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I. Abdoli,[1,a),b)] H. Löwen,[2] J.-U. Sommer,[1,a)] and A. Sharma[1,a)]

AFFILIATIONS

[1] Leibniz-Institut für Polymerforschung Dresden, Institut Theorie der Polymere, 01069 Dresden, Germany
[2] Institut für Theoretische Physik II: Weiche Materie, Heinrich-Heine-Universität Düsseldorf, Düsseldorf 40225, Germany

[a)] **Also at:** Technische Universität Dresden, Institut für Theoretische Physik, 01069 Dresden, Germany.
[b)] **Author to whom correspondence should be addressed:** abdoli@ipfdd.de

ABSTRACT

The probability per unit time for a thermally activated Brownian particle to escape over a potential well is, in general, well-described by Kramers's theory. Kramers showed that the escape time decreases exponentially with increasing barrier height. The dynamics slow down when the particle is charged and subjected to a Lorentz force due to an external magnetic field. This is evident via a rescaling of the diffusion coefficient entering as a prefactor in the Kramers's escape rate without any impact on the barrier-height-dependent exponent. Here, we show that the barrier height can be effectively changed when the charged particle is subjected to a vortex flow. While the vortex alone does not affect the mean escape time of the particle, when combined with a magnetic field, it effectively pushes the fluctuating particle either radially outside or inside depending on its sign relative to that of the magnetic field. In particular, the effective potential over which the particle escapes can be changed to a flat, a stable, and an unstable potential by tuning the signs and magnitudes of the vortex and the applied magnetic field. Notably, the last case corresponds to enhanced escape dynamics.

I. INTRODUCTION

A Brownian particle undergoes an erratic motion as a result of its collisions with the solvent molecules. If the particle is being initially put at the bottom of a potential well, the thermal activation of the particle may cause an escape from the potential well over an energetic barrier. Using the flux-over-population method,[1] Kramers first derived the escape rate of a Brownian particle over an energy barrier moving in a bistable potential, regardless of what happens after this escape.[2] He showed that the probability per unit time for the particle to escape the potential well exponentially decays with the height of the energy barrier. Kramers derived limiting expressions for weak friction and strong damping and realized a global maximum at some intermediate value of the damping, which is known as *Kramers's turnover*.[3–5] The problem has been generalized to include memory friction[6–8] and athermal fluctuations[9–13] and was extended to quantum field theory.[14,15]

While Kramers's framework and its extensions have thoroughly been studied with the relevant deterministic potential force fields,[16–21] much less is known when the deterministic force is nonconservative, namely when it is not of potential type.[22–24] Recently, by taking into account a nonconservative force in the form of Lorentz force, we have studied the escape dynamics of a two-dimensional Brownian system with a broken spatial symmetry via two noises with different strengths.[25] We have shown that while the escape process becomes anisotropic (i.e., particles tend to escape the potential well more along the axis with a larger noise strength) due to two different noises, when subjected to an external magnetic field, the spatial symmetry can be restored.[25] However, to our knowledge, it is expected that the escape process is reduced (or unaffected in the direction of the applied magnetic field) by external constant magnetic fields,[24,25] which is evident via a rescaling of the diffusion coefficient. It has been shown that the combined influence of a nonconservative force and a magnetic field may cause an instability in the system.[26] Here, taking advantage of such an instability, we show that the Lorentz force due to a constant magnetic field can result in enhanced escape dynamics.

