classification discussions more complex. In addition, we find that the optimal investment strategies are the same as [23], but the optimal reinsurance strategies are different, although their classification intervals are similar.

4. Numerical analysis

In this section, we provide some numerical analysis to study the influencing factors of the optimal reinsurance-investment strategy and explain the results for better understanding in the economic sense. Throughout this section, unless otherwise stated, we put $r_0 = 0.1$, $r_1 = 0.2$, $r_2 = 0.3$, $\beta_1 = -1$, $\beta_2 = -0.8$, $T = 5$, $m_1 = 1.8$, $m_2 = 1.3$, $s_1 = 5$, $s_2 = 8$, $\sigma_1 = 1$ and $\sigma_2 = 2$. We assume that the claim size Z_i are i.i.d. with common exponential distribution $E(\varphi)$, the intensity $\varphi = 1$, then $\mu = 1$, $\mu_2 = 2$ and $\sigma_0 = \sqrt{2}$. The number of claims $N(t)$ is assumed to be a Poisson process with intensity $\lambda = 1$.

Figure 1 shows that the amount invested by the insurer in risky asset is larger than the reinsurer at a certain time. This may be due to the fact that, the relative risk aversion coefficient of the insurer is larger than that of the reinsurer in this example, i.e., $m_1 > m_2$. As time passes, we find that the amount of risky assets invested by the insurer and reinsurer increases gradually over time when other parameters are fixed. The main reason is that, the rates of return of the risky assets are always larger than the interest rate of the risk-free asset, both the insurer and reinsurer will gain more profits from investment in the risky asset as time goes.

Figure 2 shows that the optimal investment strategy decreases with the risk aversion coefficient. The reason is that, the insurer will increase the reinsurance proportion and reduce the amount of investment in risky asset when the risk aversion coefficient becomes larger.

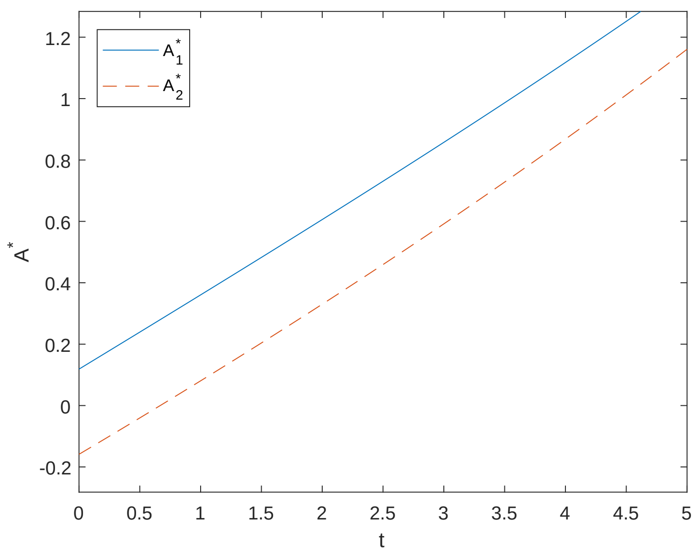

Figure 1. The optimal investment strategy A_1^*, A_2^* change over time.

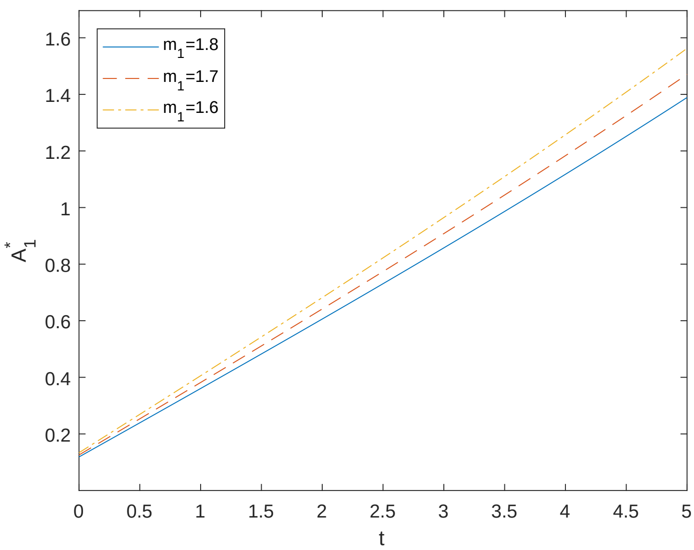

Figure 2. The effect of m_1 on the optimal investment strategy A_1^*.

In Figure 3, we find that near the initial time, the larger the risk aversion coefficient of reinsurer is, the more the amount of investment in risky asset is. This is because the larger the risk aversion coefficient is, the more reinsurance premiums are charged by reinsurers, and they can invest more money in risky asset. Furthermore, we also find that the amount of the reinsurer's investment in risky asset increases more gently with larger risk aversion coefficient.

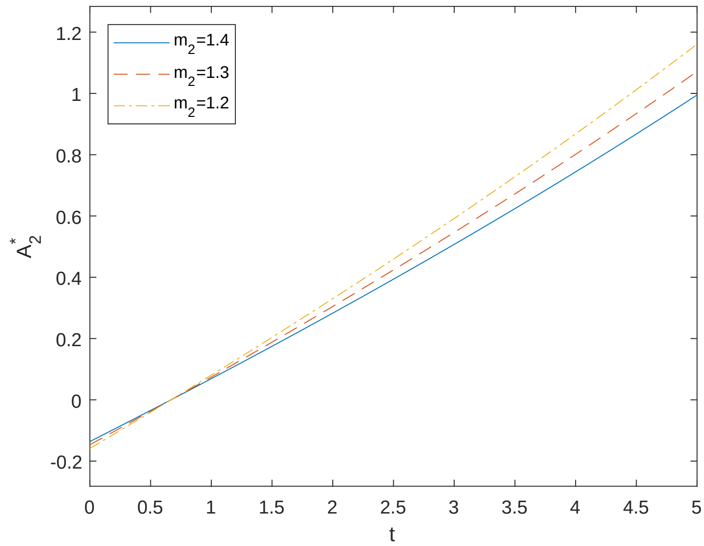

Figure 3. The effect of m_2 on the optimal investment strategy A_2^*.

For a fixed risk-free interest rate r_0, Figures 4 and 5 show that both the insurer and reinsurer will invest more in risky assets when the instantaneous rates of return of the risky assets increase. This coincides with our intuition.

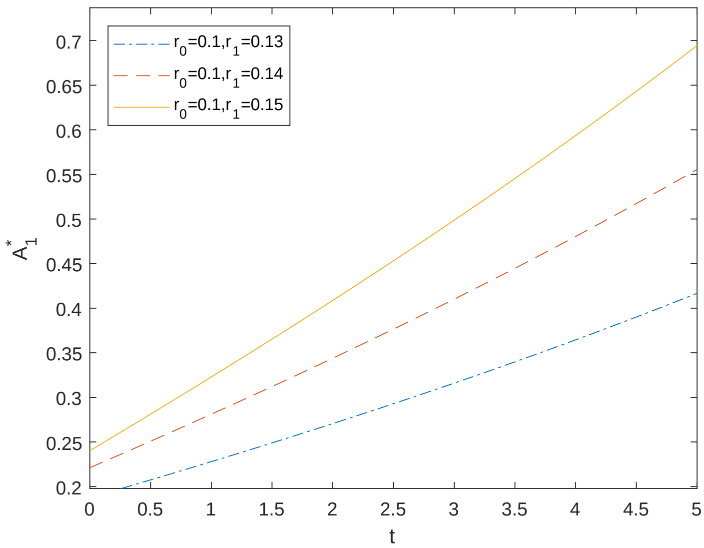

Figure 4. The effect of $r_1 - r_0$ on the optimal investment strategy A_1^*.