In this paper, we study the escape dynamics of a Brownian particle from a harmonic trap, which is cut off at a certain distance, in the presence of a vortex and the Lorentz force due to an external constant magnetic field. Taking advantage of the spatial isotropy in the system, we derive an exact expression for the mean first passage time. While the vortex alone does not affect the escape dynamics, we observe a nontrivial result when an external magnetic field is present: the mean first passage time can be reduced or enhanced. This is attributed to the shape change of the effective potential well. By tuning the external magnetic field or alternatively the strength of the vortex, the effective potential can change shape to a flat, a stable, or an unstable potential. This means that by tuning either parameter, the barrier energy over which the particle may escape can be effectively altered to a smaller or larger one whose origin can be understood as follows: the combination of the vortex flow and the magnetic field effectively pushes the fluctuating particle either radially outside or inside depending on their signs. In other words, the combination of the two fields, which individually induces no radial force, gives rise to a radial force. There are systems where exactly this kind of dynamics can be studied, including skyrmions, which have a strong Magnus force, which can lead to driven particles more easily escaping a trap than the overdamped case, and act like charged particles in magnetic fields[27–31] and dusty plasmas.[32,33] In what follows, we first introduce the model. Next, we calculate the mean first passage time, which can be written in terms of an effective potential. We then study the trends of the escape time with respect to the magnetic field strength and the vortex flow, and finally, we discuss several experimental realizations of the setup considered here.

II. MODEL

We consider an overdamped charged Brownian particle with the charge q subjected to an external magnetic field B in the $-\hat{z}$ direction. Since the Lorentz force due to the field does not affect the motion of the particle in the z direction, we effectively reduce the system to a two-dimensional one and study the motion of the particle in the xy plane. The particle is trapped in an isotropic potential $U(x, y) = k(x^2 + y^2)/2$ and undergoes a vortex flow due to the nonconservative force $\boldsymbol{F}_{nc} = \varepsilon(-y, x)^\top$. Here, k and ε are the stiffness of the potential and the strength of the nonconservative force, respectively. A schematic of the system is shown in Fig. 1. It is experimentally and theoretically known that even statically optically trapped Brownian particles in the overdamped limit represent nonequilibrium behavior characterized by Brownian vortices. This is due to the nonconservative forces generated by optical scattering forces.[34–37] Moreover, by applying a prescribed vortex flow field such as a rotating bucket to an underdamped Brownian particle, one can induce similar terms to the nonconservative force, i.e., $-\varepsilon y$ and εx.[38] The combination of the vortex flow and the magnetic field gives rise to a radial force, resulting in a quasipotential that, when combined with the isotropic potential, acts as an effective potential. The effective potential can be a stable one, a flat one, or an unstable one, which can be quantitatively exactly derived as follows.

It has been shown that the overdamped dynamics of the particle derived by simply setting the inertia term to zero can yield an incorrect description in the presence of a magnetic field.[39] In this case, the overdamped Langevin equation describing the dynamics of the system can be derived using the low-mass approach,[25,40,41] which can be written as

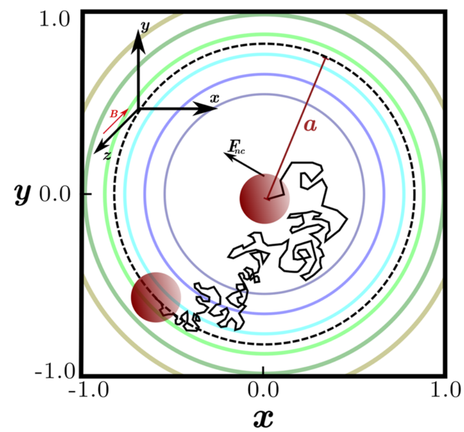

FIG. 1. A single charged particle diffusing in a two-dimensional harmonic potential $U(x, y) = k(x^2 + y^2)/2$, shown by concentric contours, with k being its stiffness. The particle is subjected to an external magnetic field B in the $-\hat{z}$ direction and a nonconservative force $\boldsymbol{F}_{nc} = \varepsilon(-y, x)$, with ε being its strength. The nonconservative force is shown for $\varepsilon > 0$. The particle can escape the trap when reaches the boundary, truncated at $r = a$, shown by a dashed circle, where $r = \sqrt{x^2 + y^2}$ is the distance from the origin.