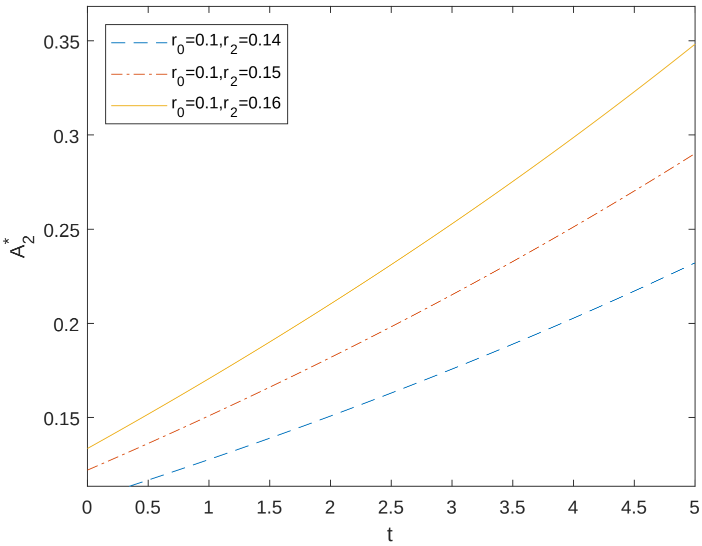

Figure 5. The effect of $r_2 - r_0$ on the optimal investment strategy A_2^*.

Let $\alpha_1 = 0.8$, then $\Delta_2 > 1$ with $m_1 = 1.8, m_2 = 1.3, \alpha_2 = 1.1, 1.2$, $\Delta_2 < 1$ with $m_1 = 2, m_2 = 1.9, \alpha_2 = 1.1, 1.2$. From Theorem 3.2, the optimal reinsurance strategy is a fixed constant $1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$ when $\Delta_2 < 1$. When $\Delta_2 > 1$, we find that in Figure 6 the larger α_2 is, the larger the initial retention level q^* is, and the earlier the optimal strategy is changed. This result can be explained by the fact that more reinsurance premium will be charged for larger α_2, so the insurer will buy less reinsurance. However, larger retention level means higher risk for the insurer, so the insurer will change the optimal reinsurance strategy earlier.

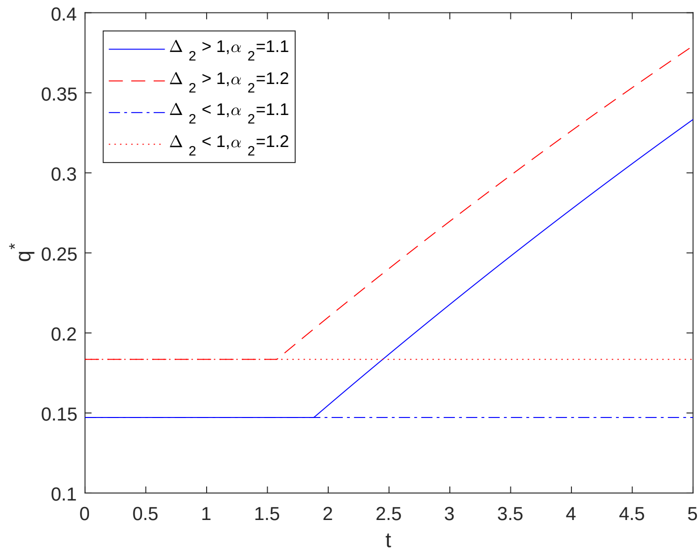

Figure 6. the optimal reinsurance retention level q^* vary over time when $m_1 > m_2$.

Figure 7 demonstrates the influence of m_1 on q^*, we vary m_1 from $1.9 \sim 2.1$. Let $m_2 = 1.2$, $\alpha_1 = 0.8$, $\alpha_2 = 1.2$, then we can calculate $\Delta_2 > 1$ with $m_1 = 1.9 \sim 2.1$, and we obtain $q^* = \overline{q}$ when $t \in \left(T - \frac{\ln \Delta_2}{r_0}, T\right)$. We find that when the risk aversion coefficient of the reinsurer is fixed, the insurer with larger risk aversion coefficient would like to buy more reinsurance.

Figure 8 shows the influence of m_2 on q^*, we vary m_2 from $1.2 \sim 1.4$. Let $m_1 = 2.0$, $\alpha_1 = 0.8$, $\alpha_2 = 1.2$, then we can calculate $\Delta_2 > 1$ with $m_2 = 1.2 \sim 1.4$, and we obtain $q^* = \overline{q}$ when $t \in \left(T - \frac{\ln \Delta_2}{r_0}, T\right)$. We find that when the risk aversion coefficient of the insurer is fixed, the reinsurer with larger risk aversion coefficient is willing to accept more claim risk. A possible reason for this phenomenon is that, the reinsurer with larger risk aversion coefficient will reduce the amount of investment in risky asset, and has more cash to hedge the claim risk.

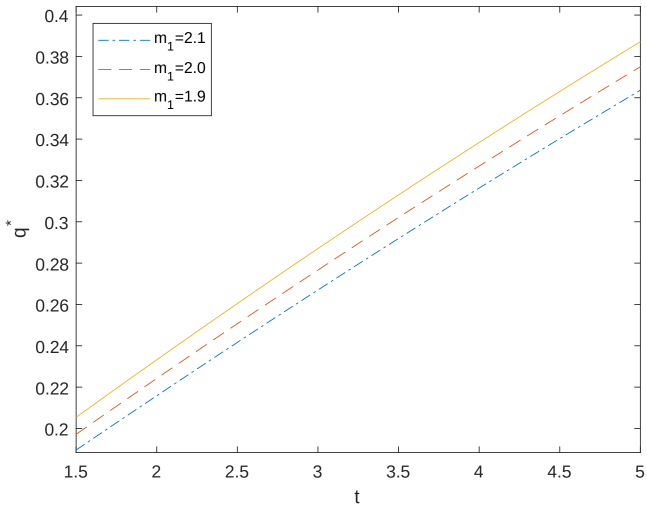

Figure 7. The effect of m_1 on the optimal reinsurance retention level q^*.

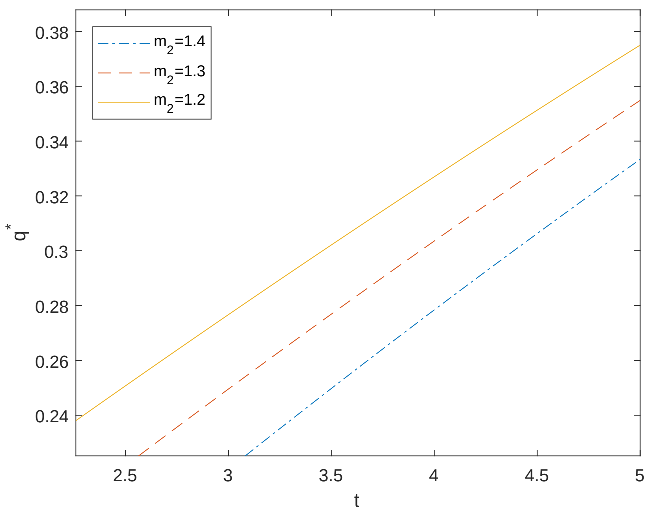

Figure 8. The effect of m_2 on the optimal reinsurance retention level q^*.

Let $m_1 = 1.8, m_2 = 1.3, \alpha_1 = 0.8$, then we can calculate $\Delta_2 > 1$ with $\alpha_2 = 1.1 \sim 1.3$. From Theorem 3.2, we obtain that $q^* = \overline{q}$ when $t \in \left(T - \frac{\ln \Delta_2}{r_0}, T\right)$. Figure 9 shows that q^* increases w.r.t time t and the safety loading of the reinsurer α_2. It can be explained that the larger the safety loading of the reinsurer, the more premium the insurer will pay, then the insurer will appropriately reduce the reinsurance proportion and increase the retention level.

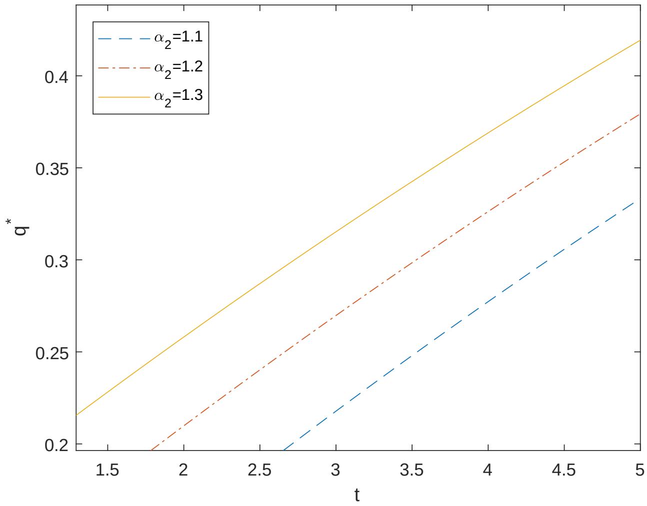

Figure 9. The effect of α_2 on the optimal reinsurance retention level q^*.

5. Conclusions

In this paper, we study the optimal investment and proportional reinsurance of the insurer and reinsurer under the joint exponential utility. We find that the optimal investment strategy is related to risk-free interest rate, the price and the instantaneous rate of return of the risky asset, and risk aversion coefficient. The value of the optimal reinsurance strategy can be divided into six different cases. Before reaching the terminal time T, the insurer needs to constantly adjust the reinsurance proportion to maximize their joint terminal wealth utility. Moreover, we get the explicit expression of the value function when the optimal reinsurance strategy q^* takes different values. Also, we use numerical simulation to illustrate the results in detail and explain the negative impact of risk aversion coefficient on risky assets investment and retention level, and the positive impact of reinsurer's safety load on retention level, etc. In the future, we will further study the excess-of-loss reinsurance with the price of the risky asset is described by Heston model, or consider different optimization criteria such as minimizing joint ruin probability of the insurer and reinsurer.

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Conflict of interest

The authors declare that there is no conflict of interest.

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Appendix A

Proof of Lemma 3.2

First, we can obtain the following equations by (2.2)

$$dS_1^{-2\beta_1}(t) = [\beta_1(2\beta_1+1)\sigma_1^2 - 2\beta_1 r_1 S_1^{-2\beta_1}(t)]dt - 2\beta_1\sigma_1\sqrt{S_1^{-2\beta_1}(t)}dW_t^{(1)}, \tag{A.1}$$

$$dS_2^{-2\beta_2}(t) = [\beta_2(2\beta_2+1)\sigma_2^2 - 2\beta_2 r_2 S_2^{-2\beta_2}(t)]dt - 2\beta_2\sigma_2\sqrt{S_2^{-2\beta_2}(t)}dW_t^{(2)}. \tag{A.2}$$

We denote $\omega(t) := \omega(t, S_1(t), S_2(t), X^{\pi^*}(t), Y^{\pi^*}(t))$, where $\omega(t, s_1, s_2, x, y)$ is given by (3.4). Applying Itô's formula to $\omega(t)$, we obtain

$$\begin{aligned}\frac{d\omega^2(t)}{\omega^2(t)} &= 2\Big\{ - m_1 q(t)\sigma_0 e^{r_0(T-t)}dW_t^{(0)} - m_1 A_1(t)S_1^{\beta_1}(t)\sigma_1 e^{r_0(T-t)}dW_t^{(1)} - m_2[1-q(t)]\sigma_0 e^{r_0(T-t)}dW_t^{(0)} \\ &\quad - m_2 A_2(t)S_2^{\beta_2}(t)\sigma_2 e^{r_0(T-t)}dW_t^{(2)} + g_{s_1}S_1^{\beta_1+1}(t)\sigma_1 dW_t^{(1)} + g_{s_2}S_2^{\beta_2+1}(t)\sigma_2 dW_t^{(2)}\Big\} + \Big\{m_1^2[q^2(t)\sigma_0^2 \\ &\quad + A_1^2(t)S_1^{2\beta_1}(t)\sigma_1^2]e^{2r_0(T-t)} + m_2^2[(1-q(t))^2\sigma_0^2 + A_2^2(t)S_2^{2\beta_2}(t)\sigma_2^2]e^{2r_0(T-t)} + g_{s_1}^2 S_1^{2\beta_1+2}(t)\sigma_1^2 \\ &\quad + g_{s_2}^2 S_2^{2\beta_2+2}(t)\sigma_2^2 + 2m_1 m_2 q(t)[1-q(t)]\sigma_0^2 e^{2r_0(T-t)} - 2m_1 A_1(t)S_1^{2\beta_1+1}(t)\sigma_1^2 g_{s_1}e^{r_0(T-t)} \\ &\quad - 2m_2 A_2(t)S_2^{2\beta_2+1}(t)\sigma_2^2 g_{s_2}e^{r_0(T-t)}\Big\}dt,\end{aligned}$$

which yields

$$\begin{aligned}\frac{d\omega^2(t)}{\omega^2(t)} &= -2\Big\{[m_1 q(t) + m_2(1-q(t))]\sigma_0 e^{r_0(T-t)}dW_t^{(0)} + \frac{r_1 - r_0}{S_1^{\beta_1}(t)\sigma_1}dW_t^{(1)} + \frac{r_2 - r_0}{S_2^{\beta_2}(t)\sigma_2}dW_t^{(2)}\Big\} \\ &\quad + \Big\{[m_1 q(t) + m_2(1-q(t))]^2\sigma_0^2 e^{2r_0(T-t)} + \frac{(r_1-r_0)^2}{S_1^{2\beta_1}(t)\sigma_1^2} + \frac{(r_2-r_0)^2}{S_2^{2\beta_2}(t)\sigma_2^2}\Big\}dt.\end{aligned}$$