$$\dot{x} = \frac{1}{\gamma(1+\kappa^2)}[-kx - \varepsilon y + k\kappa y - \varepsilon\kappa x] + \xi_x(t), \tag{1}$$

$$\dot{y} = \frac{1}{\gamma(1+\kappa^2)}[-ky + \varepsilon x - k\kappa x - \varepsilon\kappa y] + \xi_y(t), \tag{2}$$

where γ is the friction coefficient and $\kappa = qB/\gamma$ is the diffusive Hall parameter quantifying the strength of the Lorentz force relative to the frictional force. We note that κ can be positive or negative depending on the sign of the applied magnetic field. Here, $\boldsymbol{\xi}(t) = (\xi_x, \xi_y)^\top$ is the Gaussian nonwhite noise with zero mean and time correlation $\langle \boldsymbol{\xi}(t)\boldsymbol{\xi}^\top(t')\rangle = T\boldsymbol{G}^{-1}\delta_+(t - t') + T(\boldsymbol{G}^{-1})^\top\delta_-(t - t')$, where T is the temperature; $\boldsymbol{G} = \gamma\begin{pmatrix} 1 & \kappa \\ -\kappa & 1 \end{pmatrix}$; and the notations $\delta_\pm(s = t - t')$ are the modified Dirac delta functions, which are zero for $s \neq 0$, while $\int_0^\infty ds\delta_+(s) = \int_{-\infty}^0 ds\delta_-(s) = 1$ and $\int_0^\infty ds\delta_-(s) = \int_{-\infty}^0 ds\delta_+(s) = 0$. Throughout this work, we set the Boltzmann constant k_B to unity. Length and time are measured in units of $\sqrt{T/k}$ and γ/k, respectively.

We use the Itô calculus to reduce the Langevin equations in Eqs. (1) and (2) to a one-dimensional problem for the variable $r = \sqrt{x^2 + y^2}$, which is given as[42]

$$\dot{r} = \frac{1}{1+\kappa^2}\left[-\frac{k + \varepsilon\kappa}{\gamma}r + \frac{D}{r}\right] + \sqrt{\frac{2D}{1+\kappa^2}}\eta(t), \tag{3}$$

where $D = T/\gamma$ is the coefficient of a freely diffusing particle and $\eta(t)$ is Gaussian white noise with zero mean and the Dirac delta time

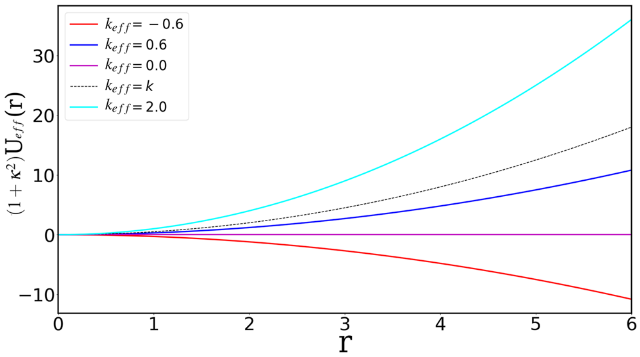

FIG. 2. Effective potential from Eq. (4) for different values of the stiffness $k_{eff} = k + \varepsilon\kappa$. By varying the parameter κ (or ε), the effective potential can change shape to a stable one if $k_{eff} > 0$, a flat one if $k_{eff} = 0$, or an unstable one if $k_{eff} < 0$.

correlation $\langle\eta(t)\eta(t')\rangle = \delta(t - t')$. The terms in the square brackets on the right-hand side of Eq. (3) describe the force on the particle due to an effective potential, given as

$$U_{eff}(r) = \frac{k_{eff}}{2\gamma(1+\kappa^2)}r^2 - \frac{D}{1+\kappa^2}\log(r), \tag{4}$$

where $k_{eff} = k + \varepsilon\kappa$ is the stiffness of the effective potential. As it is evident from the effective stiffness, in the absence of the vortex flow, the potential simply gets rescaled by the factor $1/(1+\kappa^2)$, as we have shown in the supplemental information of Ref. 25. Moreover, in the absence of the magnetic field, there is no effect of the vortex flow on the effective potential and Eq. (4) reduces to the well-known results in Ref. 42 for a rotationally symmetric Ornstein–Uhlenbeck process in two dimensions. The second term on the right-hand side comes from the transformation to r and corresponds to an extremely repulsive potential at the origin due to the reduced number of states on the circle of radius r. This term influences the motion of the particle only near the origin and is negligible for larger distances as compared to the first term.