The solution of (A.3) is

$$\begin{aligned}\frac{\omega^2(t)}{\omega^2(0)} = \exp\Big\{ \int_0^t -2[m_1 q(u) + m_2(1-q(u))]\sigma_0 e^{r_0(T-u)} dW_u^{(0)} - \frac{1}{2}\int_0^t 4[m_1 q(u) + m_2(1-q(u))]^2 \\ \times \sigma_0^2 e^{2r_0(T-u)} du - \int_0^t \frac{2(r_1-r_0)}{S_1^{\beta_1}(u)\sigma_1} dW_u^{(1)} - \frac{1}{2}\int_0^t \frac{4(r_1-r_0)^2}{S_1^{2\beta_1}(u)\sigma_1^2} du - \int_0^t \frac{2(r_2-r_0)}{S_2^{\beta_2}(u)\sigma_2} dW_u^{(2)} \\ - \frac{1}{2}\int_0^t \frac{4(r_2-r_0)^2}{S_2^{2\beta_2}(u)\sigma_2^2} du + \int_0^t [\frac{3(r_1-r_0)^2}{S_1^{2\beta_1}(u)\sigma_1^2} + \frac{3(r_2-r_0)^2}{S_2^{2\beta_2}(u)\sigma_2^2} + 3(m_1 q(u) + m_2(1-q(u)))^2 \\ \sigma_0^2 e^{2r_0(T-u)}] du\Big\}.\end{aligned} \tag{A.3}$$

Furthermore, by Lemma 1 of Gu et al. [7], we know that

$$\begin{aligned}\exp\Big\{ \int_0^t -\frac{2(r_1-r_0)}{S_1^{\beta_1}(u)\sigma_1} dW_u^{(1)} - \frac{1}{2}\int_0^t \frac{4(r_1-r_0)^2}{S_1^{2\beta_1}(u)\sigma_1^2} du\Big\}, \\ \exp\Big\{ \int_0^t -\frac{2(r_2-r_0)}{S_2^{\beta_2}(u)\sigma_2} dW_u^{(2)} - \frac{1}{2}\int_0^t \frac{4(r_2-r_0)^2}{S_2^{2\beta_2}(u)\sigma_2^2} du\Big\},\end{aligned}$$

are martingales. According to Lemma 3.1, we obtain

$$E\Big\{\exp\Big[\int_0^t \frac{3(r_1-r_0)^2}{S_1^{2\beta_1}(u)\sigma_1^2} du\Big]\Big\} < \infty, \quad E\Big\{\exp\Big[\int_0^t \frac{3(r_2-r_0)^2}{S_2^{2\beta_2}(u)\sigma_2^2} du\Big]\Big\} < \infty.$$

Then taking the expectation on both sides of Eq (A.3), we obtain

$$E[\omega^2(t)] = \omega^2(0) E\Big\{\int_0^t \Big[3(m_1 q(u) + m_2(1-q(u)))^2 \sigma_0^2 e^{2r_0(T-u)} + \frac{3(r_1-r_0)^2}{S_1^{2\beta_1}(u)\sigma_1^2} + \frac{3(r_2-r_0)^2}{S_2^{2\beta_2}(u)\sigma_2^2}\Big] du\Big\} < \infty.$$

This ends the proof of Lemma 3.2.

Appendix B

Proof of Theorem 3.1

We first prove that $\omega = V$, and then prove that π^* is the optimal policy.

(i) Since ω is a function in $C^{1,2,2,2,2}([0,T) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$, for all $t \in [0,T)$, $\pi \in \Pi$, and any stopping time $\tau \in [t, \infty)$, applying Itô's formula to ω between t and $T \wedge \tau$, we obtain that

$$\begin{aligned}&\omega(T\wedge\tau, S_1(T\wedge\tau), S_2(T\wedge\tau), X^\pi(T\wedge\tau), Y^\pi(T\wedge\tau)) \\ &= \omega(t, s_1, s_2, x, y) + \int_t^{T\wedge\tau} \mathcal{A}^\pi \omega(u, S_1(u), S_2(u), X^\pi(u), Y^\pi(u)) du \\ &+ \int_t^{T\wedge\tau} \Big\{q(u)\omega_x + [1-q(u)]\,\omega_y\Big\} \sigma_0 dW_u^{(0)} + \int_t^{T\wedge\tau} [\omega_{s_1} S_1^{\beta_1+1}(u) + \omega_x A_1(u) S_1^{\beta_1}(u)]\sigma_1 dW_u^{(1)} \\ &+ \int_t^{T\wedge\tau} [\omega_{s_2} S_2^{\beta_2+1}(u) + \omega_y A_2(u) S_2^{\beta_2}(u)]\sigma_2 dW_u^{(2)}.\end{aligned} \tag{B.1}$$

Since $\omega \in C^{1,2,2,2,2}$, ω_x, ω_y, ω_{s_1} and ω_{s_2} are bounded and continuous in $[t, T \wedge \tau]$, then we have

$$E\left[\int_t^{T\wedge\tau} [q(u)\omega_x + (1-q(u))\omega_y]^2 \sigma_0^2 du\right] \leq k_1 \sigma_0^2 (T-t),$$

$$\begin{aligned}\int_t^{T\wedge\tau} \left[\omega_{s_1} S_1^{\beta_1+1}(u) + \omega_x A_1(u) S_1^{\beta_1}(u)\right]^2 \sigma_1^2 du &\leq k_2 \int_t^{T\wedge\tau} \left[S_1^{\beta_1+1}(u) + A_1(u) S_1^{\beta_1}(u)\right]^2 \sigma_1^2 du \\ &\leq 2k_2 \int_t^{T\wedge\tau} [S_1^{2\beta_1+2}(u) + A_1^2(u) S_1^{2\beta_1}(u)] \sigma_1^2 du,\end{aligned}$$

and

$$\int_t^{T\wedge\tau} \left[\omega_{s_2} S_2^{\beta_2+1}(u) + \omega_y A_2(u) S_2^{\beta_2}(u)\right]^2 \sigma_2^2 du \leq 2k_3 \int_t^{T\wedge\tau} [S_2^{2\beta_2+2}(u) + A_2^2(u) S_2^{2\beta_2}(u)] \sigma_2^2 du,$$

where k_1, k_2 and k_3 are positive constants.
Let

$$\tau_n = T \wedge \inf\{s \geq t : H_1(s) \geq n\} \wedge \inf\{s \geq t : H_2(s) \geq n\} \tag{B.2}$$

where

$$H_1(s) = \int_t^s S_1^{2(\beta_1+1)}(u)\sigma_1^2 du, \quad H_2(s) = \int_t^s S_2^{2(\beta_2+1)}(u)\sigma_2^2 du. \tag{B.3}$$