Figure 2 represents the scaled effective potential from Eq. (4) for different values of the parameter k_{eff} without the logarithmic term. By tuning the diffusive Hall parameter or, alternatively, the strength of the nonconservative force, the effective potential changes shape: the potential is stable if $k_{eff} > 0$, flat if $k_{eff} = 0$, and unstable if $k_{eff} < 0$. It becomes a simple quadratic potential in the absence of ε and/or κ.

III. MEAN ESCAPE TIME

We consider a particle that is trapped in an isotropic potential $U(x, y)$ which takes advantage of the spatial symmetry whose distance from the origin, $r = |\boldsymbol{r}|$, can be described by Eq. (3). We are interested in the mean time at which the particle reaches the boundary, truncated at $r = a$, as shown in Fig. 1. As we show in the Appendix, the mean escape time can be exactly calculated from Eq. (3), which reads

$$\langle t\rangle = \frac{\gamma(1+\kappa^2)}{2k_{eff}}\left[\mathrm{Ei}(\beta\Delta E_{eff}) - \log(\beta\Delta E_{eff}) - \gamma_{EM}\right], \tag{5}$$

if $k_{eff} > 0$ corresponding to the effective stable potential, and

$$\langle t\rangle = \frac{\gamma(1+\kappa^2)}{2k_{eff}}\left[-\mathrm{Ei}(-\beta|\Delta E_{eff}|) + \log(\beta|\Delta E_{eff}|) + \gamma_{EM}\right], \tag{6}$$

if $k_{eff} < 0$ corresponds to the unstable effective potential, where β is the inverse of the temperature, γ_{EM} is the Euler–Mascheroni constant, and $\mathrm{Ei}(x)$ is the exponential integral. Here, $\Delta E_{eff} = \Delta E + \varepsilon\kappa a^2/2$ is the effective barrier energy, which is the real barrier height $\Delta E = ka^2/2$ augmented by the coupling between the magnitude of the applied magnetic field and the strength of the vortex. Using the series expansion of the exponential integral at $k_{eff} = 0$ for Eqs. (5) and (6), the mean escape time for the effective flat potential reads

$$\langle t\rangle \sim \frac{(1+\kappa^2)}{4D}a^2, \tag{7}$$

which is the mean escape time for a freely diffusing particle scaled by $1+\kappa^2$. In the limit of large barrier heights, the exponential integral in Eq. (5) can be expanded and, as a consequence, the mean escape time reduces to $\langle t\rangle \sim \gamma(1+\kappa^2)\exp(\beta\Delta E_{eff})/(2k_{eff}\beta\Delta E_{eff})$. In the absence of the vortex, which corresponds to $\varepsilon = 0$, the result reduces to the Kramers's result rescaled by $1+\kappa^2$ arising from the trivial rescaling of the diffusion coefficient. The expression becomes the same as the Kramers's one when the magnetic field is absent $\kappa = 0$. This confirms that the vortex field alone does not affect the mean escape time. The intuitive reason for that can be understood as follows: for $\kappa = 0$, the presence of a vortex field only changes the azimuthal motion but not the radial one, which leaves the radial particle escape unaffected.

Figure 3 shows the mean escape time with respect to the diffusive Hall parameter κ. Obviously, it takes a longer time for the

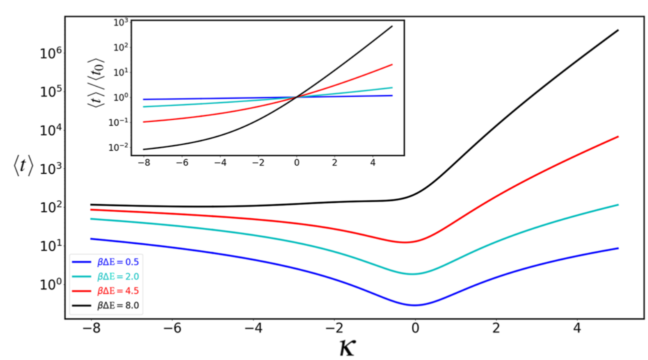

FIG. 3. Mean escape time as a function of the diffusive Hall parameter κ from Eqs. (5) and (7) for different values of the scaled barrier height $\beta\Delta E$ with $\beta = 1.0$, $\gamma = 1.0$, and $\varepsilon = 0.2$. Obviously, the mean escape time increases with increasing barrier height. It can increase or decrease by tuning the parameter κ: the presence of a vortex field can work together with the applied magnetic field to effectively push the fluctuating particle either radially outside, if $\kappa < -k/\varepsilon$, or inside, if $\kappa > -k/\varepsilon$. The former corresponds to the case in which the combination helps the particle to escape. The point $\kappa = 0$ corresponds to unaffected escape time by the vortex (i.e., $k_{eff} = k$). In the inset, we show the mean escape time which is scaled by the mean escape time in the absence of the vortex $\langle t_0\rangle$, where the subscript 0 indicates the zero strength length of the vortex flow. It implies that the mean escape time can decrease with increasing κ as compared to the mean escape time without the vortex flow.

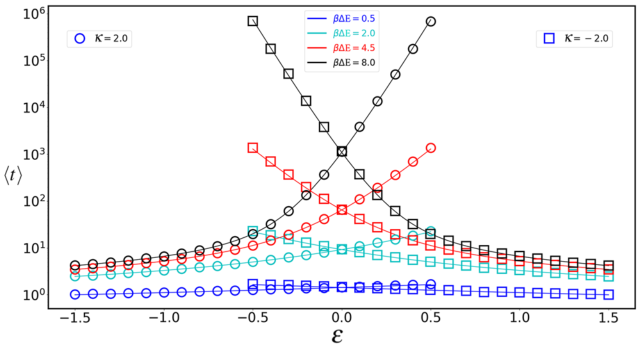

FIG. 4. Mean escape time with respect to the strength of the conservative force ε from Eqs. (5) and (7) for different values of the scaled barrier heights with $\beta = 1.0$ and $\gamma = 1.0$. The lines with circles and squares correspond to the results with $\kappa = 2.0$ and $\kappa = -2.0$, respectively. The mean escape time can increase or decrease with increasing strength of the vortex flow, which depends on its sign relative to that of κ and their magnitude compared to the stiffness of the potential k.

particle to escape over larger barrier heights as is evident in the figure. The magnetic field together with the vortex flow creates additional fluctuations in the radial direction which can be directed either outward or inward depending on its sign. The former corresponds to the case in which the combination of the vortex flow and the magnetic field helps the particle to escape. The inset shows the mean escape time scaled by the mean escape time in the absence of the vortex, which is indicated by the subscript 0. The mean escape time can decrease with increasing magnetic field as compared to the mean escape time without the vortex flow and remains almost constant for small barrier heights.

In Fig. 4, we show that tuning the strength of the vortex flow is an alternative way to vary the mean escape time, which is evident in Eqs. (5) and (6) via the product of the two parameters, i.e., $\varepsilon\kappa$. Therefore, similar trends are expected. The figure represents the mean escape time with respect to the parameter ε for a system with $\kappa = 2.0$, denoted by lines with circles, and a system with $\kappa = -2.0$, denoted by lines with squares. Our results imply that the mean escape time can be decreased or increased by tuning the vortex flow strength depending on its sign relative to that of the magnetic field and their magnitude compared to the stiffness of the potential k.