Hence, by (B.2), (B.3) and $E[\int_0^T A_i^2(t) S_i^{2\beta_i}(t) dt] < \infty, i = 1, 2$, the last three terms of (B.1) are square-integrable martingales for $\tau = \tau_n$. Then taking conditional expectation given (t, s_1, s_2, x, y) on both sides of (B.1), we get

$$\begin{aligned}&E^{t,s_1,s_2,x,y}[\omega(T\wedge\tau_n, S_1(T\wedge\tau_n), S_2(T\wedge\tau_n), X^\pi(T\wedge\tau_n), Y^\pi(T\wedge\tau_n))] \\ &= \omega(t, s_1, s_2, x, y) + E^{t,s_1,s_2,x,y}\left[\int_t^{T\wedge\tau_n} \mathcal{A}^\pi \omega(u) du\right] \\ &\leq \omega(t, s_1, s_2, x, y),\end{aligned} \tag{B.4}$$

where $\omega(u) = \omega(u, S_1(u), S_2(u), X^\pi(u), Y^\pi(u))$. By Eq (3.18) in Lemma 3.2, we know that $E[\omega(T \wedge \tau_n, S_1(T\wedge\tau_n), S_2(T\wedge\tau_n), X^\pi(s\wedge\tau_n), Y^\pi(T\wedge\tau_n))]$, $n = 1, 2, \ldots$ are uniformly integrable. As a result, for any $\pi \in \Pi$, we have

$$\begin{aligned}&E^{t,s_1,s_2,x,y}[U(X^\pi(T), Y^\pi(T))] \\ &= \lim_{n\to\infty} E^{t,s_1,s_2,x,y}[\omega(T\wedge\tau_n, S_1(T\wedge\tau_n), S_2(T\wedge\tau_n), X^\pi(T\wedge\tau_n), Y^\pi(T\wedge\tau_n))] \\ &\leq \omega(t, s_1, s_2, x, y).\end{aligned}$$

Therefore,

$$\begin{aligned}V(t, s_1, s_2, x, y) &= \sup_{\pi\in\Pi} E^{t,s_1,s_2,x,y}[U(X^\pi(T), Y^\pi(T))] \\ &\leq \omega(t, s_1, s_2, x, y).\end{aligned}$$

(ii) Suppose that π^* is a measurable function valued in Π such that

$$\begin{aligned} & -\frac{\partial \omega}{\partial t}(t, s_1, s_2, x, y) - \sup_{\pi \in \Pi} \mathcal{L}^{\pi} \omega(t, s_1, s_2, x, y) \\ = & -\frac{\partial \omega}{\partial t}(t, s_1, s_2, x, y) - \mathcal{L}^{\pi^*} \omega(t, s_1, s_2, x, y) = 0. \end{aligned}$$

Then it is easy to see that the inequality in (B.4) becomes an equality with $\pi = \pi^*$, which in turn yields

$$\begin{aligned} \omega(t, s_1, s_2, x, y) &= E^{t,s_1,s_2,x,y}[\omega(T, S_1(T), S_2(T), X^{\pi^*}(T), Y^{\pi^*}(T)] \\ &\le V(t, s_1, s_2, x, y). \end{aligned}$$

Above all, we have proved that $\omega(t, s_1, s_2, x, y) = V(t, s_1, s_2, x, y)$, and π^* is an optimal Markov control.

Appendix C

Proof of Theorem 3.2

We first show how to classify the optimal reinsurance strategy and provide the value of the optimal investment $A_1^*(t)$, $A_2^*(t)$ when $q^*(t)$ takes three different values, finally we give the proof of the explicit expression of the value function $V(t, s_1, s_2, x, y)$.

Inserting (3.10) and (3.11) into (3.6), we obtain

$$\begin{aligned} & [r_0 d(t) - d_t - m_1 \lambda \alpha_1 \mu_2] e^{r_0(T-t)} + g_t + \frac{1}{2} s_1^{2\beta_1+2} \sigma_1^2 g_{s_1 s_1} + r_0 s_1 g_{s_1} - \frac{(r_1 - r_0)^2}{2 s_1^{2\beta_1} \sigma_1^2} \\ & + \frac{1}{2} s_2^{2\beta_2+2} \sigma_2^2 g_{s_2 s_2} + r_0 s_2 g_{s_2} - \frac{(r_2 - r_0)^2}{2 s_2^{2\beta_2} \sigma_2^2} + \inf_q [f(q, t)] = 0. \end{aligned} \tag{C.1}$$

In order to find the optimal value of q that minimizes $f(q, t)$ given by (3.9), we need to discuss the concavity/convexity of $f(q, t)$ and the size relationship of $\bar{q}(t)$, $1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$ and 1.

Note that $\bar{q}(t) \le 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$ iff

$$\begin{cases} m_1 > m_2, \begin{cases} \Delta_2 \ge 1, 0 \le t \le T - \frac{\ln \Delta_2}{r_0}, \\ \Delta_2 < 1, \end{cases} \\ m_1 < m_2, \begin{cases} \Delta_1 > \Delta_2 \ge 1, T - \frac{\ln \Delta_1}{r_0} \le t \le T - \frac{\ln \Delta_2}{r_0}, \\ \Delta_1 > 1 > \Delta_2, T - \frac{\ln \Delta_1}{r_0} \le t \le T, \end{cases} \end{cases} \tag{C.2}$$

$1 - \sqrt{\frac{\alpha_1}{\alpha_2}} < \bar{q}(t) < 1$ iff

$$\begin{cases} m_1 > m_2, \Delta_2 \ge 1, T - \frac{\ln \Delta_2}{r_0} < t < T, \\ m_1 < m_2, \Delta_1 > \Delta_2 \ge 1, T - \frac{\ln \Delta_2}{r_0} < t < T, \end{cases} \tag{C.3}$$

and $\bar{q}(t) > 1$ iff

$$m_1 < m_2, \begin{cases} \Delta_1 \geq 1, 0 < t < T - \frac{\ln \Delta_1}{r_0}, \\ \Delta_1 < 1. \end{cases} \tag{C.4}$$

On the other hand, $\frac{\partial^2 f(q,t)}{\partial q^2} > 0$ iff

$$\begin{cases} m_1 > m_2, \\ m_1 < m_2, \begin{cases} \Delta_1 \geq 1, 0 \leq t \leq T - \frac{ln\Delta_1}{r_0}, \\ \Delta_1 < 1, \end{cases} \end{cases} \tag{C.5}$$

and $\frac{\partial^2 f(q,t)}{\partial q^2} < 0$ iff

$$m_1 < m_2,\ \Delta_1 \geq 1,\ T - \frac{ln\Delta_1}{r_0} \leq t \leq T. \tag{C.6}$$

Based on the above analysis, we have the following conclusion.

(1) If $\bar{q}(t) \leq 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$ and $\frac{\partial^2 f(q,t)}{\partial q^2} > 0$, then $q^* = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$.
Combining (C.2) and (C.5), we obtain $m_1 > m_2, \begin{cases} \Delta_2 \geq 1, 0 \leq t \leq T - \frac{\ln \Delta_2}{r_0}, \\ \Delta_2 < 1. \end{cases}$

(2) If $\bar{q}(t) \leq 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$ and $\frac{\partial^2 f(q,t)}{\partial q^2} < 0$, then $q^* = 1$.
Combining (C.2) and (C.6), we obtain $m_1 < m_2, \begin{cases} \Delta_1 > \Delta_2 \geq 1, T - \frac{\ln \Delta_1}{r_0} \leq t \leq T - \frac{\ln \Delta_2}{r_0}, \\ \Delta_1 > 1 > \Delta_2, T - \frac{\ln \Delta_1}{r_0} \leq t \leq T. \end{cases}$

(3) If $1 - \sqrt{\frac{\alpha_1}{\alpha_2}} < \bar{q}(t) < 1$ and $\frac{\partial^2 f(q,t)}{\partial q^2} > 0$, then $q^* = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$.
Combining (C.3) and (C.5), we obtain
$m_1 > m_2, \Delta_2 \geq 1, T - \frac{\ln \Delta_2}{r_0} < t < T$.

(4) If $1 - \sqrt{\frac{\alpha_1}{\alpha_2}} < \bar{q}(t) < 1$ and $\frac{\partial^2 f(q,t)}{\partial q^2} < 0$, then $q^* = 1$.
Combining (C.3) and (C.6), we obtain $m_1 < m_2, \Delta_1 > \Delta_2 \geq 1, T - \frac{\ln \Delta_2}{r_0} < t < T$.

(5) If $\bar{q}(t) > 1$ and $\frac{\partial^2 f(q,t)}{\partial q^2} > 0$, then $q^*(t) = \bar{q}(t)$.
Combining (C.4) and (C.5), we obtain $m_1 < m_2, \begin{cases} \Delta_1 \geq 1, 0 < t < T - \frac{\ln \Delta_1}{r_0}, \\ \Delta_1 < 1. \end{cases}$

(6) If $\bar{q}(t) > 1$ and $\frac{\partial^2 f(q,t)}{\partial q^2} < 0$, then $q^* = 1$ or $1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$.
Combining (C.4) and (C.6), we find the intersection of the two is empty.

Combining above (1)–(6), we obtain the result of optimal reinsurance strategy shown in Theorem 3.2. Next, we proof the value of the optimal investment strategy $A_1^*(t)$ and $A_2^*(t)$, when $q^*(t)$ takes different values.

(1) When $q^* = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$, substituting it into (C.1) yields

$$\begin{aligned} &\{r_0 d(t) - d_t - m_1 \lambda \alpha_1 \mu_2 + \Big[\Big(\frac{1}{2} - \sqrt{\frac{\alpha_1}{\alpha_2}}\Big) {m_1}^2 + \frac{\alpha_1}{2\alpha_2}(m_1 - m_2)^2 + \sqrt{\frac{\alpha_1}{\alpha_2}} m_1 m_2\Big] \sigma_0^2 e^{r_0(T-t)}\} e^{r_0(T-t)} \\ &+ g_t + r_0 s_1 g_{s_1} + \frac{1}{2} s_1^{2\beta_1+2} \sigma_1^2 g_{s_1 s_1} - \frac{(r_1 - r_0)^2}{2 s_1^{2\beta_1} \sigma_1^2} + r_0 s_2 g_{s_2} + \frac{1}{2} s_2^{2\beta_2+2} \sigma_2^2 g_{s_2 s_2} - \frac{(r_2 - r_0)^2}{2 s_2^{2\beta_2} \sigma_2^2} = 0, \end{aligned} \tag{C.7}$$

which can be split into following two equations

$$\Big[\Big(\frac{1}{2}-\sqrt{\frac{\alpha_1}{\alpha_2}}\Big){m_1}^2+\frac{\alpha_1}{2\alpha_2}(m_1-m_2)^2+\sqrt{\frac{\alpha_1}{\alpha_2}}m_1m_2\Big]\sigma_0^2e^{r_0(T-t)}+r_0d(t)-d_t-m_1\lambda\alpha_1\mu_2=0, \quad (C.8)$$

and

$$g_t+r_0s_1g_{s_1}+\frac{1}{2}s_1^{2\beta_1+2}\sigma_1^2g_{s_1s_1}-\frac{(r_1-r_0)^2}{2s_1^{2\beta_1}\sigma_1^2}+r_0s_2g_{s_2}+\frac{1}{2}s_2^{2\beta_2+2}\sigma_2^2g_{s_2s_2}-\frac{(r_2-r_0)^2}{2s_2^{2\beta_2}\sigma_2^2}=0. \quad (C.9)$$

Note that (C.8) is a linear ordinary differential equation with the boundary condition $d(T)=0$, it is not difficult to derive that

$$\begin{aligned} d(t)=&-\frac{m_1\lambda\alpha_1\mu_2}{r_0}[e^{-r_0(T-t)}-1]+\frac{\sigma_0^2}{2r_0}\Big[\Big(\frac{1}{2}-\sqrt{\frac{\alpha_1}{\alpha_2}}\Big){m_1}^2+\frac{\alpha_1}{2\alpha_2}(m_1-m_2)^2+\sqrt{\frac{\alpha_1}{\alpha_2}}m_1m_2\Big] \\ &\times[e^{-r_0(T-t)}-e^{r_0(T-t)}]. \end{aligned} \quad (C.10)$$

Trying to solve (C.9), we put

$$g(t,s_1,s_2)=m(t,k_1,k_2),\ k_1=s_1^{-2\beta_1},\ k_2=s_2^{-2\beta_2}, \quad (C.11)$$

with the boundary condition given by $m(T,k_1,k_2)=0$. Then we can obtain the relationship between the partial derivatives of m and g as follows

$$\begin{aligned} &g_t=m_t,\ g_{s_1}=-2\beta_1s_1^{-2\beta_1-1}m_{k_1},\ g_{s_2}=-2\beta_2s_2^{-2\beta_2-1}m_{k_2}, \\ &g_{s_2s_2}=2\beta_2(2\beta_2+1)s_2^{-2\beta_2-2}m_{k_2}+4\beta_2^2s_2^{-4\beta_2-2}m_{k_2k_2}, \\ &g_{s_1s_1}=2\beta_1(2\beta_1+1)s_1^{-2\beta_1-2}m_{k_1}+4\beta_1^2s_1^{-4\beta_1-2}m_{k_1k_1}. \end{aligned} \quad (C.12)$$

Substituting the above first and second partial derivatives into Eq (C.9), and after simplification we have

$$\begin{aligned} &m_t+[\beta_1(2\beta_1+1)\sigma_1^2-2r_0\beta_1k_1]m_{k_1}+2\beta_1^2\sigma_1^2k_1m_{k_1k_1}-\frac{(r_1-r_0)^2k_1}{2\sigma_1^2}+[\beta_2(2\beta_2+1)\sigma_2^2 \\ &-2r_0\beta_2k_2]m_{k_2}+2\beta_2^2\sigma_2^2k_2m_{k_2k_2}-\frac{(r_2-r_0)^2k_2}{2\sigma_2^2}=0. \end{aligned} \quad (C.13)$$

Motivated by Li et al. [23], we assume that a solution of Eq (C.13) has a following form

$$m(t,k_1,k_2)=I(t)+J_1(t)k_1+J_2(t)k_2, \quad (C.14)$$

with the boundary condition $I(T)=J_1(T)=J_2(T)=0$. Then, we obtain the partial derivatives of m as

$$m_t=I_t+J_{1t}k_1+J_{2t}k_2,\ m_{k_1}=J_1(t),\ m_{k_2}=J_2(t),\ m_{k_1k_1}=m_{k_2k_2}=0, \quad (C.15)$$