IV. DISCUSSION

In this work, we studied the effect of a vortex flow on the escape dynamics of a Brownian magneto-system made of a single charged Brownian particle subjected to an external magnetic field. We expressed the potential in an effective form, which can change the shape to a stable, a flat, or an unstable potential depending on the stiffness of the effective potential. Taking advantage of the spatial isotropy in the system, we obtained an exact expression for the mean escape time. In the absence of the vortex, exerted by the nonconservative force, the Lorentz force due to the external magnetic field slows down the dynamics of the system without any qualitative change, which is evident via the trivial rescaling of the diffusion coefficient. We showed that while the vortex alone does not affect the mean escape time, when coupled to the magnetic field, it can enhance or reduce the escape time: this is intuitive as the magnetic field together with the vortex flow creates additional fluctuations in the radial direction, which can be directed either outward or inward depending on its sign. In other words, the combination of the two fields, which individually induce no radial force, gives rise to a radial force. We showed that the barrier over which the particle escapes can be effectively changed to a larger or smaller one depending on the relative signs of the strength of the vortex flow and the applied magnetic field and their magnitude compared to the stiffness of the potential in which the particle is trapped. Moreover, the trap can be effectively switched off by an appropriate sign and value of the magnetic field.

A possible experimental realization is to trap the particle using optical tweezers either in a radio-frequency plasma sheath with a vertical magnetic field[44,45] or in a rotating frame of reference. By rotating the reference frame, a Coriolis force can be induced, which acts the same as the Lorentz force due to an external magnetic field.[46–48] As it has been shown that even statically optically trapped Brownian particles undergo a nonconservative force induced by optical scattering forces,[34–37,43] we expect that the study of the enhanced escape dynamics does not require an additional vortex. Another possibility is to apply a rotating bucket to an underdamped Brownian particle, which induces similar terms to the nonconservative force in the overdamped limit.[38]

From a future perspective, it could be interesting to study the escape dynamics of an oppositely charged dimer whose center exhibits strongly enhanced dynamics.[49] We have recently shown that the diffusive dynamics of a charged particle can be enhanced by collisions. It is interesting to investigate how crowding effects can facilitate the escape dynamics in an interacting crowded system of charged Brownian particles.[50] Some of the next features could be to study skyrmions in nonconservative traps or landscapes that could be useful for steering and controlling their motion.[27–31] In the limit of a low persistence length, an active chiral particle follows curved trajectories, similar to the Brownian motion of a charged particle[51,52] under a magnetic field. Therefore, another study of interest would be the escape dynamics of a chiral active Brownian particle in the presence of a vortex. It could be interesting to study how an external magnetic field can affect an active turnover for an active particle in a bistable potential[53]—an optimal correlation time where the transition rate is maximized—and how a vortex influences new turnovers observed in the presence of a fluctuating magnetic field.[22,23] Finally, it is interesting to study the escape dynamics in a sheared, charged colloidal system.[54,55]

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

I. Abdoli: Data curation (lead); Formal analysis (lead); Writing – original draft (lead). **H. Löwen**: Conceptualization (equal); Writing – review & editing (equal). **J.-U. Sommer**: Conceptualization (equal); Writing – review & editing (equal). **A. Sharma**: Formal analysis (equal); Supervision (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX: DERIVATION OF THE MEAN ESCAPE TIME

The main purpose of this Appendix is to derive the mean escape time in Eq. (5) to Eq. (7). We start with the underdamped Langevin equation describing the dynamics of a charged Brownian particle with mass m and charge q subjected to a magnetic field B in the $-\hat{z}$ direction. The velocity Langevin equation for the position $\boldsymbol{r} = (x, y)^\top$ and the velocity $\boldsymbol{v} = (v_x, v_y)^\top$ of the particle under the effect of the linear nonconservative force $\boldsymbol{F}_{nc} = \varepsilon(-y, x)^\top$ and the conservative force $\boldsymbol{F}_c = -k(x, y)^\top$ due to the isotropic potential $U(x, y) = k(x^2 + y^2)/2$ can be written as

$$m\dot{\boldsymbol{v}} = -\boldsymbol{K}\boldsymbol{r} - \boldsymbol{G}\boldsymbol{v}(t) + \sqrt{2\gamma T}\boldsymbol{\eta}(t), \tag{A1}$$