Substituting (C.15) into (C.13), we have

$$\begin{aligned} &I_t+J_{1t}k_1+J_{2t}k_2+[\beta_1(2\beta_1+1)\sigma_1^2-2r_0\beta_1k_1]J_1(t)-\frac{(r_1-r_0)^2k_1}{2\sigma_1^2} \\ &+[\beta_2(2\beta_2+1)\sigma_2^2-2r_0\beta_2k_2]J_2(t)-\frac{(r_2-r_0)^2k_2}{2\sigma_2^2}=0, \end{aligned} \quad (C.16)$$

which can be simplified into

$$
\begin{aligned}
&k_1[J_{1t} - 2r_0\beta_1 J_1(t) - \frac{(r_1 - r_0)^2}{2\sigma_1^2}] + k_2[J_{2t} - 2r_0\beta_2 J_2(t) - \frac{(r_2 - r_0)^2}{2\sigma_2^2}] \\
&\quad + I_t + J_1(t)\beta_1(2\beta_1 + 1)\sigma_1^2 + J_2(t)\beta_2(2\beta_2 + 1)\sigma_2^2 = 0,
\end{aligned}
\tag{C.17}
$$

with $k_1 = s_1^{-2\beta_1} \neq 0, k_2 = s_2^{-2\beta_2} \neq 0, J_1(T) = J_2(T) = 0$. We can split Eq (C.17) into three equations:

$$
J_{1t} - 2r_0\beta_1 J_1(t) - \frac{(r_1 - r_0)^2}{2\sigma_1^2} = 0, \tag{C.18}
$$

$$
J_{2t} - 2r_0\beta_2 J_2(t) - \frac{(r_2 - r_0)^2}{2\sigma_2^2} = 0, \tag{C.19}
$$

and

$$
I_t + J_1(t)\beta_1(2\beta_1 + 1)\sigma_1^2 + J_2(t)\beta_2(2\beta_2 + 1)\sigma_2^2 = 0. \tag{C.20}
$$

Since Eqs (C.18) and (C.19) are linear ordinary differential equations with the boundary condition $J_1(T) = J_2(T) = 0$, we derive that

$$
J_1(t) = \frac{(r_1 - r_0)^2}{4r_0\beta_1\sigma_1^2}[e^{-2r_0\beta_1(T-t)} - 1], \tag{C.21}
$$

$$
J_2(t) = \frac{(r_2 - r_0)^2}{4r_0\beta_2\sigma_2^2}[e^{-2r_0\beta_2(T-t)} - 1]. \tag{C.22}
$$

Combining (C.20), (C.21) and (C.22), we have

$$
\begin{aligned}
I(t) = &\frac{(r_1 - r_0)^2(2\beta_1 + 1)}{4r_0}\Big[\frac{1 - e^{-2r_0\beta_1(T-t)}}{2r_0\beta_1} - (T - t)\Big] + \frac{(r_2 - r_0)^2(2\beta_2 + 1)}{4r_0} \\
&\times \Big[\frac{1 - e^{-2r_0\beta_2(T-t)}}{2r_0\beta_2} - (T - t)\Big].
\end{aligned}
\tag{C.23}
$$

Using Eqs (C.15), (C.21), (C.22), (3.10) and (3.11), we obtain

$$
A_1^*(t) = \frac{r_1^2 - r_0^2 - (r_1 - r_0)^2 e^{-2r_0\beta_1(T-t)}}{2r_0 m_1 s_1^{2\beta_1}\sigma_1^2} e^{-r_0(T-t)}, \tag{C.24}
$$

$$
A_2^*(t) = \frac{r_2^2 - r_0^2 - (r_2 - r_0)^2 e^{-2r_0\beta_2(T-t)}}{2r_0 m_2 s_2^{2\beta_2}\sigma_2^2} e^{-r_0(T-t)}. \tag{C.25}
$$

Above all, we present the expression of $d(t)$, $g(t, s_1, s_2)$, $m(t, k_1, k_2)$, $I(t)$, $J_1(t)$, and $J_2(t)$ by (C.10), (C.12), (C.14), (C.23), (C.21) and (C.22), then we can get the explicit expression of the value function $V(t, s_1, s_2, x, y)$.

(2) When $q^*(t)=\bar{q}(t)$, substituting it into (C.1) yields

$$\begin{aligned}&[r_0d(t)-d_t-m_1\lambda\alpha_1\mu_2]e^{r_0(T-t)}+g_t+\frac{1}{2}s_1^{2\beta_1+2}\sigma_1^2g_{s_1s_1}+r_0s_1g_{s_1}-\frac{(r_1-r_0)^2}{2s_1^{2\beta_1}\sigma_1^2}\\&+\frac{1}{2}s_2^{2\beta_2+2}\sigma_2^2g_{s_2s_2}+r_0s_2g_{s_2}-\frac{(r_2-r_0)^2}{2s_2^{2\beta_2}\sigma_2^2}+f(\bar{q},t)=0.\end{aligned}\tag{C.26}$$

It's easy to see that Eq (3.9) is independent of g and its partial derivatives. Thus, (C.26) can be split into following two equations

$$r_0d(t)-d_t-m_1\lambda\alpha_1\mu_2=0,\tag{C.27}$$

and

$$\begin{aligned}&g_t+\frac{1}{2}s_1^{2\beta_1+2}\sigma_1^2g_{s_1s_1}+r_0s_1g_{s_1}-\frac{(r_1-r_0)^2}{2s_1^{2\beta_1}\sigma_1^2}+\frac{1}{2}s_2^{2\beta_2+2}\sigma_2^2g_{s_2s_2}\\&+r_0s_2g_{s_2}-\frac{(r_2-r_0)^2}{2s_2^{2\beta_2}\sigma_2^2}+f(\bar{q},t)=0,\end{aligned}\tag{C.28}$$

Note that (C.27) is a linear ordinary differential equation with the boundary condition $d(T)=0$, we have

$$d(t)=-\frac{m_1\lambda\alpha_1\mu_2}{r_0}[e^{-r_0(T-t)}-1].\tag{C.29}$$

Since (C.28) is similar with (C.9), we can get the expression of $I(t)$ which is similar with (C.23), and it is given by

$$\begin{aligned}I(t)=&\frac{(r_1-r_0)^2(2\beta_1+1)}{4r_0}\Big[\frac{1-e^{-2r_0\beta_1(T-t)}}{2r_0\beta_1}-(T-t)\Big]+\int_t^T f(\bar{q},\tau)d\tau+\frac{(r_2-r_0)^2(2\beta_2+1)}{4r_0}\\&\times\Big[\frac{1-e^{-2r_0\beta_2(T-t)}}{2r_0\beta_2}-(T-t)\Big],\end{aligned}\tag{C.30}$$

where

$$\begin{aligned}f(\bar{q},t)=&\lambda\alpha_2(1-\bar{q})^2\mu_2(m_1-m_2)e^{r_0(T-t)}+\bar{q}(1-\bar{q})m_1m_2\sigma_0^2e^{2r_0(T-t)}\\&+\frac{1}{2}\sigma_0^2e^{2r_0(T-t)}[\bar{q}^2m_1^2+(1-\bar{q})^2m_2^2]\\=&\frac{\lambda\alpha_2\mu_2m_1^2(m_1-m_2)\sigma_0^4e^{3r_0(T-t)}+2\lambda^2\alpha_2^2\mu_2^2m_1^2\sigma_0^2e^{2r_0(T-t)}}{[2\lambda\alpha_2\mu_2+(m_1-m_2)\sigma_0^2e^{r_0(T-t)}]^2}\\=&\frac{\lambda\alpha_2\mu_2m_1^2(m_1-m_2)\sigma_0^4e^{3r_0(T-t)}+(4-2)\lambda^2\alpha_2^2\mu_2^2m_1^2\sigma_0^2e^{2r_0(T-t)}}{[2\lambda\alpha_2\mu_2+(m_1-m_2)\sigma_0^2e^{r_0(T-t)}]^2}\\=&-\frac{1}{r_0}\left[\frac{\lambda\alpha_2\mu_2m_1^2\sigma_0^2e^{2r_0(T-t)}}{2\lambda\alpha_2\mu_2+(m_1-m_2)\sigma_0^2e^{r_0(T-t)}}\right]'+\frac{e^{r_0(T-t)}}{r_0}\left[\frac{\lambda\alpha_2\mu_2m_1^2\sigma_0^2e^{r_0(T-t)}}{2\lambda\alpha_2\mu_2+(m_1-m_2)\sigma_0^2e^{r_0(T-t)}}\right]'.\end{aligned}$$

Then

$$\int_t^T f(\bar{q},\tau)d\tau = \int_t^T -\frac{1}{r_0}\left[\frac{\lambda\alpha_2\mu_2 m_1^2\sigma_0^2 e^{2r_0(T-\tau)}}{2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2 e^{r_0(T-\tau)}}\right]' d\tau + \int_t^T \frac{e^{r_0(T-\tau)}}{r_0}\left[\frac{\lambda\alpha_2\mu_2 m_1^2\sigma_0^2 e^{r_0(T-\tau)}}{2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2 e^{r_0(T-\tau)}}\right]' d\tau. \tag{C.31}$$

Furthermore,

$$\begin{aligned}
&\int_t^T \frac{e^{r_0(T-\tau)}}{r_0}\left[\frac{\lambda\alpha_2\mu_2 m_1^2\sigma_0^2 e^{r_0(T-\tau)}}{2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2 e^{r_0(T-\tau)}}\right]' d\tau \\
&= \left[\frac{e^{r_0(T-\tau)}}{r_0} \times \frac{\lambda\alpha_2\mu_2 m_1^2\sigma_0^2 e^{r_0(T-\tau)}}{2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2 e^{r_0(T-\tau)}}\right]\Bigg|_t^T + \int_t^T \frac{\lambda\alpha_2\mu_2 m_1^2\sigma_0^2 e^{2r_0(T-\tau)}}{2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2 e^{r_0(T-\tau)}} d\tau \\
&= \frac{\lambda\alpha_2\mu_2 m_1^2\sigma_0^2}{2r_0\lambda\alpha_2\mu_2 + r_0(m_1-m_2)\sigma_0^2} - \frac{\lambda\alpha_2\mu_2 m_1^2\sigma_0^2 e^{2r_0(T-t)}}{2r_0\lambda\alpha_2\mu_2 + r_0(m_1-m_2)\sigma_0^2 e^{r_0(T-t)}} - \frac{\lambda\alpha_2\mu_2 m_1^2\sigma_0^2}{r_0} \\
&\quad \times \int_{e^{r_0(T-t)}}^1 \frac{\delta}{2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2\delta} d\delta,
\end{aligned} \tag{C.32}$$

and

$$\begin{aligned}
&\int_{e^{r_0(T-t)}}^1 \frac{\delta}{2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2\delta} d\delta \\
&= \frac{1}{(m_1-m_2)^2\sigma_0^4}[(m_1-m_2)\sigma_0^2\delta - 2\lambda\alpha_2\mu_2 \ln|2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2\delta|]|_{e^{r_0(T-t)}}^1 \\
&= -\frac{1}{(m_1-m_2)^2\sigma_0^4}\left\{(m_1-m_2)\sigma_0^2[e^{r_0(T-t)}-1] + 2\lambda\alpha_2\mu_2 \ln\left|\frac{2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2}{2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2 e^{r_0(T-t)}}\right|\right\}.
\end{aligned} \tag{C.33}$$

Combined (C.31), (C.32) and (C.33) , we get

$$\int_t^T f(\bar{q},\tau)d\tau = \frac{\lambda\alpha_2\mu_2 m_1^2}{r_0(m_1-m_2)}[e^{r_0(T-t)}-1] + \frac{2\lambda^2\alpha_2^2\mu_2^2 m_1^2}{r_0(m_1-m_2)^2\sigma_0^2}\ln\left|\frac{2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2}{2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2 e^{r_0(T-t)}}\right|.$$

As a result, (C.30) is converted to

$$\begin{aligned}
I(t) =& \frac{(r_1-r_0)^2(2\beta_1+1)}{4r_0}\left[\frac{1-e^{-2r_0\beta_1(T-t)}}{2r_0\beta_1} - (T-t)\right] + \frac{(r_2-r_0)^2(2\beta_2+1)}{4r_0} \\
&\times \left[\frac{1-e^{-2r_0\beta_2(T-t)}}{2r_0\beta_2} - (T-t)\right] + \frac{\lambda\alpha_2\mu_2 m_1^2}{r_0(m_1-m_2)}[e^{r_0(T-t)}-1] \\
&+ \frac{2\lambda^2\alpha_2^2\mu_2^2 m_1^2}{r_0(m_1-m_2)^2\sigma_0^2}\ln\left|\frac{2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2}{2\lambda\alpha_2\mu_2 + (m_1-m_2)\sigma_0^2 e^{r_0(T-t)}}\right|.
\end{aligned} \tag{C.34}$$

Above all, we obtain the expression of $d(t)$, $I(t)$, $J_1(t)$, and $J_2(t)$ by (C.29), (C.34), (C.21) and (C.22), then we can get the explicit expression of the value function $V(t, s_1, s_2, x, y)$. Similarly, we can obtain A_1^* and A_2^* are given by (C.24) and (C.25).

(3) When $q^* = 1$, substituting it into (C.1) yields

$$[r_0 d(t) - d_t - m_1\lambda\alpha_1\mu_2 + \frac{1}{2}m_1^2\sigma_0^2 e^{r_0(T-t)}]e^{r_0(T-t)} + g_t + \frac{1}{2}s_1^{2\beta_1+2}\sigma_1^2 g_{s_1 s_1} + r_0 s_1 g_{s_1} - \frac{(r_1-r_0)^2}{2s_1^{2\beta_1}\sigma_1^2} + \frac{1}{2}s_2^{2\beta_2+2}\sigma_2^2 g_{s_2 s_2} + r_0 s_2 g_{s_2} - \frac{(r_2-r_0)^2}{2s_2^{2\beta_2}\sigma_2^2} = 0. \quad (C.35)$$

Also, (C.35) can be split into (C.9) and

$$r_0 d(t) - d_t - m_1\lambda\alpha_1\mu_2 + \frac{1}{2}m_1^2\sigma_0^2 e^{r_0(T-t)} = 0. \quad (C.36)$$

Solving the above linear ordinary differential equation (C.36) with the boundary condition $d(T) = 0$, we obtain

$$d(t) = \frac{m_1^2\sigma_0^2}{4r_0}[e^{-r_0(T-t)} - e^{r_0(T-t)}] + \frac{m_1\lambda\alpha_1\mu_2}{r_0}[1 - e^{-r_0(T-t)}].$$

The solution of (C.9) is given by (C.11)–(C.23). So, we can get the explicit expression of the value function $V(t, s_1, s_2, x, y)$. Similarly, we can obtain $A_1^*(t)$ and $A_2^*(t)$ are given by (C.24) and (C.25).

Above all, the proof of Theorem 3.2 is completed.