where $\boldsymbol{\eta}(t) = (\eta_x(t), \eta_y(t))^\top$ is the Gaussian white noise with zero mean and Dirac delta correlation $\langle \boldsymbol{\eta}(t)\boldsymbol{\eta}^\top(t')\rangle = \delta(t - t')$ with γ being the friction coefficient and T being the temperature. The matrices $\boldsymbol{G}$ and $\boldsymbol{K}$ are defined as

$$\boldsymbol{G} = \gamma\begin{pmatrix} 1 & \kappa \\ -\kappa & 1 \end{pmatrix}, \quad \boldsymbol{K} = \begin{pmatrix} k & \varepsilon \\ -\varepsilon & k \end{pmatrix}, \tag{A2}$$

with $\kappa = qB/\gamma$ being the diffusive Hall parameter that quantifies the strength of the Lorentz force relative to the frictional force. Using the low-mass approach, the corresponding overdamped Langevin equation can be written as[25,40,41]

$$\dot{\boldsymbol{r}} = \boldsymbol{A}\boldsymbol{r} + \boldsymbol{\xi}(t), \tag{A3}$$

where $\boldsymbol{A} = \boldsymbol{G}^{-1}\boldsymbol{K}$ and $\boldsymbol{\xi}(t) = (\xi_x, \xi_y)^\top$ is the Gaussian nonwhite noise with

$$\langle \boldsymbol{\xi}(t)\rangle = 0, \tag{A4}$$

$$\langle \boldsymbol{\xi}(t)\boldsymbol{\xi}^\top(t')\rangle = T\boldsymbol{G}^{-1}\delta_+(t - t') + T(\boldsymbol{G}^{-1})^\top\delta_-(t - t'), \tag{A5}$$

where $\delta_\pm(s = t - t')$ are the modified Dirac delta functions, which are zero for $s \neq 0$, while $\int_0^\infty \mathrm{d}s\delta_+(s) = \int_{-\infty}^0 \mathrm{d}s\delta_-(s) = 1$ and $\int_0^\infty \mathrm{d}s\delta_-(s) = \int_{-\infty}^0 \mathrm{d}s\delta_+(s) = 0$.

Equation (A3) can be rewritten as Eqs. (1) and (2) in the Cartesian coordinates and, thereafter, using the Itô calculus, can be reduced to a one-dimensional equation for the variable r, which is the distance from the origin and is given by Eq. (3). It has been shown that using the backward Fokker–Planck equation, the mean escape time for a Brownian particle from the initial position x_i to the final position x_f can be written as[42]

$$\langle t\rangle_{x_i \to x_f} = \frac{2}{D}\int_{x_i}^{x_f} \exp\left[\frac{U(y)}{D}\right]\mathrm{d}y\int_{-\infty}^{y} \exp\left[-\frac{U(z)}{D}\right]\mathrm{d}z, \tag{A6}$$

where $D = T/\gamma$ is the diffusion coefficient for a freely moving particle and $U(x)$ is a double well potential. Thus, the mean time for the particle to escape the trap, truncated at $r = a$, can be obtained from Eq. (A6) with the following replacements:

$$U(x) \to \frac{k_{eff}}{2\gamma(1 + \kappa^2)}r^2 - \frac{D}{1 + \kappa^2}\log(r), \tag{A7}$$

$$D \to \frac{2D}{1 + \kappa^2}, \tag{A8}$$

$$x_i \to 0, \tag{A9}$$

$$x_f \to a, \tag{A10}$$

$$-\infty \to 0, \tag{A11}$$

which reads

$$\langle t\rangle = \frac{1 + \kappa^2}{D}\int_0^a y^{-1} \exp\left[\frac{k_{eff}}{2\gamma D}y^2\right]dy\int_0^y z \exp\left[-\frac{k_{eff}}{2\gamma D}z^2\right]dz. \tag{A12}$$

This equation can be exactly solved: using a change of variables, the second integral on the right-hand side gives $(\gamma D/k_{eff})\left[1 - \exp\left(-k_{eff}y^2/2\gamma D\right)\right]$. By substitution of this solution into Eq. (A12), the resulting integral can be exactly solved, which gives Eqs. (5) and (6).

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